Optimal Rates for Zero-Order Convex Optimization: The Power of Two Function Evaluations

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Abstract—We consider derivative-free algorithms for stochastic and nonstochastic convex optimization problems that use only function values rather than gradients. Focusing on nonasymptotic bounds on convergence rates, we show that if pairs of function values are available, algorithms for d-dimensional optimization that use gradient estimates based on random perturbations suffer a factor of at most $\sqrt{d}$ in convergence rate over traditional stochastic gradient methods. We establish such results for both smooth and nonsmooth cases, sharpening previous analyses that suggested a worse dimension dependence, and extend our results to the case of multiple ($m \geq 2$) evaluations. We complement our algorithmic development with information-theoretic lower bounds on the minimax convergence rate of such problems, establishing the sharpness of our achievable results up to constant (sometimes logarithmic) factors.

Index Terms—Optimization, zero-order optimization, on-line learning, statistical learning, lower bounds.

I. INTRODUCTION

DERIVATIVE-FREE optimization schemes have a long history in optimization; for instance, see the book by Spall [1] for an overview. Such procedures are desirable in settings in which explicit gradient calculations may be computationally infeasible, expensive, or impossible. Classical techniques in stochastic and non-stochastic optimization, including Kiefer-Wolfowitz-type procedures (see [2]), use function difference information to approximate gradients of the function to be minimized rather than calculating gradients. There has recently been renewed interest in optimization problems with only functional (zero-order) information available—rather than first-order gradient information—in optimization, machine learning, and statistics [3]–[7].

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In machine learning and statistics, this interest has centered around bandit optimization [3], [4], [8], where a player and adversary compete, with the player choosing points θ in some domain Θ and an adversary choosing a point x, forcing the player to suffer a loss $F(\theta; x)$. The goal is to choose an optimal point $\theta \in \Theta$ based only on observations of function values $F(\theta; x)$. Applications of such bandit problems include online auctions and advertisement selection for search engines. Similarly, the field of simulation-based optimization provides many examples of problems in which optimization is performed based only on function values [1], [5], [9]. Additionally, in many problems in statistics—including graphical model inference [10] and structured-prediction [11]—the objective is defined variationally (as the maximum of a family of functions), so explicit differentiation may be difficult.

Despite the long history and recent renewed interest in such procedures, a precise understanding of their convergence behavior remains elusive. In this paper, we study algorithms for solving stochastic convex optimization problems of the form

$$\underset{\theta \in \Theta}{\text{minimize}} \; f(\theta) := \mathbb{E}_P[F(\theta; X)] = \int_{\mathcal{X}} F(\theta; x) dP(x), \qquad (1)$$

where $\Theta \subseteq \mathbb{R}^d$ is a compact convex set, P is a distribution over the space $\mathcal{X}$, and for P-almost every $x \in \mathcal{X}$, the function $F(\cdot; x)$ is closed and convex. We focus on the convergence rates of algorithms observing only stochastic realizations of the function values $f(\theta)$, though our algorithms naturally apply in the non-stochastic case as well.

One related body of work focuses on problems where, for a given value $x \in \mathcal{X}$ (or sample $X \sim P$), it is only possible to observe $F(\theta; x)$ at a single location θ. Nemirovski and Yudin [12, Ch. 9.3] develop a randomized sampling strategy that estimates the gradient $\nabla F(\theta; x)$ via randomized evaluations of function values at points θ on the surface of an ℓ_2-sphere. Flaxman et al. [3] build on this approach and establish some implications for bandit convex optimization problems. The convergence rates given in these early papers are sub-optimal, as more recent work shows [7]. For instance, Agarwal et al. [7] provide algorithms that achieve convergence rates after k iterations of $\mathcal{O}(d^{16}/\sqrt{k})$; however, as the authors themselves note, the algorithms are quite complicated. Jamieson et al. [13] present simpler comparison-based algorithms for solving a subclass of such problems, and Shamir [14] gives optimal algorithms for quadratic objectives, as well as providing some lower bounds on optimization error when only single function values are available.

Some of the difficulties inherent in optimization using only a single function evaluation are alleviated when the function $F(\cdot; x)$ can be evaluated at *two* points, as noted independently by Agarwal et al. [4] and Nesterov [5]. Such multi-point settings prove useful for optimization problems in which observations X are available, yet we only have black-box access to objective values $F(\theta; X)$; examples of such problems include simulation-based optimization [5], [9] and variational approaches to graphical models and classification [10], [11]. The essential insight underlying multi-point schemes is as follows: for small non-zero scalar u and a vector $Z \in \mathbb{R}^d$, the quantity $(F(\theta+uZ; x) - F(\theta; x))/u$ approximates a directional derivative of $F(\theta; x)$ in the direction Z that first-order schemes may exploit. Relative to schemes based on only a single function evaluation at each iteration, such two-sample-based gradient estimators exhibit faster convergence rates [4]–[6]. In the current paper, we take this line of work further, in particular by characterizing *optimal* rates of convergence over all procedures based on multiple noisy function evaluations. Moreover, adopting the two-point perspective, we present simple randomization-based algorithms that achieve these optimal rates.

More formally, we study algorithms that receive a vector of paired observations, $Y(\theta, \tau) \in \mathbb{R}^2$, where θ and τ are points selected by the algorithm. The tth observation takes the form

$$Y^t(\theta^t, \tau^t) := \begin{bmatrix} F(\theta^t; X^t) \\ F(\tau^t; X^t) \end{bmatrix}, \qquad (2)$$

where X^t is an independent sample drawn from the distribution P. After k iterations, the algorithm returns a vector $\widehat{\theta}(k) \in \Theta$. In this setting, we analyze stochastic gradient and mirror-descent procedures [12], [15]–[17] that construct gradient estimators using the two-point observations Y^t (as well as the natural extension to $m \geq 2$ observations). By a careful analysis of the dimension dependence of certain random perturbation schemes, we show that the convergence rate attained by our stochastic gradient methods is roughly a factor of $\sqrt{d}$ worse than that attained by stochastic methods that observe the full gradient $\nabla F(\theta; X)$. Under appropriate conditions, our convergence rates are a factor of $\sqrt{d}$ better than those attained in past work [4], [5]. For smooth problems, Ghadimi and Lan [6] provide results sharper than those in the papers [4], [5], but do not show optimality of their methods nor consider non-Euclidean problems. In addition, although we present our results in the framework of stochastic optimization, our analysis also applies to (multi-point) bandit online convex optimization problems [3], [4], [8], where our results are the sharpest provided to date. Our algorithms apply in both smooth and non-smooth cases as well as to non-stochastic problems [5], [12], where our procedures give the fastest known convergence guarantees for the non-smooth case. Finally, by using information-theoretic techniques for proving lower bounds in statistical estimation, we establish that our explicit achievable rates are sharp up to constant factors or (in some cases) factors at most logarithmic in the dimension.

The remainder of this paper is organized as follows: in the next section, we present our multi-point gradient estimators and their convergence rates, providing results in Section II-A and II-B for smooth and non-smooth objectives F, respectively. In Section III, we provide information-theoretic minimax lower bounds on the best possible convergence rates, uniformly over all schemes based on function evaluations. We devote Sections IV and Section V to proofs of the achievable convergence rates and the lower bounds, respectively, deferring more technical arguments to appendices.

Notation: For sequences indexed by d, the inequality $a_d \lesssim b_d$ indicates that there is a universal numerical constant c such that $a_d \leq c \cdot b_d$. For a convex function $f : \mathbb{R}^d \to \mathbb{R}$, we let

$$\partial f(\theta) := \{g \in \mathbb{R}^d \mid f(\tau) \geq f(\theta) + \langle g, \tau - \theta \rangle \quad \forall \ \tau \in \mathbb{R}^d\}$$

denote the subgradient set of f at θ. We say a function f is λ-strongly convex with respect to the norm $\|\cdot\|$ if for all θ, $\tau \in \mathbb{R}^d$, we have $f(\tau) \geq f(\theta) + \langle g, \tau - \theta \rangle + (\lambda/2) \|\theta - \tau\|^2$ for all $g \in \partial f(\theta)$. Given a norm $\|\cdot\|$, we denote its dual norm by $\|\cdot\|_*$. We let $\mathsf{N}(0, I_{d \times d})$ denote the standard normal distribution on $\mathbb{R}^d$. We denote the ℓ_2-ball in $\mathbb{R}^d$ with radius r centered at v by $\mathbb{B}^d(v, r)$, and $\mathbb{S}^{d-1}(v, r)$ denotes the $(d-1)$-dimensional ℓ_2-sphere in $\mathbb{R}^d$ with radius r centered at v. We also use the shorthands $\mathbb{B}^d = \mathbb{B}^d(0, 1)$ and $\mathbb{S}^{d-1} = \mathbb{S}^{d-1}(0, 1)$, and $\mathbb{1}$ for the all-ones vector.

II. ALGORITHMS

We begin by providing some background on the class of stochastic mirror descent methods for solving the problem $\min_{\theta \in \Theta} f(\theta)$. They are based on a *proximal function* ψ, meaning a differentiable and strongly convex function defined over Θ. The proximal function defines a Bregman divergence $D_\psi : \Theta \times \Theta \to \mathbb{R}_+$ via

$$D_\psi(\theta, \tau) := \psi(\theta) - \psi(\tau) - \langle \nabla \psi(\tau), \theta - \tau \rangle.$$

The mirror descent (MD) method generates a sequence of iterates $\{\theta^t\}_{t=1}^\infty$ contained in Θ, using stochastic gradient information to perform the update from iterate to iterate. The algorithm is initialized at some point $\theta^1 \in \Theta$. At iterations $t = 1, 2, 3, \ldots$, the MD method receives a (subgradient) vector $g^t \in \mathbb{R}^d$, which it uses to compute the next iterate via

$$\theta^{t+1} = \arg\min_{\theta \in \Theta} \left\{ \langle g^t, \theta \rangle + \frac{1}{\alpha(t)} D_\psi(\theta, \theta^t) \right\}, \qquad (3)$$

where $\{\alpha(t)\}_{t=1}^\infty$ is a non-increasing sequence of positive stepsizes.

Throughout the paper, we impose two assumptions that are standard in analysis of mirror descent methods [12], [16], [17]. Letting θ^* denote a minimizer of the problem (1), the first assumption concerns properties of the proximal function ψ and the optimizaton domain Θ.

Assumption A: The proximal function ψ is 1-strongly convex with respect to the norm $\|\cdot\|$. There exists $R < \infty$ such that $D_\psi(\theta^*, \theta) \leq \frac{1}{2} R^2$ for $\theta \in \Theta$, and the domain Θ is compact.

Our second assumption is standard for almost all first-order stochastic gradient methods [5], [17], [18], and it holds whenever the functions $F(\cdot; x)$ are G-Lipschitz with respect to

the norm $\|\cdot\|$. We use $\|\cdot\|_*$ to denote the dual norm to $\|\cdot\|$, and let $g : \Theta \times \mathcal{X} \to \mathbb{R}^d$ denote a measurable subgradient selection for the functions F; that is, $g(\theta; x) \in \partial F(\theta; x)$ with $\mathbb{E}[g(\theta; X)] \in \partial f(\theta)$.

Assumption B: There is a constant $G < \infty$ such that the (sub)gradient selection g satisfies $\mathbb{E}[\|g(\theta; X)\|_*^2] \le G^2$ for $\theta \in \Theta$.

When Assumptions A and B hold, the convergence rates of stochastic mirror descent methods are well understood. In detail, suppose that the variables $X^t \in \mathcal{X}$ are sampled i.i.d. according to P. With the assignment $g^t = g(\theta^t; X^t)$, let the sequence $\{\theta^t\}_{t=1}^\infty$ be generated by the mirror descent iteration (3). Then for a stepsize $\alpha(t) = \alpha/\sqrt{t}$, the average $\widehat{\theta}(k) = \frac{1}{k}\sum_{t=1}^k \theta^t$ satisfies

$$\mathbb{E}[f(\widehat{\theta}(k))] - f(\theta^*) \le \frac{1}{2\alpha\sqrt{k}}R^2 + \frac{\alpha}{\sqrt{k}}G^2. \tag{4}$$

We refer to the papers [16], [17, Sec. 2.3] for results of this type.

For the remainder of this section, we explore the use of function difference information to obtain subgradient estimates that can be used in mirror descent methods to achieve statements similar to the convergence guarantee (4). We begin by analyzing the smooth case—when the instantaneous functions $F(\cdot; x)$ have Lipschitz gradients—and proceed to the more general (non-smooth) case in the subsequent section.

A. Two-Point Gradient Estimates and Convergence Rates: Smooth Case

Our first step is to show how to use two function values to construct nearly unbiased estimators of the gradient of the objective function f under a smoothness condition. Using analytic methods different from those from past work [4], [5], we are able to obtain optimal dependence with the problem dimension d. In more detail, our procedure is based on a non-increasing sequence of positive smoothing parameters $\{u_t\}_{t=1}^\infty$ and a distribution μ on $\mathbb{R}^d$, to be specified, satisfying $\mathbb{E}_\mu[ZZ^\top] = I$. Given a smoothing constant u, vector z, and observation x, we define the directional gradient estimate at the point θ as

$$\mathsf{G}_{\mathrm{sm}}(\theta; u, z, x) := \frac{F(\theta + uz; x) - F(\theta; x)}{u}z. \tag{5}$$

Using the estimator (5), we then perform the following two steps. First, upon receiving the point $X^t \in \mathcal{X}$, we sample an independent vector Z^t from μ and set

$$\begin{aligned} g^t &= \mathsf{G}_{\mathrm{sm}}(\theta^t; u_t, Z^t, X^t) \\ &= \frac{F(\theta^t + u_t Z^t; X^t) - F(\theta^t; X^t)}{u_t}Z^t. \end{aligned} \tag{6}$$

In the second step, we apply the mirror descent update (3) to the quantity g^t to obtain the next parameter θ^{t+1}.

Intuition for the estimator (5) follows by considering directional derivatives. The directional derivative $f'(\theta, z)$ of the function f at the point θ in the direction z is

$$f'(\theta, z) := \lim_{u\downarrow 0}\frac{1}{u}(f(\theta + uz) - f(\theta)).$$

This limit always exists when f is convex [19, Ch. VI], and if f is differentiable at θ, then $f'(\theta, z) = \langle \nabla f(\theta), z \rangle$. With this background, the estimate (5) is motivated by the following fact [5, eq. (32)]: whenever $\nabla f(\theta)$ exists, we have

$$\mathbb{E}[f'(\theta, Z)Z] = \mathbb{E}[\langle \nabla f(\theta), Z \rangle Z] = \nabla f(\theta),$$

where the final equality uses our unbiasedness assumption that $\mathbb{E}[ZZ^\top] = I$. Consequently, given sufficiently small choices of u_t, the vector (6) should be a nearly unbiased estimator of the gradient $\nabla f(\theta^t)$.

In addition to the condition $\mathbb{E}_\mu[ZZ^\top] = I$, we require a few additional assumptions on μ. The first ensures that the estimator g^t is well-defined.

Assumption C: The domain of the functions F and support of μ satisfy

$$\mathrm{dom}\, F(\cdot; x) \supset \Theta + u_1 \operatorname{supp}\mu \quad \text{for } x \in \mathcal{X}. \tag{7}$$

If we apply smoothing with Gaussian perturbation, the containment (7) implies $\mathrm{dom}\, F(\cdot; x) = \mathbb{R}^d$, though we still optimize over the compact set Θ in the update (3). We remark in passing that if the condition (7) fails, it is possible to optimize instead over the smaller domain $(1-\epsilon)\Theta$, assuming w.l.o.g. that Θ has non-empty interior, so long as μ has compact support (cf. Agarwal et al. [4, Algorithm 2]). We also impose the following properties on the smoothing distribution:

Assumption D: For $Z \sim \mu$, the quantity $M(\mu) := \sqrt{\mathbb{E}[\|Z\|^4 \|Z\|_*^2]}$ is finite, and moreover, there is a function $s : \mathbb{N} \to \mathbb{R}_+$ such that

$$\mathbb{E}[\|\langle g, Z \rangle Z\|_*^2] \le s(d)\|g\|_*^2 \quad \text{for any } g \in \mathbb{R}^d. \tag{8}$$

Although the quantity $M(\mu)$ is required to be finite, its value does not appear explicitly in our theorem statements. On the other hand, the dimension-dependent quantity $s(d)$ from condition (8) appears explicitly in our convergence rates. As an example of these two quantities, suppose that we take μ to be the distribution of the standard normal $\mathsf{N}(0, I_{d\times d})$, and use the ℓ_2-norm $\|\cdot\| = \|\cdot\|_2$. In this case, a straightfoward calculation shows that $M(\mu)^2 \lesssim d^3$ and $s(d) \lesssim d$.

Finally, as previously stated, the analysis of this section requires a smoothness assumption:

Assumption E: There is a function $L : \mathcal{X} \to \mathbb{R}_+$ such that for P-almost every $x \in \mathcal{X}$, the function $F(\cdot; x)$ has $L(x)$-Lipschitz continuous gradient with respect to the norm $\|\cdot\|$, and the quantity $L(P) := \sqrt{\mathbb{E}[(L(X))^2]}$ is finite.

Essential to stochastic gradient procedures—recall Assumption B and the result (4)—is that the gradient estimator g^t be nearly unbiased and have small norm. Accordingly, the following lemma provides quantitative guarantees on the error associated with the gradient estimator (5).

