Math 103A Midterm 1

Q1: ① check binary operation:

Let $a, b \in \mathbb{Z}$, $a * b = a+b+2 \in \mathbb{Z}$.

$\langle \mathbb{Z}, * \rangle$ is closed

G_1: for all $a, b, c \in \mathbb{Z}$,

$$(a*b)*c = (a+b+2)*c = a+b+2+c+2 = a+b+c+4$$

$$a*(b*c) = a*(b+c+2) = a+b+c+2+2 = a+b+c+4$$

$(a*b)*c = a*(b*c)$ ✓

G_2: identity is -2.

for all $x \in \mathbb{Z}$, $(-2)*x = -2+x+2 = x-2+2 = x*(-2) = x$

$-2 \in \mathbb{Z}$. ✓

G_3: The inverse of $a \in \mathbb{Z}$ is $-4-a$

$$(-4-a)*a = -4-a+a+2 = -2$$

$$\parallel$$

$$a*(-4-a) = a-4-a+2 = -2$$

Since $-4 \in \mathbb{Z}$, $-a \in \mathbb{Z}$

Thus, $(-4-a) \in \mathbb{Z}$ ✓

Q2: G is abelian group with identity e.
wanna show subset $\{x \in G \mid x^3 = e\}$ is a subgroup of G.

① show $M = \{x \in G \mid x^3 = e\}$ is closed under binary operation

Let $x, y \in M$
$x^3 = e, y^3 = e$
Since G is abelian $\Rightarrow xy = yx$.
$(xy)(xy)(xy) = x^3 y^3$ — G is commutative. (can change order)
$= e \cdot e$
$= e$.
Hence, $xy \in M$ — closed.

② G has identity e, $e^3 = e \in M$. identity exists.

③ $x^3 = e$ $\quad xxx = e$.

$x^{-1}xxx = x^{-1}$
$xx = x^{-1}$ multiply x^{-1} on left
$x^{-1}xx = x^{-1}x^{-1}$ multiply x^{-1} on left
$x = x^{-1}x^{-1}$
$x^{-1}x = x^{-1}x^{-1}x^{-1}$ multiply x^{-1} on left
$e = (x^{-1})^3$

Hence $x^{-1} \in M$.
Therefore, M is a subgroup of G.

Q3:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 1 & 2 \end{pmatrix}$$

(a) $\sigma = (1,3)(2,4)(5,6)$

$\tau = (1,6,2,5)(3,4)$

(b) $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix}$

$= (1,3)(2,4)(5,6)$

$\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 4 & 3 & 2 & 1 \end{pmatrix}$

$= (1,5,2,6)(3,4)$

(c) $\sigma^2 \sigma^{-1} \tau^{-1} = \sigma^2 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 4 & 3 \end{pmatrix}$

$= i \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 4 & 3 \end{pmatrix}$

$= (1,6,3,2,5,4)$

4:

$$\mu\rho^k \mu\rho^{-2} \mu^{-1}$$

$$= \mu \rho^k \mu\rho^{-2} \mu \qquad \mu^{-1} = \mu$$

$$= \mu \rho^k \mu (\mu \rho^{n+2})$$

$$= \mu\rho^k (\mu\mu) \rho^{n+2}$$

$$= \mu\rho^k \rho^{n+2}$$

$$= \mu\rho^k \rho^n \rho^2 \qquad \rho^n = i$$

$$= \mu\rho^k \rho^2$$

$$= \mu\rho^{k+2} \qquad 0 \leq k < n$$

Q5:

$f: U_6 \to \mathbb{Z}_6 \quad z = e^{i(\frac{\pi}{3})} \longleftrightarrow 4 \; \Delta$

$z^0 = 0$

$z = 4$

$z^2 = zz = 4 +_6 4 = 2$

$z^3 = z^2 z = 2 +_6 4 = 0$

$z^4 = z^3 z = 0 +_6 4 = 4 \quad \Delta$

$z^5 = z^4 z = 4 +_6 4 = 2$

From Δ, we know this shows $f(x_1) = f(x_1)$ $x_1 \neq x_2$, which is not injective.

Hence, they are not isomorphic.