MVCOMP1, Ex.8

To be submitted by December 13th, 2024, 6PM

Algorithmic regularization *(24 points)*

The Mikkola's regularization algorithm (Mikkola & Merritt 2006) is possibly the most used regularization scheme for a system of N bodies. **The exercise consists in writing a Python scripts that implements Mikkola's regularization for a system of two bodies.**

Let's start by writing down the equations of motion for the reduced particle as:

$$\begin{aligned} \dot{\vec{v}} &= -G\, m_{\rm TOT}\, \frac{\vec{x}}{x^3} + \vec{f}(t, \vec{x}, \vec{v}) \\ \dot{\vec{x}} &= \vec{v}. \end{aligned} \tag{1}$$

In the above equations, $\vec{x} \equiv \vec{x}_i - \vec{x}_j$ and $\vec{v} \equiv \vec{v}_i - \vec{v}_j$ are the relative distance and velocity of a system of two particles with indexes i and j, $m_{\rm TOT} = m_i + m_j$ is their total mass, G is the gravity constant, and $\vec{f}(t, \vec{x}, \vec{v})$ is an external acceleration.

The algorithm to be implemented, with a Verlet drift-kick-drift (DKD) method, is the following.

$$\begin{aligned} t_{1/2} &= t_0 + \frac{h}{2} \left(\frac{1}{2} v_0^2 + b_0 \right)^{-1} \\ \vec{x}_{1/2} &= \vec{x}_0 + \frac{h}{2}\, \vec{v}_0 \left(\frac{1}{2} v_0^2 + b_0 \right)^{-1} \\ \vec{v}_1 &= \vec{v}_0 - h\, \frac{\vec{x}_{1/2}}{x_{1/2}^2} + h\, \vec{g}_{1/2} \\ b_1 &= b_0 - h\, (\vec{v}_{1/2} \cdot \vec{g}_{1/2}) \\ t_1 &= t_{1/2} + \frac{h}{2} \left(\frac{1}{2} v_1^2 + b_1 \right)^{-1} \\ \vec{x}_1 &= \vec{x}_{1/2} + \frac{h}{2} \vec{v}_1 \left(\frac{1}{2} v_1^2 + b_1 \right)^{-1}. \end{aligned} \tag{2}$$

In the above equation, the indexes 0, 1/2 and 1 refer to the beginning, the middle and the end of the time-step, while

$$\vec{g}_{1/2} \equiv \vec{g}(t_{1/2}, \vec{x}_{1/2}, \vec{v}_{1/2}) = \frac{x_{1/2}}{G\,(m_i + m_j)}\, \vec{f}(t_{1/2}, \vec{x}_{1/2}, \vec{v}_{1/2}). \tag{3}$$

Consider the following steps.

(a) Integrate the above scheme assuming $\vec{g} = (0, 0, 0)$. This should result in the unperturbed motion of a binary system subject to gravity force. Assume the following initial conditions for the binary system

$$\begin{aligned} m_i,\, m_j &= 1,\, 1\, M_\odot \quad \text{(masses)} \\ a &= 1.0\,\text{au} \quad \text{(semi-major axis, in astronomical units)} \\ e &= 0.90 \quad \text{(eccentricity)} \\ phi &= 0 \quad \text{(initial phase of the binary system).} \end{aligned} \tag{4}$$

Import the code init_cond.py to properly initialize $\vec{x}_i$, $\vec{x}_j$, $\vec{v}_i$, $\vec{v}_j$ (at time zero) starting from the above quantities in units where $G = 1$.The conversions are already handled by the code init_cond.py.

Integrate the system from physical time $t_0 = 0$ to $t_f = 50$. Assume a fixed regularized time step $h = 0.1$.

Hint: note that for the unperturbed binary system $b_0 = G\, m_{\mathrm{TOT}}/(2\, a)$ is the specific binding energy of the reduced particle. **(10 points)**

(b) Plot the trajectory of the reduced particle in the orbital plane (x, y), and plot $x = |\vec{x}|$ as a function of physical time t. **(2 points)**

(c) Repeat the integration with a non-regularized Verlet method (*), assuming $t_0 = 0$, $t_f = 50$ and $\tilde{h} = 0.01$ (note that $\tilde{h}$ in this case is the physical time-step). Plot the trajectory of the reduced particle in the orbital plane (x, y), and plot $x = |\vec{x}|$ as a function of physical time t. What are the differences compared to the regularized algorithm?

(*) Note: "non-regularized" Verlet method means *the integration of the system with the velocity Verlet method with physical time-step, no change of coordinates.* This is the same algorithm as you already implemented for the Pythagorean problem (Exercise Sheet 3), just copy and paste it from your previous script, adapting it to the fact that here we are using 1 single effective body. **(4 points)**

(d) Now add a perturbing acceleration to the regularized Mikkola's algorithm (the same as point (a) of this exercise) in the form of

$$\vec{f}(t, \vec{x}, \vec{v}) = \vec{f}(t, \vec{x}) = -\vec{x}. \tag{5}$$

Calculate the integral with $t_0 = 0$, $t_f = 50$ and $h = 0.1$ (where h is the regularized time-step). Plot the trajectory of the reduced particle in the orbital plane (x, y), and plot $x = |\vec{x}|$ as a function of physical time t. What do you find? Why?**(6 points)**

(e) Is the Mikkola regularization algorithm symplectic? Why? **(2 points)**