Homework 04

Exercise 1

A bivariate Gibbs sampler for a vector $x = (x_1, x_2)$ draws iteratively from the conditional distributions in the following way:

- choose a starting value $p(x_1|x_2^{(0)})$
- for each iteration i:
  - draw $x_2^{(i)}$ from $p(x_2|x_1^{(i-1)})$
  - draw $x_1^{(i)}$ from $p(x_1|x_2^{(i)})$

As in other MCMC algorithms, samples from the first few iterations are usually discarded. This is known as the "warmup" phase.

Suppose that the conditional distributions are

$$\begin{aligned} x_1|x_2 &\sim \mathcal{N}\Big(\mu_1 + \frac{\rho}{\sigma_2^2}(x_2 - \mu_2), \sigma_1^2 - \frac{\rho^2}{\sigma_2^2}\Big) \\ x_2|x_1 &\sim \mathcal{N}\Big(\mu_2 + \frac{\rho}{\sigma_1^2}(x_1 - \mu_1), \sigma_2^2 - \frac{\rho^2}{\sigma_1^2}\Big) \end{aligned},$$

where $\rho, \mu_1, \mu_2, \sigma_1, \sigma_2$ are real valued parameters and $|\rho| < 1$.

1. (**code**) Implement a Gibbs sampler which takes as inputs the number of warmup draws `warmup` , the number of iterations `iters` and the parameters `rho, mu1, mu2, sigma1, sigma2` .
2. (**code**) Set $\rho = 0.3, \mu_1 = 0.5, \mu_2 = 0.1, \sigma_1 = 1, \sigma_2 = 1.5$ and plot the distributions of x_1 and x_2.
3. (**theory**) What is the joint distribution of $x = (x_1, x_2)$?

Exercise 2

(**code**)

Let θ_1 and θ_2 be real valued parameters of the function

$$f(x, \theta_1, \theta_2) = \frac{e^{\theta_1 x} + \theta_2}{x}.$$

We observe some noisy realizations (x_{obs}, y_{obs}) of this function and want to infer the posterior distribution of θ_1 and θ_2.

1. Generate 30 observations (x_{obs}, y_{obs}) as follows:

$$\begin{aligned} \theta_1 &= 0.5 \\ \theta_2 &= 3 \\ \rho &\sim \mathcal{N}(0, 0.3^2) \\ x_{obs} &\sim \mathcal{U}(0, 10) \\ y_{obs} &= f(x_{obs}, \theta_1, \theta_2) + \rho \end{aligned}$$

and plot them using seaborn `sns.scatterplot` function.

2. Given the observations from (1.) and the generative model

$$\begin{aligned} \theta_1 &\sim \text{Exp}(2.) \\ \theta_2 &\sim \text{Exp}(0.2) \\ \gamma &\sim \mathcal{U}(0, 0.5) \\ \hat{y} &= f(x, \theta_1, \theta_2) \\ y &\sim \mathcal{N}(\hat{y}, \gamma) \end{aligned}$$

use pyro `NUTS` to infer the posterior distribution of $\theta_1, \theta_2, \gamma$. Set `warmup_steps=2000` and `num_samples=100` .

3. Look at convergence checks on the traces and discuss the quality of your estimates. Remeber that you can extract posterior samples from an `MCMC` object using `get_samples()` method (notebook 04).

4. Use `sns.scatterplot` to plot the observed couples (x_{obs}, y_{obs}) and `sns.lineplot` to plot the learned function f with the posterior estimates θ_1, θ_2.