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1.1 Geometry of immersed hypersurfaces

The study of immersed hypersurfaces is a fundamental topic in differential geometry, and is especially important in the field of geometric analysis. An immersed hypersurface $X : M^n \to \overline{M}^{n+1}$ is a submanifold of a higher-dimensional space that is embedded in that space in such a way that the submanifold has the same dimension as the space in which it is embedded minus one. In other words, it is a submanifold of codimension one. We will always assume the embedding to be smooth. The most common case is $\overline{M}^{n+1} = \mathbb{R}^{n+1}$, usually extensively studied in undergraduate courses when $n = 2$. We will be assuming that the embedding is also isometric, i.e. the metric g on M^n is the one induced by $(\overline{M}^{n+1}, \overline{g})$.

Symbols referring to $(\overline{M}^{n+1}, \overline{g})$ will have a line on top, otherwise the symbol will refer to (M^n, g).

The pullback of the tangent bundle of $\overline{M}^{n+1}$ to M^n is a smooth vector bundle on M^n:

$$X^*T\overline{M}^{n+1} = T\overline{M}^{n+1}|_{M^n} = \amalg_{p\in M^n} T_p\overline{M}^{n+1}$$

One of the most important concepts in the geometry of immersed hypersurfaces is the concept of a normal vector field. The normal vector field ν is a section of the pullback vector bundle X^*TM^{n+1} on the manifold M^n that is perpendicular to the tangent space of M^n at each point. At each point p:

$$T_p\overline{M}^{n+1} = T_pM^n \oplus N_pM^n$$

where N_pM^n is the normal vector bundle generated by the normal vector. This also allows us to define the tangent and normal projection on $T\overline{M}^{n+1}|_{M^n}$ by taking the two respective components.

Clearly, taking $\overline{\nabla}$ to be the Levi-Civita connection on $(\overline{M}^{n+1}, \overline{g})$, we can decompose it as:

$$\overline{\nabla}_v w = (\overline{\nabla}_v w)^\top + (\overline{\nabla}_v w)^\perp$$

Definition 1.1. *The* second fundamental form *is then defined as:*

$$\mathrm{II}(v,w) = (\overline{\nabla}_v w)^\perp$$

It is a bilinear symmetric tensor because TM^n is involutive in $T\overline{M}^{n+1}$ and depends only on the local value of v and w by symmetry. we can therefore write it as

$$\mathrm{II}(v,w) = -(h_{ij}v^i w^j)\nu$$

for some matrix $A(p) = \{h_{ij}\}$. we can define the principal curvatures of the hypersurface, as the eigenvalues of this matrix.

It is also possible to check that $(\overline{\nabla}_v w)^\top$ satisfies the definition the Levi-Civita connection on (M^n, g), therefore, from its uniqueness:

$$\begin{aligned}\nabla_v w &= (\overline{\nabla}_v w)^\top \\ \overline{\nabla}_v w &= \nabla_v w + \mathrm{II}(v,w)\end{aligned}$$

where ∇ is the Levi-Civita connection on (M^n, g). This result is known as the *Gauss Formula*. It has to be noted however that we are implicitly considering tangent vectors that are not in the same space. Indeed, making that more explicit, the formula should be:

$$\overline{\nabla}_{X_* v} X_* w = X_*(\nabla_v w) + \mathrm{II}(v,w)$$

Proposition 1.2. (The Weingarten Equation) *If $v, w \in TM^n$ and $\nu \in NM^n$, if one considers the corresponding derivations in $T\overline{M}^{n+1}$ the following equation holds:*

$$\left\langle \overline{\nabla}_v \nu, w \right\rangle_{\overline{g}} = -\left\langle \nu, \mathrm{II}(v,w) \right\rangle_{\overline{g}}$$

Proof. As $\langle \nu, w \rangle_{\overline{g}} \equiv 0$ on M,

$$\begin{aligned}0 &= v \left\langle \nu, w \right\rangle_{\overline{g}} \\ &= \left\langle \overline{\nabla}_v \nu, w \right\rangle_{\overline{g}} + \left\langle \nu, \overline{\nabla}_v w \right\rangle_{\overline{g}} \\ &= \left\langle \overline{\nabla}_v \nu, w \right\rangle_{\overline{g}} + \left\langle \nu, \mathrm{II}(v,w) \right\rangle_{\overline{g}}\end{aligned}$$

applying the Gauss Formula and the fact that $\nabla_v w \in TM^n$ in the last step □

It is also usual to define the associated Weingarten map, which is the linear map between sections of M $s : \Gamma(M) \to \Gamma(M)$ satisfying:

$$\langle s(v), w\rangle_g = \langle \nu, \mathrm{I\!I}(v,w)\rangle_{\overline{g}}$$

the linear map s is also known as the *shape operator of M*.

Combining this with the formula above, taking into account that it holds for a generic $w \in TM$:

$$s(v) = -(\overline{\nabla}_v \nu)^\top$$

We are going to use these equations in local coordinates, in the form shown below.

Proposition 1.3. *The above equations in local coordinates are equivalent to the following equations:*

$$\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial X^\alpha}{\partial x^k} + \overline{\Gamma}^\alpha_{\beta\delta}\frac{\partial X^\beta}{\partial x^i}\frac{\partial X^\delta}{\partial x^k} = -h_{ij}\nu^\alpha \tag{1.1}$$

$$\frac{\partial \nu^\alpha}{\partial x^i} + \overline{\Gamma}^\alpha_{\beta\delta}\frac{\partial X^\beta}{\partial x^i}\nu^\delta = h_{ij}g^{jl}\frac{\partial X^\alpha}{\partial x^l} \tag{1.2}$$

where ν is the normal unit vector at the point and $A = \{h_{ij}\}$ is the second fundamental form, thus $h_{ij} = \left\langle \nu, \overline{\nabla}_{\overline{\partial_i}}\overline{\partial_j}\right\rangle_{\overline{g}}$

Proof. For any connection ∇. and any derivations $v = v^i\partial_i$ and $w = w^j\partial_j$:

$$\nabla_v w = \nabla_{(v^i\partial_i)}(w^j\partial_j) = v(w^k)\partial_k + (v^i w^j \Gamma^k_{ij})\partial_k$$

Let $\partial_1, \dots \partial_n$ be a basis of TM^n at a point, and let $\overline{\partial_i} = X_*\partial_i$, $\overline{\partial_{n+1}} = \nu$. Let's consider the Gauss Formula for two generic ∂_i, ∂_j, using roman letters for indices varying between 1 and n and greek letters for indices varying between 1 and $n+1$, and δ^{ij} the Kronecker delta:

$$\overline{\nabla}_{X_*\partial_i}X_*\partial_j = X_*(\nabla_{\partial_i}\partial_j) + \mathrm{I\!I}(\partial_i, \partial_j)$$

$$(\overline{\partial_i}(X_*\partial_j)^\alpha)\partial_\alpha + ((X_*\partial_i)^\beta(X_*\partial_j)^\delta\overline{\Gamma}^\alpha_{\beta\delta})\partial_\alpha = X_*(\cancel{(\partial_i\delta^{jk})}\partial_k + \delta^{ij}\Gamma^k_{ij}\partial_k) - h_{ij}\nu^\alpha\overline{\partial_\alpha}$$

$$(\overline{\partial_i}(X_*\partial_j)^\alpha)\partial_\alpha + ((X_*\partial_i)^\beta(X_*\partial_j)^\delta\overline{\Gamma}^\alpha_{\beta\delta})\partial_\alpha = X_*(\Gamma^k_{ij}\partial_k) - h_{ij}\nu^\alpha\overline{\partial_\alpha}$$

$$\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} + \overline{\Gamma}^\alpha_{\beta\delta}\frac{\partial X^\beta}{\partial x^i}\frac{\partial X^\delta}{\partial x^k} = \Gamma^k_{ij}\frac{\partial X^\alpha}{\partial x^k} - h_{ij}\nu^\alpha$$

