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# THE ALEXANDROV MOVING PLANE METHOD AND APPLICATIONS TO GEOMETRIC FLOWS


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# Introduction

To be confirmed

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## 1.1 Geometry of immersed hypersurfaces

The study of immersed hypersurfaces is a fundamental topic in differential geometry, and is especially important in the field of geometric analysis. An immersed hypersurface $X : M^n \to \overline{M}^{n+1}$ is a submanifold of a higher-dimensional space that is embedded in that space in such a way that the submanifold has the same dimension as the space in which it is embedded minus one. In other words, it is a submanifold of codimension one. We will always assume the embedding to be smooth. The most common case is $\overline{M}^{n+1} = \mathbb{R}^{n+1}$, usually extensively studied in undergraduate courses when $n = 2$. We will be assuming that the embedding is also isometric, i.e. the metric $g$ on $M^n$ is the one induced by $(\overline{M}^{n+1}, \overline{g})$.

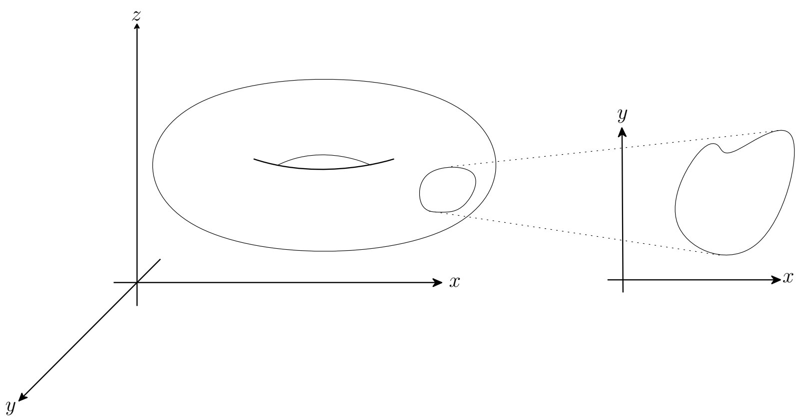


Figure 1.1: A (hyper)surface immersed in $\mathbb{R}^3$

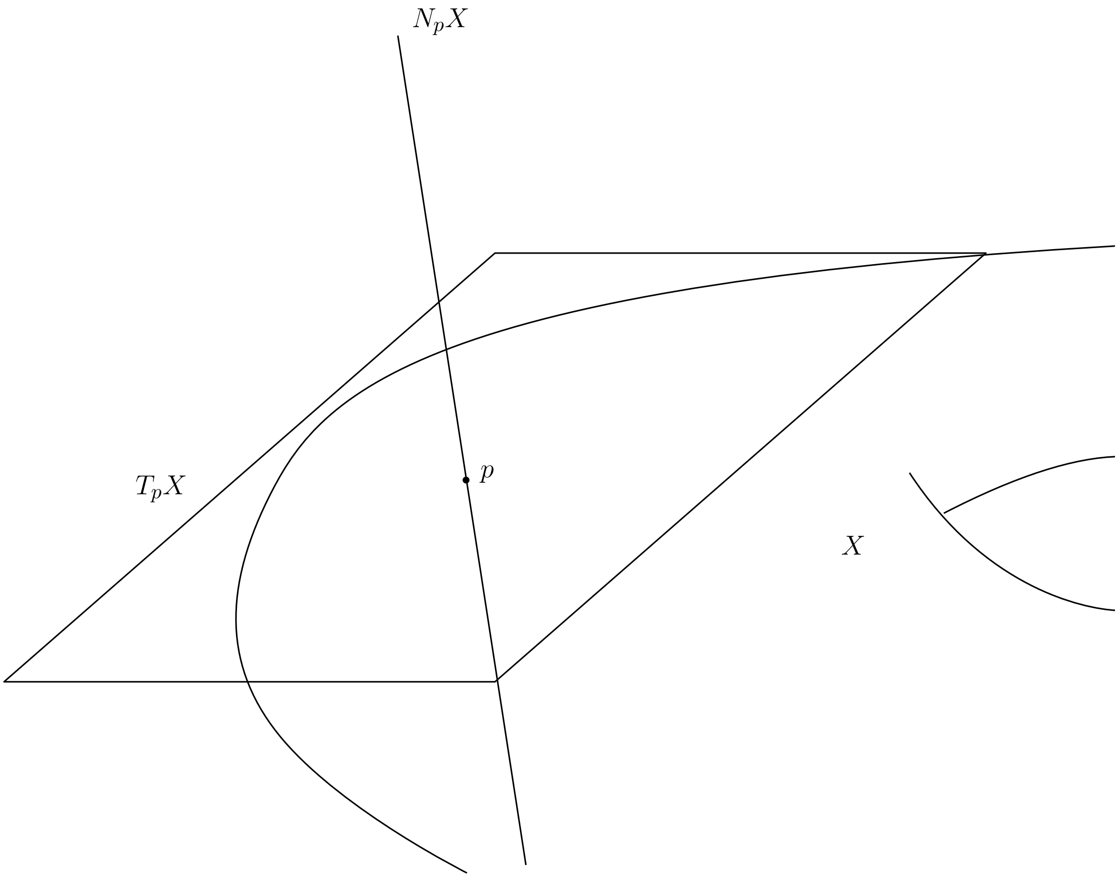


Figure 1.2: The tangent and the normal bundle at a point $p$

Symbols referring to $(\overline{M}^{n+1}, \overline{g})$ will have a line on top, otherwise the symbol will refer to $(M^n, g)$.

The pullback of the tangent bundle of $\overline{M}^{n+1}$ to $M^n$ is a smooth vector bundle on $M^n$:

$$X^*T\overline{M}^{n+1} = T\overline{M}^{n+1}|_{M^n} = \amalg_{p\in M^n} T_p\overline{M}^{n+1}$$

One of the most important concepts in the geometry of immersed hypersurfaces is the concept of a normal vector field. The normal vector field $\nu$ is a section of the pullback vector bundle $X^*TM^{n+1}$ on the manifold $M^n$ that is perpendicular to the tangent space of $M^n$ at each point. At each point $p$:

$$T_p\overline{M}^{n+1} = T_pM^n \oplus N_pM^n$$

where $N_pM^n$ is the normal vector bundle generated by the normal vector. If the manifold is oriented, there is usually a choice we can make on the normal vector:

we will generally chose the outward pointing normal vector. This also allows us to define the tangent and normal projection on $T\overline{M}^{n+1}|_{M^n}$ by taking the two respective components.

Clearly, taking $\overline{\nabla}$ to be the Levi-Civita connection on $(\overline{M}^{n+1}, \overline{g})$, we can decompose it as:

$$\overline{\nabla}_v w = (\overline{\nabla}_v w)^\top + (\overline{\nabla}_v w)^\perp$$

**Definition 1.1.** *The* second fundamental form *is then defined as:*

$$\mathrm{I\!I}(v, w) = (\overline{\nabla}_v w)^\perp$$

It is a bilinear symmetric tensor because $TM^n$ is involutive in $T\overline{M}^{n+1}$ and depends only on the local value of $v$ and $w$ by symmetry. we can therefore write it as

$$\mathrm{I\!I}(v, w) = -(h_{ij} v^i w^j)\nu$$

for some matrix $A(p) = \{h_{ij}\}$. we can define the principal curvatures of the hypersurface, as the eigenvalues of this matrix.

It is also possible to check that $(\overline{\nabla}_v w)^\top$ satisfies the definition the Levi-Civita connection on $(M^n, g)$, therefore, from its uniqueness:

$$\begin{aligned} \nabla_v w &= (\overline{\nabla}_v w)^\top \\ \overline{\nabla}_v w &= \nabla_v w + \mathrm{I\!I}(v, w) \end{aligned}$$

where $\nabla$ is the Levi-Civita connection on $(M^n, g)$. This result is known as the *Gauss Formula*. It has to be noted however that we are implicitly considering tangent vectors that are not in the same space. Indeed, making that more explicit, the formula should be:

$$\overline{\nabla}_{X_* v} X_* w = X_*(\nabla_v w) + \mathrm{I\!I}(v, w)$$

**Proposition 1.2. (The Weingarten Equation)** *If $v, w \in TM^n$ and $\nu \in NM^n$, if one considers the corresponding derivations in $T\overline{M}^{n+1}$ the following equation holds:*

$$\left\langle \overline{\nabla}_v \nu, w \right\rangle_{\overline{g}} = -\left\langle \nu, \mathrm{I\!I}(v, w) \right\rangle_{\overline{g}}$$

*Proof.* As $\langle \nu, w \rangle_{\overline{g}} \equiv 0$ on $M$,

$$\begin{aligned} 0 &= v \left\langle \nu, w \right\rangle_{\overline{g}} \\ &= \left\langle \overline{\nabla}_v \nu, w \right\rangle_{\overline{g}} + \left\langle \nu, \overline{\nabla}_v w \right\rangle_{\overline{g}} \\ &= \left\langle \overline{\nabla}_v \nu, w \right\rangle_{\overline{g}} + \left\langle \nu, \mathrm{I\!I}(v, w) \right\rangle_{\overline{g}} \end{aligned}$$

applying the Gauss Formula and the fact that $\nabla_v w \in TM^n$ in the last step $\square$

It is also usual to define the associated Weingarten map, which is the linear map between sections of $M$ $s : \Gamma(M) \to \Gamma(M)$ satisfying:

$$\langle s(v), w\rangle_g = \langle \nu, \mathrm{II}(v, w)\rangle_{\overline{g}}$$

the linear map $s$ is also known as the *shape operator of* $M$.

Combining this with the formula above, taking into account that it holds for a generic $w \in TM$:

$$s(v) = -(\overline{\nabla}_v \nu)^\top$$

We are going to use these equations in local coordinates, in the form shown below.

**Proposition 1.3.** *The above equations in local coordinates are equivalent to the following equations:*

$$\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial X^\alpha}{\partial x^k} + \overline{\Gamma}^\alpha_{\beta\delta}\frac{\partial X^\beta}{\partial x^i}\frac{\partial X^\delta}{\partial x^k} = -h_{ij}\nu^\alpha \tag{1.1}$$

$$\frac{\partial \nu^\alpha}{\partial x^i} + \overline{\Gamma}^\alpha_{\beta\delta}\frac{\partial X^\beta}{\partial x^i}\nu^\delta = h_{ij}g^{jl}\frac{\partial X^\alpha}{\partial x^l} \tag{1.2}$$

*where $\nu$ is the normal unit vector at the point and $A = \{h_{ij}\}$ is the second fundamental form, thus $h_{ij} = \left\langle \nu, \overline{\nabla}_{\overline{\partial_i}}\overline{\partial_j}\right\rangle_{\overline{g}}$*

*Proof.* For any connection $\nabla$. and any derivations $v = v^i\partial_i$ and $w = w^j\partial_j$:

$$\nabla_v w = \nabla_{(v^i\partial_i)}(w^j\partial_j) = v(w^k)\partial_k + (v^i w^j \Gamma^k_{ij})\partial_k$$

Let $\partial_1, \dots \partial_n$ be a basis of $TM^n$ at a point, and let $\overline{\partial_i} = X_*\partial_i$, $\overline{\partial_{n+1}} = \nu$. Let's consider the Gauss Formula for two generic $\partial_i$, $\partial_j$, using roman letters for indices varying between 1 and $n$ and greek letters for indices varying between 1 and $n+1$, and $\delta^{ij}$ the Kronecker delta:

$$\begin{aligned}
\overline{\nabla}_{X_*\partial_i} X_*\partial_j &= X_*(\nabla_{\partial_i}\partial_j) + \mathrm{II}(\partial_i, \partial_j)\\
(\overline{\partial_i}(X_*\partial_j)^\alpha)\partial_\alpha + ((X_*\partial_i)^\beta (X_*\partial_j)^\delta \overline{\Gamma}^\alpha_{\beta\delta})\overline{\partial_\alpha} &= X_*(\Gamma^k_{ij}\partial_k) - h_{ij}\nu^\alpha\overline{\partial_\alpha}\\
\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} + \overline{\Gamma}^\alpha_{\beta\delta}\frac{\partial X^\beta}{\partial x^i}\frac{\partial X^\delta}{\partial x^k} &= \Gamma^k_{ij}\frac{\partial X^\alpha}{\partial x^k} - h_{ij}\nu^\alpha
\end{aligned}$$

Which is the formula (1.1). To get the second formula, first note that $s(v) = -(\overline{\nabla}_v\nu)^\top$. We then compute $-\langle s(\partial_i), \overline{\partial_\alpha}\rangle_{\overline{g}}$:

$$\begin{aligned}
\left\langle \overline{\nabla}_{\overline{\partial_i}}\nu, \overline{\partial_\alpha}\right\rangle_{\overline{g}} &= -\left\langle s(\partial_i), \overline{\partial_\alpha}\right\rangle_{\overline{g}}\\
\left\langle \left(\frac{\partial \nu^\alpha}{\partial x^i} + \overline{\Gamma}^\alpha_{\beta\delta}\frac{\partial X^\beta}{\partial x^i}\nu^\delta\right)\overline{\partial_\alpha}, \overline{\partial_\alpha}\right\rangle_{\overline{g}} &= h_{ij}g^{jl}\frac{\partial X^\alpha}{\partial x^l}
\end{aligned}$$

leading to (1.2). □

## 1.2 Local representation as a graph

A well known result, Dini's Theorem[1], states that, given a smooth function $F$ defined on an open subset of the product space $\mathbb{R}^n \times \mathbb{R}^m$, if $F(x, y) = 0$ and the partial derivative of $F$ with respect to $y$ is nonzero at a point $(x_0, y_0)$, then there exists an open neighborhood of $x_0$ in $\mathbb{R}^n$ and a unique smooth function $y = g(x)$ defined on that neighborhood such that $y_0$ is a regular value of $g$ and $(x, g(x))$ is a smooth solution to the equation $F(x, y) = 0$.

Consequence of the Dini's theorem is a powerful result that allows one to locally represent a submanifold of $\mathbb{R}^n \times \mathbb{R}^m$ as the graph of a smooth function. This theorem is widely used in differential geometry, geometric analysis, and many other fields of mathematics and physics. We provide a version of this theorem below:

**Theorem 1.4** (Local representation as a graph). *Let $X^n$ be a submanifold $X^n \subset \mathbb{R}^{n+1}$ and let $x_0 \in X$. Then there exists a neighbourhood of $x_0$, $U \subset X^n$, such that $U$ is the graph of a function. Moreover, this function can be of the form*

$$f : \pi(U) \subset \mathbb{R}^n \to U$$
$$U = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | x_0 = f(x_1, ..., x_n)\}$$

*for any of the possible orders of the usual basis for $\mathbb{R}^n$, $(e_0, \ldots, e_n)$, as long as $e_0 \notin T_x M$, where $\pi(U)$ is the projection on the last $n$ coordinates ($(x_0, \ldots, x_n) \mapsto (x_1, \ldots, x_n)$).*

A proof of the 2D-case of the version of the theorem can be found in [17] which extends naturally to the $n$ dimensional case, with almost no changes. This immediately extends to:

**Corollary 1.5** (Local representation as a graph on the tangent). *Let $X^n$ be a submanifold $X^n \subset \mathbb{R}^{n+1}$ and let $x_0 \in X$. Then there exists a neighbourhood of $x_0$ $U \subset X^n$ and a smooth function $f : T_x X^n \to \mathbb{R}$ such that any $x_0 \in U$ can be expressed as*

$$x_0 = p + f(p)\nu$$

*where $\nu$ is the vector normal to $T_{x_0} X^n$, for an appropriate point $p \in T_{x_0} X^n$. In other words, every submanifold $X^n \subset \mathbb{R}^{n+1}$ is locally expressible as a graph on its tangent space.*

*Proof.* By rotation, we may assume $T_x X^n$ orthogonal to $e_1$. Then one can just apply the previous theorem. □

We will use this later to prove Theorem 2.3.

[1] also known as Implicit Function Theorem

## 1.3 Some well established results from analysis

We now introduce some well known results from analysis. The first result we introduce is the maximum principle.

The maximum principle is a classical result of mathematical analysis, and it is usually introduced in a first course on partial differential equations. It is a fundamental tool in the theory of partial differential equations. It is a statement about the behavior of solutions to certain types of PDEs and provides a method for obtaining upper and lower bounds on the solutions. The principle states that the maximum and minimum values of a solution to elliptic or parabolic PDE occur on the boundary of the domain unless the function is constant.

