Exact solution:

$$x(t+h) = x + h\dot{x}(t) + \frac{1}{2!} h^2 \ddot{x}(t) + \frac{1}{3!} h^3 \dddot{x}(t) + \mathcal{O}(h^4)$$

$\dot{x}(t) = f(x(t), t)$ subst.

$$x(t+h) = x(t) + h f(x(t), t) + \frac{1}{2!}(f_x f + f_t) h^2$$

$$+ \frac{1}{3!} \frac{d}{dt}(f_x f + f_t) h^3 + \mathcal{O}(h^4)$$

- RK2 solution

$k_1 = f(x(t), t)$

$k_2 = f(x(t) + k_1 h, t + h)$

- $x(t+h) = x(t) + \frac{1}{2}(k_1 + k_2) h$

expanding k_2:

$$k_2 = \underbrace{f(x(t), t) + h(f_x f + f_t)}_{} + \mathcal{O}(h^2)$$

substituting

From RK2

- $x(t+h) = x(t) + \frac{1}{2}\left(f + f + h(f_x f + f_t)\right) h + \mathcal{O}(h^3)$
- $x(t+h) = \underbrace{x(t) + hf + \frac{1}{2} h^2 (f_x f + f_t)}_{} + \mathcal{O}(h^3)$

matches the exact solution up to h^2 term!

→ RK2 is a second order method.