Lemma 1: Under Assumptions D and E, the gradient estimate (5) has expectation

$$\mathbb{E}[\mathsf{G}_{\mathrm{sm}}(\theta; u, Z, X)] = \nabla f(\theta) + uL(P)v(\theta, u) \tag{9}$$

for $v = v(\theta, u) \in \mathbb{R}^d$ such that $\|v\|_* \le \frac{1}{2}\mathbb{E}[\|Z\|^2 \|Z\|_*]$. Its expected squared norm has the bound

$$\mathbb{E}[\|\mathsf{G}_{\mathrm{sm}}(\theta; u, Z, X)\|_*^2]$$
$$\le 2s(d)\mathbb{E}\left[\|g(\theta; X)\|_*^2\right] + \frac{1}{2}u^2 L(P)^2 M(\mu)^2. \tag{10}$$

See Section IV-B for the proof. The bound (9) shows that the estimator g^t is unbiased for the gradient up to a correction term of order u_t, while the second inequality (10) shows that the second moment is—up to an order u_t^2 correction—within a factor $s(d)$ of the standard second moment $\mathbb{E}[\|g(\theta; X)\|_*^2]$. We note in passing that the parameter u in the lemma can be taken arbitrarily close to 0, which only makes G_{sm} a better estimate of g. The intuition is straightforward: with two points, we can obtain arbitrarily accurate estimates of the directional derivative.

Our main result in this section is the following theorem on the convergence rate of the mirror descent method using the gradient estimator (6).

Theorem 1: Under Assumptions A, B, C, D, and E, consider a sequence $\{\theta^t\}$ generated according to the mirror descent update (3) *using the gradient estimator* (6), *with step and perturbation sizes*

$$\alpha(t) = \alpha \frac{R}{2G\sqrt{s(d)}\sqrt{t}} \quad \text{and} \quad u_t = u \frac{G\sqrt{s(d)}}{L(P)M(\mu)} \cdot \frac{1}{t}$$

for $t = 1, 2, \ldots$. Then for all k,

$$\mathbb{E}\left[f(\widehat{\theta}(k)) - f(\theta^*)\right] \leq 2\frac{RG\sqrt{s(d)}}{\sqrt{k}} \max\left\{\alpha, \alpha^{-1}\right\}$$
$$+ \alpha u^2 \frac{RG\sqrt{s(d)}}{k} + u \frac{RG\sqrt{s(d)}\log(2k)}{k}, \quad (11)$$

where $\widehat{\theta}(k) = \frac{1}{k}\sum_{t=1}^{k}\theta^t$, and the expectation is taken with respect to the samples X and Z.

The proof of Theorem 1 builds on convergence proofs developed in the analysis of online and stochastic convex optimization [4], [5], [15], [17], but requires additional technical care, since we never truly receive unbiased gradients. We provide the proof in Section IV-A.

Before continuing, we make a few remarks. First, the method is reasonably robust to the selection of the step-size multiplier α; Nemirovski et al. [17] previously noted this robustness for gradient-based MD methods. As long as $\alpha(t) \propto 1/\sqrt{t}$, mis-specifying the multiplier α results in a scaling at worst linear in $\max\{\alpha, \alpha^{-1}\}$. We may also use multiple independent random samples $Z^{t,i}$, $i = 1, 2, \ldots, m$, in the construction of the gradient estimator (6) to obtain more accurate estimates of the gradient via $g^t = \frac{1}{m}\sum_{i=1}^{m}\mathsf{G}_{sm}(\theta^t; u_t, Z^{t,i}, X^t)$. See Corollary 2 to follow for an example of this construction. In addition, the convergence rate of the method is independent of the Lipschitz continuity constant $L(P)$ of the instantaneous gradients $\nabla F(\cdot; X)$, because, as noted following Lemma 1, we may take u arbitrarily close to 0. This suggests that similar results may hold for non-differentiable functions; indeed, as we show in the next section, a slightly more complicated construction of the estimator g^t leads to analogous guarantees for general non-smooth functions.

Although we have provided bounds on the expected convergence rate, it is possible to give high-probability convergence guarantees (see [17], [20]) under additional tail conditions on g—for example, under the boundedness condition $\|g(\theta; X)\|_* \leq G$—though obtaining sharp dimension-dependence requires care. Additionally, while we have presented our results as convergence guarantees for stochastic optimization problems, an inspection of our analysis in Section IV-A shows that we also obtain (expected) regret bounds for bandit online convex optimization problems (see [3], [4], [8]).

1) Examples and Corollaries: We now provide examples of sampling strategies giving bounds for the mirror descent algorithm based on the subgradient estimator (6). For each corollary, we specify the norm $\|\cdot\|$, proximal function ψ, and distribution μ. We then compute the values that the distribution μ implies in Assumption E and apply Theorem 1 to obtain a convergence rate.

We begin with a corollary that characterizes the convergence rate of our algorithm with the proximal function $\psi(\theta) := \frac{1}{2}\|\theta\|_2^2$ under a Lipschitz condition:

Corollary 1: Given domain $\Theta \subseteq \{\theta \in \mathbb{R}^d \mid \|\theta\|_2 \leq R\}$, suppose that μ is uniform on the surface of the ℓ_2-ball of radius $\sqrt{d}$, and that $\mathbb{E}[\|g(\theta; X)\|_2^2] \leq G^2$. Then

$$\mathbb{E}\left[f(\widehat{\theta}(k)) - f(\theta^*)\right] \leq 2\frac{RG\sqrt{d}}{\sqrt{k}} \max\{\alpha, \alpha^{-1}\}$$
$$+ \alpha u^2 \frac{RG\sqrt{d}}{k} + u \frac{RG\sqrt{d}\log(2k)}{k}.$$

Proof: By the assumption that $\mathbb{E}[ZZ^\top] = I$, we see that for any $g \in \mathbb{R}^d$ we have

$$\mathbb{E}[\|\langle g, Z\rangle Z\|_2^2] = d\mathbb{E}[\langle g, Z\rangle^2] = d\mathbb{E}[g^\top ZZ^\top g],$$

and as $\|Z\|_2 = \sqrt{d}$, we have $M(\mu) = \sqrt{\mathbb{E}[\|Z\|_2^6]} = d^{3/2}$. Thus Assumption D holds with $s(d) = d$, and the claim follows from Theorem 1. $\qquad\square$

The rate Corollary 1 provides is the fastest derived to date for zero-order stochastic optimization using two function evaluations; both Agarwal et al. [4] and Nesterov [5] achieve rates of convergence of order $RGd/\sqrt{k}$. In concurrent work, Ghadimi and Lan [6] provide a result (their Corollary 3.3) that achieves a similar rate to that above, but their primary focus is on non-convex problems. Moreover, we show in the sequel that this convergence rate is actually optimal.

Using multiple function evaluations yields faster convergence rates, as we obtain more accurate estimates of the instantaneous gradients $g(\theta; X)$. The following extension of Corollary 1 illustrates this effect:

Corollary 2: In addition to the conditions of Corollary 1, let $Z^{t,i}$, $i = 1, \ldots, m$ be sampled independently according to μ, and at each iteration of mirror descent use the gradient estimate $g^t = \frac{1}{m}\sum_{i=1}^{m}\mathsf{G}_{sm}(\theta^t; u_t, Z^{t,i}, X^t)$ with the step and perturbation sizes

$$\alpha(t) = \frac{\alpha R}{2G\max\{\sqrt{\frac{d}{m}}, 1\}} \cdot \frac{1}{\sqrt{t}} \quad \text{and} \quad u_t = \frac{uG}{L(P)d^{3/2}} \cdot \frac{1}{t}.$$

There exists a universal constant $C \leq 5$ such that for all k,

$$\mathbb{E}\left[f(\widehat{\theta}(k)) - f(\theta^*)\right]$$
$$\leq C\frac{RG\sqrt{1+\frac{d}{m}}}{\sqrt{k}}\left[\max\{\alpha, \alpha^{-1}\} + \frac{\alpha u^2}{\sqrt{k}} + u\frac{\log(2k)}{k}\right].$$

Corollary 2 shows the intuitive result that, with a number of evaluations linear in the dimension d, it is possible to attain the standard (full-information) convergence rate $RG/\sqrt{k}$ (see [21]) using only function evaluations; we are (essentially) able to estimate the gradient $g(\theta; X)$. We provide a proof of Corollary 2 in Section IV-C.

In high-dimensional scenarios, appropriate choices for the proximal function ψ yield better scaling on the norm of the gradients [12], [17], [22]. In the setting of online learning or stochastic optimization, suppose that one observes gradients $g(\theta; X)$. If the domain Θ is the simplex, then exponentiated gradient algorithms [16], [23] using the proximal function $\psi(\theta) = \sum_j \theta_j \log \theta_j$ obtain rates of convergence dependent on the ℓ_∞-norm of the gradients $\|g(\theta; X)\|_\infty$. This scaling is more palatable than bounds that depend on Euclidean norms applied to the gradient vectors, which may be a factor of $\sqrt{d}$ larger. Similar results apply using proximal functions based on ℓ_p-norms [16], [24]. In our case, if we make the choice $p = 1 + \frac{1}{\log(2d)}$ and $\psi(\theta) = \frac{1}{2(p-1)} \|\theta\|_p^2$, we obtain the following corollary, which holds under the conditions of Theorem 1.

Corollary 3: Suppose that $\mathbb{E}[\|g(\theta; X)\|_\infty^2] \leq G^2$, the optimization domain Θ is contained in the ℓ_1-ball $\{\theta \in \mathbb{R}^d \mid \|\theta\|_1 \leq R\}$, and μ is uniform on the hypercube $\{-1, 1\}^d$. There is a universal constant $C \leq 2 \exp(1)$ such that

$$\mathbb{E}\left[f(\widehat{\theta}(k)) - f(\theta^*) \right] \leq C \frac{RG\sqrt{d \log(2d)}}{\sqrt{k}} \max\left\{ \alpha, \alpha^{-1} \right\}$$
$$+ C \frac{RG\sqrt{d \log(2d)}}{k} \left(\alpha u^2 + u \log k \right).$$

Proof: The chosen proximal function ψ is strongly convex with respect to the norm $\|\cdot\|_p$ (see [12, Appendix 1]). In addition, the choice $q = 1 + \log(2d)$ implies $1/p + 1/q = 1$, and $\|v\|_q \leq \exp(1) \|v\|_\infty$ for any $v \in \mathbb{R}^d$. Consequently, we have $\mathbb{E}[\|\langle g, Z \rangle Z\|_q^2] \leq \exp(2)\mathbb{E}[\|\langle g, Z \rangle Z\|_\infty^2]$, which allows us to apply Theorem 1 with the norm $\|\cdot\| = \|\cdot\|_1$ and the dual norm $\|\cdot\|_* = \|\cdot\|_\infty$.

We claim that Assumption D is satisfied with $s(d) \leq d$. Since $Z \sim \text{Uniform}(\{-1, 1\}^d)$, for $g \in \mathbb{R}^d$ we have

$$\mathbb{E}\left[\|\langle g, Z \rangle Z\|_\infty^2 \right] = \mathbb{E}\left[\langle g, Z \rangle^2 \right] = \|g\|_2^2 \leq d \|g\|_\infty^2.$$

Finally, we have $M(\mu) = \sqrt{\mathbb{E}[\|Z\|_1^4 \|Z\|_\infty^2]} = d^2$, which is finite as needed. By the inclusion of Θ in the ℓ_1-ball of radius R and our choice of proximal function, we have

$$(p - 1)D_\psi(\theta, \tau) \leq \frac{1}{2} \|\theta\|_p^2 + \frac{1}{2} \|\tau\|_p^2 + \|\theta\|_p \|\tau\|_p.$$

(For instance, see Lemma 3 in the paper [22].) We thus find that $D_\psi(\theta, \tau) \leq 2R^2 \log(2d)$ for any $\theta, \tau \in \Theta$, and using the step and perturbation size choices of Theorem 1 gives the result. $\square$

Corollary 3 attains a convergence rate that scales with dimension as $\sqrt{d \log d}$, which is a much worse dependence on dimension than that of (stochastic) mirror descent using full gradient information [12], [17]. As in Corollaries 1 and 2, which have similar additional $\sqrt{d}$ factors, the additional dependence on d suggests that while $\mathcal{O}(1/\epsilon^2)$ iterations are

required to achieve ϵ-optimization accuracy for mirror descent methods, the two-point method requires $\mathcal{O}(d/\epsilon^2)$ iterations to obtain the same accuracy. In Section III we show that this dependence is sharp: apart from logarithmic factors, no algorithm can attain better convergence rates, including the problem-dependent constants R and G.

B. Two-Point Gradient Estimates and Convergence Rates: General Case

We now turn to the general setting in which the function $F(\cdot; x)$, rather than having a Lipschitz continuous gradient, satisfies only the milder condition of Lipschitz continuity. The difficulty in this non-smooth case is that the simple gradient estimator (6) may have overly large norm. For instance, a naive calculation using only the G-Lipschitz continuity of the function f gives the bound

$$\mathbb{E}\left[\|(f(\theta + uZ) - f(\theta))Z/u\|_2^2 \right]$$
$$\leq G^2 \mathbb{E}\left[\|u \|Z\|_2 Z/u\|_2^2 \right] = G^2 \mathbb{E}[\|Z\|_2^4]. \quad (12)$$

This upper bound always scales at least quadratically in the dimension, since we have the lower bound $\mathbb{E}[\|Z\|_2^4] \geq (\mathbb{E}[\|Z\|_2^2])^2 = d^2$, where the final equality uses the assumption $\mathbb{E}[ZZ^\top] = I_{d \times d}$. This quadratic dependence on dimension leads to a sub-optimal convergence rate. Moreover, this scaling appears to be unavoidable using a single perturbing random vector: taking $f(\theta) = G \|\theta\|_2$ and setting $\theta = 0$ shows that the bound (12) may hold with equality.

Nevertheless, the convergence rate in Theorem 1 shows that *near* non-smoothness is effectively the same as being smooth. This suggests that if we can smooth the objective f slightly, we may achieve a rate of convergence even in the non-smooth case that is roughly the same as that in Theorem 1. The idea of smoothing the objective has been used to obtain faster convergence rates in both deterministic and stochastic optimization [25], [26]. In the stochastic setting, Duchi et al. [26] leverage the well-known fact that convolution is a smoothing operation, and they consider minimization of a sequence of smoothed functions

$$f_u(\theta) := \mathbb{E}[f(\theta + uZ)] = \int f(\theta + uz)d\mu(z), \quad (13)$$

where $Z \in \mathbb{R}^d$ has density with respect to Lebesgue measure. In this case, f_u is always differentiable; moreover, if f is Lipschitz, then ∇f_u is Lipschitz under mild conditions.

The smoothed function (13) leads us to a *two-point* strategy: we use a random direction as in the smooth case (6) to estimate the gradient, but we introduce an extra step of randomization for the point at which we evaluate the function difference. Similar double-randomization schemes were proposed by Gupal [27] (see also the book of Polyak [28, Ch. 5.6.2]), though understanding of their convergence behavior remained limited. Roughly speaking, this randomness has the effect of making it unlikely that the perturbation vector Z is near a point of non-smoothness, which allows us to apply results similar to those in the smooth case.

More precisely, our construction uses two non-increasing sequences of positive parameters $\{u_{1,t}\}_{t=1}^\infty$ and $\{u_{2,t}\}_{t=1}^\infty$ with

$u_{2,t} \leq u_{1,t}/2$, and two smoothing distributions μ_1, μ_2 on $\mathbb{R}^d$. Given smoothing constants u_1, u_2, vectors z_1, z_2, and observation x, we define the (non-smooth) gradient estimate at the point θ as

$$
\begin{aligned}
& \mathsf{G}_{\mathrm{ns}}(\theta; u_1, u_2, z_1, z_2, x) \\
& \quad := \frac{F(\theta + u_1 z_1 + u_2 z_2; x) - F(\theta + u_1 z_1; x)}{u_2} z_2. \quad (14)
\end{aligned}
$$

Using G_{ns} we may define our gradient estimator, which follows the same intuition as our construction of the stochastic gradient (6) from the smooth estimator (5). Now, upon receiving the point X^t, we sample independent vectors $Z_1^t \sim \mu_1$ and $Z_2^t \sim \mu_2$, and set

$$
g^t = \mathsf{G}_{\mathrm{ns}}(\theta^t; u_{1,t}, u_{2,t}, Z_1^t, Z_2^t, X^t). \quad (15)
$$

We then proceed as in the preceding section, using this estimator in the mirror descent method.

To demonstrate the convergence of gradient-based schemes with gradient estimator (15), we require a few additional assumptions. For simplicity, in this section we focus on results for the Euclidean norm $\|\cdot\|_2$. We impose the following condition on the Lipschitzian properties of $F(\cdot; x)$, which is a slight strengthening of Assumption B.

Assumption B′: There is a function $G: \mathcal{X} \to \mathbb{R}_+$ such that for P-a.e. $x \in \mathcal{X}$, the function $F(\cdot; x)$ is $G(x)$-Lipschitz with respect to the ℓ_2-norm $\|\cdot\|_2$, and the quantity $G(P) := \sqrt{\mathbb{E}[G(X)^2]}$ is finite.

We also impose the following assumption on the smoothing distributions μ_1 and μ_2.