The maximum principle can be used to prove the existence, uniqueness, and regularity of solutions to elliptic and parabolic PDEs. It can also be used to obtain estimates on the behavior of solutions and to study the asymptotic behavior of solutions as the domain becomes large. The principle is widely used in many fields of mathematics and physics, such as geometric analysis, mathematical physics, and fluid dynamics. One of the many versions of this well know theorem is this:

**Theorem 1.6** (Maximum principle for parabolic equations)**.** *Let $\Omega$ be an open, bounded, connected set. Assume $u \in C_1^2(\Omega \times [0,T]) \cap C^1(\overline{\Omega} \times [0,T])$. Suppose $u$ satisfies:*

$$-\frac{\partial u}{\partial t} + \left( \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i}^{n} b_i \frac{\partial}{\partial x_i} + c \right) u = -u_t + Lu \geq 0 \tag{1.3}$$

*where $L$ is an elliptic differential operator, i.e. there exists $\theta > 0$ such that $\sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|$ for all $\xi \in \mathbb{R}^n$ and $(x,t) \in \Omega \times [0,T]]$. Suppose also that $c \equiv 0$ in $\Omega$. Then:*

- *if $u$ attains its maximum in an interior point $(x_0, t_0) \in \Omega \times [0,T]$, then $u$ is constant in $\Omega \times [0, t_0]$.*
- *If, instead, under the same conditions, $u_t + Lu \geq 0$ and attains its minimum in an interior point of $\Omega \times [0,T]$, then $u$ is constant in $\Omega \times [0, t_0]$*

A proof of this result can be found, for example, in [15]. The theorem extends also to situations where the condition holds in a convex bounded connected region $R \subseteq \Omega \times [0,T]$: in that case, if $u$ attains its maximum in an interior point then $u$ has the same value at any point in $R$ that can be connected to it through a segment going in the backwards direction of time and a "horizontal" line contained in $\Omega$. This version of the theorem can be found for example in [12]:

**Theorem 1.7.** *Let $u$ satisfy the uniformly parabolic differential inequality (1.3) with $c(x) \leq 0$ in a region $R_T = \{(x_1, x_2, \ldots, x_n, t) | t \leq T\}$ where $R$ is a non-empty connected open set, and suppose that the coefficients of $L$ are bounded. Suppose that the maximum of $u$ in $R_T$ is $M$ and that it is attained at a point $(x, t)$ of $R_T$. Thus if $(y, s)$ is a point of $R$ which can be connected to $(x, t)$ by a path in $R$ consisting only of horizontal segments and upward vertical segments, then $u(y, s) = M$.*

Hopf's boundary point lemma is another important classical tool in the study of PDEs that provides a criterion for determining the behavior of solutions to certain types of elliptic or parabolic PDEs near the boundary of the domain. The lemma states that if one has a solutions to some kinds of partial differential inequalities, then the normal derivative of the solution at that point is strictly positive.

It is often used to obtain estimates on the behavior of solutions near the boundary, and to prove the existence and uniqueness of solutions to boundary value problems. The lemma is named after the German mathematician Eberhard Hopf, who first formulated it in the 1950s. In [12] we find the following version of the Hopf's boundary point lemma:

**Theorem 1.8.** *Let $u$ be a solution to the parabolic inequality*

$$-u_t + Lu \geq 0$$

*with $L$ an elliptic linear differential operator with bounded coefficients such that $c(x) \leq 0$,in a domain $E$, and let $E_t = \{(x, s) \in E | s \leq t\}$. Suppose the maximum $M$ of $u$ is attained at a point $P = (x, t)$ on the boundary $\partial E$.*
*Assume that a sphere through $P$ can be constructed which is in $E_s$ such that*

- *tangent to $\partial E$ at $P$*
- *the set of point of its interior $(y, s)$ such that $s \leq t$ lies in $E_s$,*
- *$u < M$ in its interior.*

*Also, suppose that the radial direction from the centre of the sphere to $P$ is not parallel to the t-axis.*
*Then, if $\frac{\partial}{\partial \nu}$ denotes any directional derivative in an outward direction from $E_s$ , we have*

$$\frac{\partial u}{\partial \nu} > 0$$

*at $P$.*

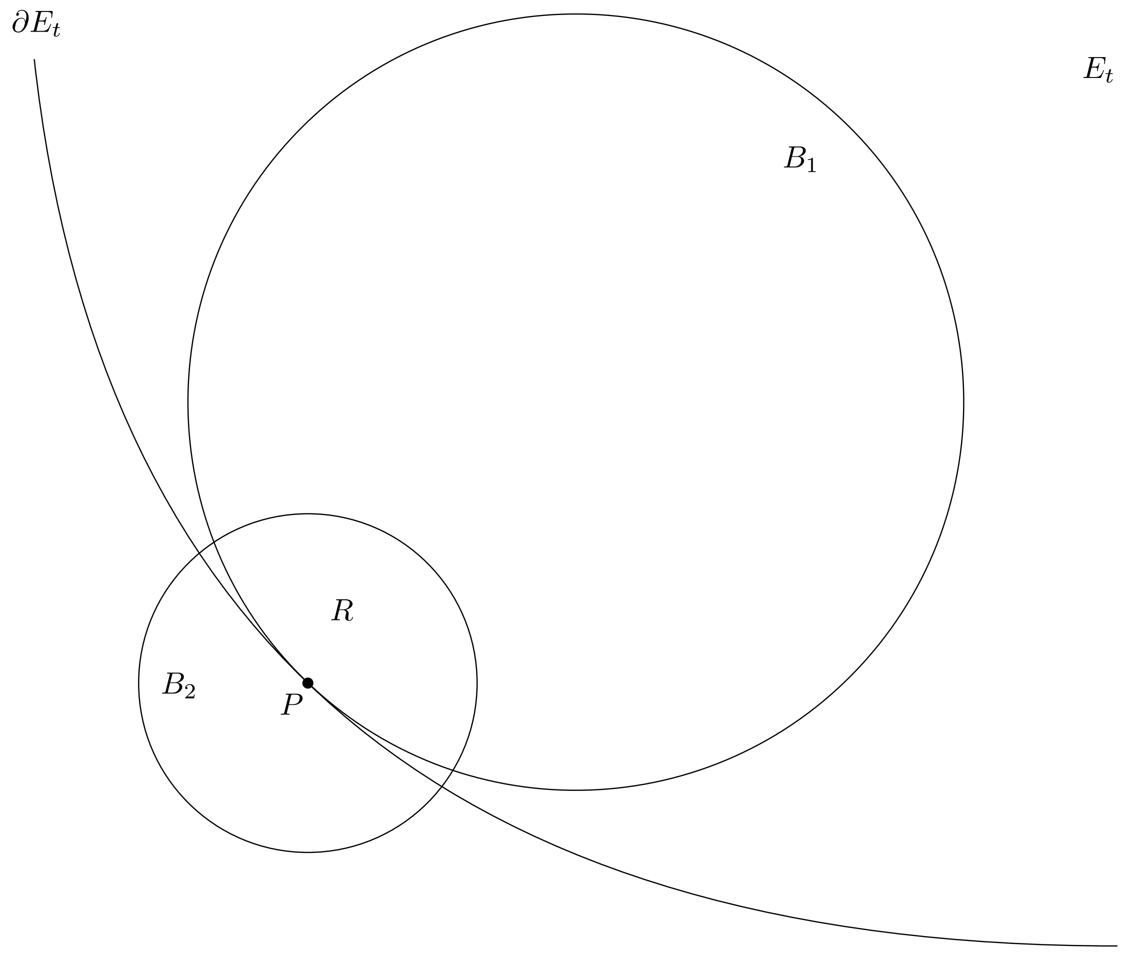


Figure 1.3: The setup in the proof at time $t$ when there are two space dimensions. The region $R$ also extends for times $s < t$ between the two spheres. The centre of $B_1$ can have time-coordinate different from $t$ if the domain $E$ is not "straight" in the time direction.

*Proof.* Let the sphere through $P$ be $B_1$. We may construct a smaller sphere $B_2$ centred at $P$. Let now:

$$\begin{aligned}
S_1 &= \partial B_1 \cap B_2 \cap E_t,\\
S_2 &= B_1 \cap \partial B_2 \cap E_t, \text{and}\\
S_3 &= B_1 \cap B_2 \cap \partial E_t = B_1 \cap B_2 \cap \{(x,s) \in E | s = t\}.
\end{aligned}$$

The three sets satisfy $S_1 \cup S_2 \cup S_3 = \partial(B_1 \cap B_2 \cap E_t)$, we may call this region $R = B_1 \cap B_2 \cap E_t$. Without loss of generality, potentially taking a smaller sphere $B_1$, we may assume that $u < M$ on $B_1$ except at $P$. As $R \subset B_1$, we also get $u < M$ on $R$. We may thus conclude that:

- $u < M$ on $R$ except at $P$
- $u \leq M - \delta$ on $S_2$ for a sufficiently small $\delta > 0$
- $u = M$ at $P$.

Now, let the centre of $B_1$ be $Q = (z, t_0)$ and let $r$ be its radius. we can now introduce the function

$$v(y,s) = exp\left(-\alpha(s-t_0)^2 - \sum_{i=1}^{n} \alpha(y_i - z_i)^2\right) - exp\left(-\alpha r^2\right)$$

This function is such that $v(y,s) = 0$ if $(y,s) \in S_1$ - including $v(x,t) = 0$, as there the first term is $e^{-\alpha r^2}$, and $v(y,s) > 0$ in the interior of $B_1$.
Thus, in the region $R$, $v(y,s) \geq 0$ and has a minimum point at the boundary on $(x,t)$, where $v(x,t) = 0$.

We can also compute $Lv$. After some calculation, we get that

$$\begin{aligned}
Lv = 2\alpha e^{\left(-\alpha(s-t_0)^2 - \sum_{i=1}^{n} \alpha(y_i - z_i)^2\right)} [2\,\alpha\,(y-z)^t A(y-z) + \\
+ \sum_{i}^{n} [b_i(y_i - z_i) + a_{i,i}] + (s-t)]
\end{aligned}$$

where $A$ is the matrix of the $a_{i,j}$. In particular, one can choose an $\alpha$ large enough, so that $Lv > 0$ in $R \cup \partial R$.

We can thus introduce $w = u + \varepsilon v$. As both $Lu$ and $Lv$ are positive in $R$, $Lw > 0$ in $R$. We can also choose $\varepsilon$ small enough so that $w < M$ on $S_2$. Also, as $v = 0$ on $S_1$, $w < M$ on $S_1$ except at $P$, and $w = M$ at $P$.

Therefore, we can apply the Strong Maximum Principle 1.7 to the region $R$ to conclude that the maximum of $w$ in $R$ is attained at $P$. Therefore:

$$\frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu} + \varepsilon \frac{\partial v}{\partial \nu} \geq 0$$

But:

$$\frac{\partial v}{\partial \nu} = \nu \cdot n \frac{\partial v}{\partial R} = -2\nu \cdot n\alpha R e^{-\alpha R} < 0$$

Where $n$ is the vector orthogonal to the sphere $S_1$. Therefore, one must have:

$$\frac{\partial u}{\partial \nu} > 0$$

as we wanted. $\square$

**Remark 1.9.** If $c(x)$ is now just bounded, we can consider, instead of $u$, $v = ue^{-\lambda t}$, thus, by change of variables

$$-v_t + Lv - \lambda v \geq 0$$

whenever $-u_t + Lu \geq 0$, and we can chose $\lambda$ large enough such that $c(x) - \lambda < 0$ and thus we can remove the hypothesis $c(x) \leq 0$ in both theorems when $c$ is bounded.

## 1.4 Applying the maximum principle to non-linear PDEs

We can now introduce an important observation shown in [12] that allows us to apply the maximum principle 1.7 and Hopf's boundary point lemma 1.8 in some non-linear settings. Firstly, we must clarify what we mean by parabolic non-linear problem.

**Definition 1.10.** *A differential non-linear problem in the form*

$$Lu = F\left(t, x, v, \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_j}\right) - v_t = f(x,t) \tag{1.4}$$

*given a smooth $F$ is parabolic if for any real vector $\xi$*

$$\sum_{i,j=1}^{n} F_{ij}\xi_i\xi_j > 0$$

*where $F_{ij}$ are the derivatives of $F$ with respect to $\frac{\partial^2 v}{\partial x_i \partial x_j}$.*

Secondly, we remind the reader of the following generalized version of the theorem of the mean, a.k.a. Lagrange's theorem:

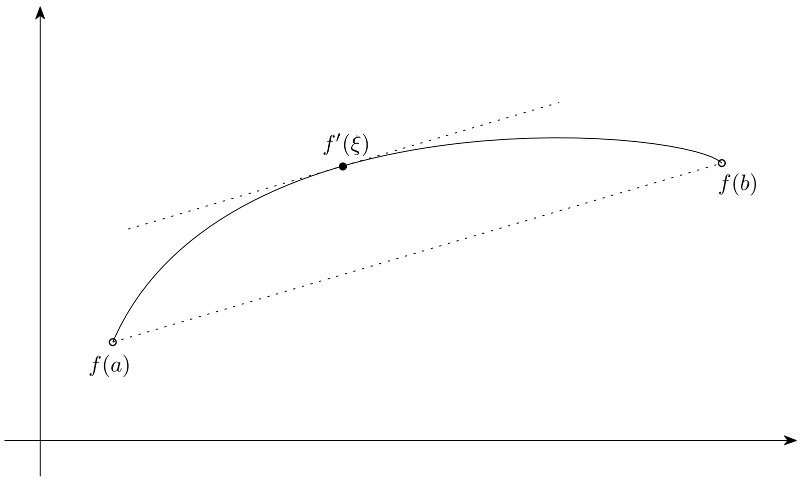


Figure 1.4: In one dimension, Lagrange's theorem states that given a smooth function $f$ on an interval $[a,b]$, there exists a $\xi$ such that $f(b)-f(a)=f'(\xi)(b-a)$.

**Theorem 1.11** (Lagrange's theorem)**.** *Given a convex open set $U \subseteq \mathbb{R}^n$ and a real function $F \in C^1(U)$, and given to points $x, y$ in $U$, there exists a point $z$ in the segment connecting $x$ and $y$ such that*

$$F(y) - F(x) = \langle \nabla F(z), (y-x) \rangle$$

Suppose that we have a solution to a non-linear parabolic problem $v$, i.e. $v$ solves (1.4):

$$Lv = F\left(t, x, v, \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_j}\right) - v_t = f(x,t)$$

for a non-linear elliptic operator $L$ in some region $E$, where we assume $F(t, x, y_i, z_{i,j})$ to be a given $C^1$ function. Suppose also that there is a $w$ which is a solution of the corresponding differential inequality:

$$Lw = F\left(t, x, w, \frac{\partial w}{\partial x_i}, \frac{\partial^2 w}{\partial x_i \partial x_j}\right) - w_t \leq f(x,t)$$

One can then consider $u = v - w$, and by combining the above we get:

$$\left(F\left(t, x, v, \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_j}\right) - F\left(t, x, w, \frac{\partial w}{\partial x_i}, \frac{\partial^2 w}{\partial x_i \partial x_j}\right)\right) - u_t \leq 0$$

Now, we can apply Lagrange's theorem to $F$ to get

$$\tilde{L}u = \left\langle \nabla F(\xi(x,t)), \left(t, x, u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}\right)\right\rangle - u_t \leq 0$$

for a fixed $\xi(x,t)$.

Thus, the difference $u$ of two sub-solutions to a non-linear differential problem is a sub-solution to a (different) *linear* parabolic problem, as the derivatives of $F$ and $\xi$ do not depend on $u$ ($\xi$ can be chosen a-priori).

We can thus see that this new problem must be parabolic, and apply the maximum principle and the Hopf's boundary point lemma to $u$. This can allow us to state the following two results which we will be using later:

**Proposition 1.12** (Maximum principle for parabolic non-linear differential equations)**.** *Suppose we have two solution $v$ and $w$ in the interval $[0,T]$ to the same parabolic non-linear differential equation (1.4) on an bounded open set $\Omega$, but with different values at $t = 0$. Suppose also that $F$ is smooth on $\overline{\Omega}$. Then, if $v > w$ in the interior of $\Omega$ at $t = 0$ and $v \geq w$ on $\partial\Omega$, $v > w$ for all $t \in [0,T]$ in the interior of $\Omega$.*

*Proof.* $u = v - w \geq 0$ is a solution of a parabolic *linear* differential equation, where the term independent of $u$ is bounded. Furthermore, if we take $c(x,t)$ it must be bounded by compactness. At $t = 0$, $u > 0$ in the interior of $\Omega$. If, at an interior point $x$, $v = w$ at a certain time $t = \tau$, $u(\tau, x) = 0$, and thus $u$ is not constant. However, it attains minimum ($u = 0$) at an interior point, thus by 1.6 it must be constant, a contradiction. □

**Proposition 1.13** (Hopf's boundary point lemma for parabolic non-linear differential equations)**.** *Suppose we have two solution $v$ and $w$ to the same parabolic non-linear differential equation (1.4) in a region $E$, but with different start conditions. Suppose also that $F$ is smooth on $\overline{\Omega}$. Let $u = v - w$ and suppose that the maximum of $u$ is attained at the point $P$. Furthermore, assume that the conditions on the shape of the region $E$ from theorem 1.8 hold. Then,*

$$\frac{\partial u}{\partial \nu}(P) > 0$$

*where we take $\nu$ as the normal to $\partial\Omega$.*

*Proof.* $u$ is a solution of a parabolic *linear* differential equation, where $c(x,t)$ is bounded by compactness. We can then apply 1.8 to $v$ (see also remark 1.9). □

## 1.5 Reflections on spheres and hyperbolic spaces

In what follows, we will focus on manifolds embedded in spaces which have constant sectional curvature. Constant curvature manifolds are classified into three types based on the sign of the curvature:

- **Positive curvature**: *Spherical geometry*, where the curvature is positive and the manifold locally resembles a sphere (e.g., the standard sphere $\mathbb{S}^n$).
- **Zero curvature**: *Flat geometry*, where the curvature is zero and the manifold locally resembles Euclidean space $\mathbb{R}^n$.
- **Negative curvature**: *Hyperbolic geometry*, where the curvature is negative and the manifold locally resembles hyperbolic space $\mathbb{H}^n$.

As in [3], we will use the symbol $\mathbb{M}^n$ to indicate a Riemannian manifold that can be replaced by any one of $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{H}^n$: the $n$-dimensional sphere, Euclidean plane or Hyperbolic space respectively.

We will also use use the symbol $\mathbb{M}^n_+$ to indicate a Riemannian manifold that can be replaced by $\mathbb{S}^n_+$, $\mathbb{R}^n$ or $\mathbb{H}^n$: the $n$-dimensional hemisphere, Euclidean plane or Hyperbolic space, respectively.