Which is the formula (1.1). To get the second formula, first note that $s(v) = -(\overline{\nabla}_v\nu)^\top$. We then compute $-\langle s(\partial_i), \overline{\partial_\alpha}\rangle_{\overline{g}}$:

$$\left\langle \overline{\nabla}_{\overline{\partial_i}}\nu, \overline{\partial_\alpha}\right\rangle_{\overline{g}} = -\left\langle s(\partial_i), \overline{\partial_\alpha}\right\rangle_{\overline{g}}$$

$$\left\langle \left(\frac{\partial \nu^\alpha}{\partial x^i} + \overline{\Gamma}^\alpha_{\beta\delta}\frac{\partial X^\beta}{\partial x^i}\nu^\delta\right)\overline{\partial_\alpha}, \overline{\partial_\alpha}\right\rangle_{\overline{g}} = h_{ij}g^{jl}\frac{\partial X^\alpha}{\partial x^l}$$

leading to (1.2). ∎

1.2 Local representation as a graph

A well known result, Dini's Theorem[1], states that, given a smooth function F defined on an open subset of the product space $\mathbb{R}^n \times \mathbb{R}^m$, if $F(x, y) = 0$ and the partial derivative of F with respect to y is nonzero at a point (x_0, y_0), then there exists an open neighborhood of x_0 in $\mathbb{R}^n$ and a unique smooth function $y = g(x)$ defined on that neighborhood such that y_0 is a regular value of g and $(x, g(x))$ is a smooth solution to the equation $F(x, y) = 0$.

Consequence of the Dini's theorem is a powerful result that allows one to locally represent a submanifold of $\mathbb{R}^n \times \mathbb{R}^m$ as the graph of a smooth function. This theorem is widely used in differential geometry, geometric analysis, and many other fields of mathematics and physics. We provide a version of this theorem below:

Theorem 1.4 (Local representation as a graph)**.** *Let X^n be a submanifold $X^n \subset \mathbb{R}^{n+1}$ and let $x_0 \in X$. Then there exists a neighbourhood of x_0, $U \subset X^n$, such that U is the graph of a function. Moreover, this function can be of the form*

$$f : \pi(U) \subset \mathbb{R}^n \to U$$
$$U = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | x_0 = f(x_1, ..., x_n)\}$$

for any of the possible orders of the usual basis for $\mathbb{R}^n$, $(e_0, \ldots, e_n)$, as long as $e_0 \notin T_xM$, where $\pi(U)$ is the projection on the last n coordinates $((x_0, \ldots, x_n) \mapsto (x_1, \ldots, x_n))$.

A proof of the 2D-case of the version of the theorem can be found in [10] which extends naturally to the n dimensional case, with almost no changes. This immediately extends to:

Corollary 1.5 (Local representation as a graph on the tangent)**.** *Let X^n be a submanifold $X^n \subset \mathbb{R}^{n+1}$ and let $x_0 \in X$. Then there exists a neighbourhood of x_0 $U \subset X^n$ and a smooth function $f : T_xX^n \to \mathbb{R}$ such that any $x_0 \in U$ can be expressed as*

$$x_0 = p + f(p)\nu$$

where ν is the vector normal to $T_{x_0}X^n$, for an appropriate point $p \in T_{x_0}X^n$. In other words, every submanifold $X^n \subset \mathbb{R}^{n+1}$ is locally expressible as a graph on its tangent space.

Proof. By rotation, we may assume T_xX^n orthogonal to e_1. Then one can just apply the previous theorem. □

We will use this later to prove Theorem 2.3.

[1]also known as Implicit Function Theorem

1.3 Some well established results from analysis

We now introduce some well known results from analysis. The first result we introduce is the maximum principle.

The maximum principle is a classical result of mathematical analysis, and it is usually introduced in a first course on partial differential equations. It is a fundamental tool in the theory of partial differential equations. It is a statement about the behavior of solutions to certain types of PDEs and provides a method for obtaining upper and lower bounds on the solutions. The principle states that the maximum and minimum values of a solution to elliptic or parabolic PDE occur on the boundary of the domain unless the function is constant.

The maximum principle can be used to prove the existence, uniqueness, and regularity of solutions to elliptic and parabolic PDEs. It can also be used to obtain estimates on the behavior of solutions and to study the asymptotic behavior of solutions as the domain becomes large. The principle is widely used in many fields of mathematics and physics, such as geometric analysis, mathematical physics, and fluid dynamics. One of the many versions of this well know theorem is this:

Theorem 1.6 (Maximum principle for parabolic equations)**.** *Let Ω be an open, bounded, connected set. Assume $u \in C_1^2(\Omega \times [0,T]) \cap C^1(\overline{\Omega} \times [0,T])$. Suppose u satisfies:*

$$-\frac{\partial u}{\partial t} + \left(\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i}^{n} b_i \frac{\partial}{\partial x_i} + c \right) u = -u_t + Lu \geq 0 \tag{1.3}$$

where L is an elliptic differential operator, i.e. there exists $\theta > 0$ such that $\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|$ for all $\xi \in \mathbb{R}^n$ and $(x,t) \in \Omega \times [0,T]]$. Suppose also that $c \equiv 0$ in Ω. Then:

- *if u attains its maximum in an interior point $(x_0,t_0) \in \Omega \times [0,T]$, then u is constant in $\Omega \times [0,t_0]$.*
- *If, instead, under the same conditions, $u_t + Lu \geq 0$ and attains its minimum in an interior point of $\Omega \times [0,T]$, then u is constant in $\Omega \times [0,t_0]$*

A proof of this result can be found, for example, in [9]. The theorem extends also to situations where the condition holds in a convex bounded connected region $R \subseteq \Omega \times [0,T]$: in that case, if u attains its maximum in an interior point then u has the same value at any point in R that can be connected to it through a segment going in the backwards direction of time and a "horizontal" line contained in Ω. This version of the theorem can be found for example in [6]:

Theorem 1.7. *Let u satisfy the uniformly parabolic differential inequality (1.3) with $c(x) \leq 0$ in a region $R_T = \{(x_1, x_2, \ldots, x_n, t) | t \leq T\}$ where R is a non-empty connected open set, and suppose that the coefficients of L are bounded. Suppose that the maximum of u in R_T is M and that it is attained at a point (x, t) of R_T. Thus if (y, s) is a point of R which can be connected to (x, t) by a path in R consisting only of horizontal segments and upward vertical segments, then $u(y, s) = M$.*

Hopf's boundary point lemma is another important classical tool in the study of PDEs that provides a criterion for determining the behavior of solutions to certain types of elliptic or parabolic PDEs near the boundary of the domain. The lemma states that if one has a solutions to some kinds of partial differential inequalities, then the normal derivative of the solution at that point is strictly positive.