Assumption F: The smoothing distributions are one of the following pairs: (1) both μ_1 and μ_2 are standard normal in $\mathbb{R}^d$ with identity covariance, (2) both μ_1 and μ_2 are uniform on the ℓ_2-ball of radius $\sqrt{d+2}$, or (3) the distribution μ_1 is uniform on the ℓ_2-ball of radius $\sqrt{d+2}$ and the distribution μ_2 is uniform on the ℓ_2-sphere of radius $\sqrt{d}$. Additionally, we assume the containment that for $x \in \mathcal{X}$,

$$
\mathrm{dom}\, F(\cdot; x) \supset \Theta + u_{1,1} \operatorname{supp} \mu_1 + u_{2,1} \operatorname{supp} \mu_2.
$$

We then have the following analog of Lemma 1, whose proof we provide in Section IV-E:

Lemma 2: Under Assumptions B′ and F, the gradient estimator (14) has expectation

$$
\begin{aligned}
& \mathbb{E}[\mathsf{G}_{\mathrm{ns}}(\theta; u_1, u_2, Z_1, Z_2, X)] \\
& \qquad = \nabla f_{u_1}(\theta) + \frac{u_2}{u_1} G(P) \upsilon(\theta, u_1, u_2), \quad (16)
\end{aligned}
$$

where $\upsilon = \upsilon(\theta, u_1, u_2)$ has bound $\|\upsilon\|_2 \leq \frac{1}{2}\mathbb{E}[\|Z_2\|_2^3]$. There exists a universal constant c such that

$$
\begin{aligned}
& \mathbb{E}\left[\|\mathsf{G}_{\mathrm{ns}}(\theta; u_1, u_2, Z_1, Z_2, X)\|_2^2\right] \\
& \qquad \leq c\, G(P)^2 d \left(\sqrt{\frac{u_2}{u_1}}\, d + 1 + \log d\right). \quad (17)
\end{aligned}
$$

Comparing Lemma 2 to Lemma 1, both show that one can obtain nearly unbiased gradient of the function f using two function evaluations, but additionally, they show that the squared norm of the gradient estimator is *at most* d times larger than the expected norm of the subgradients $\partial F(\theta; x)$, as captured by the quantity G^2 from Assumption B or B′.

In our approach, non-smoothness introduces an additional logarithmic penalty in the dimension; it may be possible to remove this factor, but we do not know how at this time. The key is that taking the second smoothing parameter u_2 to be small enough means that, aside from the dimension penalty, the gradient estimator g^t is essentially unbiased for $\nabla f_{u_{1,t}}(\theta^t)$ and has squared norm at most $G^2 d \log d$. This bound on size is essential for our main result, which we now state.

Theorem 2: Under Assumptions A, B′, and F, consider a sequence $\{\theta^t\}_{t=1}^\infty$ generated according to the mirror descent update (3) using the gradient estimator (15) with step and perturbation sizes

$$
\alpha(t) = \frac{\alpha R}{G(P)\sqrt{d \log(2d)}\sqrt{t}}, \quad u_{1,t} = \frac{uR}{t}, \quad u_{2,t} = \frac{uR}{d^2 t^2}.
$$

Then there exists a universal (numerical) constant c such that for all k,

$$
\begin{aligned}
& \mathbb{E}\left[f(\widehat{\theta}(k)) - f(\theta^*)\right] \\
& \quad \leq c \frac{RG(P)\sqrt{d}}{\sqrt{k}} \left[\max\{\alpha, \alpha^{-1}\}\sqrt{\log(2d)} + u\frac{\log(2k)}{\sqrt{k}}\right],
\end{aligned}
$$
(18)

where $\widehat{\theta}(k) = \frac{1}{k}\sum_{t=1}^k \theta^t$, and the expectation is taken with respect to the samples X and Z.

The proof of Theorem 2 roughly follows that of Theorem 1, except that we prove that the sequence θ^t approximately minimizes the sequence of smoothed functions $f_{u_{1,t}}$ rather than f. However, for small $u_{1,t}$, these two functions are quite close, which combined with the estimates from Lemma 2 gives the result. We give the full argument in Section IV-D.

Theorem 2 shows that the convergence rate of our two-point stochastic gradient algorithm for general non-smooth functions is (at worst) a factor of $\sqrt{\log d}$ worse than the rate for smooth functions in Corollary 1. Notably, the rate of convergence here has substantially better dimension dependence than previously known results [4]–[6].

III. LOWER BOUNDS ON ZERO-ORDER OPTIMIZATION

Thus far, we have presented two main results (Theorems 1 and 2) that provide achievable rates for perturbation-based gradient procedures. It is natural to wonder whether or not these rates are sharp. In this section, we show that our results are—in general—unimprovable by more than a constant factor (a logarithmic factor in dimension in the setting of Corollary 3). These results show that *no* algorithm exists that can achieve a faster convergence rate than those we have presented under the oracle model (2).

We begin by describing the notion of minimax error. Let $\mathcal{F}$ be a collection of pairs (F, P), each of which defines an objective function of the form (1). Let $\mathbb{A}_k$ denote the collection of all algorithms that observe a sequence of data points $(Y^1, \ldots, Y^k) \subset \mathbb{R}^2$ with $Y^t = [F(\theta^t, X^t)\ F(\tau^t, X^t)]$ and return an estimate $\widehat{\theta}(k) \in \Theta$. Given an algorithm $\mathcal{A} \in \mathbb{A}_k$ and a pair $(F, P) \in \mathcal{F}$, we define the optimality gap

$$
\begin{aligned}
\epsilon_k(\mathcal{A}, F, P, \Theta) & := f(\widehat{\theta}(k)) - \inf_{\theta \in \Theta} f(\theta) \\
& = \mathbb{E}_P\left[F(\widehat{\theta}(k); X)\right] - \inf_{\theta \in \Theta} \mathbb{E}_P\left[F(\theta; X)\right],
\end{aligned}
$$

where $\widehat{\theta}(k)$ is the output of algorithm $\mathcal{A}$ on the sequence of observed function values. The expectation of this random variable defines the *minimax error*

$$\epsilon_k^*(\mathcal{F}, \Theta) := \inf_{\mathcal{A} \in \mathbb{A}_k} \sup_{(F,P) \in \mathcal{F}} \mathbb{E}[\epsilon_k(\mathcal{A}, F, P, \Theta)], \quad (19)$$

where the expectation is taken over the observations $(Y^1, \ldots, Y^k)$ and any additional randomness in $\mathcal{A}$. This quantity measures the performance of the best algorithm in $\mathbb{A}_k$, where performance is required to be uniformly good over the class $\mathcal{F}$.

We now turn to the statement of our lower bounds, which are based on simple choices of the classes $\mathcal{F}$. For a given ℓ_p-norm $\|\cdot\|_p$, we consider the class of linear functionals

$$\mathcal{F}_{G,p} := \{(F, P) \mid F(\theta; x) = \langle \theta, x \rangle, \ \mathbb{E}_P[\|X\|_p^2] \le G^2\}.$$

Each of these function classes satisfy Assumption B′ by construction, and moreover, $\nabla F(\cdot; x)$ has Lipschitz constant 0 for all x. We state each of our lower bounds assuming that the domain Θ is equal to some ℓ_q-ball of radius R, that is, $\Theta = \{\theta \in \mathbb{R}^d \mid \|\theta\|_q \le R\}$. Our first result considers the case $p = 2$ with domain Θ an arbitrary ℓ_q-ball with $q \ge 1$, so we measure gradients in the ℓ_2-norm.

Proposition 1: For the class $\mathcal{F}_{G,2}$ and $\Theta = \{\theta \in \mathbb{R}^d \mid \|\theta\|_q \le R\}$, we have

$$\epsilon_k^*(\mathcal{F}_{G,2}, \Theta) \ge \frac{1 - \frac{1}{q}}{12} \frac{GR}{\sqrt{k}} \min\left\{ d^{1-\frac{1}{q}}, k^{1-\frac{1}{q}} \right\}. \quad (20)$$

Combining the lower bound (20) with our algorithmic schemes in Section II shows that they are optimal up to constant factors. More specifically, for $q \ge 2$, the ℓ_2-ball of radius $d^{1/2-1/q}R$ contains the ℓ_q-ball of radius R, so Corollary 1 provides an upper bound on the minimax rate of convergence of order $RG\sqrt{d}d^{1/2-1/q}/\sqrt{k} = RGd^{1-1/q}/\sqrt{k}$ in the smooth case, while for $k \ge d$, Proposition 1 provides the lower bound $RGd^{1-1/q}/\sqrt{k}$. Theorem 2, providing a rate of $RG\sqrt{d \log d}/\sqrt{k}$ in the general (non-smooth) case, is also tight to within logarithmic factors. Consequently, the stochastic gradient descent algorithm (3) coupled with the sampling strategies (6) and (15) is optimal for stochastic problems with two-point feedback.

We can prove a parallel lower bound that applies when using multiple ($m \ge 2$) function evaluations in each iteration, that is, in the context of Corollary 2. In this case, an inspection of the proof of Proposition 1 shows that we have the bound

$$\epsilon_k^*(\mathcal{F}_{G,2}, \Theta) \ge \frac{1 - \frac{1}{q}}{10} \frac{GR}{\sqrt{mk}} \min\left\{ d^{1-\frac{1}{q}}, k^{1-\frac{1}{q}} \right\}. \quad (21)$$

We show this in the remarks following the proof of Proposition 1 in Section V-A. In particular, we see that the minimax rate of convergence over the ℓ_2-ball is $RG\sqrt{d/m}/\sqrt{k}$, which approaches the full information minimax rate of convergence, $RG/\sqrt{k}$, as $m \to d$.

For our second lower bound, we investigate the minimax rates at which it is possible to solve stochastic convex optimization problems in which the objective is Lipschitz continuous in the ℓ_1-norm, or equivalently, in which the gradients are bounded in ℓ_∞-norm. As noted earlier,

such scenarios are suitable for high-dimensional problems (see [17]).

Proposition 2: For the class $\mathcal{F}_{G,\infty}$ with $\Theta = \{\theta \in \mathbb{R}^d \mid \|\theta\|_1 \le R\}$, we have

$$\epsilon_k^*(\mathcal{F}_{G,\infty}, \Theta) \ge \frac{1}{24} \frac{GR}{\sqrt{k}} \min\left\{ \frac{\sqrt{k}}{\sqrt{\log(3k)}}, \frac{\sqrt{d}}{\sqrt{\log(3d)}} \right\}.$$

This result also demonstrates the optimality of our mirror descent algorithms up to logarithmic factors. Recalling Corollary 3, the MD algorithm (3) with $\psi(\theta) = \frac{1}{2(p-1)} \|\theta\|_p^2$, where $p = 1 + 1/\log(2d)$, implies that $\epsilon_k^*(\mathcal{F}_G, \Theta) \lesssim GR\sqrt{d \log(2d)}/\sqrt{k}$. On the other hand, Proposition 2 provides the lower bound $\epsilon_k^*(\mathcal{F}_G, \Theta) \gtrsim GR\sqrt{d}/\sqrt{k \log d}$. These upper and lower bounds match up to logarithmic factors in dimension.

It is worth comparing these lower bounds to the achievable rates of convergence when full gradient information is available—that is, when one has access to the subgradient selection $g(\theta; X)$—and when one has access to only a single function evaluation $F(\theta; X)$ at each iteration. We begin with the latter, presenting a minimax lower bound essentially due to Shamir [14] for comparison. We denote the minimax optimization error using a single function evaluation in each of k iterations by $\epsilon_k^{\text{single}}$. For the lower bound, we impose both Lipschitz conditions on the functions F and a variance condition on the observations $F(\theta; X)$: for a given variance σ^2, Lipschitz constant G, and ℓ_p-norm, we consider the family of optimization problems defined by the class of convex losses

$$\mathcal{F}_{\sigma,G,p} := \Big\{(F, P) \mid \mathbb{E}_P[\|\partial F(\theta; X)\|_p^2] \le G^2 \ \text{and} \\ \mathbb{E}[(F(\theta; X) - f(\theta))^2] \le \sigma^2\Big\}.$$

In our proofs, we restrict this class to functions of the form $F(\theta; x) = c_1 \|\theta - c_2 x\|_1$, where c_1, c_2 are constants chosen to guarantee the above inclusions. By an extension of techniques of Shamir [14, Th. 7], we have the following proposition.

Proposition 3: For any $p, q \ge 1$, the class $\mathcal{F}_{\sigma,G,p}$, and any $R > 0$ with $\Theta \supset \{\theta \in \mathbb{R}^d \mid \|\theta\|_q \le R\}$, we have

$$\epsilon_k^{\text{single}}(\mathcal{F}_{\sigma,G,q}, \Theta) \ge \frac{1}{4} \min\left\{ \frac{d\sigma}{\sqrt{k}}, GRd^{1-\frac{1}{p}-\frac{1}{q}} \right\}.$$

Proposition 3 shows that the asymptotic difficulty of optimization grows at least quadratically with the dimension d. Indeed, consider the Euclidean case ($p = q = 2$), and consider minimizing 1-Lipschitz convex functions over the ℓ_2-ball. Assuming that observations have variance 1, the minimax lower bound becomes $\frac{1}{4} \min\{d/\sqrt{k}, 1\}$, so achieving ϵ accuracy requires $\Omega(d^2/\epsilon^2)$ iterations. This is substantially worse than the complexity possible when using two function evaluations: Corollary 1 implies that the minimax rate of convergence scales as $\sqrt{d/k}$, so d/ϵ^2 iterations are necessary and sufficient to achieve ϵ accuracy. By comparing with Corollary 2 and Proposition 1, we see a phase transition: with only a single function evaluation per sample X, the minimax rate is $d/\sqrt{k}$, yet with $m \ge 2$ function evaluations, it is possible to obtain rates of convergence scaling as $\sqrt{d/m}/\sqrt{k}$.

For the case of *linear losses*—that is, when $F(\theta; x) = \langle \theta, x \rangle$—there is a smaller gap between convergence rates possible using single function evaluation and those attainable with multiple evaluations. In the Euclidean case of the preceding paragraph, the minimax convergence rate for linear losses scales (up to logarithmic factors) as $\sqrt{d/k}$ when only a single evaluation is available (see, e.g., Bubeck and Cesa-Bianchi [29], Th. 5.11]). Similarly, optimization of linear losses over the simplex (or the ℓ_1-ball)—the classical bandit problem of Lai and Robbins [30], [31]—scales to within logarithmic factors as $\sqrt{d/k}$ in the paired evaluation case (Corollary 3 and Proposition 2) and as $\sqrt{d/k}$ in the single evaluation case as well [32, Th. 5]. In the linear case, then, there is not (generally) a phase transition between single and multiple evaluations.

Returning now to a comparison with the full information case, each of Propositions 1 and 2 includes an additional $\sqrt{d}$ factor as compared to analogous minimax rates [16], [17], [21] applicable to the case of full gradient information. These $\sqrt{d}$ factors disappear from the achievable convergence rates in Corollaries 1 and 3 when one uses $g^t = g(\theta; X)$ in the mirror descent updates (3). Consequently, our analysis shows that in the zero-order setting—in addition to dependence on the radius R and second moment G^2—any algorithm must suffer at least an additional $\mathcal{O}(\sqrt{d})$ penalty in convergence rate, and optimal algorithms suffer precisely this penalty. In models for optimization in which there is a unit cost for each function evaluation and a unit cost for obtaining a single dimension of the gradient, the cost of using full gradient information and that for using only function evaluations is identical; in cases where performing d function evaluations is substantially more expensive than computing a single gradient, however, it is preferable to use full gradient information if possible, even when the cost of obtaining the gradients is somewhat nontrivial.

IV. CONVERGENCE PROOFS

We provide the proofs of the convergence results from Section II in this section, deferring more technical arguments to the appendices.

A. Proof of Theorem 1

Before giving the proof of Theorem 1, we state a standard lemma on the mirror descent iterates (see, for example, Nemirovski et al. [17, Sec. 2.3] or Beck and Teboulle [16, eq. (4.21)]).