**Definition 1.14.** $\mathbb{H}^n$ *is the* $n$-dimensional hyperbolic plane*. We can define it as the half space* $\{x \in \mathbb{R}^n | x_n > 0\}$ *with the Riemannian metric*

$$g_x = \frac{1}{x_n^2}\langle \cdot, \cdot \rangle$$

*where* $\langle \cdot, \cdot \rangle$ *is the Euclidean dot product on* $\mathbb{R}^n$

**Remark 1.15.** This is not the only way we could define the hyperbolic space $\mathbb{H}^n$. Another alternative is taking the $n$-dimensional disc $D^n = \{x \in \mathbb{R}^n | \|x\| < 1\}$ with the Riemannian metric

$$g_x = \frac{4}{(1 - \|x\|^2)^2}\langle \cdot, \cdot \rangle$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean dot product on $\mathbb{R}^n$.

**Remark 1.16.** Let $\mathbb{S}^n \setminus \{P\}$ be the standard $n$-dimensional unitary sphere minus a point, with the standard induced Euclidean metric. Through stereographic projection it is isometric to $\mathbb{R}^n$ with metric:

$$g_x = \frac{4}{(1 + \|x\|^2)^2}\langle \cdot, \cdot \rangle$$

where $\langle \cdot, \cdot \rangle$ is again the Euclidean dot product on $\mathbb{R}^n$.

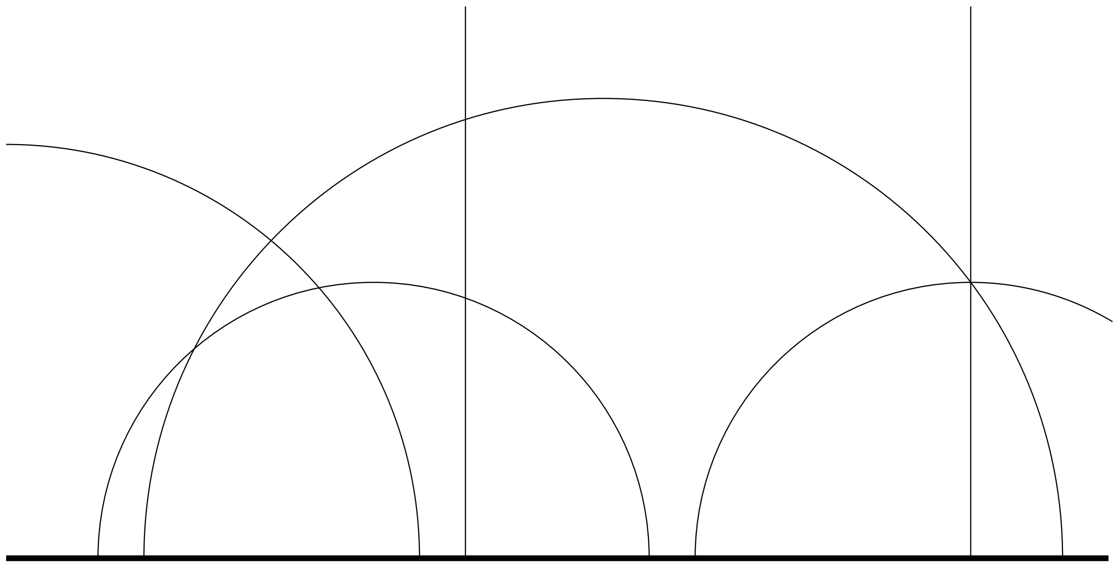

Figure 1.5: On the half-space model of $\mathbb{H}^n$, geodesics are half lines parallel to the $x_n$ axis and circumferences with centre on the $x_n = 0$ hyperplane (see [13]).

**Remark 1.17.** The models above have curvature $\pm 1$. For other values of the curvature one has to consider the following Riemannian metrics

- for the half-space model of hyperbolic space:

$$g_x = \frac{R^2}{x_n^2}\langle \cdot, \cdot \rangle$$

- for the Poincaré ball model of hyperbolic space one has to take the disk with radius $R$ and:

$$g_x = \frac{4R^4}{(R^2 - \|x\|^2)^2}\langle \cdot, \cdot \rangle$$

- For the stereographic projection of a sphere with radius $R$:

$$g_x = \frac{4R^4}{(R^2 + \|x\|^2)^2}\langle \cdot, \cdot \rangle$$

We will now define reflections in $\mathbb{M}^n$:

- On $\mathbb{R}^n$ we choose an hyperplane and use the usual reflections.

- On $S^n$, reflections are those induced by $\mathbb{R}^{n+1}$ when the fixed plane passes through the origin. Each one can be identified by vector orthogonal to the plane we chose. Each hyperplane through the origin in $\mathbb{R}^{n+1}$ defines a $(n-1)$-sphere through intersection with $S^n$, which can all be rotated into each other. We can parametrize each point in $S^n_+$ the following way:
  - choose a point $O$ and a direction $v \in T_O S^n$, and consider the geodesic $\gamma_v$ such that $\gamma_v(0) = O$ and $\dot{\gamma_v}(0) = v$. Assume that $\gamma$ is parametrised by arc-length.
  - Consider the $(n-1)$-sphere $\pi_0$ passing through $\gamma_v(0)$ and orthogonal to $\dot{\gamma_v}(0)$. One can rotate that sphere along the geodesic $\gamma_v$ so that it touches each point in $S^n_+$. We will call the sphere passing through $\gamma_v(t)$ $\pi_{v,t}$. Each point $x$ is in a unique $\pi_{v,t}$.
  - We can then assign to each point in $S^n_+$ a unique couple of coordinates $(x, t)$, where $x$ is the coordinate in $S^{n-1} \cap S^n_+$ when rotating it back to $\pi_0$, and $t$ is the unique $t$ such that the point is in $\pi_{v,t}$.

  Observe that a reflection about $\pi_{v,s}$ is $(x, t) \mapsto (x, 2s - t)$. This assignment can actually be extended to any point on $S^n_-$ as well (the other hemisphere) minus any point on the plane orthogonal to $v$ passing through $O$.

- On $\mathbb{H}^n$ we can also construct a similarly constructed foliation:
  - As a first step, choose a point a point $O \in \mathbb{H}^n$. $\mathbb{M}^n$ is a homogeneous space, so the construction does not depend on the choice we make.
  - Choose any direction in $v \in T_O\mathbb{H}^n$ and consider the geodesic $\gamma_v : \mathbb{R} \to \mathbb{H}^n$ satisfying $\gamma_v(0) = O$ and $\dot{\gamma_v}(0) = v$. Assume that $\gamma$ is parametrised by arc-length.
  - Consider the hyperplane $\pi_0$ passing through $\gamma_v(0)$ and orthogonal to $\dot{\gamma_v}(0)$. Then consider the 1-parameter group of isometries of $\mathrm{H}^n$ such that $g_t(\gamma_v(0)) = \gamma_v(t)$ and such that the curves $t \mapsto g_t(x)$ are orthogonal to $\pi_0$ for each $x \in \pi_0$. This allows us to assign to each point in $\mathrm{H}^n$ coordinates $(x, t)$ where $x \in \pi_0$ and $t \in \mathbb{R}$.
  - consider now any hyperplane $\pi_t$ passing through $\gamma_v(t)$ and orthogonal to $\dot{\gamma_v}(t)$. The reflection fixing $\pi_t$ will be the one given by the formula $(x, t) \mapsto (x, 2s - t)$.

  One may find this insufficient, as there is no guarantee that this construction exists, but it is easy to convince oneself that it does: through rotations and translations it is possible to assume without loss of generality that the point we chose is $e_n$ and the direction is $e_1$ (assuming the half-space model in the

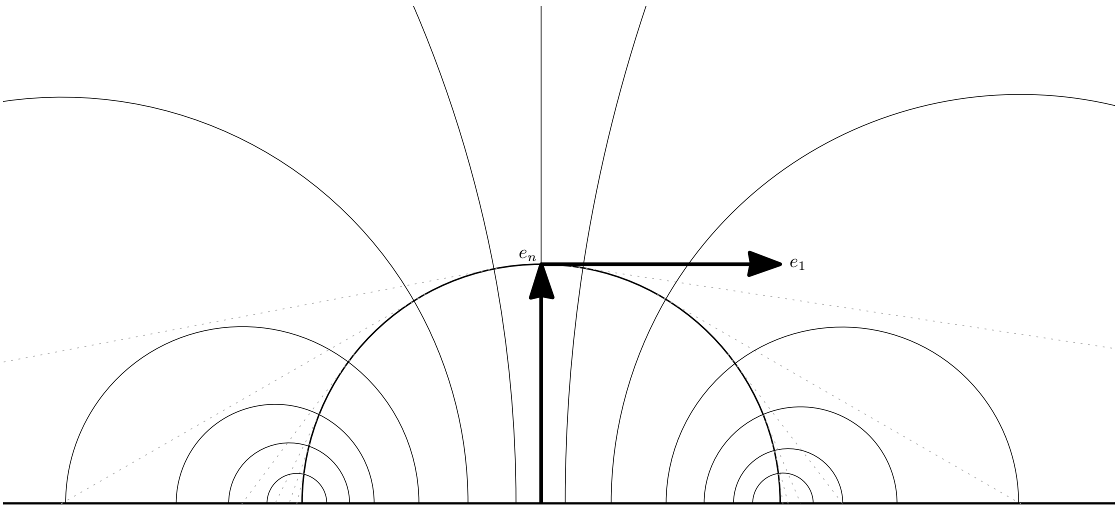


Figure 1.6: The moving hyperplanes in $\mathbb{H}^n$ when the point we chose is $e_n$ and the direction is $e_1$, projected onto the plane spanned by these two vectors. The centres of the hemispheres are the intersection of the $x_n = 0$ plane with the line spanned by the tangent vector.

definition above). We can do this because $\mathbb{H}^n$ is frame-homogenous. Then the geodesic is the unit circle in the plane spanned by $e_1$ and $e_n$, and the (euclidean) hemispheres centred on the $x_n = 0$ plane and normal to the geodesic at one of its points are the hyperplanes (see figure 1.6). For a proof that $\mathbb{H}^n$ is frame-homogenous, see proposition 3.9 in [13].

**Remark 1.18.** Another choice of direction on the hyperbolic plane allows one to write an explicit formula for the reflection about each hyperplanes, which may be more enlightening than the construction above. One can repeat the construction, taking the point $e_n$ and direction $e_n$. The geodesic is then the vertical $x_n$ axis and the totally geodesic hyperplanes are therefore the euclidean spheres centred at the origin. Assume without loss of generality that curvature is $\pm 1$, and take the sphere of radius $r$. We claim that the reflection about the sphere of radius $r$ is the spherical inversion about the sphere. We remind the reader that a spherical inversion about a sphere of centre $O$ is a transformation mapping a point $P$ to the point $P'$ on the (euclidean) line through $O$ and $P$ such that $\overline{OP} \cdot \overline{OP'} = r^2$. Therefore the point $(\underline{0}, a)$ on the geodesic is mapped to $(\underline{0}, \frac{r^2}{a})$ by the inversion.

Computing the geodesic distance of these two points to $(\underline{0}, r)$:

$$\int_r^a \frac{1}{x}dx = [\ln x]_r^a = \ln a - \ln r$$
$$\int_{\frac{r^2}{a}}^r \frac{1}{x}dx = [\ln x]_{\frac{r^2}{a}}^r = \ln r - 2\ln r + \ln a - \ln r = \ln a - \ln r$$

we see that the points are the equidistant from $(\underline{0}, r)$ and therefore the corresponding spheres/hyperplanes have to map to each other. On the other hand, it is a inversions are orientation-reversing isometries in hyperbolic space. Please note however that, if we consider $I_r(P)$, for a fixed a point $P \in \mathbb{H}^n$, and let $r$ vary, it does not move along a geodesic unless it is on the geodesic generating the motion. Therefore, as the geodesic is the shortest path, taking $r$ to be varying between $R_0$ and $R_1$

$$\mathrm{dist}(I_{R_0}(P), I_{R_1}(P)) \le \mathrm{length}(I_r(P))$$

in general, with a strict inequality unless $P$ is on the geodesic. This is unlike the situation on the plane, where equality holds everywhere.

**Remark 1.19.** The inequality

$$\mathrm{dist}(I_{R_0}(P), I_{R_1}(P)) \le \mathrm{length}(I_r(P)) \tag{1.5}$$

taking corresponding objects, holds also on $S^n$.

## 1.6 The Method of Moving Planes

To provide some justification and context to the next chapters, we describe Alexandrov reflection, also known as Method of the Moving Planes in $\mathbb{M}_+^{n+1}$.

Let $X : M^n \to \mathbb{M}_+^{n+1}$ be a hypersurface in a constant curvature ambient space. If the ambient space is a sphere, $X$ must be contained in a hemisphere to avoid issues with multiple self-intersections. Assume also that $X = \partial\Omega$ for a bounded domain $\Omega$ in $\mathbb{M}_+^{n+1}$.

- As a first step, choose a point a point $O \in \mathbb{M}^n$. As $\mathbb{M}^n$ is a homogeneous space, so the construction does not depend on the choice we make. Without loss of generality, we choose the origin in $\mathbb{R}^n$, $e_n$ in $\mathbb{H}^n$ and the north pole in $S^n$.

- Choose any direction in $v \in T_O\mathbb{M}^n$ and consider the geodesic $\gamma_v : I \to \mathbb{M}^n$ satisfying $\gamma_v(0) = O$ and $\dot{\gamma_v}(0) = v$. Assume that $\gamma$ is parametrised by arclength. Here $I = \mathbb{R}$ if $\mathbb{M}_+^n$ is flat or hyperbolic, and $I = (-\frac{\pi}{2}, \frac{\pi}{2})$ if $\mathbb{M}_+^n$ is a hemisphere.

- Consider the hyperplanes $\pi_{v,s}$ passing through $\gamma_v(s)$ and orthogonal to $\dot{\gamma_v}(s)$.

The method consists of reflecting the part of $X$ "below" the hyperplane into the top part, and using properties of both halves of the hypersurface together to prove some statement about the non-reflected hypersurface.

To make this more precise, we can define:

$$X_{v,s} = \{p \in X \mid p \in \pi_{v,t} \text{ for some } t < s\}$$

Let $X^{\pi}_{v,s}$ be the reflection of $X_{v,s}$ about $\pi_{v,s}$. We can then define:

$$\begin{aligned} m_v &= \sup\left\{s \in I \mid X^{\pi}_{v,s} \subset \Omega \text{ for every } t < s\right\} \\ &= \sup\left\{s \in I \mid X \cap X^{\pi}_{v,s} = \emptyset \text{ for every } t < s\right\} \end{aligned}$$

the last time at which the hypersurface and its reflection do not touch internally. Please note that at $m_v$ the two surfaces are tangent at some point, which can be either in the interior of X, or on the boundary. The hyperplane $\pi_{v,m_v}$ is the critical hyperplane.

## 1.7 The Alexandrov soap-bubble theorem

To give some justification to the method we descrived, we will outline the proof an important result that uses this method, the so-called Alexandrov soap bubble theorem. For more details, see [3].

**Theorem 1.20.** *The only $C^2$-regular connected hypersurfaces embedded in $\mathbb{M}^{n+1}_+$ and such that the mean curvature is constant are the distance spheres.*

In [3] this theorem is proved more generally: let $H_X$ be a $C^2$ function of the ordered principal curvatures $H_X = f(\kappa_1, \ldots, \kappa_n)$, and

$$f : \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \leq x_2 \leq \cdots \leq x_n\} \to \mathbb{R}$$

is such that

$$f(x) > 0 \text{ if } x_i > 0 \text{ for every } i = 1, \ldots, n$$

and it is concave on the component of $\{x \in \mathbb{R}^n \mid f(x) > 0\}$ containing $\{x \in \mathbb{R}^n \mid x_i > 0\}$. Then the following more general theorem holds:

**Theorem 1.21.** *The only $C^2$-regular connected hypersurfaces embedded in $\mathbb{M}^{n+1}_+$ and such that $H_X$ is constant are the distance spheres.*

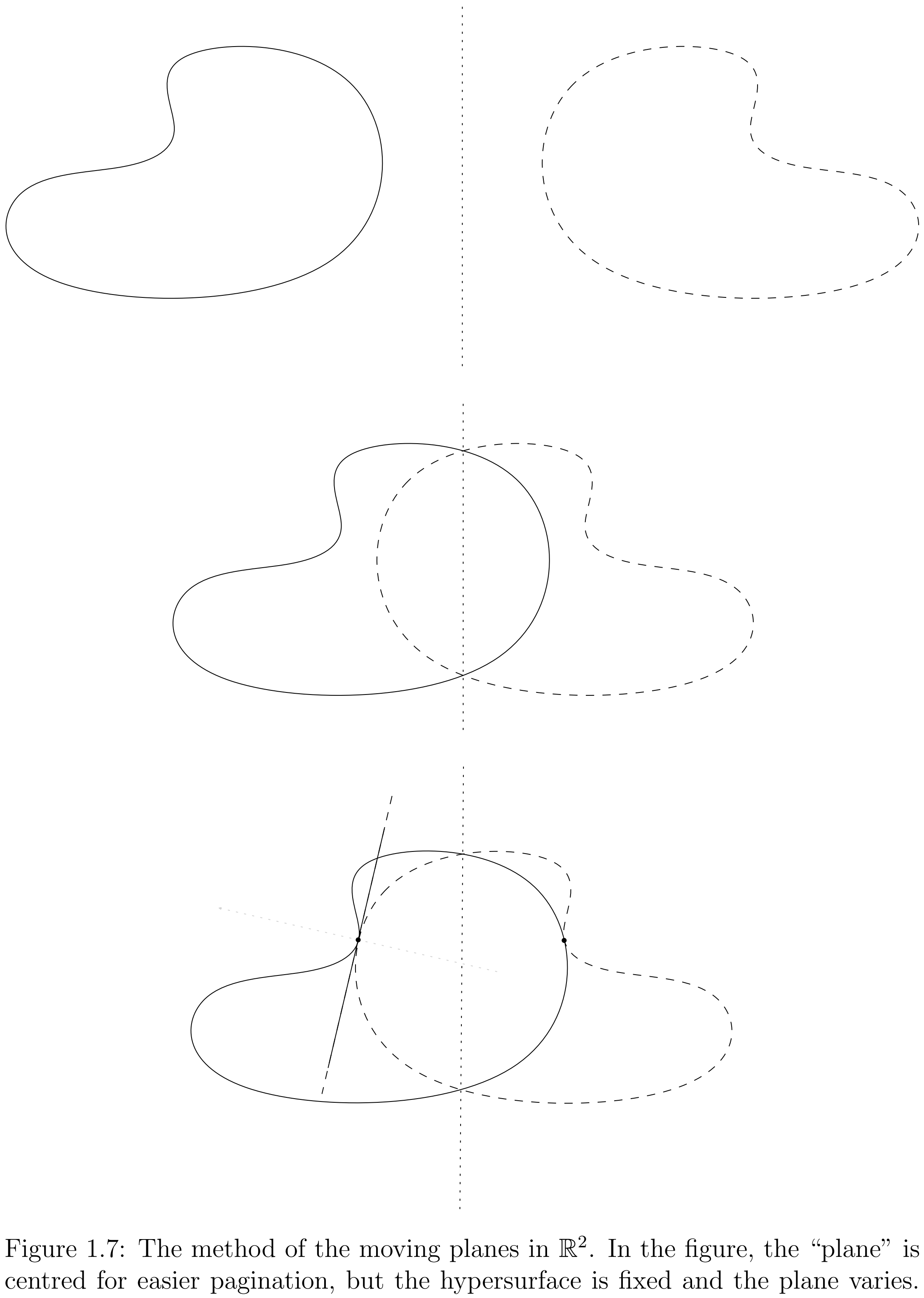

Figure 1.7: The method of the moving planes in $\mathbb{R}^2$. In the figure, the "plane" is centred for easier pagination, but the hypersurface is fixed and the plane varies. The third figure shows the critical time $m_v$ when the hypersurface and its reflection touch internally.