It is often used to obtain estimates on the behavior of solutions near the boundary, and to prove the existence and uniqueness of solutions to boundary value problems. The lemma is named after the German mathematician Eberhard Hopf, who first formulated it in the 1950s. In [6] we find the following version of the Hopf's boundary point lemma:

Theorem 1.8. *Let u be a solution to the parabolic inequality*

$$-u_t + Lu \geq 0$$

with L an elliptic linear differential operator with bounded coefficients such that $c(x) \leq 0$,in a domain E, and let $E_t = \{(x, s) \in E | s \leq t\}$. Suppose the maximum M of u is attained at a point $P = (x, t)$ on the boundary ∂E.
Assume that a sphere through P can be constructed which is in E_s such that

- *tangent to ∂E at P*
- *the set of point of its interior (y, s) such that $s \leq t$ lies in E_s,*
- *$u < M$ in its interior.*

Also, suppose that the radial direction from the centre of the sphere to P is not parallel to the t-axis.
Then, if $\frac{\partial}{\partial \nu}$ denotes any directional derivative in an outward direction from E_s , we have

$$\frac{\partial u}{\partial \nu} > 0$$

at P.

Proof. Let the sphere through P be B_1. We may construct a smaller sphere B_2 centred at P. Let now:

$$\begin{aligned} S_1 &= \partial B_1 \cap B_2 \cap E_t, \\ S_2 &= B_1 \cap \partial B_2 \cap E_t, \text{and} \\ S_3 &= B_1 \cap B_2 \cap \partial E_t = B_1 \cap B_2 \cap \{(x,s) \in E | s = t\}. \end{aligned}$$

The three sets satisfy $S_1 \cup S_2 \cup S_3 = \partial(B_1 \cap B_2 \cap E_t)$, we may call this region $R = B_1 \cap B_2 \cap E_t$. Without loss of generality, potentially taking a smaller sphere B_1, we may assume that $u < M$ on B_1 except at P. As $R \subset B_1$, we also get $u < M$ on R. We may thus conclude that:

- $u < M$ on R except at P
- $u \leq M - \delta$ on S_2 for a sufficiently small $\delta > 0$
- $u = M$ at P.

Now, let the centre of B_1 be $Q = (z, t_0)$ and let r be its radius. we can now introduce the function

$$v(y,s) = exp\left(-\alpha(s-t_0)^2 - \sum_{i=1}^{n} \alpha(y_i - z_i)^2\right) - exp\left(-\alpha r^2\right)$$

This function is such that $v(y,s) = 0$ if $(y,s) \in S_1$ - including $v(x,t) = 0$, as there the first term is $e^{-\alpha r^2}$, and $v(y,s) > 0$ in the interior of B_1.
Thus, in the region R, $v(y,s) \geq 0$ and has a minimum point at the boundary on (x,t), where $v(x,t) = 0$.

We can also compute Lv. After some calculation, we get that

$$\begin{aligned} Lv = 2\alpha e^{\left(-\alpha(s-t_0)^2 - \sum_{i=1}^{n} \alpha(y_i - z_i)^2\right)} [2\,\alpha\,(y-z)^t A(y-z) + \\ + \sum_{i}^{n} [b_i(y_i - z_i) + a_{i,i}] + (s-t)] \end{aligned}$$

where A is the matrix of the $a_{i,j}$. In particular, one can choose an α large enough, so that $Lv > 0$ in $R \cup \partial R$.

We can thus introduce $w = u + \varepsilon v$. As both Lu and Lv are positive in R, $Lw > 0$ in R. We can also choose ε small enough so that $w < M$ on S_2. Also, as $v = 0$ on S_1, $w < M$ on S_1 except at P, and $w = M$ at P.

Therefore, we can apply the Strong Maximum Principle 1.7 to the region R to conclude that the maximum of w in R is attained at P. Therefore:

$$\frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu} + \varepsilon \frac{\partial v}{\partial \nu} \geq 0$$

But:

$$\frac{\partial v}{\partial \nu} = \nu \cdot n \frac{\partial v}{\partial R} = -2\nu \cdot n\alpha R e^{-\alpha R} < 0$$

Where n is the vector orthogonal to the sphere S_1. Therefore, one must have:

$$\frac{\partial u}{\partial \nu} > 0$$

as we wanted. □

Remark 1.9. If $c(x)$ is now just bounded, we can consider, instead of u, $v = ue^{-\lambda t}$, thus, by change of variables

$$-v_t + Lv - \lambda v \geq 0$$

whenever $-u_t + Lu \geq 0$, and we can chose λ large enough such that $c(x) - \lambda < 0$ and thus we can remove the hypothesis $c(x) \leq 0$ in both theorems when c is bounded.

1.4 Applying the maximum principle to non-linear PDEs

We can now introduce an important observation shown in [6] that allows us to apply the maximum principle 1.7 and Hopf's boundary point lemma 1.8 in some non-linear settings. Firstly, we must clarify what we mean by parabolic non-linear problem.

Definition 1.10. *A differential non-linear problem in the form*

$$Lu = F\left(t, x, v, \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_j}\right) - v_t = f(x,t) \tag{1.4}$$

given a smooth F is parabolic if for any real vector ξ

$$\sum_{i,j=1}^{n} F_{ij}\xi_i\xi_j > 0$$

where F_{ij} are the derivatives of F with respect to $\frac{\partial^2 v}{\partial x_i \partial x_j}$.

Secondly, we remind the reader of the following generalized version of the theorem of the mean, a.k.a. Lagrange's theorem:

Theorem 1.11 (Lagrange's theorem). *Given a convex open set $U \subseteq \mathbb{R}^n$ and a real function $F \in C^1(U)$, and given to points x, y in U, there exists a point z in the segment connecting x and y such that*

$$F(y) - F(x) = \langle \nabla F(z), (y - x) \rangle$$

Suppose that we have a solution to a non-linear parabolic problem v, i.e. v solves (1.4):

$$Lu = F\left(t, x, v, \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_j}\right) - v_t = f(x, t)$$

for a non-linear elliptic operator L in some region E, where we assume $F(t, x, y_i, z_{i,j})$ to be a given C^1 function. Suppose also that there is a w which is a solution of the corresponding differential inequality:

$$Lw = F\left(t, x, w, \frac{\partial w}{\partial x_i}, \frac{\partial^2 w}{\partial x_i \partial x_j}\right) - w_t \leq f(x, t)$$

One can then consider $u = v - w$, and by combining the above we get:

$$Lv = \left(F\left(t, x, v, \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_j}\right) - F\left(t, x, w, \frac{\partial w}{\partial x_i}, \frac{\partial^2 w}{\partial x_i \partial x_j}\right)\right) - u_t \leq 0$$

Now, we can apply Lagrange's theorem to F to get

$$Lv = \left\langle \nabla F(\xi(x, t)), \left(t, x, u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}\right)\right\rangle - u_t \leq 0$$

for a fixed $\xi(x, t)$.