Lemma 3: Let $\{g^t\}_{t=1}^k \subset \mathbb{R}^d$ be a sequence of vectors, and let θ^t be generated by the mirror descent iteration (3). If Assumption A holds, then for any $\theta^* \in \Theta$ we have

$$\sum_{t=1}^{k} \langle g^t, \theta^t - \theta^* \rangle \leq \frac{1}{2\alpha(k)} R^2 + \sum_{t=1}^{k} \frac{\alpha(t)}{2} \left\| g^t \right\|_*^2.$$

Defining the error vector $e^t := \nabla f(\theta^t) - g^t$, Lemma 3 implies that

$$\sum_{t=1}^{k} \left(f(\theta^t) - f(\theta^*) \right) \leq \sum_{t=1}^{k} \langle \nabla f(\theta^t), \theta^t - \theta^* \rangle$$

$$= \sum_{t=1}^{k} \langle g^t, \theta^t - \theta^* \rangle + \sum_{t=1}^{k} \langle e^t, \theta^t - \theta^* \rangle.$$

$$\leq \frac{1}{2\alpha(k)} R^2 + \sum_{t=1}^{k} \frac{\alpha(t)}{2} \left\| g^t \right\|_*^2 + \sum_{t=1}^{k} \langle e^t, \theta^t - \theta^* \rangle. \quad (22)$$

For each iteration $t = 2, 3, \ldots$, let $\mathcal{F}_{t-1}$ denote the σ-field of $X^1, \ldots, X^{t-1}$ and $Z^1, \ldots, Z^{t-1}$. Then Lemma 1 implies $\mathbb{E}[e^t \mid \mathcal{F}_{t-1}] = u_t L(P) v_t$, where $v_t \equiv v(\theta^t, u_t)$ satisfies $\|v_t\|_* \leq \frac{1}{2} M(\mu)$. Since $\theta^t \in \mathcal{F}_{t-1}$, we can first take an expectation conditioned on $\mathcal{F}_{t-1}$ to obtain

$$\sum_{t=1}^{k} \mathbb{E}[\langle e^t, \theta^t - \theta^* \rangle] \leq L(P) \sum_{t=1}^{k} u_t \mathbb{E}[\|v_t\|_* \|\theta^t - \theta^*\|]$$

$$\leq \frac{1}{2} M(\mu) R L(P) \sum_{t=1}^{k} u_t,$$

where in the last step above we have used the relation $\|\theta^t - \theta^*\| \leq \sqrt{2D_\psi(\theta^*, \theta)} \leq R$. Statement (10) of Lemma 1 coupled with the assumption that $\mathbb{E}[\|g(\theta^t; X)\|_*^2 \mid \mathcal{F}_{t-1}] \leq G^2$ yields

$$\mathbb{E}\left[\left\|g^t\right\|_*^2\right] = \mathbb{E}\left[\mathbb{E}\left[\left\|g^t\right\|_*^2 \mid \mathcal{F}_{t-1}\right]\right]$$

$$\leq 2s(d)G^2 + \frac{1}{2} u_t^2 L(P)^2 M(\mu)^2.$$

Applying the two estimates above to our initial bound (22) yields that

$$\sum_{t=1}^{k} \mathbb{E}\left[f(\theta^t) - f(\theta^*)\right] \leq \frac{1}{2\alpha(k)} R^2 + s(d)G^2 \sum_{t=1}^{k} \alpha(t)$$

$$+ \frac{1}{4} L(P)^2 M(\mu)^2 \sum_{t=1}^{k} u_t^2 \alpha(t)$$

$$+ \frac{1}{2} M(\mu) R L(P) \sum_{t=1}^{k} u_t. \quad (23)$$

Now we use our choices of the sample size $\alpha(t)$ and u_t to complete the proof. For the former, we have $\alpha(t) = \alpha R/(2G\sqrt{s(d)}\sqrt{t})$. Since $\sum_{t=1}^k t^{-\frac{1}{2}} < \int_0^k t^{-\frac{1}{2}} dt = 2\sqrt{k}$, we have

$$\frac{R^2}{2\alpha(k)} + s(d)G^2 \sum_{t=1}^{k} \alpha(t) \leq RG\sqrt{s(d)}\sqrt{k}(\alpha + \alpha^{-1}).$$

For the second summation in the quantity (23), we have the bound

$$\alpha u^2 \left(\frac{G^2 s(d)}{L(P)^2 M(\mu)^2}\right) \frac{R L(P)^2 M(\mu)^2}{4G\sqrt{s(d)}} \sum_{t=1}^{k} \frac{1}{t^{5/2}}$$

$$\leq \alpha u^2 RG\sqrt{s(d)}$$

since $\sum_{t=1}^k t^{-5/2} \leq 4$. The final term in the inequality (23) is similarly bounded by

$$u\left(\frac{G\sqrt{s(d)}}{L(P)M(\mu)}\right) \frac{R L(P) M(\mu)}{2} (\log k + 1)$$

$$= u \frac{RG\sqrt{s(d)}}{2} (\log k + 1) \leq u RG\sqrt{s(d)} \log(2k).$$

Combining the preceding inequalities with Jensen's inequality yields the claim (11).

B. Proof of Lemma 1

Let h be an arbitrary convex function with L_h-Lipschitz continuous gradient with respect to the norm $\|\cdot\|$. Using the tangent plane lower bound for a convex function and the L_h-Lipschitz continuity of the gradient, for any $u > 0$ we have

$$
h'(\theta, z) = \frac{\langle \nabla h(\theta), uz \rangle}{u} \leq \frac{h(\theta + uz) - h(\theta)}{u}
$$
$$
\leq \frac{\langle \nabla h(\theta), uz \rangle + (L_h/2)\|uz\|^2}{u}.
$$

Consequently, for any point $\theta \in \operatorname{relint dom} h$ and for any $z \in \mathbb{R}^d$, we have

$$
\frac{h(\theta + uz) - h(\theta)}{u} z = h'(\theta, z)z + \frac{L_h u}{2}\|z\|^2 \gamma(u, \theta, z)z, \tag{24}
$$

where γ is some function with range contained in $[0, 1]$. Since $\mathbb{E}[ZZ^\top] = I_{d \times d}$ by assumption, equality (24) implies

$$
\mathbb{E}\left[\frac{h(\theta + uZ) - h(\theta)}{u} Z\right]
$$
$$
= \mathbb{E}\left[h'(\theta, Z)Z + \frac{L_h u}{2}\|Z\|^2 \gamma(u, \theta, Z)Z\right]
$$
$$
= \nabla h(\theta) + uL_h v(\theta, u), \tag{25}
$$

where $v(\theta, u) \in \mathbb{R}^d$ is an error vector with $\|v(\theta, u)\|_* \leq \frac{1}{2}\mathbb{E}[\|Z\|^2 \|Z\|_*]$.

We now turn to proving the statements of the lemma. Recalling the definition (5) of the gradient estimator, we see that for P-almost every $x \in \mathcal{X}$, expression (25) implies that

$$
\mathbb{E}[\mathsf{G}_{\mathrm{sm}}(\theta; u, Z, x)] = \nabla F(\theta; x) + uL(x)v(\theta, u)
$$

for $v = v(\theta, u)$ with $2\|v\|_* \leq \mathbb{E}[\|Z\|^2 \|Z\|_*]$. We have $\mathbb{E}[\nabla F(\theta; X)] = \nabla f(\theta^t)$, and independence implies that

$$
\mathbb{E}[L(X)\|v(\theta, u)\|_*] \leq \sqrt{\mathbb{E}[L(X)^2]}\sqrt{\mathbb{E}[\|v\|_*^2]}
$$
$$
\leq \frac{1}{2}L(P)\mathbb{E}[\|Z\|^2 \|Z\|_*],
$$

from which the bound (9) follows.

For the second statement (10) of the lemma, apply equality (24) to $F(\cdot; X)$, obtaining

$$
\mathsf{G}_{\mathrm{sm}}(\theta; u, Z, X) = \langle g(\theta, X), Z \rangle Z + \frac{L(X)u}{2}\|Z\|^2 \gamma Z
$$

for some function $\gamma \equiv \gamma(u, \theta, Z, X) \in [0, 1]$. The relation $(a + b)^2 \leq 2a^2 + 2b^2$ then gives

$$
\mathbb{E}[\|\mathsf{G}_{\mathrm{sm}}(\theta; u, Z, X)\|_*^2]
$$
$$
\leq \mathbb{E}\left[\left(\|\langle g(\theta, X), Z \rangle Z\|_* + \frac{1}{2}\left\|L(X)u\|Z\|^2 \gamma Z\right\|_*\right)^2\right]
$$
$$
\leq 2\mathbb{E}\left[\|\langle g(\theta, X), Z \rangle Z\|_*^2\right] + \frac{u^2}{2}\mathbb{E}\left[L(X)^2 \|Z\|^4 \|Z\|_*^2\right].
$$

Finally, Assumption D coupled with the independence of X and Z gives the bound (10).

C. Proof of Corollary 2

We show that averaging multiple directional estimates gives a gradient estimator whose expected squared norm is smaller by a factor of m than that attained using a single vector Z. Fixing x, let $g = \nabla F(\theta; x) + uL(x)v(\theta, u, x)$ denote the expectation of $\mathsf{G}_{\mathrm{sm}}(\theta; u, Z, x)$ taken over Z uniform on $\sqrt{d}\mathbb{B}^d$, where $2\|v\|_2 \leq d^{3/2}$, by equation (25). In this case, for Z^i drawn i.i.d. μ, we obtain

$$
\mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^m \mathsf{G}_{\mathrm{sm}}(\theta; u, Z^i, x)\right\|_2^2\right]
$$
$$
= \|g\|_2^2 + \mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^m \mathsf{G}_{\mathrm{sm}}(\theta; u, Z^i, x) - g\right\|_2^2\right]
$$
$$
= \|g\|_2^2 + \frac{1}{m}\mathbb{E}[\left\|\mathsf{G}_{\mathrm{sm}}(\theta; u, Z^1, x) - g\right\|_2^2].
$$

Now, taking an expectation over X, we have

$$
\mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^m \mathsf{G}_{\mathrm{sm}}(\theta; u, Z^i, X)\right\|_2^2\right]
$$
$$
\leq \mathbb{E}[\|\nabla F(\theta; X) + uL(X)v(\theta, u, X)\|_2^2]
$$
$$
+ \frac{1}{m}\mathbb{E}[\left\|\mathsf{G}_{\mathrm{sm}}(\theta; u, Z^1, X)\right\|_2^2]
$$
$$
\overset{(i)}{\leq} 2\mathbb{E}[\|\nabla F(\theta; X)\|_2^2] + \frac{1}{2}u^2 d^3 \mathbb{E}[L(X)^2]
$$
$$
+ \frac{1}{m}\left(2d\mathbb{E}[\|\nabla F(\theta; X)\|_2^2] + \frac{1}{2}u^2 L(P)^2 d^3\right)
$$
$$
= 2\frac{m + d}{m}\mathbb{E}[\|\nabla F(\theta; X)\|_2^2] + \frac{1}{2}\frac{m + 1}{m}u^2 L(P)^2 d^3,
$$

where inequality (i) follows from Lemma 1 and Jensen's inequality. Noting that the non-u-dependent terms scale as $(1 + d/m)\mathbb{E}[\|\nabla F(\theta; X)\|_2^2]$, comparison of the last display with Lemma 1's application in Theorem 1 and Corollary 1 shows that the stepsizes specified in the corollary give the desired guarantee.

D. Proof of Theorem 2

The proof of Theorem 2 is similar to that of Theorem 1. To simplify our proof, we first state a lemma bounding the moments of vectors that satisfy Assumption F.

Lemma 4: Let the random vector Z be distributed as $N(0, I_{d \times d})$, uniformly on the ℓ_2-ball of radius $\sqrt{d + 2}$, or uniformly on the ℓ_2-sphere of radius $\sqrt{d}$. For any $k \in \mathbb{N}$, there is a constant c_k (dependent only on k) such that

$$
\mathbb{E}\left[\|Z\|_2^k\right] \leq c_k d^{\frac{k}{2}}.
$$

In all cases we have $\mathbb{E}[ZZ^\top] = I_{d \times d}$, and $c_k \leq 3$ for $k = 4$ and $c_k \leq \sqrt{3}$ for $k = 3$.

See Appendix A for the proof. We now turn to the proof proper. From Lemmas E.2 and E.3 of the paper [26], the function f_u defined in (13) satisfies $f(\theta) \leq f_u(\theta) \leq f(\theta) + uG\sqrt{d + 2}$ for $\theta \in \Theta$. Defining the error vector $e^t := \nabla f_{u_{1,t}}(\theta^t) - g^t$ and noting that $\sqrt{d + 2} \leq \sqrt{3d}$, we

thus have

$$
\sum_{t=1}^{k} \left(f(\theta^t) - f(\theta^*) \right)
$$

$$
\leq \sum_{t=1}^{k} \left(f_{u_{1,t}}(\theta^t) - f_{u_{1,t}}(\theta^*) \right) + \sqrt{3} G \sqrt{d} \sum_{t=1}^{k} u_{1,t}
$$

$$
\leq \sum_{t=1}^{k} \langle \nabla f_{u_{1,t}}(\theta^t), \theta^t - \theta^* \rangle + \sqrt{3} G \sqrt{d} \sum_{t=1}^{k} u_{1,t}
$$

$$
= \sum_{t=1}^{k} \left(\langle g^t, \theta^t - \theta^* \rangle + \langle e^t, \theta^t - \theta^* \rangle + \sqrt{3} G \sqrt{d} u_{1,t} \right),
$$

where we have used the convexity of f_u and the definition of e^t. Applying Lemma 3 to the summed $\langle g^t, \theta^t - \theta^* \rangle$ terms as in the proof of Theorem 1, we obtain

$$
\sum_{t=1}^{k} \left(f(\theta^t) - f(\theta^*) \right) \leq \frac{R^2}{2\alpha(k)} + \frac{1}{2} \sum_{t=1}^{k} \alpha(t) \left\| g^t \right\|_2^2
$$

$$
+ \sum_{t=1}^{k} \langle e^t, \theta^t - \theta^* \rangle + \sqrt{3} G \sqrt{d} \sum_{t=1}^{k} u_{1,t}.
$$
(26)

The proof from this point is similar to the proof of Theorem 1 (cf. inequality (22)). Specifically, we bound the squared gradient $\|g^t\|_2^2$ terms, the error $\langle e^t, \theta^t - \theta^* \rangle$ terms, and then control the summed u_t terms. For the remainder of the proof, we let $\mathcal{F}_{t-1}$ denote the σ-field generated by the random variables $X^1, \ldots, X^{t-1}, Z_1^1, \ldots, Z_1^{t-1}$, and $Z_2^1, \ldots, Z_2^{t-1}$.

Bounding $\langle e^t, \theta^t - \theta^* \rangle$: Our first step is note that Lemma 2 implies $\mathbb{E}[e^t \mid \mathcal{F}_{t-1}] = \frac{u_{2,t}}{u_{1,t}} G v_t$, where the vector $v_t \equiv v(\theta^t, u_{1,t}, u_{2,t})$ satisfies $\|v_t\|_2 \leq \frac{1}{2} \mathbb{E}[\|Z_2\|_2^3]$. As in the proof of Theorem 1, this gives

$$
\sum_{t=1}^{k} \mathbb{E}[\langle e^t, \theta^t - \theta^* \rangle] \leq G \sum_{t=1}^{k} \frac{u_{2,t}}{u_{1,t}} \mathbb{E}[\|v_t\|_2 \|\theta^t - \theta^*\|_2]
$$

$$
\leq \frac{1}{2} \mathbb{E}[\|Z_2\|_2^3] R G \sum_{t=1}^{k} \frac{u_{2,t}}{u_{1,t}}.
$$

When Assumption F holds, Lemma 4 implies the expectation bound $\mathbb{E}[\|Z_2\|_2^3] \leq \sqrt{3} d^{3/2}$. Thus

$$
\sum_{t=1}^{k} \mathbb{E}[\langle e^t, \theta^t - \theta^* \rangle] \leq \frac{\sqrt{3} d \sqrt{d}}{2} R G \sum_{t=1}^{k} \frac{u_{2,t}}{u_{1,t}}.
$$

Bounding $\|g^t\|_2^2$: Turning to the squared gradient terms from the bound (26), Lemma 2 gives

$$
\mathbb{E}[\|g^t\|_2^2] = \mathbb{E}[\mathbb{E}[\|g^t\|_2^2 \mid \mathcal{F}_{t-1}]]
$$

$$
\leq c\, G^2 d \left(\sqrt{\frac{u_{2,t}}{u_{1,t}}} d + 1 + \log d \right)
$$

$$
\leq c'\, G^2 d \left(\sqrt{\frac{u_{2,t}}{u_{1,t}}} d + \log(2d) \right),
$$

where $c, c' > 0$ are numerical constants independent of $\{u_{1,t}\}, \{u_{2,t}\}$.

Summing Out the Smoothing Penalties: Applying the preceding estimates to our earlier bound (26), we get that for a numerical constant c,

$$
\sum_{t=1}^{k} \mathbb{E}[f(\theta^t) - f(\theta^*)]
$$

$$
\leq \frac{R^2}{2\alpha(k)} + c G^2 d \log(2d) \sum_{t=1}^{k} \alpha(t) + c G^2 d^2 \sum_{t=1}^{k} \sqrt{\frac{u_{2,t}}{u_{1,t}}} \alpha(t)
$$

$$
+ \frac{\sqrt{3}}{2} R G d \sqrt{d} \sum_{t=1}^{k} \frac{u_{2,t}}{u_{1,t}} + \sqrt{3} G \sqrt{d} \sum_{t=1}^{k} u_{1,t}.
$$
(27)

We bound the right hand side above using our choices of $\alpha(t)$, $u_{1,t}$, and $u_{2,t}$. We also use the relations $\sum_{t=1}^{k} t^{-\frac{1}{2}} \leq 2\sqrt{k}$ and $\sum_{t=1}^{k} t^{-1} \leq 1 + \log k \leq 2 \log k$ for $k \geq 3$. With the setting $\alpha(t) = \alpha R / (G \sqrt{d \log(2d)} \sqrt{t})$, the first two terms in (27) become

$$
\frac{R^2}{2\alpha(k)} + c G^2 d \log(2d) \sum_{t=1}^{k} \alpha(t)
$$

$$
\leq \frac{R G \sqrt{d \log(2d)}}{2\alpha} \sqrt{k} + 2c\alpha R G \sqrt{d \log(2d)} \sqrt{k}
$$

$$
\leq c' \max\{\alpha, \alpha^{-1}\} R G \sqrt{d \log(2d)} \sqrt{k}
$$

for a universal constant c'. Since we have chosen $u_{2,t}/u_{1,t} = 1/(d^2 t)$, we may bound the third term in expression (27) by

$$
G^2 d^2 \sum_{t=1}^{k} \sqrt{\frac{u_{2,t}}{u_{1,t}}} \alpha(t)
$$

$$
= G^2 d \left(\frac{\alpha R}{G \sqrt{d \log(2d)}} \right) \sum_{t=1}^{k} \frac{1}{t} \leq \frac{2\alpha R G \sqrt{d}}{\sqrt{\log(2d)}} \log(2k).
$$

Similarly, the fourth term in the bound (27) becomes

$$
\frac{\sqrt{3}}{2} R G d \sqrt{d} \sum_{t=1}^{k} \frac{u_{2,t}}{u_{1,t}} \leq \frac{\sqrt{3} R G}{\sqrt{d}} \log(2k).
$$

Finally, since $u_{1,t} = uR/t$, we may bound the last term in expression (27) with

$$
\sqrt{3} G \sqrt{d} \sum_{t=1}^{k} u_{1,t} \leq 2\sqrt{3} u R G \sqrt{d} \log(2k).
$$

Using Jensen's inequality to note that $\mathbb{E}[f(\widehat{\theta}(k))] \leq \frac{1}{k} \sum_{t=1}^{k} \mathbb{E}[f(\theta^t)]$ and eliminating lower-order terms, we obtain the claim (18).