The following proposition holds:

**Proposition 1.22.** *Let $X = \partial\Omega$ be a $C^2$-regular, connected, closed hypersurface embedded in $\mathbb{M}_+^{n+1}$, where $\Omega$ is a bounded domain. Assume that for every geodesic $\gamma : \mathbb{R} \to \mathbb{M}^{n+1}$ there exists a hyperplane orthogonal to $\gamma$ such that $X$ is symmetric about $\pi$. Then $X$ is a distance sphere about its center of mass $O$, i.e. the unique minimum of*

$$P_\Omega(x) = \int_\Omega d(x,a)^2 da$$

The proof of this proposition is in [3] and is omitted here. We now move on to the proof of Theorem 1.20:

*Proof.* Assume that $X$ is a manifold with constant mean curvature. We want to show that, for any point $O$ and for every direction $v \in T_O\mathbb{M}^{n+1}$, $X$ is symmetric about a plane perpendicular to the geodesic $\exp_O(tv)$. We put ourselves in the hypothesis of the method of the moving planes described in section 1.6. At $m_v$, $X \cap X^\pi_{v,m_v}$ is non-empty by definition of $m_v$ and therefore it is closed in $X^\pi_{v,m_v}$. We want to show that $X \cap X^\pi_{v,m_v}$ is an open set in $X^\pi_{v,m_v}$.

Let $p \in X \cap X^\pi_{v,m_v}$. As the two manifolds are tangent

$$T_pX = T_pX^\pi_{v,m_v}.$$

By Theorem 1.5, we can represent both manifold as the Euclidean graph of two functions $C^2$ functions $u$ and $\tilde{u}$ defined in a neighbourhood of $p$ inside the tangent space. Consider now the differential equation:

$$K(u(x)) = K(\tilde{u}(x)) = \text{ constant}$$

where $K$ is the mean-curvature operator. As $X$ is a manifold with constant mean curvature, it holds everywhere. Looking at equation (1.1), we see that, given that the principal curvature are a function of the $h_{ij}$, the operator $K$ is an elliptic operator. Reasoning exactly like we did in section 1.4 for parabolic differential equations, we see that $u - \tilde{u}$ is the solution of a linear elliptic differential equation of the form $L(u - \tilde{u}) = 0$, with $u(p) = \tilde{u}(p) = 0$.

Without loss of generality, we can assume that, in the neighbourhood where we defined $u$ and $\tilde{u}$, $u - \tilde{u} \geq 0$.

If $p$ is an interior point in $X^\pi_{v,m_v}$, we can then apply the maximum principle for elliptic equations (see section 6.4 in [15]) and obtain that $u = \tilde{u}$ in the neighbourhood.

Otherwise, if $p \in \pi_{v,s}$, $\nabla u(p) = \nabla\tilde{u}(p) = 0$ and one can apply Hopf's boundary point lemma for elliptic equations (see section 6.4 in [15]) to conclude again that $u = \tilde{u}$ in the neighbourhood.

Therefore, the whole neighbourhood is in $X \cap X^{\pi}_{v,m_v}$, hence the intersection is open, as every point $p$ is contained in an open ball. This proves that $X$ is symmetric about $\pi_{v,m_v}$. The theorem is then consequence of Proposition 1.22 $\square$

**Remark 1.23.** To prove the theorem in the aforementioned more general setting, one would need to only prove that the differential equation $H_X(u(x)) = \text{constant}$ is an elliptic equation. Its ellipticity is a standard result in Geometric Analysis, and can be found in the literature. It is an immediate consequence of the fact that equation (2.2) is parabolic (see section 2.2).

# Chapter 2

# The Chow-Gulliver Critical Planes Result

The main result we want to establish in this chapter is theorem 2.6, a result about critical hyperplanes when applying the method of the moving planes to solution of a large class of non-linear parabolic partial differential equations, and whose proof is somewhat similar to Theorem 1.20.

We will first describe the differential equations we are analysing, then prove that they are parabolic, then prove the theorem.

After proving the theorem we will include some corollaries of the result and use it to find some estimates for the gradient of the support function and the gradient radial function.

Finally, an application of the result is presented, showing that the condition of *coming out of a point* is particularly rigid on expansive flows.

## 2.1 Class of Equations we analyze

We consider manifolds $M^n$ embedded in $\mathbb{R}^{n+1}$, i.e. there is an embedding $X_0 : M^n \to \mathbb{R}^{n+1}$ parametrizing the hypersurface $X_0(M^n)$.

Let $F : \{(\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n | \kappa_1 \leq \cdots \leq \kappa_n\} \to \mathbb{R}$ be a $C^1$ function satisfying:

$$\frac{\partial F}{\partial \kappa_i} > 0 \text{ for all } i = 1, \ldots, n \tag{2.1}$$

and consider the evolution equation

$$\begin{cases} \dfrac{\partial X_t}{\partial t} = -F(\kappa_1(x), \ldots, \kappa_n(x))\nu \\ X(0) = X_0 \end{cases} \tag{2.2}$$

where $\nu$ is the outward normal to $X_t(M^n)$ at the point $X_t(x)$ and $\kappa_1 \leq \cdots \leq \kappa_n$ are the principal curvatures at $X_t(x)$.

## 2.2 Parabolicity of the differential equation (2.2)

The condition (2.1) will guarantee that equation (2.2) is a parabolic equation. This may be confusing, as (2.2) does not make it obvious how to apply definition 1.10.

In order to classify a non-linear partial differential equation one has to understand how it behaves "close to a solution" in the solutions space. We want to prove that very close to any solution, "moving in any direction", the change in the equation is always a parabolic PDE. This will then tell us that our equation is parabolic, and that the theorems that apply to solutions of parabolic partial differential equations apply to our equation as well. To do so, we are going to "linearise" the differential equation about a solution.

Like in [8], as $F$ is a symmetric function in the principal curvatures, we may interchangeably take $F$ to be a function of the Weingarten map tensor or of the second fundamental form, and thus we get:

$$\frac{\partial X_t}{\partial t} = -F(h_{ij}(X_t))\nu$$

To understand the behaviour close to a solution, we can substitute in our equation $X_t$ with a $X_t + \varepsilon u_t$ to get:

$$\frac{\partial X_t}{\partial t} + \varepsilon \frac{\partial u_t}{\partial t} = -F(h_{ij}(X_t + \varepsilon u_t))\nu_{(X_t+\varepsilon u_t)} \tag{2.3}$$

where we mean that $\nu_{(X_t+\varepsilon u_t)}$ is the normal to the perturbed immersion. We are interested in the behaviour of this equation for a small $\varepsilon$. This equation when taking the limit for $\varepsilon \to 0$ is the so-called linearisation of the PDE; we want this PDE to be a parabolic equation to apply our results.

We can use the Weingarten equation (1.1) to write the RHS explicitly:

$$
\begin{aligned}
h_{ij}(X_t+\varepsilon u_t) &= -\left\langle v, \nu_{(X_t+\varepsilon u_t)}\right\rangle \text{ where} \\
v^\alpha &= \frac{\partial^2 X_t^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial X_t^\alpha}{\partial x^k} + \overline{\Gamma}_{\beta\delta}^\alpha \frac{\partial X_t^\beta}{\partial x^i}\frac{\partial X_t^\delta}{\partial x^k} + \\
&\quad + \varepsilon\left(\frac{\partial^2 u_t^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u_t^\alpha}{\partial x^k} + \overline{\Gamma}_{\beta\delta}^\alpha\left(\frac{\partial X_t^\beta}{\partial x^i}\frac{\partial u_t^\delta}{\partial x^k} + \frac{\partial u_t^\beta}{\partial x^i}\frac{\partial X_t^\delta}{\partial x^k}\right)\right) + \\
&\quad + \varepsilon^2\left(\overline{\Gamma}_{\beta\delta}^\alpha \frac{\partial u_t^\beta}{\partial x^i}\frac{\partial u_t^\delta}{\partial x^k}\right) \\
v^\alpha &= w + \varepsilon\left(\frac{\partial^2 u_t^\alpha}{\partial x^i \partial x^j} + \text{lower order terms}\right) + o(\varepsilon)
\end{aligned}
$$

where $h_{ij}(X_t) = -\langle w, \nu_{X_t}\rangle$.

Putting it all together in the first line:

$$
\begin{aligned}
h_{ij}(X_t+\varepsilon u_t) &= -\left\langle w + \varepsilon\left(\frac{\partial^2 u_t}{\partial x^i \partial x^j} + \text{lower order terms}\right) + o(\varepsilon), \nu_{(X_t+\varepsilon u_t)}\right\rangle \\
&= \left\langle w, \nu_{(X_t+\varepsilon u_t)}\right\rangle - \varepsilon\left\langle\left(\frac{\partial^2 u_t}{\partial x^i \partial x^j} + \text{lower order terms}\right), \nu_{(X_t+\varepsilon u_t)}\right\rangle + o(\varepsilon) \\
&= h_{ij}(X_t) + \left\langle w, \nu_{(X_t+\varepsilon u_t)} - \nu_{X_t}\right\rangle - \varepsilon H_{ij} + o(\varepsilon) \\
&= h_{ij}(X_t) - \varepsilon H_{ij} + o(\varepsilon)
\end{aligned}
$$

Were on the last step we are using the fact that $\nu_{(X_t+\varepsilon u_t)} - \nu_{X_t} = O(\varepsilon)$, and as this gets smaller the component of w parallel to the difference also is $O(\varepsilon)$, as $w$ is parallel to $\nu_{X_t}$. We can then expand F in the RHS of the equation (2.3) to the first order, as it is a $C^1$ function:

$$
\begin{aligned}
\cancel{\frac{\partial X_t}{\partial t}} + \varepsilon\frac{\partial u_t}{\partial t} &= -\left(\cancel{F(h_{ij}(X_t))} + \varepsilon\langle DF, H\rangle + o(\varepsilon)\right)\nu_{(X_t+\varepsilon u_t)} \\
\varepsilon\frac{\partial u_t}{\partial t} &= \varepsilon\left(\frac{\partial F}{\partial h_{ij}} g^{ik} g^{jl}\left\langle\frac{\partial^2 u_t}{\partial x^k \partial x^l}, \nu_{(X_t+\varepsilon u_t)}\right\rangle + \text{lower order terms}\right)\nu_{(X_t+\varepsilon u_t)} + o(\varepsilon) \\
\frac{\partial u_t}{\partial t} &= \frac{\partial F}{\partial h_{ij}} g^{ik} g^{jl}\left\langle\frac{\partial^2 u_t}{\partial x^k \partial x^l}, \nu_{(X_t+\varepsilon u_t)}\right\rangle \nu_{(X_t+\varepsilon u_t)} + \text{lower order terms} + o(1)
\end{aligned}
$$

Letting $\varepsilon \to 0$, we get to:

$$
\frac{\partial u_t}{\partial t} = \frac{\partial F}{\partial h_{ij}} g^{ik} g^{jl}\left\langle\frac{\partial^2 u_t}{\partial x_k \partial x_l}, \nu\right\rangle \nu + \text{lower order terms}
$$

For the second order term to be positive definite, then,

$$\left(\frac{\partial F}{\partial h_{ij}}\right)_{i,j}$$

must be positive definite. Or equivalently, as the principal curvatures are the eigenvalues of the matrix $(h_{ij})_{i,j}$,

$$\frac{\partial F}{\partial \kappa_i} > 0$$

A more general approach proving that the differential equation is parabolic can be found in [16].

**Remark 2.1.** While in this chapter we are using the standard metric of $\mathbb{R}^n$, the proof above is valid using any other metric.

## 2.3 An existence result

In [8] one finds the following comforting existence result for the class of equations we are analysing under very broad hypothesis. The same paper also includes a proof of the result with some more restrictive hypotheses. A complete proof is quite more involved. We will not use it, nor prove it, but it is included here for completeness and peace of mind:

**Theorem 2.2** (Short term existence of a solution for (2.2))**.** *Suppose $X_0 : M^n \to \mathbb{R}^{n+1}$ is a smooth, closed hypersurface in $\mathbb{R}^{n+1}$, such that (2.1) holds at all points in $X_0$, i.e. for all the values of the principal curvatures $\kappa_i$ realized at some point on $X_0$. Then, (2.2) has a smooth solution, at least on some short time interval $[0, T)$, $T > 0$.*

A much more extended analysis of the problem of the existence of solution to the equation can be found in [16].

## 2.4 Local representation as a graph of a solution of (2.2)

As a first step, from the Corollary 1.5, we can establish the following:

**Theorem 2.3** (Local representation as a graph of a solution of (2.2))**.** *Let $X^n$ be a submanifold $X^n \subset \mathbb{R}^{n+1}$ and let $F : X^n \times (0, T) \to \mathbb{R}^{n+1}$ be a solution of (2.2). Also let $t \in (0, T)$ and $x \in F(X^n, t)$. Then there exists a neighbourhood of $(x, t)$,*

*$U \subset F(X^n \times (0,T))$, and a smooth function $f : T_xF(X^n,t) \times (0,T) \to \mathbb{R}$ such that any $(x_0,t_0) \in U$ can be expressed as*

$$x_0 = p + f(p,t_0)\nu$$

*where $\nu$ is a vector normal to $T_xF(X^n,t)$, for an appropriate point $p \in T_xF(X^n,t)$.*

*Proof.* We can consider the image of $F : X^n \times (0,T) \to \mathbb{R}^{n+1}$ as a manifold in $\mathbb{R}^{n+2}$ by considering $G := (F(x,t),t)$. Moreover $\frac{\partial G_t}{\partial e_j} \equiv 0$ for all possible vectors of the canonical basis of $\mathbb{R}^{n+1}\times\mathbb{R}$ except for the one corresponding to the time coordinate, where it is 1. Also, $\frac{\partial G_i}{\partial e_j} \equiv \frac{\partial F_i}{\partial e_j}$ and the first $n$ coordinates of $\frac{\partial G}{\partial t}$ form a vector normal to $T_xX^n$ by (2.2). Thus, the tangent space of $\mathrm{Im}(G)$ is $T_xX^n \times \{0\} \oplus \mathrm{span}\langle(\nu,1)\rangle$ and we can apply corollary 1.5 to $\mathrm{Im}(G)$ to get a function $\tilde{f}$ such that

$$(x_0,t) = [(p,0)+(s\nu,s)] + \tilde{f}(p,s)(\nu,-\|\nu\|)$$

as $(\nu,-\|\nu\|)$ is the vector orthogonal to $T_xX^n\times\{0\}\oplus\mathrm{span}\langle(\nu,1)\rangle$. Let $\sigma_p : I \subset \mathbb{R} \to \mathbb{R}$ be the function that associates to $t$ the appropriate $s$ in the expression above. Projecting to the first $n+1$ coordinates and calling $f(p,t) = \sigma_p(t) + \tilde{f}(p,\sigma_p(t))$ one gets:

$$x_0 = p + f(p,t)\nu$$

which is our thesis, as long as $\sigma_p(t)$ is smooth. This is indeed the case, as the graph function $\Gamma_f : x \mapsto (x,f(x))$ is smooth for any smooth function $f$ and has a smooth inverse (and thus, the inverse $(x_0,t) \mapsto ((p,0)+(s\nu,s))$ is smooth). □

One can show through direct calculation (see [18], Exercise 1.1.2) that, if an immersed hypersurface $\phi : M \to \mathbb{R}^{n+1}$ is locally the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$ (i.e., locally, $(x,f(x)) = \phi$), then:

$$g_{ij} = \delta_{ij} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}$$
$$\nu = -\frac{(\nabla f,-1)}{\sqrt{1+|\nabla f|^2}}$$
$$h_{ij} = -\frac{H_{ij}}{\sqrt{1+|\nabla f|^2}}$$

Where the matrix $(H_{ij})_{ij}$ is the hessian of $f$. Thus, if one considers the principal curvatures, they are closely related to the eigenvalues of the hessian of $f$.