Thus, the difference u of two sub-solutions to a non-linear differential problem is a sub-solution to a (different) *linear* parabolic problem, as the derivatives of F and ξ do not depend on u (ξ can be chosen a-priori).

We can thus see that this new problem must be parabolic, and apply the maximum principle and the Hopf's boundary point lemma to u. This can allow us to state the following two results which we will be using later:

Proposition 1.12 (Maximum principle for parabolic non-linear differential equations). *Suppose we have two solution v and w in the interval $[0, T]$ to the same parabolic non-linear differential equation (1.4) on an bounded open set Ω, but with different start conditions. Suppose also that F is smooth on $\overline{\Omega}$. Then, if $v > w$ in the interior of Ω at $t = 0$ and $v \geq w$ on $\partial\Omega$, $v > w$ for all $t \in [0, T]$ in the interior of Ω.*

Proof. $u = v - w \geq 0$ is a solution of a parabolic *linear* differential equation, where the term independent of u is bounded. Furthermore, if we take $c(x,t)$ it must be bounded by compactness. At $t = 0$, $u > 0$ in the interior of Ω. If, at an interior point x, $v = w$ at a certain time $t = \tau$, $u(\tau, x) = 0$, and thus u is not constant. However, it attains minimum ($u = 0$) at an interior point, thus by 1.6 it must be constant, a contradiction. □

Proposition 1.13 (Hopf's boundary point lemma for parabolic non-linear differential equations)**.** *Suppose we have two solution v and w to the same parabolic non-linear differential equation (1.4) in a region E, but with different start conditions. Suppose also that F is smooth on $\overline{\Omega}$. Let $u = v - w$ and suppose that the maximum of u is attained at the point P. Furthermore, assume that the conditions on the shape of the region E from theorem 1.8 hold. Then,*

$$\frac{\partial u}{\partial \nu}(P) > 0$$

where we take ν as the normal to $\partial\Omega$.

Proof. u is a solution of a parabolic *linear* differential equation, where $c(x,t)$ is bounded by compactness. We can then apply 1.8 to v (see also remark 1.9). □

1.5 Reflections on spheres and hyperbolic spaces

In what follows, we will focus on manifolds embedded in spaces which have constant sectional curvature. Constant curvature manifolds are classified into three types based on the sign of the curvature:

- **Positive curvature**: *Spherical geometry*, where the curvature is positive and the manifold locally resembles a sphere (e.g., the standard sphere $\mathbb{S}^n$).
- **Zero curvature**: *Flat geometry*, where the curvature is zero and the manifold locally resembles Euclidean space $\mathbb{R}^n$.
- **Negative curvature**: *Hyperbolic geometry*, where the curvature is negative and the manifold locally resembles hyperbolic space $\mathbb{H}^n$.

As in [2], we will use the symbol $\mathbb{M}^n$ to indicate a Riemannian manifold that can be replaced by any one of $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{H}^n$: the n-dimensional sphere, Euclidean plane or Hyperbolic space respectively.

We will also use use the symbol $\mathbb{M}^n_+$ to indicate a Riemannian manifold that can be replaced by $\mathbb{S}^n_+$, $\mathbb{R}^n$ or $\mathbb{H}^n$: the n-dimensional hemisphere, Euclidean plane or Hyperbolic space, respectively.

Definition 1.14. $\mathbb{H}^n$ *is the* n-dimensional hyperbolic plane. *We can define it as the half space* $\{x \in \mathbb{R}^n | x_n > 0\}$ *with the Riemannian metric*

$$g_x = \frac{1}{x_n^2}\langle \cdot, \cdot \rangle$$

where $\langle \cdot, \cdot \rangle$ *is the Euclidean dot product on* $\mathbb{R}^n$

Remark 1.15. This is not the only way we could define the hyperbolic space $\mathbb{H}^n$. Another alternative is taking the n-dimensional disc $D^n = \{x \in \mathbb{R}^n | \|x\| < 1\}$ with the Riemannian metric

$$g_x = \frac{4}{(1 - \|x\|^2)^2}\langle \cdot, \cdot \rangle$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean dot product on $\mathbb{R}^n$.

Remark 1.16. Let $\mathbb{S}^n \setminus \{P\}$ be the standard n-dimensional unitary sphere minus a point, with the standard induced Euclidean metric. Through stereographic projection it is isometric to $\mathbb{R}^n$ with metric:

$$g_x = \frac{4}{(1 + \|x\|^2)^2}\langle \cdot, \cdot \rangle$$

where $\langle \cdot, \cdot \rangle$ is again the Euclidean dot product on $\mathbb{R}^n$.

Remark 1.17. The models above have curvature ± 1. For other values of the curvature one has to consider the following Riemannian metrics

- for the half-space model of hyperbolic space:

$$g_x = \frac{R^2}{x_n^2}\langle \cdot, \cdot \rangle$$

- for the Poincaré ball model of hyperbolic space one has to take the disk with radius R and:

$$g_x = \frac{4R^4}{(R^2 - \|x\|^2)^2}\langle \cdot, \cdot \rangle$$

- For the stereographic projection of a sphere with radius R:

$$g_x = \frac{4R^4}{(R^2 + \|x\|^2)^2}\langle \cdot, \cdot \rangle$$

We will now define reflections in $\mathbb{M}^n$:

- On $\mathbb{R}^n$ we choose an hyperplane and use the usual reflection
- On S^n, reflections are those induced by $\mathbb{R}^n + 1$ when the fixed plane passes through the origin. Each one can be identified by vector orthogonal to the plane we chose.
- On $\mathbb{H}^n$ we have to be more careful:
  - As a first step, choose a point a point $O \in \mathbb{H}^n$. $\mathbb{M}^n$ is a homogeneous space, so the construction does not depend on the choice we make.
  - Choose any direction in $v \in T_O\mathbb{H}^n$ and consider the geodesic $\gamma_v : \mathbb{R} \to \mathbb{H}^n$ satisfying $\gamma_v(0) = O$ and $\dot{\gamma_v}(0) = v$. Assume that γ is parametrised by arc-length.
  - Consider the hyperplane π_0 passing through $\gamma_v(0)$ and orthogonal to $\dot{\gamma_v}(0)$. Then consider the 1-parameter group of isometries of H^n such that $g_t(\gamma_v(0)) = \gamma_v(t)$ and such that the curves $t \mapsto g_t(x)$ are orthogonal to π_0 for each $x \in \pi_0$. This allows us to assign to each point in H^n coordinates (x, t) where $x \in \pi_0$ and $t \in \mathbb{R}$.
  - consider now any hyperplane π_t passing through $\gamma_v(t)$ and orthogonal to $\dot{\gamma_v}(t)$. The reflection fixing π_t will be the one given by the formula $(x, t) \mapsto (x, 2s - t)$.

1.6 The Method of Moving Planes

To provide some justification and context to the next chapters, we describe Alexandrov reflection, also known as Method of the Moving Planes in $\mathbb{M}^{n+1}_+$.