E. Proof of Lemma 2

The proof of Lemma 2 relies on the following key technical result:

Lemma 5: Let $k \geq 1$ and $u \geq 0$. Let $Z_1 \sim \mu_1$ and $Z_2 \sim \mu_2$ be independent random variables in $\mathbb{R}^d$, where μ_1 and μ_2 satisfy Assumption F. There exists a constant c_k, depending only on k, such that for every 1-Lipschitz convex function h,

$$
\mathbb{E}\left[|h(Z_1 + uZ_2) - h(Z_1)|^k \right]
$$

$$
\leq c_k u^k \left[u d^{\frac{k}{2}} + 1 + \log^{\frac{k}{2}}(d + 2k) \right].
$$

The proof is fairly technical, so we defer it to Appendix B. It is based on the dimension-free concentration of Lipschitz functions of standard Gaussian vectors and vectors uniform on $\mathbb{B}^d$.

We return now to the proof of Lemma 2 proper, providing arguments for inequalities (16) and (17). For convenience we recall the definition $G(x)$ as the Lipschitz constant of $F(\cdot; x)$ (Assumption B$'$) and the definition (14) of the non-smooth directional gradient

$$
\begin{aligned}
&\mathsf{G}_{\mathrm{ns}}(\theta; u_1, u_2, z_1, z_2, x) \\
&\quad = \frac{F(\theta + u_1 z_1 + u_2 z_2; x) - F(\theta + u_1 z_1; x)}{u_2} z_2.
\end{aligned}
$$

We begin with the second statement (17) of Lemma 2. By applying Lemma 5 to the 1-Lipschitz convex function $h(\tau) = \frac{1}{u_1 G(X)} F(\theta + u_1 \tau; X)$ and setting $u = u_2/u_1$, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\|\mathsf{G}_{\mathrm{ns}}(\theta; u_1, u_2, Z_1, Z_2, x)\|_2^2\right] \\
&= \frac{u_1^2 G(x)^2}{u_2^2} \mathbb{E}\left[(h(Z_1 + (u_2/u_1)Z_2) - h(Z_1))^2 \|Z_2\|_2^2\right] \\
&\leq \frac{G(x)^2}{u^2} \mathbb{E}\left[(h(Z_1 + u Z_2) - h(Z_1))^4\right]^{\frac{1}{2}} \mathbb{E}\left[\|Z_2\|_2^4\right]^{\frac{1}{2}}. \quad (28)
\end{aligned}
$$

Lemma 4 implies that $\mathbb{E}[\|Z_2\|_2^4]^{\frac{1}{2}} \leq \sqrt{3} d$ for smoothing distributions satisfying Assumption F.

It thus remains to bound the first expectation in the product (28). By Lemma 5,

$$
\mathbb{E}\left[(h(Z_1 + u Z_2) - h(Z_1))^4\right] \leq c u^4 \left[u d^2 + 1 + \log^2 d\right]
$$

for a numerical constant $c > 0$. Taking the square root of both sides of the preceding display, then applying inequality (28), yields

$$
\begin{aligned}
&\mathbb{E}\left[\|\mathsf{G}_{\mathrm{ns}}(\theta; u_1, u_2, Z_1, Z_2, x)\|_2^2\right] \\
&\qquad\qquad\qquad \leq c \frac{G(x)^2}{u^2} u^2 d \left[\sqrt{u} d + 1 + \log d\right].
\end{aligned}
$$

Integrating over x using the Lipschitz Assumption B$'$ proves the inequality (17) in Lemma 2.

For the first statement of the lemma, we define the shorthand $F_u(\theta; x) = \mathbb{E}[F(\theta + u Z_1; x)]$, where the expectation is over $Z_1 \sim \mu_1$, and note that by Fubini's theorem, $\mathbb{E}[F_u(\theta; X)] = f_u(\theta)$. By taking the expectation of G_{ns} with respect to Z_1 only, we get

$$
\begin{aligned}
&\mathbb{E}\left[\mathsf{G}_{\mathrm{ns}}(\theta; u_1, u_2, Z_1, z_2, x)\right] \\
&\qquad\qquad = \frac{F_{u_1}(\theta + u_2 z_2; x) - F_{u_1}(\theta; x)}{u_2} z_2.
\end{aligned}
$$

Since $\theta \mapsto F(\theta; x)$ is $G(x)$-Lipschitz, Lemmas E.2(iii) and E.3(iii) of the paper by Duchi et al. [26] imply $F_u(\cdot; x)$ is $G(x)$-Lipschitz, has $G(x)/u$-Lipschitz continuous gradient, and satisfies the unbiasedness condition $\mathbb{E}[\nabla F_u(\theta; X)] = \nabla f_u(\theta)$. Therefore, the same argument bounding the bias (9) in the proof of Lemma 1 (recall inequalities (24) and (25)) yields the claim (16).

V. PROOFS OF LOWER BOUNDS

We now present the proofs for our lower bounds on the minimax error (19). Our lower bounds are based on several techniques from the statistics and information-theory literature (see [33]–[35]). Our basic strategy is to reduce the optimization problem to several binary hypothesis testing problems: we choose a finite set of functions, show that optimizing well implies that one can solve each of the binary hypothesis tests, and then, as in statistical minimax theory [33], [34], apply divergence-based lower bounds for the probability of error in hypothesis testing problems.

Before embarking on our proofs, we assume all optimization algorithms are deterministic (see [35]). This is no loss of generality, as each of our lower bounds applies to average (rather than worst) case problems. Indeed, recall the definition (19) of the minimax error, and let $\mathcal{V}$ be a finite set indexing a subset $\{(F_v, P_v)\}_{v \in \mathcal{V}} \subset \mathcal{F}$ with $f_v(\theta) = \mathbb{E}_{P_v}[F(\theta; X)]$. Then

$$
\sup_{(F, P)} \mathbb{E}[\epsilon_k(\mathcal{A}, F, P, \Theta)] \geq \frac{1}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} \mathbb{E}_{P_v}\left[f_v(\widehat{\theta}) - \inf_{\theta \in \Theta} f_v(\theta)\right].
$$

At each iteration of any algorithm $\mathcal{A}$, we may write $(\theta^t, \tau^t) = H_t(Y^{1:t-1}, U^t)$, where U^t are random variables, independent of $Y^{1:t-1}$, encoding the randomness in $\mathcal{A}$, and H_t is a deterministic function. By iterating expectations, we have

$$
\begin{aligned}
&\frac{1}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} \mathbb{E}_{P_v}\left[f_v(\widehat{\theta}) - \inf_{\theta \in \Theta} f_v(\theta)\right] \\
&= \frac{1}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} \mathbb{E}\left[\mathbb{E}_{P_v}[f_v(\widehat{\theta}) - \inf_{\theta \in \Theta} f_v(\theta) \mid U^{1:k}]\right] \\
&\geq \inf_{u^{1:k}} \frac{1}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} \mathbb{E}_{P_v}[f_v(\widehat{\theta}) - \inf_{\theta \in \Theta} f_v(\theta) \mid U^{1:k} = u^{1:k}].
\end{aligned}
$$

Thus, so long as we prove lower bounds for the averaged optimality gap, we may incorporate the "best" randomness into the algorithm $\mathcal{A}$ and assume it is deterministic.

A. Proof of Proposition 1

The basic outline of our proofs is similar. At a high level, for each binary vector v in the Boolean hypercube $\mathcal{V} = \{-1, 1\}^d$, we construct a linear function f_v that is "well-separated" from the other functions $\{f_w, w \neq v\}$. Our notion of separation enforces the following property: if θ^v minimizes f_v over Θ, then for each coordinate $j \in [d]$ for which $\mathrm{sgn}(\widehat{\theta}_j) \neq \mathrm{sgn}(\theta_j^v)$, there is an additive penalty in the optimization accuracy $f_v(\widehat{\theta}) - f_v(\theta^v)$. Consequently, we can lower bound the optimization accuracy by the testing error in the following *canonical testing problem*: nature chooses an index $v \in \mathcal{V}$ uniformly at random, and we must identify the indices v_j based on the observations $Y^1, \ldots, Y^k$. By applying lower bounds on the testing error related to the Assouad and Le Cam techniques for lower bounding minimax error [34], we thus obtain lower bounds on the optimization error.

In more detail, consider (instantaneous) objective functions of the form $F(\theta; x) = \langle \theta, x \rangle$. For each $v \in \mathcal{V}$, let P_v denote the Gaussian distribution $\mathsf{N}(\delta v, \sigma^2 I_{d \times d})$, where $\delta > 0$ is a

parameter to be chosen, so that

$$f_v(\theta) := \mathbb{E}_{P_v}[F(\theta; X)] = \delta \langle \theta, v \rangle.$$

For each $v \in \mathcal{V}$, let θ^v minimize $f_v(\theta)$ over $\Theta := \{\theta \in \mathbb{R}^d \mid \|\theta\|_q \leq R\}$. A calculation shows that $\theta^v = -R d^{1/q} v$, so that $\mathrm{sgn}(\theta_j^v) = -v_j$. Next we claim that, for any vector $\widehat{\theta} \in \mathbb{R}^d$,

$$f_v(\widehat{\theta}) - f_v(\theta^v) \geq \frac{1 - \frac{1}{q}}{d^{1/q}} \delta R \sum_{j=1}^{d} \mathbf{1} \left\{ \mathrm{sgn}(\widehat{\theta}_j) \neq \mathrm{sgn}(\theta_j^v) \right\}. \quad (29)$$

Inequality (29) shows that if it is possible to optimize well—that is, to find a vector $\widehat{\theta}$ with a relatively small optimality gap—then it is also possible to estimate the signs of v. To establish inequality (29), we state a lemma providing a gap in optimality for solutions of related problems:

Lemma 6: For a given integer $i \in [d]$, consider the two optimization problems (over $\theta \in \mathbb{R}^d$)

$$(A) \quad \begin{matrix} \text{minimize } \theta^\top \mathbf{1} \\ \text{subject to } \|\theta\|_q \leq 1 \end{matrix} \quad \text{and}$$

$$(B) \quad \begin{matrix} \text{minimize } \theta^\top \mathbf{1} \\ \text{subject to } \|\theta\|_q \leq 1, \quad \theta_j \geq 0 \text{ for } j \in [i], \end{matrix}$$

with optimal solutions θ^A and θ^B, respectively. Then $\langle \mathbf{1}, \theta^A \rangle \leq \langle \mathbf{1}, \theta^B \rangle - (1 - 1/q) i/d^{1/q}$.

See Appendix C for a proof. Returning to inequality (29), we note that $f_v(\widehat{\theta}) - f_v(\theta^v) = \delta \langle v, \widehat{\theta} - \theta^v \rangle$. By symmetry, Lemma 6 implies that for every coordinate j such that $\mathrm{sgn}(\widehat{\theta}_j) \neq \mathrm{sgn}(\theta_j^v)$, the objective value $f_v(\widehat{\theta})$ must be at least a quantity $(1 - 1/q)\delta R/d^{1/q}$ larger than the optimal value $f_v(\theta^v)$, which yields inequality (29).

Now we use inequality (29) to give a probabilistic lower bound. Consider the mixture distribution $\mathbb{P} := (1/|\mathcal{V}|) \sum_{v \in \mathcal{V}} P_v$. For any estimator $\widehat{\theta}$, we have

$$\max_v \mathbb{E}_{P_v}[f_v(\widehat{\theta}) - f_v(\theta^v)] \geq \frac{1}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} \mathbb{E}_{P_v}[f_v(\widehat{\theta}) - f_v(\theta^v)]$$

$$\geq \frac{1 - 1/q}{d^{1/q}} \delta R \sum_{j=1}^{d} \mathbb{P}(\mathrm{sgn}(\widehat{\theta}_j) \neq -V_j).$$

Consequently, the minimax error is lower bounded as

$$\epsilon_k^*(\mathcal{F}_{G,2}, \Theta) \geq \frac{1 - \frac{1}{q}}{d^{1/q}} \delta R \left\{ \inf_{\widehat{v}} \sum_{j=1}^{d} \mathbb{P}(\widehat{v}_j(Y^{1:k}) \neq V_j) \right\}, \quad (30)$$

where $\widehat{v}$ denotes any (deterministic) testing function mapping from the observations $\{Y^t\}_{t=1}^k$ to $\{-1, 1\}^d$.

Next we lower bound the testing error by a total variation distance. By Le Cam's inequality, for any set A and distributions P, Q, we have $P(A) + Q(A^c) \geq 1 - \|P - Q\|_{\mathrm{TV}}$. We apply this inequality to the "positive jth coordinate" and "negative jth coordinate" sampling distributions

$$P_{+j} := \frac{1}{2^{d-1}} \sum_{v \in \mathcal{V}: v_j = 1} P_v \quad \text{and} \quad P_{-j} := \frac{1}{2^{d-1}} \sum_{v \in \mathcal{V}: v_j = -1} P_v,$$

corresponding to conditional distributions over Y^t given the events $\{v_j = 1\}$ or $\{v_j = -1\}$. Applying Le Cam's inequality

yields

$$\mathbb{P}(\widehat{v}_j(Y^{1:k}) \neq V_j)$$
$$= \frac{1}{2} P_{+j}(\widehat{v}_j(Y^{1:k}) \neq 1) + \frac{1}{2} P_{-j}(\widehat{v}_j(Y^{1:k}) \neq -1)$$
$$\geq \frac{1}{2} \left(1 - \|P_{+j} - P_{-j}\|_{\mathrm{TV}} \right).$$

Combined with the bound $\sum_{j=1}^{d} \|P_{+j} - P_{-j}\|_{\mathrm{TV}} \leq \sqrt{d} (\sum_{j=1}^{d} \|P_{+j} - P_{-j}\|_{\mathrm{TV}}^2)^{\frac{1}{2}}$ (from the Cauchy-Schwartz inequality), we obtain

$$\epsilon_k^*(\mathcal{F}_{G,2}, \Theta)$$
$$\geq \frac{(1 - \frac{1}{q})\delta R}{2d^{1/q}} \sum_{j=1}^{d} \left(1 - \|P_{+j} - P_{-j}\|_{\mathrm{TV}} \right)$$
$$\geq \frac{(1 - \frac{1}{q})d^{1 - \frac{1}{q}} \delta R}{2} \left(1 - \left(\frac{1}{d} \sum_{j=1}^{d} \|P_{+j} - P_{-j}\|_{\mathrm{TV}}^2 \right)^{\frac{1}{2}} \right). \quad (31)$$

The remainder of the proof provides sharp enough bounds on $\sum_j \|P_{+j} - P_{-j}\|_{\mathrm{TV}}^2$ to leverage inequality (31). Define the covariance matrix

$$\Sigma := \sigma^2 \begin{bmatrix} \|\theta\|_2^2 & \langle \theta, \tau \rangle \\ \langle \theta, \tau \rangle & \|\tau\|_2^2 \end{bmatrix} = \sigma^2 [\theta \; \tau]^\top [\theta \; \tau], \quad (32)$$

with the corresponding shorthand Σ^t for the covariance computed for the tth pair (θ^t, τ^t). We have:

Lemma 7: For each $j \in \{1, \ldots, d\}$, the total variation norm is bounded as

$$\|P_{+j} - P_{-j}\|_{\mathrm{TV}}^2 \leq \delta^2 \sum_{t=1}^{k} \mathbb{E} \left[\begin{bmatrix} \theta_j^t \\ \tau_j^t \end{bmatrix}^\top (\Sigma^t)^{-1} \begin{bmatrix} \theta_j^t \\ \tau_j^t \end{bmatrix} \right]. \quad (33)$$

See Appendix D for a proof of this lemma.