## 2.5 The Moving Planes Method and the Chow result

We will now present the Chow-Gulliver result, presented more or less as in paragraph 2 in [1]. Suppose that we have a hypersurface embedded in $\mathbb{R}^{n+1}$ evolving according to equation (2.2). For a fixed time $t$, we can apply the method of the moving planes as described in section 1.6: we can take parallel hyperplanes $\pi_{v,s}$ orthogonal to $v$ intersecting $X$, and consider the reflection $X^{\pi}_{v,s}$. There will be a hyperplane $\pi_{v,m_v}$ where $X$ and $X^{\pi}_{v,s}$ are tangent. As the hypersurface evolves, we may wonder how this critical threshold changes over time. If it behaves in a predictable way, we can hope to use the technique on the evolving manifold, otherwise, it may be hopeless if the critical plane moves back and forth multiple times. In the next section, we are going to prove a result in this general direction, to show a form of "regularity" in this sense. First, however, we need to introduce a marginally stricter definition for the concept of "reflecting inside itself".

Let $\pi$ be a hyperplane in $\mathbb{R}^{n+1}$. We may assume $\pi$ orthogonal to a unit vector $v \in \mathbb{R}^{n+1}$, i.e. $\langle x, v\rangle = C$ for all $x \in \pi$ for some constant $C$. In our notation for the method of moving planes, assuming we pick the origin as a starting point, this means that $\pi = \pi_{v,C}$.

Then, $\mathbb{R}^{n+1}$ is divided by $\pi$ into two half-spaces, which we will name

$$H^+(\pi) = \{x \in \mathbb{R}^{n+1} : \langle x, v\rangle > C\} = \bigcup_{s>C} \pi_{v,s} \text{ and}$$
$$H^-(\pi) = \{x \in \mathbb{R}^{n+1} : \langle x, v\rangle < C\} = \bigcup_{s<C} \pi_{v,s}.$$

**Definition 2.4.** *We say* we can reflect $X : M^n \to \mathbb{R}^{n+1}$ strictly with respect to $\pi$ *if both:*

- $X^\pi \cap H^-(\pi) \subset \mathrm{int}(X) \cap H^-(\pi)$ *where* $X^\pi$ *is the reflection of* $X$ *about* $\pi$ *and* $\mathrm{int}(X)$ *is the region inside* $X$.
- $V \notin T_xM$ *for all* $x \in M^n \cap \pi$

This fundamentally means that the reflection of one of the halves of $X$ on the other side of $\pi$ is contained in the region inside $M^n$ and the tangent spaces of $X$ and of the half-reflection do not form a ninety degree angle with $\pi$, at all points on $\pi \cap X$. As the two tangent spaces are one the reflection of the other, this means that they do not coincide.

**Definition 2.5.** *We say* we can reflect $X : M^n \to \mathbb{R}^{n+1}$ strictly up to $(\pi, v)$ *if we can reflect* $M^n$ *strictly with respect to* $\pi_{v,s}$ *for all hyperplanes* $\pi_{v,s}$ *such that* $s < C$.

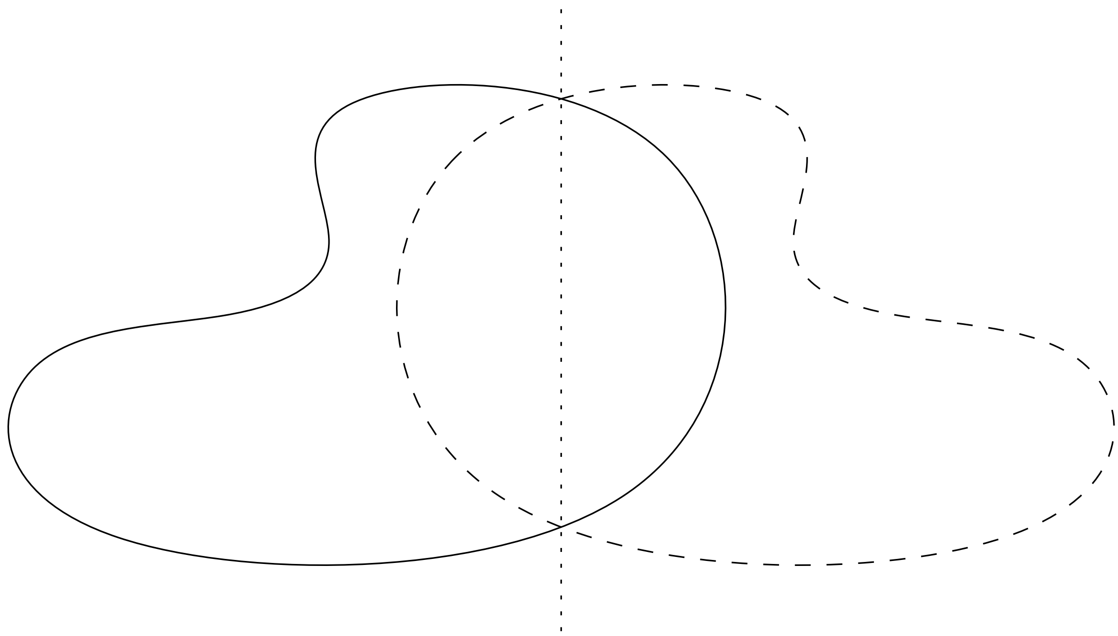

Figure 2.1: Example: We can reflect $X : M^n \to \mathbb{R}^{n+1}$ strictly with respect to $\pi$

The key idea of the main result in the next section is as follows: suppose we have an embedded smooth hypersurface $X$ evolving according to (2.2) and a fixed hyperplane $\pi$, intersecting $X$. Suppose that, at some time $t$, $X$ and $X_\pi$ touch outside of $\pi$. We can consider $X$ and $X_\pi$ as local graphs over the same hyperplane $\pi$, and we can show that these function evolve according to the same differential equation. Using the strong maximum principle and the Hopf boundary point lemma, then, one can conclude that the two functions coincide, and have been coinciding up until that point. We can then conclude that if $X$ and $X_\pi$ only touch in $X \cap \pi$ at the beginning of the evolution, then they will never touch elsewhere.

## 2.6 The Chow-Gulliver result

The main theorem is the following:

**Theorem 2.6** (Chow-Gulliver). *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a $C^2$ solution to equation (2.2). Then, if we can reflect $X(M^n, 0) = X_0$ strictly with respect to $\pi$, then for all $t \in [0, T)$ we can reflect $X(M^n, t) = X_t$ strictly with respect to $\pi$.*

*Proof.* By contradiction, suppose that there is a time $t$ such that the thesis is false, and that it is the smallest such $t$. Then, for all $\tau \in [0, t)$, $X_{\tau,\pi} \cap H^-(\pi) \subset \mathrm{int}(X_\tau) \cap H^-(\pi)$; the unit vector orthogonal to $\pi$, $V$, is such that $V \notin T_x X_\tau$ for all $x \in X_\tau \cap \pi$ and $\tau \in [0, t)$; and either of the conditions fails at $t$, i.e. either:

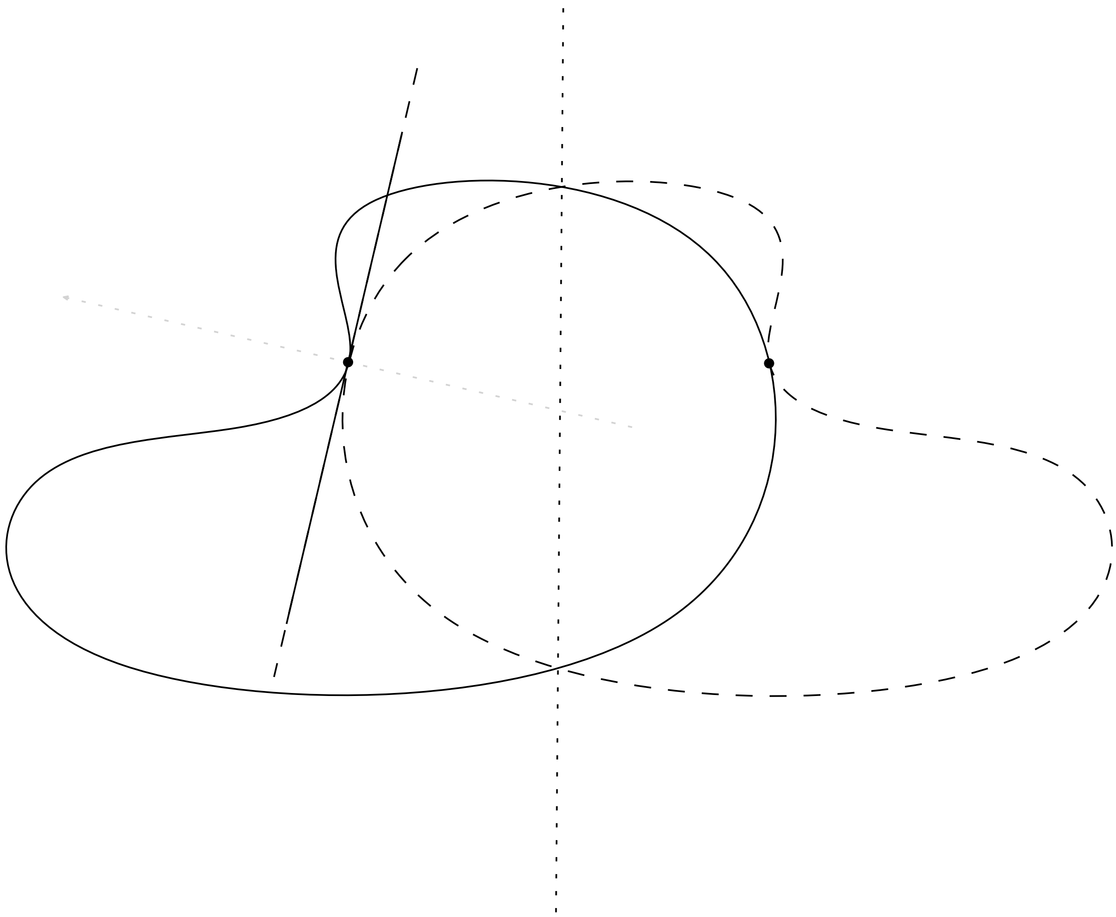

Figure 2.2: Example: We cannot reflect $X : M^n \to \mathbb{R}^{n+1}$ strictly with respect to $\pi$, because there is an interior contact

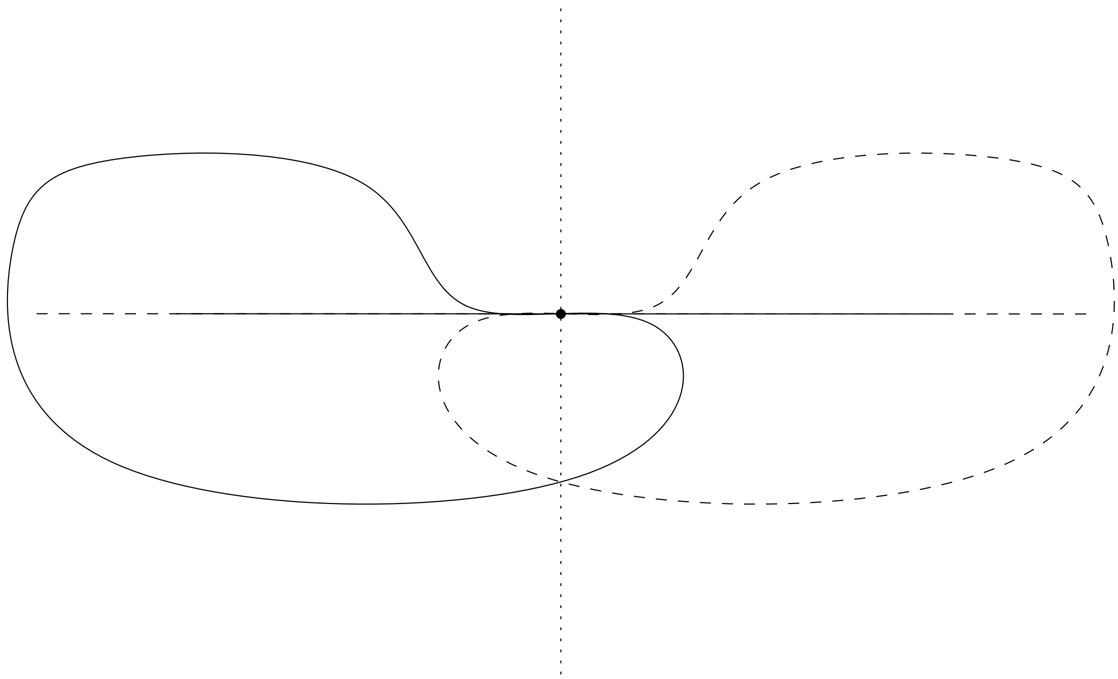

Figure 2.3: Example: We cannot reflect $X : M^n \to \mathbb{R}^{n+1}$ strictly with respect to $\pi$, because the tangent spaces coincide at a point about which we are reflecting

(i) $X_{t,\pi} \cap H^-(\pi) \cap X_t \neq \emptyset$

(ii) $V \in T_x X_t$ for some $x \in \pi$.

(i) Suppose the first case is true. Then, there exists $x_0 \in X_{t,\pi} \cap H^-(\pi) \cap X_t$ such that at $x_0$ the two manifolds are tangent.
We can take a neighbourhood of $(x_0, t) \in X_t \times \mathbb{R}$ such that both $X_{t,\pi}$ and $X_t$ are graphs over $T_{x_0}X_t$ by 2.3.
We can explicitly write the functions $f : U \times (t-\varepsilon, t+\varepsilon) \to X_t$, where $U \subset T_{x_0}X_t$, and the corresponding $f_\pi$ for $X_{t,\pi}$. We can also write

$$\begin{aligned} f &: (x,t) \mapsto x + \tilde{f}(x,t)\nu \\ f_\pi &: (x,t) \mapsto x + \tilde{f}_\pi(x,t)\nu \end{aligned}$$

for appropriate functions $\tilde{f} : U \times (t-\varepsilon, t+\varepsilon) \to \mathbb{R}$ and $\tilde{f}_\pi : U \times (t-\varepsilon, t+\varepsilon) \to \mathbb{R}$, where $\nu$ is a fixed unit vector normal to $T_{x_0}X_t$. $\tilde{f}$ and $\tilde{f}_\pi$ are solutions to the same second order PDE, which is parabolic by what was discussed in paragraph 2.2, hence we can apply Proposition 1.12 to conclude that $\tilde{f} \equiv \tilde{f}_\pi$, and thus $X_{t,\pi}$ and $X_t$ coincide in a neighbourhood of $(x,t)$, a contradiction as we assumed that $t$ is the first $t$ where the flows touch.

(ii) Suppose instead that $V \in T_x X_t$ for some $t \in [0,t)$ and some $x \in X_t \cap \pi$. Then $T_x X_t = T_x X_{t,\pi}$ and in a neighbourhood of $(x,t)$ both $X_t$ and $X_{t,\pi}$ are graphs of two smooth functions over $T_x X_t$ by 2.3, i.e. again

$$\begin{aligned} f &: (x,t) \mapsto x + \tilde{f}(x,t)\nu \\ f_\pi &: (x,t) \mapsto x + \tilde{f}_\pi(x,t)\nu \end{aligned}$$

Moreover, in $\overline{H^-(\pi)}$, $f_\pi \geq f$, because $M^n_\pi \cap H^-(\pi) \subset \mathrm{int}(M^n) \cap H^-(\pi)$. Finally, $f(x,t) = f_\pi(x,t)$, hence $f_\pi - f(x,t) = 0$, and thus $(x,t)$ is a minimum point on the boundary for $f_\pi - f$. Also, we must have

$$\frac{\partial f}{\partial V}(x,t) = \frac{\partial f_\pi}{\partial V}(x,t)$$

because the graphs are both tangent to $T_x X_t$, and $V$ here is the outward pointing normal to the boundary by definition of the reflection. Thus,

$$\frac{\partial (f - f_\pi)}{\partial V}(x,t) = 0$$

But we must have

$$\frac{\partial (f - f_\pi)}{\partial V}(x,t) > 0$$

at a minimum on the boundary by Proposition 1.13, a contradiction. □

## 2.7 Some corollaries of the result

In this section we collect a number of corollaries to the main result above. This first one is an immediate consequence of the main result:

**Corollary 2.7.** *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a $C^2$ embedded solution to equation (2.2). Then, if we can reflect $X_0$ strictly up to $(\pi_{v,C}, v)$, for all $t \in [0,T)$ $v \notin T_x X_t$ for all $x \in X_t \cap \overline{H^+(\pi)}$. In particular, $X_t \cap \overline{H^+(\pi)}$ is a graph over $\pi$ for all $t \in [0,T)$.*

Another immediate consequence of theorem 2.6 is the following:

**Corollary 2.8.** *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a $C^2$ solution to equation (2.2). Then, if we can reflect $X_0$ strictly up to $(\pi_{v,C}, v)$, for all $t \in [0,T)$ we can reflect $X(M,t)$ strictly up to $(\pi_{v,C}, v)$.*

*Proof.* The hypothesis of the theorem are true for each $\pi_K$ in the definition, thus we can reflect strictly with respect to each $\pi_K$ for all $t \in [0,T)$, and thus we can reflect $X(M,t)$ strictly up to $(\pi_{v,C}, v)$. □

