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TITOLO

THE ALEXANDROV MOVING PLANES METHOD AND APPLICATIONS TO GEOMETRIC FLOWS

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Introduction

This Master's thesis analyses the Alexandrov Moving Planes Method and its applications to Geometric Flows.

The main original result in this thesis is the extension of a result from Chow and Gulliver about flows in Euclidean space to the case where the ambient space is constant curvature spaces. Moreover, in the last chapter, we include a new proof - using the moving planes method - of the well known result stating that some area-preserving and volume-preserving flows do not leave a large enough compact.

The Moving Planes Method is a technique in Analysis that can be used to prove radial symmetry of certain solutions to some differential equations. The method was originally introduced by Alexandrov to characterize the sphere as the only hypersurfaces with constant curvature (see for example [1]), and then used by Serrin on elliptic PDEs (see [20]) and Gidas-Ni-Nirenberg (see [10]).

To apply the method, one considers a solution and its reflections about a foliation of the ambient space by geodesic hyperplanes. The method consists of reflecting the part of a solution "below" the hyperplane into the top part, and using properties of both copies of the solution together, generally some form of maximum principle, to prove some property of the non-reflected solutions. The method was originally used to analyse geometric elliptic PDEs, and it is now an important tool in Geometric Analysis.

The central result is a theorem by Chow and Gulliver (theorem 3.8) contained in a 1997 paper [4], who proved that the method *behaves well* with respect to certain parabolic geometric flows.

We consider manifolds M^n embedded in $\mathbb{R}^{n+1}$, i.e. there is an embedding $X_0 : M^n \to \mathbb{R}^{n+1}$ parametrizing the hypersurface $X_0(M^n)$ evolving according to a geometric flow in the form:

$$\begin{cases} \dfrac{\partial X_t}{\partial t} = -F(\kappa_1(x), \dots, \kappa_n(x))\nu \\ X(0) = X_0 \end{cases}$$

where ν is the outward normal to $X_t(M^n)$ at the point $X_t(x)$, $\kappa_1 \leq \cdots \leq \kappa_n$ are the principal curvatures at $X_t(x)$, and

$$\frac{\partial F}{\partial \kappa_i} > 0 \text{ for all } i = 1, \dots, n$$

The Chow-Gulliver result states:

Theorem (Chow-Gulliver)**.** *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a C^2 solution to equation the equation above. Then, if we can reflect $X(M^n, 0) = X_0$ strictly with respect to π, then for all $t \in [0,T)$ we can reflect $X(M^n, t) = X_t$ strictly with respect to π.*

Here by strict reflection we mean that the following two conditions are met:

- The half of the manifold above the plane reflects inside the other half of the manifold, without touching it.
- The tangent spaces of the manifold and its reflection about the plane do not coincide at any point on the plane other than the points where the manifold and the plane are tangent to each other.

The key idea in the proof of the result is as follows: suppose we have an embedded smooth hypersurface X evolving according to the equation above and a fixed hyperplane π, intersecting X. Suppose that, at some time t, X and its reflection about the hyperplane X_π touch at a point not on π. We can consider X and X_π as local graphs over the same hyperplane π, and we can show that these function evolve according to the same differential equation. Using the strong maximum principle and the Hopf boundary point lemma, then, one can conclude that the two functions coincide, and have been coinciding up until that point. In particular, if at some time t a solution and its reflection do not touch, they will not touch for all subsequent times.

In the first chapter, some foundational results are collected. Firstly, some equations in local coordinates for immersed hypersurfaces are derived and some well known theorems in differential geometry and on parabolic differential equations are stated, and proved when necessary. A version of the maximum principle and of Hopf's boundary point lemma for non-linear equations is introduced in section 1.4.

After that, in section 2.1 and 2.2 we introduce the notation for reflections in constant-curvature spaces and the Alexandrov Moving Planes Method. A sketch of the proof of the Alexandrov soap-bubble theorem is included in the last section of the chapter, to show an application of the technique to elliptic differential equations.

In the third chapter, in the first few sections we give justification to why these equations are parabolic, and introduce the Moving Planes Method for parabolic flows. In section 3.6 the proof of the theorem from Chow and Gulliver is included. Some corollaries are then proved in section 3.7, and applied in section 3.8 to find gradient estimates for the the support function and the radial function. Finally, in section 3.9 a result on ancient solutions to expansive flows from [19] is included.