Now we use the bound (33) to provide a further lower bound on inequality (31). We first note the identity

$$\sum_{j=1}^{d} \begin{bmatrix} \theta_j \\ \tau_j \end{bmatrix} \begin{bmatrix} \theta_j \\ \tau_j \end{bmatrix}^\top = \begin{bmatrix} \|\theta\|_2^2 & \langle \theta, \tau \rangle \\ \langle \theta, \tau \rangle & \|\tau\|_2^2 \end{bmatrix}.$$

Recalling the definition (32) of the covariance matrix Σ, Lemma 7 implies that

$$\sum_{j=1}^{d} \|P_{+j} - P_{-j}\|_{\mathrm{TV}}^2$$
$$\leq \delta^2 \sum_{t=1}^{k} \mathbb{E} \left[\sum_{j=1}^{d} \mathrm{tr} \left((\Sigma^t)^{-1} \begin{bmatrix} \theta_j^t \\ \tau_j^t \end{bmatrix} \begin{bmatrix} \theta_j^t \\ \tau_j^t \end{bmatrix}^\top \right) \right]$$
$$= \frac{\delta^2}{\sigma^2} \sum_{t=1}^{k} \mathbb{E} \left[\mathrm{tr} \left((\Sigma^t)^{-1} \Sigma^t \right) \right] = 2 \frac{k\delta^2}{\sigma^2}. \quad (34)$$

Returning to the estimation lower bound (31), we thus find the nearly final lower bound

$$\epsilon_k^*(\mathcal{F}_{G,2}, \Theta) \geq \left(1 - \frac{1}{q} \right) \frac{d^{1 - 1/q} \delta R}{2} \left(1 - \left(\frac{2k\delta^2}{d\sigma^2} \right)^{\frac{1}{2}} \right). \quad (35)$$

Enforcing $(F, P) \in \mathcal{F}_{G,2}$ amounts to choosing the parameters σ^2 and δ^2 so that $\mathbb{E}[\|X\|_2^2] \leq G^2$ for $X \sim \mathsf{N}(\delta v, \sigma^2 I_{d \times d})$, after which we may use inequality (35)

to complete the proof of the lower bound. By construction, we have $\mathbb{E}[\|X\|_2^2] = (\delta^2 + \sigma^2)d$, so choosing $\sigma^2 = 8G^2/9d$ and $\delta^2 = (G^2/9)\min\{1/k, 1/d\}$ guarantees that

$$1 - \left(\frac{2k\delta^2}{d\sigma^2}\right)^{\frac{1}{2}} \geq 1 - \left(\frac{18}{72}\right)^{\frac{1}{2}} = \frac{1}{2}$$

and

$$\mathbb{E}[\|X\|_2^2] = \frac{8G^2}{9} + \frac{G^2 d}{9}\min\left\{\frac{1}{k}, \frac{1}{d}\right\} \leq G^2.$$

Substituting these choices of δ and σ^2 in inequality (35) gives the lower bound

$$\epsilon_k^*(\mathcal{F}_{G,2}, \Theta) \geq \frac{1 - \frac{1}{q}}{12} d^{1-1/q} RG \min\left\{\frac{1}{\sqrt{k}}, \frac{1}{\sqrt{d}}\right\}$$

$$= \frac{1 - \frac{1}{q}}{12} \frac{d^{1-1/q} RG}{\sqrt{k}} \min\left\{1, \sqrt{k/d}\right\}.$$

To complete the proof of the claim (20), we note that the above lower bound also applies to any d_0-dimensional problem for $d_0 \leq d$. More rigorously, we choose $\mathcal{V} = \{-1, 1\}^{d_0} \times \{0\}^{d-d_0}$, and define the sampling distribution P_v on X so that given $v \in \mathcal{V}$, the coordinate distributions of X are independent with $X_j \sim \mathsf{N}(\delta v_j, \sigma^2)$ for $j \leq d_0$ and $X_j = 0$ for $j > d_0$. A reproduction of the preceding proof, substituting $d_0 \leq d$ for each appearance of the dimension d, then yields the claimed bound (20) when we choose $d_0 = \min\{k, d\}$.

Remarks on Multiple Evaluations: By an extension of Lemma 7, we may consider the case in which at each iteration, the method may query for function values at the m points $\theta_{(1)}, \ldots, \theta_{(m)} \in \mathbb{R}^d$. Let $\theta_{j,(i)}^t$ denote the jth coordinate of the ith query point in iteration t. In this case, an immediate analogue of Lemma 7 implies

$$\|P_{+j} - P_{-j}\|_{\text{TV}}^2 \leq \delta^2 \sum_{t=1}^k \mathbb{E}\begin{bmatrix} \theta_{j,(1)}^t \\ \vdots \\ \theta_{j,(m)}^t \end{bmatrix}^\top (\Sigma^t)^{-1} \begin{bmatrix} \theta_{j,(1)}^t \\ \vdots \\ \theta_{j,(m)}^t \end{bmatrix},$$

where $\Sigma^t = \sigma^2[\theta_{(1)}^t \cdots \theta_{(m)}^t]^\top[\theta_{(1)}^t \cdots \theta_{(m)}^t]$ denotes a covariance matrix as in equation (32). Following the calculation of inequality (34), we obtain

$$\sum_{j=1}^d \|P_{+j} - P_{-j}\|_{\text{TV}}^2 \leq \frac{\delta^2}{\sigma^2} \sum_{t=1}^k \mathbb{E}\left[\text{tr}\left((\Sigma^t)^{-1}\Sigma^t\right)\right] = \frac{mk\delta^2}{\sigma^2}.$$

Substituting this inequality in place of (34) and following the subsequent proof implies the lower bound $\frac{1-q^{-1}}{10} d^{1-1/q} RG/\sqrt{mk} \cdot \min\{1, \sqrt{k/d}\}$. Replacing d with $\min\{k, d\}$ gives inequality (21).

B. Proof of Proposition 2

The proof is similar to that of Proposition 1, except instead of using the set $\mathcal{V} = \{-1, 1\}^d$, we use the $2d$ standard basis vectors and their negatives, that is, $\mathcal{V} = \{\pm e_j\}_{j=1}^d$. We use the same sampling distributions as in the proof of Proposition 1, so under P_v the random vectors $X \sim \mathsf{N}(\delta v, \sigma^2 I_{d \times d})$, and we have $f_v = \mathbb{E}_{P_v}[F(\theta; X)] = \delta\langle\theta, v\rangle$. Let us define P_j to be

the distribution P_v for $v = e_j$ and similarly for P_{-j}, and let $\theta^v = \arg\min_\theta\{f_v(\theta) \mid \|\theta\|_1 \leq R\} = -Rv$.

We now provide the reduction from optimization to testing. First, if $v = \pm e_j$, then any estimator $\widehat{\theta}$ satisfying $\text{sgn}(\widehat{\theta}_j) \neq \text{sgn}(\theta_j^v)$ must have $f_v(\widehat{\theta}) - f_v(\theta^v) \geq \delta R$. We thus see that for $v \in \{\pm e_j\}$,

$$f_v(\widehat{\theta}) - f_v(\theta^v) \geq \delta R \, \mathbf{1}\left\{\text{sgn}(\widehat{\theta}_j) \neq \text{sgn}(\theta_j^v)\right\}.$$

Consequently, we obtain the multiple binary hypothesis testing lower bound

$$\max_v \mathbb{E}_{P_v}[f_v(\widehat{\theta}) - f_v(\theta^v)] \geq \frac{1}{2d}\sum_{v \in \mathcal{V}} \mathbb{E}_{P_v}[f_v(\widehat{\theta}) - f_v(\theta^v)]$$

$$\geq \frac{\delta R}{2d}\sum_{j=1}^d \left[P_j(\text{sgn}(\widehat{\theta}_j) \neq -1) + P_{-j}(\text{sgn}(\widehat{\theta}_j) \neq 1)\right]$$

$$\overset{(i)}{\geq} \frac{\delta R}{2d}\sum_{j=1}^d \left[1 - \|P_j - P_{-j}\|_{\text{TV}}\right].$$

For the final inequality (i), we applied Le Cam's inequality as in the proof of Proposition 1. Thus, as in the derivation of inequality (31) from the Cauchy-Schwarz inequality, this yields

$$\epsilon_k^*(\mathcal{F}_{G,\infty}, \Theta) \geq \frac{\delta R}{2}\left(1 - \frac{1}{\sqrt{d}}\left(\sum_{j=1}^d \|P_j - P_{-j}\|_{\text{TV}}^2\right)^{\frac{1}{2}}\right). \tag{36}$$

We now turn to providing a bound on $\sum_{j=1}^d \|P_j - P_{-j}\|_{\text{TV}}^2$ analogous to that in the proof of Proposition 1. We claim that

$$\sum_{j=1}^d \|P_j - P_{-j}\|_{\text{TV}}^2 \leq 2\frac{k\delta^2}{\sigma^2}. \tag{37}$$

Inequality (37) is nearly immediate from Lemma 7. Indeed, given the pair $W = [\theta \ \tau] \in \mathbb{R}^{d \times 2}$, the observation $Y = W^\top X$ is distributed (conditional on v and W) as $\mathsf{N}(\delta W^\top v, \Sigma)$ where $\Sigma = \sigma^2 W^\top W$ is the covariance (32). For $v = e_j$ and $w = -e_j$, we know that $\langle\theta, v - w\rangle = 2\theta_j$ and so

$$D_{\text{kl}}\left(\mathsf{N}(\delta W^\top v, \Sigma)\|\mathsf{N}(\delta W^\top w, \Sigma)\right) = 2\delta^2 \begin{bmatrix} \theta_j \\ \tau_j \end{bmatrix}^\top \Sigma^{-1}\begin{bmatrix} \theta_j \\ \tau_j \end{bmatrix}.$$

By analogy with the proof of Lemma 7, we may repeat the derivation of inequalities (33) and (34) *mutatis mutandis* to obtain inequality (37). Combining inequalities (36) and (37) then gives the lower bound

$$\epsilon_k^*(\mathcal{F}_{G,\infty}, \Theta) \geq \frac{\delta R}{2}\left(1 - \left(\frac{2\delta^2 k}{d\sigma^2}\right)^{\frac{1}{2}}\right).$$

It thus remains to choose δ and σ^2 to guarantee the containment $(F, P) \in \mathcal{F}_{G,\infty}$. Equivalently, we must establish the gradient bound $\mathbb{E}[\|X\|_\infty^2] \leq G^2$, with which the next lemma helps.

Lemma 8: Given any vector with $\|v\|_\infty \leq 1$, and the random vector $X \sim \mathsf{N}(\delta v, \sigma^2 I_{d \times d})$, we have

$$\mathbb{E}[\|X\|_\infty^2] \leq 3\sigma^2 \log(3d) + 4\delta^2.$$

Proof: The vector $Z = X - \delta v$ has $\mathsf{N}(0, \sigma^2 I_{d \times d})$ distribution. By Jensen's inequality, for all $\epsilon \geq 0$ we have

$$\|X\|_\infty^2 \leq (1 + \epsilon) \|Z\|_\infty^2 + (1 + \epsilon^{-1}) \delta^2 \|v\|_\infty^2$$
$$\leq (1 + \epsilon) \|Z\|_\infty^2 + (1 + \epsilon^{-1}) \delta^2.$$

Classical results on Gaussian vectors [36, Ch. 2] imply $\mathbb{E}[\|Z\|_\infty^2] \leq \sigma^2(\frac{1}{\lambda} \log d + \frac{1}{2\lambda} \log \frac{1}{1-2\lambda})$ for all $\lambda \in [0, 1/2]$, so taking $\epsilon = 1/3$ and $\lambda = 4/9$ implies the lemma. $\square$

As a consequence of Lemma 8, by taking

$$\sigma^2 = \frac{2G^2}{9 \log(3d)} \quad \text{and} \quad \delta^2 = \frac{G^2}{36 \log(3d)} \min\left\{1, \frac{d}{k}\right\},$$

we obtain the bounds

$$\mathbb{E}[\|X\|_\infty^2] \leq \frac{2}{3}G^2 + \frac{4}{36}G^2 < G^2$$

and

$$1 - \left(\frac{2\delta^2 k}{d\sigma^2}\right)^{\frac{1}{2}} \geq 1 - \left(\frac{18}{72}\right)^{\frac{1}{2}} = \frac{1}{2}.$$

Substituting into the lower bound on ϵ_k^* yields

$$\epsilon_k^* \geq \frac{\delta R}{4} \geq \frac{1}{24\sqrt{\log(3d)}} \frac{GR}{\sqrt{k}} \min\left\{\sqrt{k}, \sqrt{d}\right\}.$$

Modulo this lower bound holding for each dimension $d_0 \leq d$, this completes the proof.

To complete the proof, we note that as in the proof of Proposition 1, we may provide a lower bound on the optimization error for any $d_0 \leq d$-dimensional problem. In particular fix $d_0 \leq d$ and let $\mathcal{V} = \{\pm e_j\}_{j=1}^{d_0} \subset \mathbb{R}^d$. Now, conditional on $v \in \mathcal{V}$, let P_v denote the distribution on X with independent coordinates whose distributions are $X_j \sim \mathsf{N}(\delta v_j, \sigma^2)$ for $j \leq d_0$ and $X_j = 0$ for $j > d_0$. As in the proof Proposition 1, we may reproduce the preceding arguments by substituting $d_0 \leq d$ for every appearance of the dimension d, giving that for all $d_0 \leq d$,

$$\epsilon_k^*(\mathcal{F}_{G,\infty}, \Theta) \geq \frac{1}{24\sqrt{\log(3d_0)}} \frac{GR}{\sqrt{k}} \min\left\{\sqrt{k}, \sqrt{d_0}\right\}.$$

Choosing $d_0 = \min\{d, k\}$ completes the proof of Proposition 2.

C. Proof of Proposition 3

This proof is somewhat similar to that of Proposition 1, in that we use the set $\mathcal{V} = \{-1, 1\}^d$ to construct a collection of functions whose minima are relatively well-separated, but for which function evaluations are hard to distinguish. In particular, for $\delta > 0$, we construct functions f_v whose minima—for different elements v, w—are all of the order $\delta \|v - w\|_1$ distant from one another, yet $\sup_\theta |f_v(\theta) - f_w(\theta)| \lesssim \delta$, so that many observations are necessary to distinguish the functions.

In more detail, for $v \in \mathcal{V} = \{-1, 1\}^d$, define the probability distribution P_v to be supported on $\{v\} \times \mathbb{R}$, where each independent draw $X = (v, \xi) \sim P_v$ contains an independent $\xi \sim \mathsf{N}(0, \sigma^2)$. Fix $\delta \in (0, Rd^{-1/q}]$, and define $G_d = Gd^{-1/p}$. Then for $x = (v, \xi)$, we define

$$F(\theta; x) = G_d \|\theta - \delta v\|_1 + \xi,$$

so

$$f_v(\theta) = G_d \|\theta - \delta v\|_1 \quad \text{and} \quad F(\theta; (v, \xi)) = f_v(\theta) + \xi.$$

Consequently, we have $\delta v = \theta^v := \arg\min_{\theta \in \Theta} f_v(\theta)$, as $\|\delta v\|_q \leq Rd^{-1/q} \|v\|_q = R$, and the variance bound $\mathbb{E}[(F(\theta; X) - f(\theta))^2] \leq \sigma^2$ is evident. Moreover,

$$\|\partial F(\theta; x)\|_p = G_d \|\text{sgn}(\theta - \delta v)\|_p \leq G_d d^{-1/p} d^{1/p} = G,$$

so the functions belong to $\mathcal{F}_{\sigma, G, p}$. By inspection, we have the separation

$$f_v(\theta) - f_v(\theta^v) \geq \delta G_d \sum_{j=1}^d \mathbf{1}\left\{\text{sgn}(\theta_j) \neq v_j\right\},$$

which is analogous to inequality (29).

Abusing notation and defining P_v to be the distribution of the k observations $F(\theta^t; X^t)$ available to the method, our earlier extension (31) of Assouad's method implies

$$\epsilon_k^{\text{single}}(\mathcal{F}_\sigma, \Theta) \geq \frac{\delta G_d}{2} \sum_{j=1}^d \left(1 - \|P_{+j} - P_{-j}\|_{\text{TV}}\right)$$
$$\geq \frac{d \delta G_d}{2}\left(1 - \left(\frac{1}{2d} \sum_{j=1}^d D_{\text{kl}}\left(P_{+j} \| P_{-j}\right)\right)^{\frac{1}{2}}\right), \tag{38}$$

where $P_{+j} = 2^{1-d} \sum_{v:v_j=1} P_v$, and similarly for $-j$. Now, note that for any $v, w \in \mathcal{V}$ such that $\|v - w\|_1 \leq 2$, we have the inequality

$$\sup_\theta |f_v(\theta) - f_w(\theta)| \leq G_d \|\delta v - \delta w\|_1 \leq 2\delta G_d$$

(compare with Lemma 10 of Shamir [14]). In particular, this uniform inequality implies that for distributions P_v and P_w, the observations $F(\theta; X) = f_v(\theta) + \xi$ are normally distributed random variables with (absolute) difference in means bounded by $\delta G_d \|v - w\|_1$ and variance σ^2. Using that the KL divergence is jointly convex in both its arguments, we have (by a completely parallel argument to the proof of Lemma 7 in Appendix D) that

$$D_{\text{kl}}\left(P_{+j} \| P_{-j}\right) \leq \frac{1}{2^d} \sum_{v \in \mathcal{V}} D_{\text{kl}}\left(P_{v,+j} \| P_{v,-j}\right)$$
$$\leq \frac{k}{2^d} \sum_{v \in \mathcal{V}} D_{\text{kl}}\left(\mathsf{N}(\delta G_d, \sigma^2) \| \mathsf{N}(-\delta G_d, \sigma^2)\right)$$
$$= \frac{k}{2^d} \sum_{v \in \mathcal{V}} \frac{1}{2\sigma^2} 4G_d^2 \delta^2 = \frac{2k G_d^2 \delta^2}{\sigma^2}.$$

Substituting the KL divergence bound in the preceding display into our inequality (38), we find

$$\epsilon_k^{\text{single}}(\mathcal{F}_\sigma, \Theta) \geq \frac{d \delta G_d}{2}\left(1 - \sqrt{k \frac{G_d^2 \delta^2}{\sigma^2}}\right)$$
$$= \frac{d \delta G_d}{2}\left(1 - \delta \frac{\sqrt{k} G_d}{\sigma}\right).$$

Choosing $\delta = \min\{Rd^{-1/q}, \sigma/2G_d\sqrt{k}\}$ and substituting $G_d = Gd^{-1/p}$ gives the proposition.