Furthermore, it is clear that for every direction $v$ there exists a hyperplane $\Pi$ perpendicular to $v$ such that we can reflect $X_0$ up to $(\Pi, v)$ and $\Pi$ intersects the interior of $X_0$. To be more precise, for every direction $v$ there exists a hyperplane $\Pi_0^v$ tangent to $X_0$ and such that $X_0 \cap H^+(\Pi_0^v) = \emptyset$. Suppose that for every $\varepsilon > 0$ we can find a plane $\pi_\varepsilon$ such $H^+(\pi_\varepsilon) \cap B_{R-\varepsilon}(C) = \emptyset$ and we cannot reflect $X_0$ strictly at $\pi_\varepsilon$, where $R$ is such that $X_0 \subset B_R(C)$. We can take the corresponding plane $\Pi_0^\varepsilon$ parallel to $\pi_\varepsilon$ and tangent to $X_0$ at the point $p_\varepsilon$. Taking a sequence $\varepsilon_n \to 0$ we find a corresponding limited sequence $p_{\varepsilon_n}$ which, by compactness, converges to a point $p \in X_0$. This point $p$ is such that arbitrarily close to it there is a point such that we cannot reflect by more than any chosen $\varepsilon > 0$ in the direction of its normal. As we can always represent $X_0$ as the graph of a function in a neighbourhood of $p$, we get a contradiction, because strict reflection in the direction of the normal by at least a fixed uniform amount $\varepsilon$ at each point is always possible locally for any graph of a smooth function. Thus, we obtain the following:

**Corollary 2.9.** *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a $C^2$ embedded solution to equation (2.2). There exists $\varepsilon > 0$ depending only on $X_0$ such that for all $t \in [0, T)$ we can reflect $X_t$ up to $(\Pi_0^v + \epsilon v, v)$ for every $v \in S^n$. In particular, if $X_0 \subset B_R(C)$, then we can always reflect $X_t$ up to $(\Pi, v)$ whenever $H^+(\Pi) \cap B_{R-\varepsilon}(C) = \emptyset$.*

In other words, we can always reflect a little $\varepsilon$ in any direction, uniformly. We can use the fact that we can always reflect about a plane outside a sphere containing $X_0$ to prove the following estimate:

**Corollary 2.10.** *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a $C^2$ embedded solution to equation (2.2). There exists $C > 0$ depending only on $X_0$ such that for all $t \in [0, T)$:*

$$\max_{x \in X_t} |x| - \min_{x \in X_t} |x| < C$$

*Proof.* We can reflect $X_0 \subset B_R(0)$ up to any plane tangent to $B_R(0)$, i.e. any plane $\pi_{v,K}$ for any $K \geq R$ and $v$ unit vector, i.e. $\pi_{v,K} = \{p \in \mathbb{R}^{n+1} : \langle p, v \rangle = K\}$. Let $x_1, x_2 \in X_t$ such that $|x_1| = \min_{x \in X_t} |x|$ and $|x_2| = \max_{x \in X_t} |x|$. Let $v = \frac{x_2 - x_1}{|x_2 - x_1|}$: we can reflect $X_t$ up to $(\pi_{v,R}, v)$ by theorem 2.6, therefore $\mathrm{dist}(x_2, \pi_{v,R}) <$

$\mathrm{dist}(x_1, \pi_{v,R})$, or in other words:

$$\begin{aligned}
\left\langle x_2, \frac{x_2 - x_1}{|x_2 - x_1|} \right\rangle - R &< \left\langle x_1, \frac{x_1 - x_2}{|x_2 - x_1|} \right\rangle + R \\
\frac{|x_2|^2}{|x_2 - x_1|} - \cancel{\frac{\langle x_2, x_1 \rangle}{|x_2 - x_1|}} - R &< \frac{|x_1|^2}{|x_2 - x_1|} - \cancel{\frac{\langle x_2, x_1 \rangle}{|x_2 - x_1|}} + R \\
\frac{|x_2|^2 - |x_1|^2}{|x_2 - x_1|} &< 2R \\
|x_2|^2 - |x_1|^2 &< 2R|x_2 - x_1| \leq 4R|x_2| \\
|x_2|^2 &< |x_1|^2 + 4R|x_2| \\
|x_2| &< |x_1|\frac{|x_1|}{|x_2|} + 4R < |x_1| + 4R \\
|x_2| - |x_1| &< 4R
\end{aligned}$$

$\square$

**Remark 2.11.** This result has an important meaning if the hypersurface uniformly expands to infinity, i.e. $\lim_{t \to T} \min_{x \in X_t} |x| = \infty$. We can then consider the rescaled hypersurfaces

$$\widetilde{X_t} = \frac{1}{\min_{x \in X_t} |x|} X_t$$

Immediately, we find that the $\widetilde{X_t}$ must converge uniformly to a sphere, because $\frac{C}{\min_{x \in X_t} |x|} \to 0$, and therefore

$$\max_{x \in \widetilde{X_t}} |x| - \min_{x \in \widetilde{X_t}} \to 0$$

Lastly:

**Corollary 2.12.** *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a $C^2$ embedded solution to equation (2.2). Let $s_v : [0, T) \to I$ be such that*

$$s_v(t) = \sup \{ s \in I \mid \text{ we can reflect } X_t \text{ strictly up to } (\pi_{v,s}, v) \} .$$

*Then $s_v(t)$ is a non-decreasing function. Also, if $X_0$ is compact, the limit*

$$\lim_{t \to T^-} s_v(t)$$

*exists and is finite.*

*Proof.* When taking $X_t$ as the starting manifold, the hypothesis of the theorem are still true, therefore we can reflect about $\pi_{c,v}$ for all $c < s_v(t)$ at all subsequent times. Thus $s_v(t)$ is non-decreasing. $\lim_{t\to T^-} s_v(t) = \sup s_v(t)$, therefore the limit exists. Also, if $X_0$ is bounded, there exists $R > 0$ such that $X_0 \subset B_R(0)$, therefore we can reflect $X_0$ strictly about any hyperplane non intersecting $B_R(0)$, as it does not touch $X_0$, and therefore we can also reflect $X_t$ strictly about the same hyperplanes by theorem 2.6. At the same time, there exists a hyperplane such that $X_t$ cannot be reflected strictly about it, because there will always be a straight line parallel to $v$ intersecting $X_t$ in multiple points, and we can always consider the hyperplane orthogonal to $v$ passing through their midpoint, about which $X_t$ cannot be reflected strictly, and $s_v(t) \neq +\infty$. Therefore $s_v(t) \in [-R, R]$, and the limit above is finite. □

## 2.8 Applying the result to find gradient estimates

In this section we collect some applications of theorem 2.6 to gradient estimates for the support function and the radial function from [1] (a more in-depth definition these two functions can be found in the introduction of [19]). Central in what will follow is this corollary providing an estimate for the tangent component of the position vector $x$ of a point on the hypersurface in its own tangent space:

**Corollary 2.13.** *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be an embedded solution to equation (2.2). There exists a constant $C$, depending only on the initial hypersurface $X_0$, such that for all points $x \in X_t$ and $t \in [0, T)$, the following inequality holds:*

$$|x - \langle x \cdot \nu\rangle \nu| \leq C,$$

*where $\nu$ is the unit normal to $X_t$ at the point $x$.*

*Proof.* : Choose $C > 0$ such that $X_0 \subset B_C(0)$. By Theorem 2.6, we can reflect $X_t$ strictly up to any plane tangent to the ball, like in the previous corollaries. Thus, for any point $x \in X_t$ and outside the ball, we know that whenever $(x, V) > C$, then $V \notin T_x X_t$. This is equivalent to saying that for all $W \in T_x X_t$, we have:

$$(x, W) \leq C.$$

If we take now the projection of $x$ on $T_x X_t$ and rescale it to be a unit vector, $W = \frac{x-(x\cdot\nu)\nu}{|x-(x\cdot\nu)\nu|} \in T_x X_t$, then we obtain:

$$C \geq (x, W) = \frac{|x - (x \cdot \nu)\nu|}{|W|} = |x - (x \cdot \nu)\nu|.$$

Thus, the corollary is proved. □

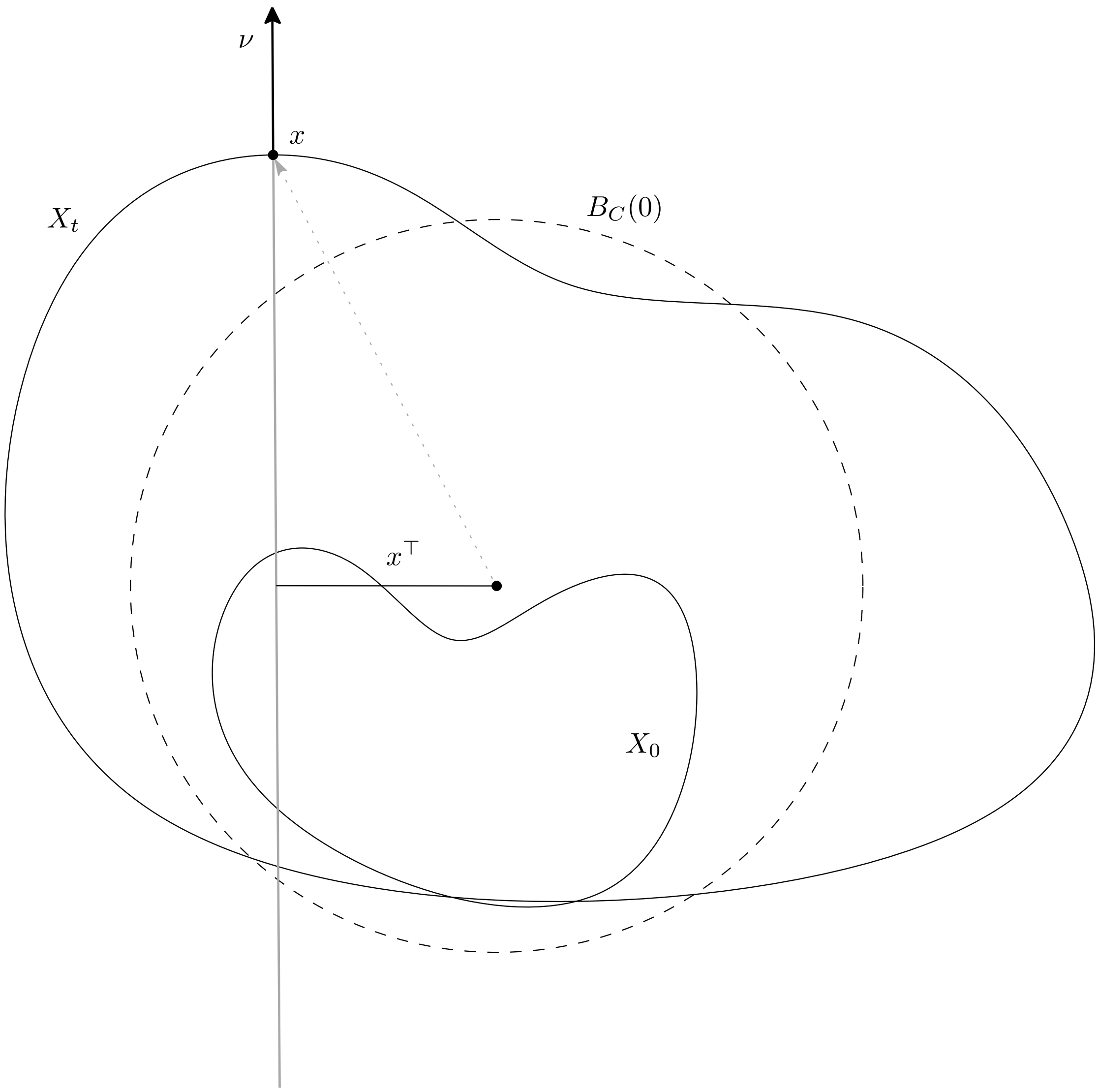


Figure 2.4: A geometric interpretation of corollary 2.13: the normal line always intersects $B_C(0)$. $x^\top = x - (x \cdot \nu)\nu$.

Let's now assume that the hypersurfaces in the solution to the equation are convex. We are going to show a gradient estimate for the support function of the hypersurface. Support functions are one of the most important concepts stemming from the study of convex sets.

**Definition 2.14.** *Let $K$ be a non-empty compact convex set in $\mathbb{R}^n$. We define the* support function $u_K : S^n \to \mathbb{R}$ *as*

$$u_K(\nu) = \sup\{\langle x, \nu\rangle : x \in K\}$$

**Remark 2.15.** Many authors define the support function on the whole euclidean space. As its gradient scales linearly with $\nabla u_K$, we will not be doing so out of simplicity. Notice that given two convex sets $K_1$ and $K_2$, $K_1 \subseteq K_2$ if and only if $u_{K_1} \leq u_{K_2}$. In this sense, a convex set is determined by its support function. Also, the tangent plane at the point in $\partial K$ which has $\nu$ as a normal vector is $h_\nu = \{x \in \mathbb{R}^n : x \cdot \nu = u_K(\nu)\}$.

**Corollary 2.16.** *Let $u : S^n \times [0, T) \to \mathbb{R}$ be the support function of convex hypersurfaces $X_t$, solving the equation (2.2). There exists a constant $C$, depending only on $u(0)$, such that:*

$$|\nabla u(\nu, t)| \leq C,$$

*for all $(\nu, t) \in S^n \times [0, T)$.*

*Proof.* For each unit normal vector $\nu \in S^n$, let $x_t \in X_t$ be the unique point such that $\nu$ is the outward unit normal to $X_t$ at $x_t$. We compute $\nabla u(\nu)$: let $R_\theta$ be a rotation of an angle $\theta$ in the direction $\partial_i \in T_\nu S^n$:

$$\begin{aligned}
\partial_i u(\nu) &= \lim_{\theta\to 0} \left[\langle x_t(R_\theta\nu), R_\theta\nu\rangle - \langle x_t(\nu), \nu\rangle\right]/\theta \\
&= \lim_{\theta\to 0} \left[\langle x_t(R_\theta\nu), \nu - \nu + R_\theta\nu\rangle - \langle x_t(\nu), \nu\rangle\right]/\theta \\
&= \lim_{\theta\to 0} \left[\langle x_t(R_\theta\nu), R_\theta\nu - \nu\rangle + \langle x_t(R_\theta\nu) - x_t(\nu), \nu\rangle\right]/\theta \\
&= \lim_{\theta\to 0} \left[\left\langle x_t(R_\theta\nu), \frac{R_\theta\nu - \nu}{\theta}\right\rangle + \left\langle \frac{x_t(R_\theta\nu) - x_t(\nu)}{\theta}, \nu\right\rangle\right] \\
&= \langle x_t(\nu), \partial_i\rangle + \cancel{\langle (dx_t)_\nu(\partial_i), \nu\rangle}
\end{aligned}$$

where $\langle (dx_t)_\nu(\partial_i), \nu\rangle = 0$ because $(dx_t)_\nu(\partial_i) \in T_{x_t}X_t$. Therefore

$$\begin{aligned}
\nabla u(\nu) &= (x_t)^\top = x_t - (x_t)^\perp \\
\nabla u(\nu) &= x_t - \langle x_t, \nu\rangle\nu = x_t - u(\nu)\nu \\
|\nabla u(\nu, t)| &= |x_t - \langle x_t, \nu\rangle\nu| \leq C
\end{aligned}$$

by applying Corollary 2.13, which completes the proof. $\square$

Now, let's consider the case where the hypersurfaces $X_t$ are starshaped for all $t \in [0, T)$. We obtain a gradient estimate for the radial function at points outside a certain compact starshaped region associated with the initial hypersurface $X_0$.