Let $X : M^n \to \mathbb{M}^{n+1}_+$ be a hypersurface in a constant curvature ambient space. If the ambient space is a sphere, X must be contained in a hemisphere to avoid issues with multiple self-intersections. Assume also that $X = \partial\Omega$ for a bounded domain Ω in $\mathbb{M}^{n+1}_+$.

- As a first step, choose a point a point $O \in \mathbb{M}^n$. As $\mathbb{M}^n$ is a homogeneous space, so the construction does not depend on the choice we make. Without loss of generality, we choose the origin in $\mathbb{R}^n$, e_n in $\mathbb{H}^n$ and the north pole in S^n.
- Choose any direction in $v \in T_O\mathbb{M}^n$ and consider the geodesic $\gamma_v : I \to \mathbb{M}^n$ satisfying $\gamma_v(0) = O$ and $\dot{\gamma_v}(0) = v$. Assume that γ is parametrised by arc-length. Here $I = \mathbb{R}$ if $\mathbb{M}^n_+$ is flat or hyperbolic, and $I = (-\frac{\pi}{2}, \frac{\pi}{2})$ if $\mathbb{M}^n_+$ is a hemisphere.

- Consider the hyperplanes $\pi_{v,s}$ passing through $\gamma_v(s)$ and orthogonal to $\dot{\gamma_v}(s)$.

The method consists of reflecting the part of X "below" the hyperplane into the top part, and using properties of both halves of the hypersurface together to prove some statement about the non-reflected hypersurface.

To make this more precise, we can define:

$$X_{v,s} = \{p \in X \mid p \in \pi_{v,t} \text{ for some } t < s\}$$

Let $X^{\pi}_{v,s}$ be the reflection of $X_{v,s}$ about $\pi_{v,s}$. We can then define:

$$\begin{aligned} m_v &= \sup\left\{s \in I \mid X^{\pi}_{v,s} \subset \Omega \text{ for every } t < s\right\} \\ &= \sup\left\{s \in I \mid X \cap X^{\pi}_{v,s} = \emptyset \text{ for every } t < s\right\} \end{aligned}$$

The last time at which the hypersurface and its reflection do not touch. Please note that at m_v the two surfaces are tangent at some point, which can be either in the interior of X, or on the boundary. The hyperplane π_{v,m_v} is the critical hyperplane.

1.7 The Alexandrov soap-bubble theorem

To give some justification to the method we descrived, we will outline the proof an important result that uses this method, the so-called Alexandrov soap bubble theorem. For more details, see [2].

Theorem 1.18. *The only C^2-regular connected hypersurfaces embedded in $\mathbb{M}^{n+1}_+$ and such that the mean curvature is constant are the distance spheres.*

In [2] this theorem is proved more generally: let H_X be a C^2 function of the ordered principal curvatures $H_X = f(\kappa_1, \dots, \kappa_n)$, and

$$f : \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \leq x_2 \leq \cdots \leq x_n\} \to \mathbb{R}$$

is such that

$$f(x) > 0 \text{ if } x_i > 0 \text{ for every } i = 1, \dots, n$$

and it is concave on the component of $\{x \in \mathbb{R}^n \mid f(x) > 0\}$ containing $\{x \in \mathbb{R}^n \mid x_i > 0\}$. Then the following more general theorem holds:

Theorem 1.19. *The only C^2-regular connected hypersurfaces embedded in $\mathbb{M}^{n+1}_+$ and such that H_X is constant are the distance spheres.*

The following proposition holds:

Proposition 1.20. *Let $X = \partial\Omega$ be a C^2-regular, connected, closed hypersurface embedded in $\mathbb{M}_+^{n+1}$, where Ω is a bounded domain. Assume that for every geodesic $\gamma : \mathbb{R} \to \mathbb{M}^{n+1}$ there exists a hyperplane orthogonal to γ such that X is symmetric about π. Then X is a distance sphere about its center of mass O, i.e. the unique minimum of*

$$P_\Omega(x) = \int_\Omega d(x,a)^2 da$$

The proof of this proposition is in [2] and is omitted here. We now move on to the proof of Theorem 1.18:

Proof. Assume that X is a manifold with constant mean curvature. We want to show that, for any point O and for every direction $v \in T_O\mathbb{M}^{n+1}$, X is symmetric about a plane perpendicular to the geodesic $\exp_O(tv)$. We put ourselves in the hypothesis of the method of the moving planes described in section 1.6. At m_v, $X \cap X^\pi_{v,m_v}$ is non-empty by definition of m_v and therefore it is closed in X^π_{v,m_v}. We want to show that $X \cap X^\pi_{v,m_v}$ is an open set in X^π_{v,m_v}.

Let $p \in X \cap X^\pi_{v,m_v}$. As the two manifolds are tangent

$$T_pX = T_pX^\pi_{v,m_v}.$$

By Theorem 1.5, we can represent both manifold as the Euclidean graph of two functions C^2 functions u and $\tilde{u}$ defined in a neighbourhood of p inside the tangent space. Consider now the differential equation:

$$K(u(x)) = K(\tilde{u}(x)) = \text{ constant}$$

where K is the mean-curvature operator. As X is a manifold with constant mean curvature, it holds everywhere. Looking at equation (1.1), we see that, given that the principal curvature are a function of the h_{ij}, the operator K is an elliptic operator. Reasoning exactly like we did in section 1.4 for parabolic differential equations, we see that $u - \tilde{u}$ is the solution of a linear elliptic differential equation of the form $L(u - \tilde{u}) = 0$, with $u(p) = \tilde{u}(p) = 0$.

Without loss of generality, we can assume that, in the neighbourhood where we defined u and $\tilde{u}$, $u - \tilde{u} \geq 0$.

If p is an interior point in X^π_{v,m_v}, we can then apply the maximum principle for elliptic equations (see section 6.4 in [9]) and obtain that $u = \tilde{u}$ in the neighbourhood.

Otherwise, if $p \in \pi_{v,s}$, $\nabla u(p) = \nabla\tilde{u}(p) = 0$ and one can apply Hopf's boundary point lemma for elliptic equations (see section 6.4 in [9]) to conclude again that $u = \tilde{u}$ in the neighbourhood.

Therefore, the whole neighbourhood is in $X \cap X^{\pi}_{v,m_v}$, hence the intersection is open, as every point p is contained in an open ball. This proves that X is symmetric about π_{v,m_v}. The theorem is then consequence of Proposition 1.20 $\square$

Remark 1.21. To prove the theorem in the aforementioned more general setting, one would need to only prove that the differential equation $H_X(u(x)) = \text{constant}$ is an elliptic equation. Its ellipticity is a standard result in Geometric Analysis, and can be found in the literature.

Chapter 2

The Chow-Gulliver Critical Planes Result

The main result we want to establish in this chapter is theorem 2.6, a result about critical hyperplanes when applying the method of the moving planes to solution of a large class of non-linear parabolic partial differential equations, and whose proof is somewhat similar to Theorem 1.18.

We will first describe the differential equations we are analysing, then prove that they are parabolic, then prove the theorem.

2.1 Class of Equations we analyze

We consider manifolds M^n embedded in $\mathbb{R}^{n+1}$, i.e. there is an embedding $X_0 : M^n \to \mathbb{R}^{n+1}$ parametrizing the hypersurface $X_0(M^n)$.