VI. DISCUSSION

We have analyzed algorithms for optimization problems that use only random function values—as opposed to gradient computations—to minimize an objective function. The algorithms we present are optimal: their convergence rates cannot be improved (in a minimax sense) by more than numerical constant factors. In addition to showing the optimality of several algorithms for smooth convex optimization without gradient information, we have also shown that the non-smooth case is no more difficult from an iteration complexity standpoint, though it requires more carefully constructed randomization schemes. As a consequence of our results, we have additionally attained sharp rates for bandit online convex optimization problems with multi-point feedback. We have also shown the necessary transition in convergence rates between gradient-based algorithms and those that compute only function values: when (sub)gradient information is available, attaining ϵ-accurate solution to an optimization problem requires $\mathcal{O}(1/\epsilon^2)$ gradient observations, while at least $\Omega(d/\epsilon^2)$ observations—but no more—are necessary using paired function evaluations, and at least $\Omega(d^2/\epsilon^2)$ are necessary using only a single function evaluation. An interesting open question is to further understand this last setting: what is the optimal iteration complexity in this case?

APPENDIX

In this appendix, we collect the proofs of the various lemmas used in our convergence arguments.

A. Proof of Lemma 4

We consider each of the distributions in turn. When Z has $\mathsf{N}(0, I_{d \times d})$ distribution, standard χ^2-distributed random variable calculations imply

$$\mathbb{E}\left[\|Z\|_2^k\right] = 2^{\frac{k}{2}} \frac{\Gamma(\frac{k}{2} + \frac{d}{2})}{\Gamma(\frac{d}{2})}.$$

That $\mathbb{E}[ZZ^\top] = I_{d \times d}$ is immediate, and the constant values c_k for $k \leq 4$ follow from direct calculations. For samples Z from the ℓ_2-sphere, it is clear that $\|Z\|_2 = \sqrt{d}$, so we may take $c_k = 1$ in the statement of the lemma. When $Z \sim \text{Uniform}(\mathbb{B}^d)$, the density $p(t)$ of $\|Z\|_2$ is given by $d \cdot t^{d-1}$; consequently, for any $k > -d$ we have

$$\mathbb{E}[\|Z\|_2^k] = \int_0^1 t^k p(t) \, dt = d \int_0^1 t^{d+k-1} \, dt = \frac{d}{d+k}. \quad (39)$$

Thus for $Z \sim \text{Uniform}(\sqrt{d+2}\,\mathbb{B}^d)$ we have $\mathbb{E}[ZZ^\top] = I_{d \times d}$, and $\mathbb{E}[\|Z\|_2^k] = (d+2)^{k/2} d/(d+k)$.

B. Proof of Lemma 5

The proof of Lemma 5 is based on a sequence of auxiliary results. Since the Lipschitz continuity of h implies the result for $d = 1$ directly, we focus on the case $d \geq 2$. First, we have the following standard result on the dimension-independent concentration of rotationally symmetric sub-Gaussian random vectors. We use this to prove that the perturbed h is close to the unperturbed h with high probability.

Lemma 9 (Rotationally Invariant Concentration): Let Z be a random variable in $\mathbb{R}^d$ having one of the following distributions: $\mathsf{N}(0, I_{d \times d})$, $\text{Uniform}(\sqrt{d+2}\,\mathbb{B}^d)$, or $\text{Uniform}(\sqrt{d}\,\mathbb{S}^{d-1})$. There is a universal (numerical) constant $c > 0$ such that for any G-Lipschitz continuous function h,

$$\mathbb{P}\left(|h(Z) - \mathbb{E}[h(Z)]| > \epsilon\right) \leq 2 \exp\left(-\frac{c\,\epsilon^2}{G^2}\right).$$

In the case of the normal distribution, we take $c = \frac{1}{2}$.

These results are standard (e.g., see Propositions 1.10 and 2.9 of Ledoux [37]).

Our next result shows that integrating out Z_2 leaves us with a smoother deviation problem, at the expense of terms of order at most $u^k \log^{k/2}(d)$. To state the lemma, we define the difference function $\Delta_u(\theta) = \mathbb{E}[h(\theta + uZ_2)] - h(\theta)$. Note that since h is convex and $\mathbb{E}[Z_2] = 0$, Jensen's inequality implies $\Delta_u(\theta) \geq 0$.

Lemma 10: Under the conditions of Lemma 5, we have

$$\mathbb{E}\left[|h(Z_1 + uZ_2) - h(Z_1)|^k\right] \leq 2^{k-1}\mathbb{E}[\Delta_u(Z_1)^k]$$
$$+ c^{-\frac{k}{2}} 2^{k-1} k^{\frac{k}{2}} u^k \log^{\frac{k}{2}}(d + 2k) + \sqrt{2}u^k$$

for any $k \geq 1$. Here c is the same constant in Lemma 9.

Proof: For each $\theta \in \Theta$, the function $\tau \mapsto h(\theta + u\tau)$ is u-Lipschitz, so Lemma 9 implies

$$\mathbb{P}\left(|h(\theta + uZ_2) - \mathbb{E}[h(\theta + uZ_2)]| > \epsilon\right) \leq 2 \exp\left(-\frac{c\,\epsilon^2}{u^2}\right).$$

On the event $A_\theta(\epsilon) := \{|h(\theta + uZ_2) - \mathbb{E}[h(\theta + uZ_2)]| \leq \epsilon\}$, we have

$$|h(\theta + uZ_2) - h(\theta)|^k$$
$$\leq 2^{k-1}|h(\theta + uZ_2) - \mathbb{E}[h(\theta + uZ_2)]|^k + 2^{k-1}\Delta_u(\theta)^k$$
$$\leq 2^{k-1}\epsilon^k + 2^{k-1}\Delta_u(\theta)^k,$$

which implies

$$\mathbb{E}\left[|h(\theta + uZ_2) - h(\theta)|^k \cdot \mathbf{1}\{A_\theta(\epsilon)\}\right]$$
$$\leq 2^{k-1}\Delta_u(\theta)^k + 2^{k-1}\epsilon^k. \quad (40a)$$

On the complement $A_\theta^c(\epsilon)$, which occurs with probability at most $2\exp(-c\epsilon^2/u^2)$, we use the Lipschitz continuity of h and Cauchy-Schwarz inequality to obtain

$$\mathbb{E}\left[|h(\theta + uZ_2) - h(\theta)|^k \cdot \mathbf{1}\{A_\theta(\epsilon)^c\}\right]$$
$$\leq \mathbb{E}\left[u^k \|Z_2\|_2^k \cdot \mathbf{1}\{A_\theta(\epsilon)^c\}\right]$$
$$\leq u^k \mathbb{E}[\|Z_2\|_2^{2k}]^{\frac{1}{2}} \cdot \mathbb{P}\left(A_\theta(\epsilon)^c\right)^{\frac{1}{2}}.$$

By direct calculations, Assumption F implies that $\mathbb{E}[\|Z_2\|_2^{2k}] \leq (d + 2k)^k$. Thus,

$$\mathbb{E}\left[|h(\theta + uZ_2) - h(\theta)|^k \cdot \mathbf{1}\{A_\theta(\epsilon)^c\}\right]$$
$$\leq u^k(d + 2k)^{\frac{k}{2}} \cdot \sqrt{2}\exp\left(-\frac{c\,\epsilon^2}{2u^2}\right). \quad (40b)$$

Combining the estimates (40a) and (40b) gives

$$\mathbb{E}\left[|h(\theta + uZ_2) - h(\theta)|^k\right] \leq 2^{k-1}\Delta_u(\theta)^k$$
$$+ 2^{k-1}\epsilon^k + \sqrt{2}u^k(d+2k)^{\frac{k}{2}}\exp\left(-\frac{c\,\epsilon^2}{2u^2}\right).$$

Setting $\epsilon^2 = \frac{k}{c}u^2\log(d+2k)$ and taking expectations over $Z_1 \sim \mu_1$ gives Lemma 10. □

By Lemma 10, it suffices to control the bias $\mathbb{E}[\Delta_u(Z_1)] = \mathbb{E}[h(Z_1 + uZ_2) - h(Z_1)]$. The following result allows us to reduce this problem to one of bounding a certain one-dimensional expectation.

Lemma 11: Let Z and W be random variables in $\mathbb{R}^d$ with rotationally invariant distributions and finite first moments. Let $\mathcal{H}$ denote the set of 1-Lipschitz convex functions $h: \mathbb{R}^d \to \mathbb{R}$, and for $h \in \mathcal{H}$, define $V(h) = \mathbb{E}[h(W) - h(Z)]$. Then

$$\sup_{h\in\mathcal{H}} V(h) = \sup_{a\in\mathbb{R}_+} \mathbb{E}\left[|\,\|W\|_2 - a| - |\,\|Z\|_2 - a|\right].$$

Proof: First, we note that $V(h) = V(h \circ U)$ for any unitary transformation U; since V is linear, if we define $\hat{h}$ as the average of $h \circ U$ over all unitary U then $V(h) = V(\hat{h})$. Moreover, for $h \in \mathcal{H}$, we have $\hat{h}(\theta) = \hat{h}_1(\|\theta\|_2)$ for some $\hat{h}_1 : \mathbb{R}_+ \to \mathbb{R}$, which is necessarily 1-Lipschitz and convex.

Letting $\mathcal{H}_1$ denote the 1-Lipschitz convex $h : \mathbb{R} \to \mathbb{R}$ satisfying $h(0) = 0$, we thus have $\sup_{h\in\mathcal{H}} V(h) = \sup_{h\in\mathcal{H}_1} \mathbb{E}[h(\|W\|_2) - h(\|Z\|_2)]$. Now, we define $\mathcal{G}_1$ to be the set of measurable non-decreasing functions bounded in $[-1,1]$. Then by first-order properties of convex functions [19, Ch. I], for any $h \in \mathcal{H}_1$, we can write $h(t) = \int_0^t g(s)ds$ for some $g \in \mathcal{G}_1$. Using this representation, we have

$$\sup_{h\in\mathcal{H}} V(h)$$
$$= \sup_{h\in\mathcal{H}_1} \{\mathbb{E}[h(\|W\|_2) - h(\|Z\|_2)]\}$$
$$= \sup_{g\in\mathcal{G}_1} \left\{\mathbb{E}[h(\|W\|_2) - h(\|Z\|_2)] \mid h(t) = \int_0^t g(s)ds\right\}. \quad (41)$$

Let g_a denote the $\{-1,1\}$-valued function with step at a, that is, $g_a(t) = -\mathbf{1}\{t \leq a\} + \mathbf{1}\{t > a\}$. We define $\mathcal{G}_1^{(n)}$ to be the set of non-decreasing step functions bounded in $[-1,1]$ with at most n steps, that is, functions of the form $g(t) = \sum_{i=1}^n b_i g_{a_i}(t)$, where $|g(t)| \leq 1$ for all $t \in \mathbb{R}$. We may then further simplify the expression (41) by replacing $\mathcal{G}_1$ with $\mathcal{G}_1^{(n)}$. The extremal points of $\mathcal{G}_1^{(n)}$ are the step functions $\{g_a \mid a \in \mathbb{R}\}$, and since the supremum (41) is linear in g, it may be taken over such g_a. Lemma 11 then follows by noting the integral equality $\int_0^t g_a(s)ds = |t - a| - |a|$. The restriction to $a \geq 0$ in the lemma follows since $\|v\|_2 \geq 0$ for all $v \in \mathbb{R}^d$. □

By Lemma 11, for any 1-Lipschitz h, the associated difference function has expectation bounded as

$$\mathbb{E}[\Delta_u(Z_1)] = \mathbb{E}[h(Z_1 + uZ_2) - h(Z_1)]$$
$$\leq \sup_{a\in\mathbb{R}_+} \mathbb{E}\left[|\,\|Z_1 + uZ_2\|_2 - a| - |\,\|Z_1\|_2 - a|\right].$$

For the distributions identified by Assumption F, we can in fact show that the preceding supremum is attained at $a = 0$.

Lemma 12: Let $Z_1 \sim \mu_1$ and $Z_2 \sim \mu_2$ be independent, where μ_1 and μ_2 satisfy Assumption F. For any $u \geq 0$, the function

$$a \mapsto \zeta(a) := \mathbb{E}\left[|\,\|Z_1 + uZ_2\|_2 - a| - |\,\|Z_1\|_2 - a|\right]$$

is non-increasing in $a \geq 0$.

We return to prove this lemma at the end of the section.

With the intermediate results above, we can complete our proof of Lemma 5. In view of Lemma 10, we only need to bound $\mathbb{E}[\Delta_u(Z_1)^k]$, where $\Delta_u(\theta) = \mathbb{E}[h(\theta + uZ_2)] - h(\theta)$. Recall that $\Delta_u(\theta) \geq 0$ since h is convex. Moreover, since h is 1-Lipschitz,

$$\Delta_u(\theta) \leq \mathbb{E}\left[|h(\theta + uZ_2) - h(\theta)|\right]$$
$$\leq \mathbb{E}[\|uZ_2\|_2] \leq u\mathbb{E}[\|Z_2\|_2^2]^{1/2} = u\sqrt{d},$$

where the last equality follows from the choices of Z_2 in Assumption F. Therefore, we have the crude but useful bound

$$\mathbb{E}[\Delta_u(Z_1)^k] \leq u^{k-1}d^{\frac{k-1}{2}}\mathbb{E}[\Delta_u(Z_1)]$$
$$= u^{k-1}d^{\frac{k-1}{2}}\mathbb{E}[h(Z_1 + uZ_2) - h(Z_1)], \quad (42)$$

where the last expectation is over both Z_1 and Z_2. Since Z_1 and Z_2 both have rotationally invariant distributions, Lemmas 11 and 12 imply that the expectation in expression (42) is bounded by

$$\mathbb{E}[h(Z_1 + uZ_2) - h(Z_1)] \leq \mathbb{E}\left[\|Z_1 + uZ_2\|_2 - \|Z_1\|_2\right].$$

Lemma 5 then follows by bounding the norm difference in the preceding display for each choice of the smoothing distributions in Assumption F. We claim that

$$\mathbb{E}\left[\|Z_1 + uZ_2\|_2 - \|Z_1\|_2\right] \leq \frac{1}{\sqrt{2}}u^2\sqrt{d}. \quad (43)$$

To see this inequality, we consider the possible distributions for the pair Z_1, Z_2 under Assumption F.

(1): Let T_d have χ^2-distribution with d degrees of freedom. Then for Z_1, Z_2 independent and $\mathsf{N}(0, I_{d\times d})$-distributed, we have the distributional identities $\|Z_1 + uZ_2\|_2 \stackrel{d}{=} \sqrt{1+u^2}\sqrt{T_d}$ and $\|Z_1\|_2 \stackrel{d}{=} \sqrt{T_d}$. Using the inequalities $\sqrt{1+u^2} \leq 1 + \frac{1}{2}u^2$ and $\mathbb{E}[\sqrt{T_d}] \leq \mathbb{E}[T_d]^{\frac{1}{2}} = \sqrt{d}$, we obtain

$$\mathbb{E}\left[\|Z_1 + uZ_2\|_2 - \|Z_1\|_2\right] = \left(\sqrt{1+u^2} - 1\right)\mathbb{E}[\sqrt{T_d}]$$
$$\leq \frac{1}{2}u^2\sqrt{d}.$$

(2): By Assumption F, if Z_1 is uniform on $\sqrt{d+2}\,\mathbb{B}^d$ then Z_2 has either Uniform($\sqrt{d+2}\,\mathbb{B}^d$) or Uniform($\sqrt{d}\,\mathbb{S}^{d-1}$) distribution. Using the inequality $\sqrt{a+b} - \sqrt{a} \leq b/(2\sqrt{a})$, valid for $a \geq 0$ and $b \geq -a$, we may write

$$\|Z_1 + uZ_2\|_2 - \|Z_1\|_2$$
$$= \sqrt{\|Z_1\|_2^2 + 2u\langle Z_1, Z_2\rangle + u^2\|Z_2\|_2^2} - \sqrt{\|Z_1\|_2^2}$$
$$\leq \frac{2u\langle Z_1, Z_2\rangle + u^2\|Z_2\|_2^2}{2\|Z_1\|_2} = u\left\langle\frac{Z_1}{\|Z_1\|_2}, Z_2\right\rangle + \frac{1}{2}u^2\frac{\|Z_2\|_2^2}{\|Z_1\|_2}.$$

Since Z_1 and Z_2 are independent and $\mathbb{E}[Z_2] = 0$, the expectation of the first term on the right hand side above

vanishes. For the second term, the independence of Z_1 and Z_2 and moment calculation (39) imply

$$\mathbb{E}\left[\|Z_1 + uZ_2\|_2 - \|Z_1\|_2\right] \leq \frac{1}{2}u^2 \mathbb{E}\left[\frac{1}{\|Z_1\|_2}\right] \mathbb{E}\left[\|Z_2\|_2^2\right]$$
$$= \frac{1}{2}u^2 \cdot \frac{1}{\sqrt{d+2}}\frac{d}{(d-1)} \cdot d \leq \frac{1}{\sqrt{2}}u^2\sqrt{d},$$

where the last inequality holds for $d \geq 2$.