**Definition 2.17.** *Suppose that $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ parametrizes starshaped hypersurfaces $X_t$ with respect to the origin. The* radial function $r : S^n \times [0, T) \to \mathbb{R}^+$ *is defined so that for each $(z, t) \in S^n \times [0, T)$, the point $r(z, t)z$ belongs to $X_t$.*

We will need the following lemma:

**Lemma 2.18.** *In the hypothesis of the definition above, there exists a constant $C$, depending only on $X_0$, such that for all points $(v, t) \in S^n \times [0, T)$, we have:*

$$r^2 |\nabla r|^2 \leq C(r^2 + |\nabla r|^2).$$

*In particular, if $r^2 > C$, then:*

$$|\nabla r|^2 \leq \frac{Cr^2}{r^2 - C}.$$

*Proof.* As $x = r(z)z$, let $\partial_i \in T_z S^n$ and $\overline{\partial_i} = \partial_i x \in T_z X_t$ for $i = 1 \dots n$ be corresponding bases in the tangent spaces. Computing this explicitly in $\mathbb{R}^{n+1}$ yields:

$$\begin{aligned}
\overline{\partial_i} &= \partial_i x \\
&= \partial_i (r(z)z) \\
&= (\partial_i r(z))z + r(z)\partial_i
\end{aligned}$$

Notice here that $z$ and the $\partial_i$ are orthogonal, therefore $|az + b^i \partial_i|^2 = a^2 + \sum_i (b^i)^2$. Computing this vector's scalar product with $r(z)z - \nabla r(z)$ yields:

$$\begin{aligned}
\langle r(z)z - \nabla r(z), \overline{\partial_i} \rangle &= \langle r(z)z - \sum_j \partial_j r(z) \partial_j, \overline{\partial_i} \rangle \\
&= \langle r(z)z, \overline{\partial_i} \rangle - \langle \sum_j \partial_j r(z) \partial_j, \overline{\partial_i} \rangle \\
&= r(z) \langle z, \overline{\partial_i} \rangle - \sum_j \partial_j r(z) \langle \partial_j, \overline{\partial_i} \rangle \\
&= r(z)(\partial_i r(z)) - \partial_i r(z) r(z) \\
&= 0
\end{aligned}$$

Moreover, notice that $\nabla r(z) \in T_z X_t$, which is orthogonal to $z$. Therefore, the following formula holds for the normal at a point with respect to the radial function:

$$\nu = \frac{r(z)z - \nabla r(z)}{\sqrt{r(z)^2 + |\nabla r(z)|^2}}$$

when taking $x = r(z)z$. Substituting the expressions above into the inequality from Corollary 2.13, we obtain:

$$\begin{aligned} C &\geq \left| r(z)z - \langle r(z)z, r(z)z - \nabla r(z)\rangle \frac{r(z)z - \nabla r(z)}{r(z)^2 + |\nabla r(z)|^2} \right| \\ &\geq \left| r(z)z - \frac{r(z)^2(r(z)z - \nabla r(z))}{r(z)^2 + |\nabla r(z)|^2} \right| \\ &\geq \left| \frac{r(z)z(r(z)^2 + |\nabla r(z)|^2) - r(z)^2(r(z)z - \nabla r(z))}{r(z)^2 + |\nabla r(z)|^2} \right| \\ &\geq \left| \frac{r(z)|\nabla r(z)|^2 z + r(z)^2 \nabla r(z)}{r(z)^2 + |\nabla r(z)|^2} \right| \end{aligned}$$

Multiplying by $r(z)^2 + |\nabla r(z)|^2$ and squaring (still using the fact that $z$ and $\nabla r(z)$ are orthogonal):

$$\begin{aligned} \left| r(z)|\nabla r(z)|^2 z + r(z)^2 \nabla r(z) \right| &\leq C(r(z)^2 + |\nabla r(z)|^2) \\ r(z)^2|\nabla r(z)|^4 + r(z)^4|\nabla r(z)|^2 &\leq C^2(r(z)^2 + |\nabla r(z)|^2)^2 \\ r^2|\nabla r|^4 + r^4|\nabla r|^2 &\leq C^2(r^2 + |\nabla r|^2)^2. \end{aligned}$$

From here, the lemma follows. □

The estimate is:

**Proposition 2.19.** *With the same hypotheses as the lemma above, there exists a constant $C$, depending only on $X_0$, such that for all $(v,t) \in S^n \times [0,T)$, taking $\Pi_0^v$ as in corollary 2.9, if $r(v,t)v \in \overline{H^+(\Pi_0^v)}$, then:*

$$|\nabla r(v,t)| \leq K.$$

*Proof.* By Corollary 2.9, there exists a constant $\epsilon > 0$, depending only on $X_0$, such that for all $(z,t) \in S^n \times [0,T)$ with $r(z,t)z \in H^+(\Pi_0^v)$, and for all $w \in S^n$ with $\langle W, z\rangle > 1 - \epsilon$, we can reflect $X_t$ up to the hyperplane $\Pi_w = \{r(z,t)z + p : \langle p, w\rangle = 0\}$. This means that $w \notin T_{r(z,t)z}X_t$, i.e., $W$ is not tangent to $X_t$. In the proof of the previous lemma we showed that $r(z)z - \nabla r(z)$ is normal to the manifold's tangent plane at $r(z)z$, so, letting $w_z$ be the projection of $w$ in the $z$ direction:

$$\begin{aligned} 0 &< \langle w, r(z)z - \nabla r(z)\rangle \\ \langle w, \nabla r(z)\rangle &< \langle w, r(z)z\rangle \\ \langle w - w_z, \nabla r(z)\rangle &< r(z)\langle w, z\rangle \\ |w - w_z||\nabla r(z)| &< r(z)\langle w, z\rangle \\ |\nabla r(z)| &< \frac{r(z)\langle w, z\rangle}{|w - w_z|} \end{aligned}$$

by the fact that $w$ is arbitrary, and estimating $|w - w_z|$ as $\sqrt{1^2 - (1-\epsilon)^2}$:

$$|\nabla r(z)| < \frac{r(z)(1-\epsilon)}{\sqrt{1^2 - (1-\epsilon)^2}}$$
$$|\nabla r(z)| < r(z)\frac{(1-\epsilon)}{\sqrt{\epsilon(2-\epsilon)}}$$

By the preceding lemma, if $r^2 > C$, then:

$$|\nabla r|^2 \leq \frac{Cr^2}{r^2 - C} < C$$
$$|\nabla r| \leq \sqrt{C}$$

otherwise,

$$|\nabla r(z)| < C\frac{(1-\epsilon)}{\sqrt{\epsilon(2-\epsilon)}}$$

Hence:

$$|\nabla r(z)| < \max\left(\sqrt{C}, C\frac{(1-\epsilon)}{\sqrt{\epsilon(2-\epsilon)}}\right)$$

□

Finally, we include the following result about the part of $X_t$ outside a ball:

**Corollary 2.20.** *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be an embedded solution to equation (2.2). Then, if, for a sphere $B$, $X_0 \subset B$, at all times $t \in [0,T)$ $X_t \backslash B$ is star-shaped with respect to the centre of $B$.*

*Proof.* $X_0 \subset B$ therefore we can reflect $X_t$ about any hyperplane tangent to $B$ by Corollary 2.9. By Corollary 2.7 $X_t \cap \overline{H^+(\pi)}$ is a graph over $\pi$, therefore, it is not possible that a normal line coming out of $B$ and orthogonal to $\pi$ intersects $X_t \cap \overline{H^+(\pi)}$ more than once. This implies that $X_t \setminus B$ is star-shaped with respect to the centre of $B$ □

## 2.9 Expansive flows and ancient solutions

In this section, roughly following [5], we consider solutions to 2.2 which are defined on a larger interval $(T_0, T_1)$, with $-\infty \leq T_0 \leq 0 < T_1 \leq \infty$.

We will limit our analysis to a sub-class of solutions to 2.2:

**Definition 2.21.** *We say that a solution to equation (2.2) is* expansive *if* $F < 0$.

An example of such a flow is the inverse curvature flow, extensively studied in the literature.

**Remark 2.22.** This assumption on $F$ implies that whenever $t < s$, $X_t \subset \mathrm{int}(X_s)$. This is implied by the fact that, in the equation, the time derivative is always an outward pointing non-zero vector. In this sense, the hypersurface is expanding in the ambient space.

**Definition 2.23.** *Let* $X : M^n \times (T_0, T_1) \to \mathbb{R}^{n+1}$ *be an embedded expansive solution to equation (2.2). We say that* $X$ comes out of a point *if there exists a point* $y_\infty$ *such that for every* $\varepsilon > 0$*, there exists a time* $\tau \in (T_0, T_1)$ *such that* $X_\tau \subset B_\varepsilon(y_\infty)$.

**Remark 2.24.** If we take the optimal $\tau$ in the definition above, Remark 2.22 implies that the function $\tau(\varepsilon)$ mapping $\varepsilon$ to the last time where $X_\tau$ is contained in the ball is non-increasing.

In particular, we will show that expansive solutions "coming out of a point" must be expanding spheres. It is easy to check that homothetically expanding spheres do satisfy the equation: Any spherical solution is completely determined by its radius at time $t$, because the equation is invariant at each point, being the principal curvatures constant: thus, solving the ordinary differential equation $r'(t) = \varphi(r(t))$ completely determines the flow, where $\varphi(r) = -F(\frac{1}{r}, \dots, \frac{1}{r})$.

We say that a solution to the equation is ancient if $T_0 = -\infty$. In the case of the spherical solution coming out of a point, there exists an ancient solution if and only if

$$+\infty = (T_1 - T_0) = \int_{T_0}^{T_1} 1 dt = \int_{T_0}^{T_1} \frac{r'(t)}{\varphi(r(t))} dt = \int_{r(T_0)}^{r(T_1)} \frac{1}{\varphi(r)} dr = \int_0^c \frac{1}{\varphi(r)} dr$$

$$\int_0^c \frac{1}{\varphi(r)} dr = +\infty$$

This equation allows us to determine, given $F$, if an ancient solution can exist (even if not coming out of a point). Solutions to (2.2) obey a comparison principle: if a solution is inside another one, it stays inside it at all times. This can be proven similarly to theorem 2.6, if at some $(x, t)$ this fails, one can apply the maximum principle to show that they coincide in a neighbourhood, which is a contradiction if one takes the first or the last $t$ where this happens. A generic solution, then, will be sandwiched between two expanding sphere solutions, one inside and one outside, and therefore cannot be ancient if no ancient expanding sphere solution exists.

The idea of ancient solutions was introduced by Richard Hamilton in his work on the Ricci flow. It has since been applied to other geometric flows as well as to other partial differential equations. Ancient solutions are significant because they capture key asymptotic features of the flow and often have unique, rigid properties that distinguish them from other solutions. For instance, in contractive flows, ancient solutions can form complex, non-spherical shapes, whereas in expansive flows, they tend to exhibit strong rigidity, commonly resulting in spherical shapes under certain conditions. These solutions thus help in understanding the geometry and topology of hypersurfaces as they evolve, especially in applications like mean curvature flow and inverse mean curvature flow in mathematics and physics.

Indeed, the property of coming out a point is particularly rigid, as shown in the following result:

**Theorem 2.25.** *Let $X : M^n \times (T_0, T_1) \to \mathbb{R}^{n+1}$ be a smooth, closed, embedded expansive solution to equation (2.2) coming out of a point. Then it is a family of expanding spheres.*

Previous results in geometric flows have shown multiple non-trivial examples of contracting flows sweeping the whole space. This could therefore be somewhat surprising, because it shows an opposite result, when running the equation in the opposite direction. It can however be explained intuitively by the idea that parabolic flows tend to *smooth things out*: it is thus possible to arrive to a point from a more irregular hypersurface, however the only thing that can "come out" of a point is something just as symmetric as a point, i.e. a sphere.

The proof is a relatively simple application of the reflection technique. The outline of the proof in [5] is the same as the one below, but using Corollary 2.9 makes it a bit simpler:

*Proof.* Fix any hyperplane $\pi$ not passing through $y_\infty$. There is $R > 0$ such that $B_{2R}(y_\infty)$ does not intersect it. By definition, there is also a time $\tau \in (T_0, T_1)$ such that $X_\tau \subset B_R(y_\infty)$. By Corollary 2.9, then, we can reflect strictly $X_t$ up to $\pi$ for any $t > \tau$.

Now consider a sequence $\epsilon_n \to 0$. Up to a subsequence, we can then find a corresponding converging non-increasing sequence $\tau_n \to \bar{t} \in [T_0, T_1)$ (here note that $\bar{t}$ can be $-\infty$) such that $X_{\tau_n} \subseteq B_{\epsilon_n}(y_\infty)$, therefore in $(\tau_n, T_1)$ I can reflect up to any hyperplane tangent to $B_{\epsilon_n}(y_\infty)$ in any direction, reasoning like we just did. At time $\bar{t}$, $X_{\bar{t}} \subseteq \cap_r B_r(y_\infty) = \{y_\infty\}$, thus we would have a singularity at $\bar{t}$ if $\bar{t} \in (T_0, T_1)$ and therefore $\bar{t} = T_0$.

On the other hand, by construction, we can reflect $X_t$ strictly about any hyperplane not intersecting $\cap_r B_r(y_\infty) = \{y_\infty\}$ at any time $t > \bar{t} = T_0$, hence we can reflect $X_t$ strictly up to any hyperplane passing through $y_\infty$, in both directions, at any time $t \in (T_0, T_1)$. We observe that in the limit, the reflection property becomes non-strict, in the sense that we have to replace the interior of $X$ in definition 2.4 with its closure, therefore $X_t$ may touch its reflection at the limit plane, i.e. the one passing through $y_\infty$. Similarly, it cannot be that the other condition is the one causing the strict reflection definition to fail, as the other condition stays instead strict.

This implies that, taking any hyperplane passing through $y_\infty$ and considering opposite directions for the reflection, $X_t$ is symmetric about said hyperplane for any time $t \in (T_0, T_1)$. By Proposition 1.22, then, we conclude that $X_t$ must be a ball. $\square$

# Chapter 3

# Extension to constant curvature spaces

In this chapter we want to extend the Chow-Gulliver result (theorem 2.6) to constant curvature spaces. We will use the same notation as in section 1.5 and 1.6. Throughout the chapter, by hyperplane we mean a totally geodesic hyperplane.

## 3.1 The equation in constant curvature spaces

As shown in [8], the equation we analysed in the previous chapter can be also considered in non-flat ambient spaces. In particular, we will analyse the case where the ambient space is one of those described in section 1.5: $\mathbb{R}^{n+1}$, $\mathbb{H}^{n+1}$, or $\mathbb{S}^n_+$. We again use the symbol $\mathbb{M}^{n+1}_+$ to indicate any of these spaces. Let $X_0 : M^n \to \mathbb{M}^{n+1}_+$ be a manifold embedded in $\mathbb{M}^{n+1}_+$. Let $F : \{(\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n | \kappa_1 \leq \cdots \leq \kappa_n\} \to \mathbb{R}$ be a $C^1$ function satisfying:

$$\frac{\partial F}{\partial \kappa_i} > 0 \text{ for all } i = 1, \ldots, n \tag{3.1}$$

and consider the evolution equation

$$\begin{cases} \dfrac{\partial X_t}{\partial t} = -F(\kappa_1(x), \ldots, \kappa_n(x))\nu \\ X(0) = X_0 \end{cases} \tag{3.2}$$

where $\nu$ is the outward normal to $X_t(M^n)$ at the point $X_t(x)$ and $\kappa_1 \leq \cdots \leq \kappa_n$ are the principal curvatures at $X_t(x)$.

As we saw in the previous chapter, it is a non-linear parabolic differential equation, as the calculation in chapter 2.2 is valid for any metric on $\mathbb{R}n + 1$, and in particular, for the metrics in our models for $\mathbb{M}^{n+1}_+$ in section 1.5. The existence result in [8] also holds as well in this case. Finally, the result in section 2.4 is not using the metric tensor, and therefore is valid in this setting as well, again because there are models on $\mathbb{R}^{n+1}$.

**Remark 3.1.** The hemisphere has a finite diameter in each direction. It is therefore possible that the flow stops in a finite time when it touches the equator. One could extend it a little bit by rotating the hypersurface in $S^{n+1}$ so that it only touches the equator at the last possible moment. It is however possible that the hypersurface touches its reflection *on the wrong side* if we let it expand past the equator, (a sphere is, after all, round). It is easier to avoid this possibility by considering only flows defined in the hemisphere.

## 3.2 Extension of theorem 2.6

Assume the hypothesis in section 1.5: $X : M^n \to \mathbb{M}_+^{n+1}$ is a hypersurface in a constant curvature ambient space, and we choose a point and a direction $v$ to foliate the ambient space. Consider a hyperplane in the foliation, $\pi = \pi_{v,C}$. As in section 1.5 we can define the reflection about $\pi$. As in the previous chapter, let $X^\pi$ be the reflection of X about $\pi$.

Then, $\mathbb{M}_+^{n+1}$ is divided by $\pi$ into two half-spaces:

$$H^+(\pi) = \bigcup_{s>C} \pi_{v,s} \text{ and } H^-(\pi) = \bigcup_{s<C} \pi_{v,s}.$$

**Definition 3.2.** *We say* we can reflect $X : M^n \to \mathbb{M}_+^{n+1}$ strictly with respect to $\pi$ *if both:*

- $X^\pi \cap H^-(\pi) \subset \operatorname{int}(X) \cap H^-(\pi)$
- *The tangent spaces $T_xX$ and $T_xX^\pi$ do not coincide for each $x \in X(M^n) \cap \pi$ (when seen as subspaces of $T_x\mathbb{M}^{n+1}$)*

This fundamentally means that the reflection of one of the halves of $X$ on the other side of $\pi$ is contained in the region inside $M^n$ and the tangent spaces of $X$ and of the half-reflection do not form a ninety degree angle with $\pi$, at all points on $\pi \cap X$. As the two tangent spaces are one the reflection of the other, this means that they do not coincide.

**Definition 3.3.** *We say* we can reflect $X : M^n \to \mathbb{R}^{n+1}$ strictly up to $(\pi, v)$ *if we can reflect $M^n$ strictly with respect to $\pi_{v,s}$ for all hyperplanes $\pi_{v,s}$ such that $s < C$.*

The result then becomes:

**Theorem 3.4** (Extended Chow-Gulliver). *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a $C^2$ solution to equation (3.2). Then, if we can reflect $X(M^n, 0) = M_0$ strictly with respect to $\pi$, then for all $t \in [0,T)$ we can reflect $X(M^n, t) = M_t$ strictly with respect to $\pi$.*

*Proof.* As before, by contradiction, suppose that there is a time $t$ such that the thesis is false, and that it is the smallest such $t$. Then, for all $\tau \in [0,t)$, $M_{\tau,\pi} \cap H^-(\pi) \subset \operatorname{int}(M_\tau) \cap H^-(\pi)$; the unit vector orthogonal to $\pi$, $V$, is such that $V \notin T_x M_\tau$ for all $x \in M_\tau \cap \pi$ and $\tau \in [0,t)$; and either of the conditions fails at $t$, i.e. either:

(i) $M_{t,\pi} \cap H^-(\pi) \cap M_t \neq \emptyset$

(ii) The tangent spaces $T_x X$ and $T_x X^\pi$ coincide for some $x \in \pi$.

(i) Suppose the first case is true. Then, there exists $x_0 \in M_{t,\pi} \cap H^-(\pi) \cap M_t$ such that at $x_0$ the two manifolds are tangent.
We can then reason as in the proof of theorem 2.6.