Let $F : \{(\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n | \kappa_1 \leq \cdots \leq \kappa_n\} \to \mathbb{R}$ be a C^1 function satisfying:

$$\frac{\partial F}{\partial \kappa_i} > 0 \text{ for all } i = 1, \ldots, n \tag{2.1}$$

and consider the evolution equation

$$\begin{cases} \dfrac{\partial X_t}{\partial t} = F(\kappa_1(x), \ldots, \kappa_n(x))\nu \\ X(0) = X_0 \end{cases} \tag{2.2}$$

where ν is the inward normal to $X_t(M^n)$ at the point $X_t(x)$ and $\kappa_1 \leq \cdots \leq \kappa_n$ are the principal curvatures at $X_t(x)$.

2.2 Parabolicity of the differential equation (3.2)

The condition (3.1) will guarantee that equation (3.2) is a parabolic equation. This may be confusing, as (3.2) does not make it obvious how to apply definition 1.10.

In order to classify a non-linear partial differential equation one has to understand how it behaves "close to a solution" in the solutions space. We want to prove that very close to any solution, "moving in any direction", the change in the equation is always a parabolic PDE. This will then tell us that our equation is parabolic, and that the theorems that apply to solutions of parabolic partial differential equations apply to our equation as well. To do so, we are going to "linearise" the differential equation about a solution.

Like in [5], as F is a symmetric function in the principal curvatures, we may interchangeably take F to be a function of the Weingarten map tensor or of the second fundamental form, and thus we get:

$$\frac{\partial X_t}{\partial t} = F(h_{ij}(X_t))\nu$$

To understand the behaviour close to a solution, we can substitute in our equation X_t with a $X_t + \varepsilon u_t$ to get:

$$\frac{\partial X_t}{\partial t} + \varepsilon \frac{\partial u_t}{\partial t} = F(h_{ij}(X_t + \varepsilon u_t))\nu_{(X_t+\varepsilon u_t)} \tag{2.3}$$

where we mean that $\nu_{(X_t+\varepsilon u_t)}$ is the normal to the perturbed immersion. We are interested in the behaviour of this equation for a small ε. This equation when taking the limit for $\varepsilon \to 0$ is the so-called linearisation of the PDE; we want this PDE to be a parabolic equation to apply our results.

We can use the Weingarten equation (1.1) to write the RHS explicitly:

$$\begin{aligned}
h_{ij}(X_t + \varepsilon u_t) &= -\left\langle v, \nu_{(X_t+\varepsilon u_t)} \right\rangle \text{ where} \\
v^\alpha &= \frac{\partial^2 X_t^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial X_t^\alpha}{\partial x^k} + \overline{\Gamma}_{\beta\delta}^\alpha \frac{\partial X_t^\beta}{\partial x^i} \frac{\partial X_t^\delta}{\partial x^k} + \\
&\quad + \varepsilon \left(\frac{\partial^2 u_t^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u_t^\alpha}{\partial x^k} + \overline{\Gamma}_{\beta\delta}^\alpha \left(\frac{\partial X_t^\beta}{\partial x^i} \frac{\partial u_t^\delta}{\partial x^k} + \frac{\partial u_t^\beta}{\partial x^i} \frac{\partial X_t^\delta}{\partial x^k} \right) \right) + \\
&\quad + \varepsilon^2 \left(\overline{\Gamma}_{\beta\delta}^\alpha \frac{\partial u_t^\beta}{\partial x^i} \frac{\partial u_t^\delta}{\partial x^k} \right) \\
v^\alpha &= w + \varepsilon \left(\frac{\partial^2 u_t^\alpha}{\partial x^i \partial x^j} + \text{lower order terms} \right) + o(\varepsilon)
\end{aligned}$$

where $h_{ij}(X_t) = -\langle w, \nu_{X_t}\rangle$.

Putting it all together in the first line:

$$\begin{aligned}
h_{ij}(X_t+\varepsilon u_t) &= -\left\langle w + \varepsilon\left(\frac{\partial^2 u_t}{\partial x^i \partial x^j} + \text{lower order terms}\right) + o(\varepsilon), \nu_{(X_t+\varepsilon u_t)}\right\rangle \\
&= \left\langle w, \nu_{(X_t+\varepsilon u_t)}\right\rangle - \varepsilon\left\langle\left(\frac{\partial^2 u_t}{\partial x^i \partial x^j} + \text{lower order terms}\right), \nu_{(X_t+\varepsilon u_t)}\right\rangle + o(\varepsilon) \\
&= h_{ij}(X_t) + \left\langle w, \nu_{(X_t+\varepsilon u_t)} - \nu_{X_t}\right\rangle - \varepsilon H_{ij} + o(\varepsilon) \\
&= h_{ij}(X_t) - \varepsilon H_{ij} + o(\varepsilon)
\end{aligned}$$

Were on the last step we are using the fact that $\nu_{(X_t+\varepsilon u_t)} - \nu_{X_t} = O(\varepsilon)$, and as this gets smaller the component of w parallel to the difference also is $O(\varepsilon)$, as w is parallel to ν_{X_t}. We can then expand F in the RHS of the equation (2.3) to the first order, as it is a C^1 function:

$$\begin{aligned}
\cancel{\frac{\partial X_t}{\partial t}} + \varepsilon\frac{\partial u_t}{\partial t} &= \left(\cancel{F(h_{ij}(X_t))} - \varepsilon\langle DF, H\rangle + o(\varepsilon)\right)\nu_{(X_t+\varepsilon u_t)} \\
\varepsilon\frac{\partial u_t}{\partial t} &= -\varepsilon\left(\frac{\partial F}{\partial h_{ij}} g^{ik}g^{jl}\left\langle\frac{\partial^2 u_t}{\partial x^k \partial x^l}, \nu_{(X_t+\varepsilon u_t)}\right\rangle + \text{lower order terms}\right)\nu_{(X_t+\varepsilon u_t)} + o(\varepsilon) \\
\frac{\partial u_t}{\partial t} &= -\frac{\partial F}{\partial h_{ij}} g^{ik}g^{jl}\left\langle\frac{\partial^2 u_t}{\partial x^k \partial x^l}, \nu_{(X_t+\varepsilon u_t)}\right\rangle\nu_{(X_t+\varepsilon u_t)} + \text{lower order terms} + o(1)
\end{aligned}$$

Letting $\varepsilon \to 0$, we get to:

$$\frac{\partial u_t}{\partial t} = \frac{\partial F}{\partial h_{ij}} g^{ik}g^{jl}\left\langle\frac{\partial^2 u_t}{\partial x_k \partial x_l}, \nu\right\rangle\nu + \text{lower order terms}$$

For the second order term to be positive definite, then,

$$\left(\frac{\partial F}{\partial h_{ij}}\right)_{i,j}$$

must be positive definite. Or equivalently, as the principal curvatures are the eigenvalues of the matrix $(h_{ij})_{i,j}$,

$$\frac{\partial F}{\partial \kappa_i} > 0$$

Remark 2.1. While in this chapter we are using the standard metric of $\mathbb{R}^n$, the proof above is valid using any other metric.