We thus obtain the claim (43), and applying inequality (43) to our earlier computation (42) yields

$$\mathbb{E}[\Delta_u(Z_1)^k] \leq \frac{1}{\sqrt{2}}u^{k+1}d^{\frac{k}{2}}.$$

Plugging in this bound on Δ_u to Lemma 10, we obtain the result

$$\mathbb{E}\left[|h(Z_1 + uZ_2) - h(Z_1)|^k\right]$$
$$\leq 2^{k-\frac{3}{2}}u^{k+1}d^{\frac{k}{2}} + c^{-\frac{k}{2}}2^{k-1}k^{\frac{k}{2}}u^k \log^{\frac{k}{2}}(d+2k) + \sqrt{2}u^k$$
$$\leq c_k u^k \left[ud^{\frac{k}{2}} + 1 + \log^{\frac{k}{2}}(d+2k)\right],$$

where c_k is a constant that only depends on k. This is the desired statement of Lemma 5.

We now return to prove the remaining intermediate lemma.

Proof of Lemma 12: Since the quantity $\|Z_1 + uZ_2\|_2$ has a density with respect to Lebesgue measure, standard results on differentiating through an expectation [38] imply

$$\frac{d}{da}\mathbb{E}\left[|\|Z_1 + uZ_2\|_2 - a|\right] = \mathbb{E}[\text{sign}(a - \|Z_1 + uZ_2\|_2)]$$
$$= \mathbb{P}(\|Z_1 + uZ_2\|_2 \leq a) - \mathbb{P}(\|Z_1 + uZ_2\|_2 > a),$$

where we used that the subdifferential of $a \mapsto |v - a|$ is $\text{sign}(a - v)$. As a consequence, we find that

$$\frac{d}{da}\zeta(a) = \mathbb{P}(\|Z_1 + uZ_2\|_2 \leq a) - \mathbb{P}(\|Z_1 + uZ_2\|_2 > a)$$
$$- \mathbb{P}(\|Z_1\|_2 \leq a) + \mathbb{P}(\|Z_1\|_2 > a)$$
$$= 2\left[\mathbb{P}(\|Z_1 + uZ_2\|_2 \leq a) - \mathbb{P}(\|Z_1\|_2 \leq a)\right]. \quad (44)$$

If we can show the quantity (44) is non-positive for all a, we obtain our desired result. It thus remains to prove that $\|Z_1 + uZ_2\|_2$ stochastically dominates $\|Z_1\|_2$ for each choice of μ_1, μ_2 satisfying Assumption F. We enumerate each of the cases below.

(1): Let T_d have χ^2-distribution with d degrees of freedom and $Z_1, Z_2 \sim \mathsf{N}(0, I_{d \times d})$. Then by definition we have $\|Z_1 + uZ_2\|_2 \stackrel{d}{=} \sqrt{1+u^2}\sqrt{T_d}$ and $\|Z_1\|_2 \stackrel{d}{=} \sqrt{T_d}$, and

$$\mathbb{P}(\|Z_1 + uZ_2\|_2 \leq a) = \mathbb{P}\left(\sqrt{T_d} \leq \frac{a}{\sqrt{1+u^2}}\right)$$
$$\leq \mathbb{P}\left(\sqrt{T_d} \leq a\right) = \mathbb{P}(\|Z_1\|_2 \leq a)$$

as desired.

(2): Now suppose Z_1, Z_2 are independent and distributed as $\text{Uniform}(r\,\mathbb{B}^d)$; our desired result will follow by setting $r = \sqrt{d+2}$. Let $p_0(t)$ and $p_u(t)$ denote the densities of $\|Z_1\|_2$ and $\|Z_1 + uZ_2\|_2$, respectively, with respect to Lebesgue measure on $\mathbb{R}$. We now compute them explicitly. For p_0, for $0 \leq t \leq r$ we have

$$p_0(t) = \frac{d}{dt}\mathbb{P}(\|Z_1\|_2 \leq t) = \frac{d}{dt}\left(\frac{t}{r}\right)^d = \frac{d\,t^{d-1}}{r^d},$$

and $p_0(t) = 0$ otherwise. For p_u, let λ denote the Lebesgue measure in $\mathbb{R}^d$ and σ denote the $(d-1)$-dimensional surface area in $\mathbb{R}^d$. The random variables Z_1 and uZ_2 have densities, respectively,

$$q_1(x) = \frac{1}{\lambda(r\,\mathbb{B}^d)} = \frac{1}{r^d\lambda(\mathbb{B}^d)} \quad \text{for } x \in r\mathbb{B}^d$$

and

$$q_u(x) = \frac{1}{\lambda(ur\,\mathbb{B}^d)} = \frac{1}{u^d r^d\lambda(\mathbb{B}^d)} \quad \text{for } x \in ur\mathbb{B}^d,$$

and $q_1(x) = q_u(x) = 0$ otherwise. The density of $Z_1 + uZ_2$ is given by the convolution

$$\tilde{q}(z) = \int_{\mathbb{R}^d} q_1(x)q_u(z-x)\,\lambda(dx)$$
$$= \int_{E(z)} \frac{1}{r^d\lambda(\mathbb{B}^d)} \cdot \frac{1}{u^d r^d\lambda(\mathbb{B}^d)}\,\lambda(dx) = \frac{\lambda(E(z))}{u^d r^{2d}\lambda(\mathbb{B}^d)^2}.$$

Here $E(z) := \mathbb{B}^d(0, r) \cap \mathbb{B}^d(z, ur)$ is the domain of integration, in which the densities $q_1(x)$ and $q_u(z-x)$ are nonzero. The volume $\lambda(E(z))$—and hence also $\tilde{q}(z)$—depend on z only via its norm $\|z\|_2$. Therefore, the density p_u of $\|Z_1 + uZ_2\|_2$ is

$$p_u(t) = \tilde{q}(te_1)\,\sigma(t\mathbb{S}^{d-1}) = \frac{\lambda(E(te_1))\,t^{d-1}\,\sigma(\mathbb{S}^{d-1})}{u^d r^{2d}\lambda(\mathbb{B}^d)^2}$$
$$= d\frac{\lambda(E(te_1))\,t^{d-1}}{u^d r^{2d}\lambda(\mathbb{B}^d)},$$

where the last equality above follows from the relation $\sigma(\mathbb{S}^{d-1}) = d\lambda(\mathbb{B}^d)$. Since $E(te_1) \subseteq \mathbb{B}^d(te_1, ur)$ by definition,

$$\lambda(E(te_1)) \leq \lambda\left(\mathbb{B}^d(te_1, ur)\right) = u^d r^d\lambda(\mathbb{B}^d),$$

so for all $0 \leq t \leq (1+u)r$ we have

$$p_u(t) = d\frac{\lambda(E(te_1))\,t^{d-1}}{u^d r^{2d}\lambda(\mathbb{B}^d)} \leq \frac{d\,t^{d-1}}{r^d},$$

and clearly $p_u(t) = 0$ for $t > (1+u)r$. In particular, $p_u(t) \leq p_1(t)$ for $0 \leq t \leq r$, which gives us our desired stochastic dominance inequality (44): for $a \in [0, r]$,

$$\mathbb{P}(\|Z_1 + uZ_2\|_2 \leq a) = \int_0^a p_u(t)\,dt$$
$$\leq \int_0^a p_0(t)\,dt = \mathbb{P}(\|Z_1\|_2 \leq a),$$

and for $a > r$ we have $\mathbb{P}(\|Z_1 + uZ_2\|_2 \leq a) \leq 1 = \mathbb{P}(\|Z_1\|_2 \leq a)$.

(3): Finally, consider the case when $Z_1 \sim \text{Uniform}(\sqrt{d+2}\,\mathbb{B}^d)$ and $Z_2 \sim \text{Uniform}(\sqrt{d}\,\mathbb{S}^{d-1})$. As in the previous case, we will show that $p_0(t) \leq p_u(t)$ for $0 \leq t \leq \sqrt{d+2}$, where $p_0(t)$ and $p_u(t)$ are the densities of $\|Z_1\|_2$ and $\|Z_1 + uZ_2\|_2$, respectively. We know that the density of $\|Z_1\|_2$ is

$$p_0(t) = \frac{d\,t^{d-1}}{(d+2)^{\frac{d}{2}}} \quad \text{for } 0 \leq t \leq \sqrt{d+2},$$

and $p_0(t) = 0$ otherwise. To compute p_u, we first determine the density $\tilde{q}(z)$ of the random variable $Z_1 + uZ_2$ with respect to the Lebesgue measure λ on $\mathbb{R}^d$. The usual convolution formula does not directly apply as Z_1 and Z_2 have densities

with respect to different base measures (λ and σ, respectively). However, as Z_1 and Z_2 are both uniform, we can argue as follows. Integrating over the surface $u\sqrt{d}\mathbb{S}^{d-1}$ (essentially performing a convolution), each point $uy \in u\sqrt{d}\,\mathbb{S}^{d-1}$ contributes the amount

$$\frac{1}{\sigma(u\sqrt{d}\,\mathbb{S}^{d-1})} \cdot \frac{1}{\lambda(\sqrt{d+2}\,\mathbb{B}^d)}$$

$$= \frac{1}{u^{d-1}\,d^{\frac{d-1}{2}}\,(d+2)^{\frac{d}{2}}\,\sigma(\mathbb{S}^{d-1})\,\lambda(\mathbb{B}^d)}$$

to the density $\tilde{q}(z)$, provided $\|z - uy\|_2 \leq \sqrt{d+2}$. For fixed $z \in (\sqrt{d+2} + u\sqrt{d})\mathbb{B}^d$, the set of such contributing points uy can be written as $E(z) = \mathbb{B}^d(z, \sqrt{d+2}) \cap \mathbb{S}^{d-1}(0, u\sqrt{d})$. Therefore, the density of $Z_1 + uZ_2$ is given by

$$\tilde{q}(z) = \frac{\sigma(E(z))}{u^{d-1}\,d^{\frac{d-1}{2}}\,(d+2)^{\frac{d}{2}}\,\sigma(\mathbb{S}^{d-1})\,\lambda(\mathbb{B}^d)}.$$

Since $\tilde{q}(z)$ only depends on z via its norm $\|z\|_2$, the formula above also gives us the density $p_u(t)$ of $\|Z_1 + uZ_2\|_2$:

$$p_u(t) = \tilde{q}(te_1)\,\sigma(t\mathbb{S}^{d-1}) = \frac{\sigma(E(z))\,t^{d-1}}{u^{d-1}\,d^{\frac{d-1}{2}}\,(d+2)^{\frac{d}{2}}\,\lambda(\mathbb{B}^d)}.$$

Noting that $E(z) \subseteq \mathbb{S}^{d-1}(0, u\sqrt{d})$ gives us

$$p_u(t) \leq \frac{\sigma(u\sqrt{d}\,\mathbb{S}^{d-1})\,t^{d-1}}{u^{d-1}\,d^{\frac{d-1}{2}}\,(d+2)^{\frac{d}{2}}\,\lambda(\mathbb{B}^d)} = \frac{d\,t^{d-1}}{(d+2)^{\frac{d}{2}}}.$$

In particular, we have $p_u(t) \leq p_0(t)$ for $0 \leq t \leq \sqrt{d+2}$, which, as we saw in the previous case, gives us the desired stochastic dominance inequality (44).

In this section, we prove the technical results necessary for the proofs of Propositions 1 and 2.

C. Proof of Lemma 6

First, note that the optimal vector $\theta^A = -d^{-1/q}\mathbb{1}$ with optimal value $-d^{1-1/q}$, and $\theta^B = -(d-i)^{-1/q}\mathbb{1}_{i+1:d}$, where $\mathbb{1}_{i+1:d}$ denotes the vector with 0 entries in its first i coordinates and 1 elsewhere. As a consequence, we have $\langle \theta^B, \mathbb{1}\rangle = -(d-i)^{1-1/q}$. Now we use the fact that by convexity of the function $x \mapsto -x^{1-1/q}$ for $q \in [1, \infty]$,

$$-d^{1-1/q} \leq -(d-i)^{1-1/q} - \frac{1-1/q}{d^{1/q}}i,$$

since the derivative of $x \mapsto -x^{1-1/q}$ at $x = d$ is given by $-(1-1/q)/d^{1/q}$ and the quantity $-x^{1-1/q}$ is non-increasing in x for $q \in [1, \infty]$.

D. Proof of Lemma 7

For notational convenience, let the distribution $P_{v,+j}$ be identical to the distribution P_v but with the jth coordinate v_j forced to be $+1$ and similarly for $P_{v,-j}$. Using Pinsker's inequality and the joint convexity of the KL-divergence, we have

$$\|P_{+j} - P_{-j}\|_{\mathrm{TV}}^2$$
$$\leq \frac{1}{4}\left[D_{\mathrm{kl}}\left(P_{+j}\|P_{-j}\right) + D_{\mathrm{kl}}\left(P_{-j}\|P_{+j}\right)\right]$$
$$\leq \frac{1}{2^{d+2}}\sum_{v\in\mathcal{V}}\left[D_{\mathrm{kl}}\left(P_{v,+j}\|P_{v,-j}\right) + D_{\mathrm{kl}}\left(P_{v,-j}\|P_{v,+j}\right)\right].$$

If we define $P_v^t(\cdot \mid y)$ to be the distribution of the tth observation Y^t conditional on v and $Y^{1:t-1} = y$, the chain-rule for KL-divergences [39] yields

$$D_{\mathrm{kl}}\left(P_{v,+j}\|P_{v,-j}\right)$$
$$= \sum_{t=1}^{k}\int_{\mathcal{Y}^{t-1}} D_{\mathrm{kl}}\left(P_{v,+j}^t(\cdot \mid y)\|P_{v,-j}^t(\cdot \mid y)\right) dP_{v,+j}(y).$$

We show how to bound the preceding sequence of KL-divergences for the observational scheme based on function-evaluations we allow. Let $W = [\theta\ \tau] \in \mathbb{R}^{d\times 2}$ denote the pair of query points, so we have by construction that the observation $Y = W^\top X$ where $X \mid V = v \sim \mathsf{N}(\delta v, \sigma^2 I_{d\times d})$. In particular, given v and the pair W, the vector $Y \in \mathbb{R}^d$ is normally distributed with mean $\delta W^\top v$ and covariance $\sigma^2 W^\top W = \Sigma$, where the covariance Σ is defined in equation (32). The KL divergence between normal distributions is $D_{\mathrm{kl}}(\mathsf{N}(\mu_1, \Sigma)\|\mathsf{N}(\mu_2, \Sigma)) = \frac{1}{2}(\mu_1 - \mu_2)^\top \Sigma^{-1}(\mu_1 - \mu_2)$. Note that if v and w differ in only coordinate j, then $\langle v - w, \theta\rangle = (v_j - w_j)\theta_j$. We thus obtain

$$D_{\mathrm{kl}}\left(P_{v,+j}^t(\cdot \mid y^{1:t-1})\|P_{v,-j}^t(\cdot \mid y^{1:t-1})\right)$$
$$\leq 2\delta^2 \begin{bmatrix}\theta_j^t\\\tau_j^t\end{bmatrix}^\top (\Sigma^t)^{-1}\begin{bmatrix}\theta_j^t\\\tau_j^t\end{bmatrix},$$

where we have used our w.l.o.g. assumption that the pair (θ^t, τ^t) is measurable $Y^{1:t-1}$. We obtain an identical bound for $D_{\mathrm{kl}}(P_{v,-j}^t(\cdot \mid y^{1:t-1})\|P_{v,+j}^t(\cdot \mid y^{1:t-1}))$. Combining the sequence of inequalities from the preceding paragraph, we thus obtain

$$\|P_{+j} - P_{-j}\|_{\mathrm{TV}}^2 \leq \frac{\delta^2}{2^d}\sum_{t=1}^{k}\sum_{v\in\mathcal{V}}\mathbb{E}\left[\begin{bmatrix}\theta_j^t\\\tau_j^t\end{bmatrix}^\top (\Sigma^t)^{-1}\begin{bmatrix}\theta_j^t\\\tau_j^t\end{bmatrix}\right]$$
$$= \delta^2\sum_{t=1}^{k}\mathbb{E}\left[\begin{bmatrix}\theta_j^t\\\tau_j^t\end{bmatrix}^\top (\Sigma^t)^{-1}\begin{bmatrix}\theta_j^t\\\tau_j^t\end{bmatrix}\right]$$

where for the equality we used the definitions of the distributions $P_{v,\pm j}$ and $P_{\pm j}$. This is the claimed inequality (33).

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