(ii) Suppose instead that $T_x M_t = T_x M_t^\pi$ and in a neighbourhood of $(x,t)$ both $M_t$ and $M_t^\pi$ are graphs of two smooth functions over $T_x M_t$ by 2.3, i.e. again

$$\begin{aligned} f &: (x,t) \mapsto x + \tilde{f}(x,t)\nu \\ f_\pi &: (x,t) \mapsto x + \tilde{f}_\pi(x,t)\nu \end{aligned}$$

We can do this because the theorem allowing us to do so is a theorem on smooth manifolds, and requires nothing on the metric, therefore the existence of models in $\mathbb{R}^{n+1}$ of the ambient spaces allows us to do the same procedure. Reasoning again as in the proof of theorem 2.6, in $\overline{H^-(\pi)}$, $f_\pi \geq f$, because $M^n_\pi \cap H^-(\pi) \subset \operatorname{int}(M^n) \cap H^-(\pi)$. Finally, $f(x,t) = f_\pi(x,t)$, hence $f_\pi - f(x,t) = 0$, and thus $(x,t)$ is a minimum point on the boundary for $f_\pi - f$. Also, for the outward pointing normal to the boundary $V$

$$\frac{\partial f}{\partial V}(x,t) = \frac{\partial f_\pi}{\partial V}(x,t)$$

because the graphs are both tangent to $T_x M_t$. We note also that the models in section 1.5 are conformal, so a vector is orthogonal to the boundary if and only if it is orthogonal to the corresponding region when seen as a region in $\mathbb{R}^{n+1}$. Thus,

$$\frac{\partial (f - f_\pi)}{\partial V}(x,t) = 0$$

But we must have

$$\frac{\partial (f - f_\pi)}{\partial V}(x,t) > 0$$

at a minimum on the boundary by Proposition 1.13, a contradiction. □

## 3.3 Extending Corollaries

We now shift the focus to extending the corollaries of theorem 2.6 in section 2.7.

The author is not aware of a standard definition of a support and a radial function in a curved setting, so we do not attempt to extend the results in section 2.8.

Clearly, corollary 2.8 has a direct equivalent in this setting:

**Corollary 3.5.** *Let $X : M^n \times [0,T) \to \mathbb{M}^{n+1}_+$ be a $C^2$ solution to equation (2.2). Then, if we can reflect $X_0$ strictly up to $(\pi_{v,C}, v)$, for all $t \in [0,T)$ we can reflect $X_t$ strictly up to $(\pi_{v,C}, v)$.*

The second result that we can extend is Corollary 2.7, although the meaning of *graph* in curved spaces is ambiguous. We adapt the corollary as follows:

**Corollary 3.6.** *Let $X : M^n \times [0,T) \to \mathbb{M}^{n+1}_+$ be a $C^2$ embedded solution to equation (3.2). Then, if we can reflect $X_0$ strictly up to $(\pi_{v,C}, v)$, then for all $t \in [0,T)$, $X_t \cap H^+(\pi_{v,C})$ is such that the projection of the coordinates of its points onto $\pi_{v,C}$, $(p,\tau) \mapsto p$, is injective.*

This condition guarantees that we can build a map from $s : \overline{\operatorname{int}(X_t \cap \pi_{v,C})} \to \mathbb{R}$ such that $(p, s(p)) \in X_t$, making $p \mapsto (p, s(p))$ act as a sort of curved graph.

*Proof.* By theorem 3.4 we can reflect up to $(\pi_{v,C}, v)$ at all times in $[0,T)$. Let $\gamma_p(s) = (p, s + C)$ be the path followed by $p \in \pi_{v,C}$ as the planes sweep through the ambient space. If two points exist with the same $p$ coordinate, say $(p, C_1)$ and $(p, C_2)$, the reflection about the hyperplane $\pi_{v,\frac{C_1+C_2}{2}}$ would map one onto the other, but as both $C_1$ and $C_2$ are greater than $C$, then we should also be able to reflect strictly about it, as we would have $\frac{C_1+C_2}{2} > C$. $\square$

Let $\tilde{B}_r(y) = \{p \in \mathbb{M}^{n+1}_+ : \operatorname{dist}(p,y) < r\}$. Reasoning exactly like in corollary 2.9, we also can extend it to this setting:

**Corollary 3.7.** *Let $X : M^n \times [0,T) \to \mathbb{M}^{n+1}_+$ be a $C^2$ embedded solution to equation (3.2). There exists $\varepsilon > 0$ depending only on $X_0$ such that for all $t \in [0,T)$ we can reflect $X_t$ up to $(\Pi^v_0 + \epsilon v, v)$ for every $v \in S^n$. In particular, if $X_0 \subset \tilde{B}_R(C)$, then we can always reflect $X_t$ up to $(\Pi, v)$ whenever $H^+(\Pi) \cap \tilde{B}_{R-\varepsilon}(C) = \emptyset$.*

Other corollaries of the section, like Corollary 2.13 and Corollary 2.10 are harder to extend, as they rely on the points at each plane moving along geodesics on the plane, which is not true in the curved case.

## 3.4 Extending theorem 2.25

It is also possible to extend the result in section 2.9 to $\mathbb{M}_+^{n+1}$. We consider *expansive* solutions to 3.2 which are defined on a larger interval $(T_0, T_1)$, with $-\infty \leq T_0 \leq 0 < T_1 \leq \infty$. Like before:

**Definition 3.8.** *We say that a solution to equation (3.2) is* expansive *if $F < 0$.*

Like in the euclidean case, expansive solutions are those where $\frac{\partial X_t}{\partial t}$ is an outward pointing vector.

**Definition 3.9.** *Let $X : M^n \times (T_0, T_1) \to \mathbb{M}_+^{n+1}$ be an embedded expansive solution to equation (3.2). We say that $X$* comes out of a point *if there exists a point $y_\infty$ such that for every $\varepsilon > 0$, there exists a time $\tau \in (T_0, T_1)$ such that $X_\tau \subset \tilde{B}_\varepsilon(y_\infty)$, where $\tilde{B}_r(y) = \{p \in \mathbb{M}_+^{n+1} : \mathrm{dist}(p, y) < r\}$.*

Like in the euclidean case, homothetically expanding distance spheres satisfy the equation: distance spheres have constant principal curvatures in $\mathbb{M}_+^{n+1}$ as well; thus, any spherical solution is completely determined by its radius at time $t$, because the equation is invariant at each point. Again, the ordinary differential equation $r'(t) = \varphi(r(t))$ completely determines the flow, where $\varphi(r) = -F(\frac{1}{r}, \ldots, \frac{1}{r})$. Like before, we can find an ancient solution if and only if

$$\int_0^c \frac{1}{\varphi(r)} dr = +\infty.$$

It is no surprise that in constant curvature spaces we can prove a statement equivalent to theorem 2.25:

**Theorem 3.10.** *Let $X : M^n \times (T_0, T_1) \to \mathbb{M}_+^{n+1}$ be a smooth, closed, embedded expansive solution to equation (3.2) coming out of a point. Then it is a family of expanding distance spheres.*

**Remark 3.11.** At a glance, it could be confusing why the shape changes, after all it is not immediately apparent what changes in equation 3.2 from the previous chapter, as we do not find the metric in it, and one might wonder if it could be seen as a version of equation 2.2 with a different $F$. This is not the case, as computing the principal curvatures interacts with the metric at a fundamental level, making it so that the two equations are not comparable.

The proof is effectively the same as before:

*Proof.* Fix any hyperplane $\pi$ not passing through $y_\infty$. There is $R > 0$ such that $\tilde{B}_{2R}(y_\infty)$ does not intersect it. By definition, there is also a time $\tau \in (T_0, T_1)$ such that $X_\tau \subset \tilde{B}_R(y_\infty)$. By Corollary 3.7, then, we can reflect strictly $X_t$ up to $\pi$ for any $t > \tau$.

Now consider a sequence $\epsilon_n \to 0$. Up to a subsequence, we can then find a corresponding converging non-increasing sequence $\tau_n \to \bar{t} \in [T_0, T_1)$ (here note that $\bar{t}$ can be $-\infty$) such that $X_{\tau_n} \subseteq B_{\epsilon_n}(y_\infty)$, therefore in $(\tau_n, T_1)$ I can reflect up to any hyperplane outside $\tilde{B}_{\epsilon_n}(y_\infty)$ in any direction, reasoning like we just did. At time $\bar{t}$, $X_{\bar{t}} \subseteq \cap_r \tilde{B}_r(y_\infty) = \{y_\infty\}$, thus we would have a singularity at $\bar{t}$ if $\bar{t} \in (T_0, T_1)$ and therefore $\bar{t} = T_0$.

On the other hand, by construction, we can reflect $X_t$ strictly about any hyperplane not intersecting $\cap_r B_r(y_\infty) = \{y_\infty\}$ at any time $t > \bar{t} = T_0$, hence we can reflect $X_t$ strictly up to any hyperplane passing through $y_\infty$, in both directions, at any time $t \in (T_0, T_1)$. We observe that in the limit, the reflection property becomes non-strict, in the sense that we have to replace the interior of $X$ in definition 2.4 with its closure, therefore $X_t$ may touch its reflection at the limit plane, i.e. the one passing through $y_\infty$. Similarly, it cannot be that the other condition is the one causing the strict reflection definition to fail, as the other condition stays instead strict.

This implies that, taking any hyperplane passing through $y_\infty$ and considering opposite directions for the reflection, $X_t$ is symmetric about said hyperplane for any time $t \in (T_0, T_1)$. By Proposition 1.22, then, we conclude that $X_t$ must be a ball. □

## 3.5 Shrinking flows on the sphere

As noted in [2], something can be said about the opposite problem for a shrinking flow on the sphere $S^{n+1}$. In fact, shrinking flows of convex manifolds on a sphere are also quite rigid, and are limited under certain condition to being shrinking spheres. This was shown in [6].

**Definition 3.12.** *We say that a solution to equation (3.2) is* shrinking *if $F \geq 0$, and $F(0) = 0$.*

The condition $F(0) = 0$ ensures that equators on the sphere are static solutions to equation (3.2). On the whole sphere one must choose a conventional direction for the unit normal.

**Theorem 3.13.** *Let $X : M^n \times (T_0, T_1) \to \mathbb{M}^{n+1}_+$ be a smooth, convex, embedded shrinking solution to equation (3.2), and $(T_0, T_1)$ be the maximum interval where the solution is defined. Suppose also that:*

$$\limsup_{t \to T_0} \max_{X_t} \sum_i \kappa_i(x) < \infty$$

*Then $X_t$ is a distance sphere for all $t \in (T_0, T_1)$.*

The proof of this result is long and involved, and the main result in [6]. It also uses a reflection argument.

**Remark 3.14.** This result acts as a sort of converse for theorem 3.10 on the hemisphere, showing that all ancient convex shrinking solutions on the sphere are the trivial shrinking spheres.

# Chapter 4

# Area- and volume-preserving flows

# [SISTEMA PARABOLICITÀ (credo serva $H > 0$ quando $k \neq 1$)]

In this chapter area-preserving and volume-preserving flows are discussed. These are flows similar to the equation (2.2), with a non-local term added. It can be shown that the theorems in chapter 2 also apply to these flows. Finally, we discuss an application of these flows.

## 4.1 The flows we consider in this chapter

Volume-preserving flows were introduced by Gerhard Huisken in 1987 (see [9]). Unlike the mean curvature flow, which contracts hypersurfaces to a point, volume-preserving flows maintain a constant enclosed volume while reducing the hypersurface area. These flows are governed by the modified evolution equation:

$$\frac{\partial X}{\partial t} = [h(t) - H(x,t)]\,\nu,$$

where $H(x,t)$ is the is the mean curvature at a point $x \in X_t$ and $h(t)$ is the average mean curvature at time $t$, a non-local term defined as

$$h(t) = \frac{\int_{X_t} H(x,t)\,d\mu}{\int_{X_t} 1\,d\mu} = \frac{\int_{M_t} H(x,t)\,d\mu}{|M_t|},$$

where $d\mu$ is the volume element on the hypersurface $X_t$. Notice that mean-curvature flows ($\frac{\partial X}{\partial t} = -H(t)\nu$) is a possible choice of $F$ in (2.2), therefore compared to the previous chapters we are just adding the global term $h(t)$. Huisken's results demonstrate that such flows lead to the convergence of the hypersurfaces to round spheres, provided the initial hypersurface is uniformly convex.

A more general family of flows is in [10]. Consider flows:

$$\begin{cases} \dfrac{\partial X_t}{\partial t} = \left[-H^k(x,t) + \phi(t)\right] \nu \\ X(0) = X_0 \end{cases} \tag{4.1}$$

where $\phi(t)$ is an appropriate non-local term and $k \in (0, \infty)$. A possible choice for $\phi$ is:

$$\phi(t) = \frac{\int_{X_t} H^k(x,t)\, d\mu}{|X_t|}, \tag{4.2}$$

corresponding to volume-preserving flows. Another possible choice is

$$\phi(t) = \frac{\int_{X_t} H^{k+1}(x,t)\, d\mu}{\int_{X_t} H(x,t)\, d\mu}, \tag{4.3}$$

which corresponds to area-preserving flows. It can be shown that if we choose $\phi$ as in (4.2), the volume of the domain enclosed by $X_t$ remains constant, while choosing $\phi$ as in (4.3) keeps the area $|X_t|$ constant.

The fact that choosing $\phi$ as in (4.2) preserves the volume is immediately apparent, as the change of the volume inside the hypersurface is the integral over $X_t$ of $\frac{\partial X_t}{\partial t} \cdot \nu$ (by Reynolds transport theorem for a constant function):

$$\int_{X_t} \frac{\partial X_t}{\partial t} \cdot \nu\, d\mu = \int_{X_t} \left[-H^k(x,t) + \phi(t)\right] d\mu = -\int_{X_t} H^k(x,t) d\mu + \phi(t)|X_t| = 0$$

On the other hand, choosing $\phi$ as in (4.3), the formula for the first variation of area says that

$$\frac{d}{dt} \int_{X_t} d\mu = -\int_{X_t} \left\langle \frac{\partial X_t}{\partial t}, H\nu \right\rangle d\mu + \cancel{\int_{\partial X_t} \nu \tfrac{\partial X_t}{\partial t}\, d\mu}$$

where the second term is cancelled because $\frac{\partial X_t}{\partial t}$ is orthogonal to the surface. Therefore

$$\begin{aligned} \frac{d}{dt} \int_{X_t} d\mu &= -\int_{X_t} \left[-H^k + \phi(t)\right] H d\mu \\ &= \int_{X_t} H^{k+1} d\mu + \phi(t) \int_{X_t} H d\mu = 0 \end{aligned}$$

Another choice for an area-preserving flow is in [11] where he considers a flow

$$\begin{cases} \dfrac{\partial X_t}{\partial t} = \left[1 - H^k(x,t)\phi(t)\right] \nu \\ X(0) = X_0 \end{cases} \tag{4.4}$$

where

$$\phi(t) = \frac{\int_{X_t} H(x,t)\, d\mu}{\int_{X_t} H^{k+1}(x,t)\, d\mu}$$

Again, computing the first variation of the area:

$$\begin{aligned} \frac{d}{dt}\int_{X_t} d\mu &= -\int_{X_t} \left[1 - H^k\phi(t)\right] H d\mu \\ &= -\int_{X_t} H d\mu + \phi(t)\int_{X_t} H^{k+1} d\mu = 0 \end{aligned}$$

## 4.2 Theorem 2.6 and corollaries

While not really parabolic flows, because of the non-local term, it is possible to apply theorems on parabolic flows to equation (4.1). Given a solution $X_t$, we can consider the evolution equation

$$\begin{cases} \dfrac{\partial X_t}{\partial t} = \left[-H^k(x,t) + \varphi_{X_t}(t)\right] \nu \\ X(0) = X_0 \end{cases} \tag{4.5}$$

where $\varphi_{X_t}$ is the constant function independent of $X$ given by $\phi$ when computed on the specific solution $X_t$. Clearly, $X_t$ is a solution also to the second equation, as the values of $\phi$ and $\varphi_{X_t}$ coincide when one considers the solution $X_t$. At the same time, equation (4.5) is a parabolic equation: the only difference with (2.2) is the addition of a constant function $\varphi_{X_t}$ independent of the solution multiplied by the normal vector, which cannot affect the second order terms (the normal vector only depends on first-order derivatives). Therefore, following this line of reasoning, one can apply theorems like the maximum principle (Proposition 1.12) and the boundary point lemma (Proposition 1.13) also to solutions of (4.1), as they must hold for solutions of the associated parabolic equation (4.5).

Immediate consequence of the above is the fact that theorem 2.6 extends to solutions of equation (4.1):

**Theorem 4.1.** *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a $C^2$ solution to equation (4.1). Then, if we can reflect $X(M^n, 0) = X_0$ strictly with respect to $\pi$, then for all $t \in [0,T)$ we can reflect $X(M^n, t) = X_t$ strictly with respect to $\pi$.*

The proof is word for word identical to that of theorem 2.6, once one notices that it is possible to apply the maximum principle and Hopf's boundary point lemma.

**Remark 4.2.** There is also a simpler approach to apply the maximum principle in this specific case: in the proof of the maximum principle one considers the difference of two solutions; in principle the extra term $\phi(t)\nu$ could be different, providing an obstruction in the proof, as there is no guarantee that $u$ is a solution to a parabolic equation, but in the situation where we need it in the proof of this theorem the two $\phi(t)$ must have the same value, as the two surfaces in this case are just one the reflection of the other, and therefore this non-local term cancels out completely. Thus, the maximum principle can be extended to this situation. Similarly, this also applies to Hopf's boundary point lemma.

As a direct consequence, all the corollaries of theorem 2.6 can be extended.

## 4.3 The solution stays inside a compact

I believe that $X_0 \subset B_R(p)$ implies $X_t \subset B_{2R}(p)$, but I have to check that.

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