2.3 An existence result

The paper from Huisken and Polden [5] states the following comforting existence result for the class of equations we are analysing under very broad hypothesis. The same paper also includes a proof of the result with some more restrictive hypotheses. A complete proof is quite more involved. We will not use it, nor prove it, but it is included here for completeness and peace of mind:

Theorem 2.2 (Short term existence of a solution for (3.2)). *Suppose $X_0 : M^n \to \mathbb{R}^{n+1}$ is a smooth, closed hypersurface in $\mathbb{R}^{n+1}$, such that (3.1) holds at all points in X_0, i.e. for all the values of the principal curvatures κ_i realized at some point on X_0. Then, (3.2) has a smooth solution, at least on some short time interval $[0, T)$, $T > 0$.*

2.4 Local representation as a graph of a solution of (3.2)

As a first step, from the Corollary 1.5, we can establish the following:

Theorem 2.3 (Local representation as a graph of a solution of (3.2)). *Let X^n be a submanifold $X^n \subset \mathbb{R}^{n+1}$ and let $F : X^n \times (0, T) \to \mathbb{R}^{n+1}$ be a solution of (3.2). Also let $t \in (0, T)$ and $x \in F(X^n, t)$. Then there exists a neighbourhood of (x, t), $U \subset F(X^n \times (0, T))$, and a smooth function $f : T_xF(X^n, t) \times (0, T) \to \mathbb{R}$ such that any $(x_0, t_0) \in U$ can be expressed as*

$$x_0 = p + f(p, t_0)\nu$$

where ν is a vector normal to $T_xF(X^n, t)$, for an appropriate point $p \in T_xF(X^n, t)$.

Proof. We can consider the image of $F : X^n \times (0, T) \to \mathbb{R}^{n+1}$ as a manifold in $\mathbb{R}^{n+2}$ by considering $G := (F(x, t), t)$. Moreover $\frac{\partial G_t}{\partial e_j} \equiv 0$ for all possible vectors of the canonical basis of $\mathbb{R}^{n+1} \times \mathbb{R}$ except for the one corresponding to the time coordinate, where it is 1. Also, $\frac{\partial G_i}{\partial e_j} \equiv \frac{\partial F_i}{\partial e_j}$ and the first n coordinates of $\frac{\partial G}{\partial t}$ form a vector normal to T_xX^n by (3.2). Thus, the tangent space of $\mathrm{Im}(G)$ is $T_xX^n \times \{0\} \oplus \mathrm{span}\langle(\nu, 1)\rangle$ and we can apply corollary 1.5 to $\mathrm{Im}(G)$ to get a function $\tilde{f}$ such that

$$(x_0, t) = [(p, 0) + (s\nu, s)] + \tilde{f}(p, s)(\nu, -\|\nu\|)$$

as $(\nu, -\|\nu\|)$ is the vector orthogonal to $T_xX^n \times \{0\} \oplus \mathrm{span}\langle(\nu, 1)\rangle$. Let $\sigma_p : I \subset \mathbb{R} \to \mathbb{R}$ be the function that associates to t the appropriate s in the expression above.

Projecting to the first $n+1$ coordinates and calling $f(p,t) = \sigma_p(t) + \tilde{f}(p, \sigma_p(t))$ one gets:

$$x_0 = p + f(p,t)\nu$$

which is our thesis, as long as $\sigma_p(t)$ is smooth. This is indeed the case, as the graph function $\Gamma_f : x \mapsto (x, f(x))$ is smooth for any smooth function f and has a smooth inverse (and thus, the inverse $(x_0, t) \mapsto ((p, 0) + (s\nu, s))$ is smooth). $\square$

One can show through direct calculation (see [12], Exercise 1.1.2) that, if an immersed hypersurface $\phi : M \to \mathbb{R}^{n+1}$ is locally the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$ (i.e., locally, $(x, f(x)) = \phi$), then:

$$\begin{aligned} g_{ij} &= \delta_{ij} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} \\ \nu &= -\frac{(\nabla f, -1)}{\sqrt{1+|\nabla f|^2}} \\ h_{ij} &= -\frac{H_{ij}}{\sqrt{1+|\nabla f|^2}} \end{aligned}$$

Where the matrix $(H_{ij})_{ij}$ is the hessian of f. Thus, if one considers the principal curvatures, they are closely related to the eigenvalues of the hessian of f.

2.5 The Moving Planes Method and the Chow result

We will now present the Chow-Gulliver result, presented more or less as in paragraph 2 in [1]. Suppose that we have a hypersurface embedded in $\mathbb{R}^{n+1}$ evolving according to equation (3.2). For a fixed time t, we can apply the method of the moving planes as described in section 1.6: we can take parallel hyperplanes $\pi_{v,s}$ orthogonal to v intersecting X, and consider the reflection $X^{\pi}_{v,s}$. There will be a hyperplane π_{v,m_v} where X and $X^{\pi}_{v,s}$ are tangent. As the hypersurface evolves, we may wonder how this critical threshold changes over time. If it behaves in a predictable way, we can hope to use the technique on the evolving manifold, otherwise, it may be hopeless if the critical plane moves back and forth multiple times. In the next section, we are going to prove a result in this general direction, to show a form of "regularity" in this sense. First, however, we need to introduce a marginally stricter definition for the concept of "reflecting inside itself".

Let π be a hyperplane in $\mathbb{R}^{n+1}$. We may assume π orthogonal to a unit vector $v \in \mathbb{R}^{n+1}$, i.e. $\langle x, v\rangle = C$ for all $x \in \pi$ for some constant C. In our notation for the method of moving planes, assuming we pick the origin as a starting point, this means that $\pi = \pi_{v,C}$.

Then, $\mathbb{R}^{n+1}$ is divided by π into two half-spaces, which we will name

$$H^+(\pi) = \{x \in \mathbb{R}^{n+1} : \langle x, v\rangle > C\} = \bigcup_{s>C} \pi_{v,s} \text{ and}$$
$$H^-(\pi) = \{x \in \mathbb{R}^{n+1} : \langle x, v\rangle < C\} = \bigcup_{s<C} \pi_{v,s}.$$

Definition 2.4. *We say* we can reflect $X : M^n \to \mathbb{R}^{n+1}$ strictly with respect to π *if both:*

- $X^\pi \cap H^-(\pi) \subset \mathrm{int}(X) \cap H^-(\pi)$ *where* X^π *is the reflection of* X *about* π *and* $\mathrm{int}(X)$ *is the region inside* X*.*
- $V \notin T_x M$ *for all* $x \in M^n \cap \pi$

This fundamentally means that the reflection of one of the halves of X on the other side of π is contained in the region inside M^n and the tangent spaces of X and of the half-reflection do not form a ninety degree angle with π, at all points on $\pi \cap X$. As the two tangent spaces are one the reflection of the other, this means that they do not coincide.

Definition 2.5. *We say* we can reflect $X : M^n \to \mathbb{R}^{n+1}$ strictly up to (π, V) *if we can reflect* M^n *strictly with respect to* π_s *for all hyperplanes* $\pi_{v,s}$ *such that* $s < C$*.*

The key idea of the main result in the next section is as follows: suppose we have an embedded smooth hypersurface X evolving according to (3.2) and a fixed hyperplane π, intersecting X. Suppose that, at some time t, X and X_π touch outside of π. We can consider X and X_π as local graphs over the same hyperplane π, and we can show that these function evolve according to the same differential equation. Using the strong maximum principle and the Hopf boundary point lemma, then, one can conclude that the two functions coincide, and have been coinciding up until that point. We can then conclude that if X and X_π only touch in $X \cap \pi$ at the beginning of the evolution, then they will never touch elsewhere.

2.6 The Chow-Gulliver result

The main theorem is the following:

Theorem 2.6 (Chow-Gulliver)**.** *Let* $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ *be a* C^2 *solution to 3.2. Then, if we can reflect* $X(M^n, 0) = M_0$ *strictly with respect to* π*, then for all* $t \in [0, T)$ *we can reflect* $X(M^n, t) = M_t$ *strictly with respect to* π*.*

Proof. By contradiction, suppose that there is a time t such that the thesis is false, and that it is the smallest such t. Then, for all $\tau \in [0,t)$, $M_{\tau,\pi} \cap H^-(\pi) \subset \mathrm{int}(M_\tau) \cap H^-(\pi)$; the unit vector orthogonal to π, V, is such that $V \notin T_x M_\tau$ for all $x \in M_\tau \cap \pi$ and $\tau \in [0,t)$; and either of the conditions fails at t, i.e. either:

(i) $M_{t,\pi} \cap H^-(\pi) \cap M_t \neq \emptyset$

(ii) $V \in T_x M_t$ for some $x \in \pi$.

(i) Suppose the first case is true. Then, there exists $x_0 \in M_{t,\pi} \cap H^-(\pi) \cap M_t$ such that at x_0 the two manifolds are tangent.
We can take a neighbourhood of $(x_0, t) \in M_t \times \mathbb{R}$ such that both $M_{t,\pi}$ and M_t are graphs over $T_{x_0}M_t$ by 2.3.
We can explicitly write the functions $f : U \times (t-\varepsilon, t+\varepsilon) \to M_t$, where $U \subset T_{x_0}M_t$, and the corresponding f_π for $M_{t,\pi}$. We can also write

$$\begin{aligned} f \ &: \ (x,t) \mapsto x + \tilde{f}(x,t)\nu \\ f_\pi \ &: \ (x,t) \mapsto x + \tilde{f}_\pi(x,t)\nu \end{aligned}$$

for appropriate functions $\tilde{f} : U \times (t-\varepsilon, t+\varepsilon) \to \mathbb{R}$ and $\tilde{f}_\pi : U \times (t-\varepsilon, t+\varepsilon) \to \mathbb{R}$, where ν is a fixed unit vector normal to $T_{x_0}M_t$. $\tilde{f}$ and $\tilde{f}_\pi$ are solutions to the same second order PDE, which is parabolic by what was discussed in paragraph 2.2, hence we can apply Proposition 1.12 to conclude that $\tilde{f} \equiv \tilde{f}_\pi$, and thus $M_{t,\pi}$ and M_t coincide in a neighbourhood of (x,t), a contradiction as we assumed that t is the first t where the flows touch.

(ii) Suppose instead that $V \in T_x M_t$ for some $t \in [0,t)$ and some $x \in M_t \cap \pi$. Then $T_x M_t = T_x M_{t,\pi}$ and in a neighbourhood of (x,t) both M_t and $M_{t,\pi}$ are graphs of two smooth functions over $T_x M_t$ by 2.3, i.e. again

$$\begin{aligned} f \ &: \ (x,t) \mapsto x + \tilde{f}(x,t)\nu \\ f_\pi \ &: \ (x,t) \mapsto x + \tilde{f}_\pi(x,t)\nu \end{aligned}$$

Moreover, in $\overline{H^-(\pi)}$, $f_\pi \geq f$, because $M^n_\pi \cap H^-(\pi) \subset \mathrm{int}(M^n) \cap H^-(\pi)$. Finally, $f(x,t) = f_\pi(x,t)$, hence $f_\pi - f(x,t) = 0$, and thus (x,t) is a minimum point on the boundary for $f_\pi - f$. Also, we must have

$$\frac{\partial f}{\partial V}(x,t) = \frac{\partial f_\pi}{\partial V}(x,t)$$

because the graphs are both tangent to $T_x M_t$, and V here is the outward pointing normal to the boundary by definition of the reflection. Thus,

$$\frac{\partial (f - f_\pi)}{\partial V}(x,t) = 0$$

But we must have

$$\frac{\partial(f - f_\pi)}{\partial V}(x,t) > 0$$

at a minimum on the boundary by Proposition 1.13, a contradiction. □

The following is an immediate consequence of the main result:

Corollary 2.7. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a C^2 solution to 3.2. Then, if we can reflect M_0 strictly up to (π, V), then for all $t \in [0,T)$ we can reflect $X(M,t)$ strictly up to (π, V).*

Proof. The hypothesis of the theorem are true for each π_K in the definition, thus we can reflect strictly with respect to each π_K for all $t \in [0,T)$, and thus we can reflect $X(M,t)$ strictly up to (π, V). □

Chapter 3

Extension to constant curvature spaces

In this chapter we want to extend the Chow-Gulliver result to constant curvature spaces. We will use the same notation as in section 1.5 and 1.6.

3.1 The equation in constant curvature spaces (da scrivere)

As shown in [5], the equation we analysed in the previous chapter can be also considered in non-flat ambient spaces. In particular, we will analyse the case where the ambient space is one of those described in section 1.5: $\mathbb{R}^{n+1}$, $\mathbb{H}^{n+1}$, or $\mathbb{S}^n_+$. We again use the symbol $\mathbb{M}^{n+1}_+$ to indicate any of these spaces. Let $X_0 : M^n \to \mathbb{M}^{n+1}_+$ be a manifold embedded in $\mathbb{M}^{n+1}_+$. Let $F : \{(\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n | \kappa_1 \leq \cdots \leq \kappa_n\} \to \mathbb{R}$ be a C^1 function satisfying:

$$\frac{\partial F}{\partial \kappa_i} > 0 \text{ for all } i = 1, \ldots, n \tag{3.1}$$

and consider the evolution equation

$$\begin{cases} \dfrac{\partial X_t}{\partial t} = F(\kappa_1(x), \ldots, \kappa_n(x))\nu \\ X(0) = X_0 \end{cases} \tag{3.2}$$

where ν is the inward normal to $X_t(M^n)$ at the point $X_t(x)$ and $\kappa_1 \leq \cdots \leq \kappa_n$ are the principal curvatures at $X_t(x)$.

As we saw in the previous chapter, it is a non-linear parabolic differential equation, as the calculation in chapter 2.2 is valid for any metric on $\mathbb{R}n + 1$, and in particular, for the metrics in our models for $\mathbb{M}_+^{n+1}$ in section 1.5. The existence result in [5] also holds as well in this case. Finally, the result in 2.4 is not using the metric tensor, and therefore is valid in this setting as well.

3.2 Extension of the result(da scrivere)

[DA SCRIVERE]

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