In the fourth chapter, we analyse the Chow-Gulliver result in spaces of constant curvature, extending some of its corollaries, as well as the result from section 3.9.

In the fifth chapter, finally, area-preserving and volume-preserving flows are introduced and their name is justified in section 5.1, and we extend the theorem from Chow and Gulliver to this new setting in section 5.2. Finally, in section 5.3, an original proof that the solution stays inside a compact is included.

Chapter 1

Preliminary Foundations

1.1 Geometry of embedded hypersurfaces

We will be assuming that the reader is familiar with the fundamental definitions and proofs in Differential and Riemannian Geometry, but we collect here some definitions and properties of immersed hypersurfaces, which are sometimes skipped in Differential Geometry courses. The study of immersed hypersurfaces is a fundamental topic in differential geometry, and is especially important in the field of geometric analysis. An hypersurface $X : M^n \to \overline{M}^{n+1}$ is a submanifold of a higher-dimensional space that is embedded in that space in such a way that the submanifold has the same dimension as the space in which it is embedded minus one. In other words, it is a submanifold of codimension one. We will always assume the embedding to be smooth. The most common case is $\overline{M}^{n+1} = \mathbb{R}^{n+1}$, usually extensively studied in undergraduate courses when $n = 2$. We will be assuming that the embedding is also isometric, i.e. the metric g on M^n is the one induced by $(\overline{M}^{n+1}, \overline{g})$.

In this chapter, symbols referring to $(\overline{M}^{n+1}, \overline{g})$ will have a line on top, otherwise the symbol will refer to (M^n, g). In particular, they do not refer to the closure of some set.

The pullback of the tangent bundle of $\overline{M}^{n+1}$ to M^n is a smooth vector bundle on M^n:

$$X^* T\overline{M}^{n+1} = T\overline{M}^{n+1}|_{M^n} = \amalg_{p \in M^n} T_p \overline{M}^{n+1}$$

The normal vector field ν is a section of the pullback vector bundle $X^* TM^{n+1}$ on the manifold M^n that is perpendicular to the tangent space of M^n at each point. At each point p:

$$T_p\overline{M}^{n+1} = T_p M^n \oplus N_p M^n$$

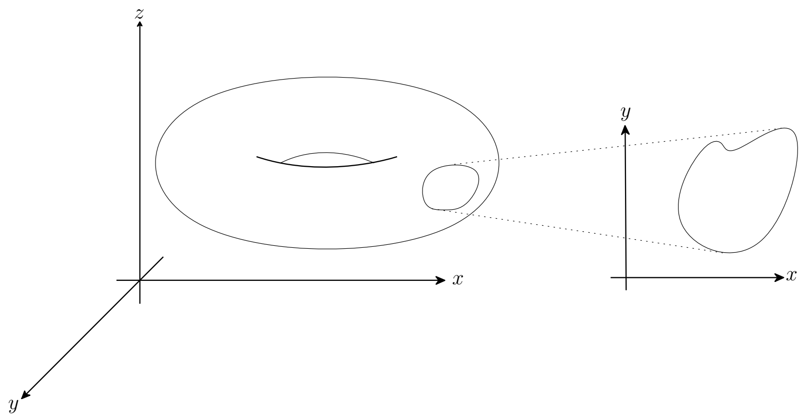

Figure 1.1: A (hyper)surface immersed in $\mathbb{R}^3$

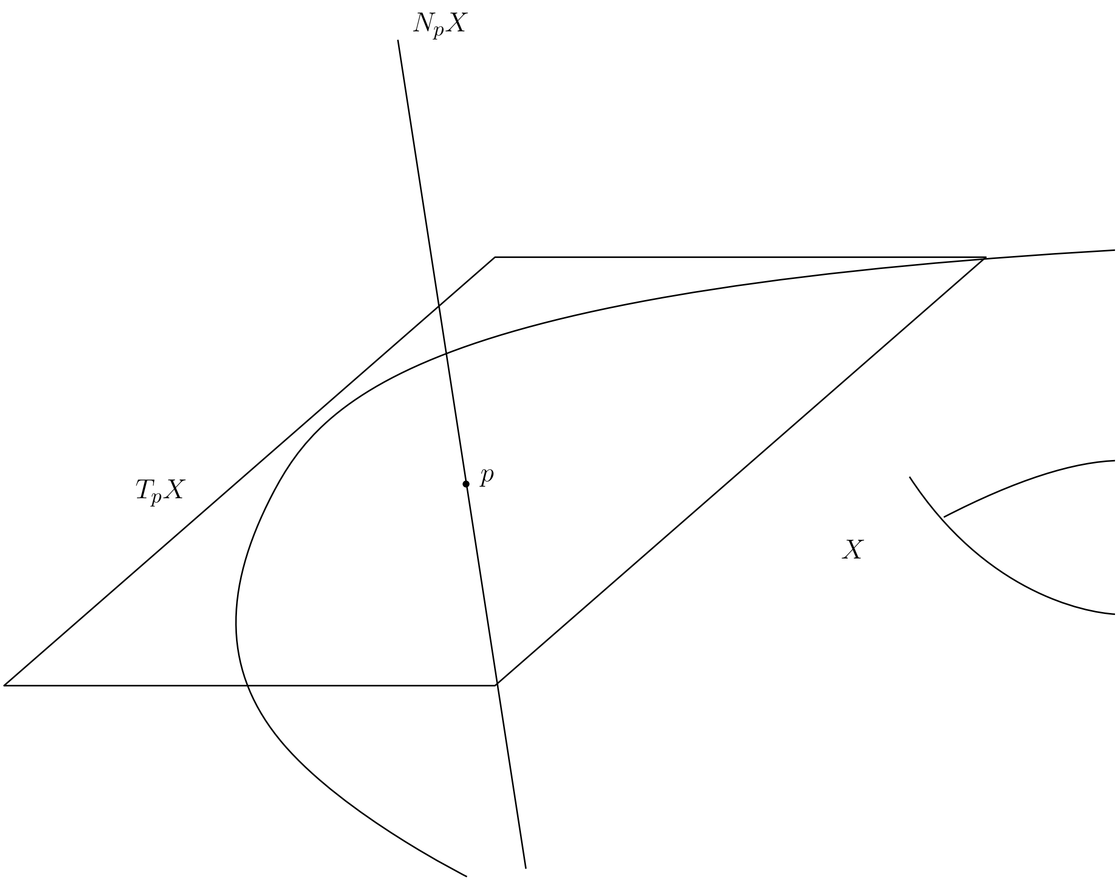

Figure 1.2: The tangent and the normal bundle at a point p

where N_pM^n is the normal vector bundle generated by the normal vector. If the manifold is oriented, there is usually a choice we can make on the normal vector: whether we choose to take the inward or outward pointing vector. We will choose the latter through unless otherwise specified. This also allows us to define the tangent and normal projection on $T\overline{M}^{n+1}|_{M^n}$ by taking the two respective components.

Clearly, taking $\overline{\nabla}$ to be the Levi-Civita connection on $(\overline{M}^{n+1}, \overline{g})$, we can decompose it as:

$$\overline{\nabla}_v w = (\overline{\nabla}_v w)^{\top} + (\overline{\nabla}_v w)^{\perp}$$

Definition 1.1. *The* second fundamental form *is then defined as:*

$$\mathrm{I\!I}(v, w) = (\overline{\nabla}_v w)^{\perp}$$

It is a bilinear symmetric tensor because TM^n is involutive in $T\overline{M}^{n+1}$ and depends only on the local value of v and w by symmetry. we can therefore write it as

$$\mathrm{I\!I}(v, w) = -(h_{ij} v^i w^j)\nu$$

for some matrix $A(p) = \{h_{ij}\}$. we can define the principal curvatures of the hypersurface, as the eigenvalues of this matrix.

It is also possible to check that $(\overline{\nabla}_v w)^{\top}$ satisfies the definition the Levi-Civita connection on (M^n, g), therefore, from its uniqueness:

$$\begin{aligned}\nabla_v w &= (\overline{\nabla}_v w)^{\top} \\ \overline{\nabla}_v w &= \nabla_v w + \mathrm{I\!I}(v, w)\end{aligned}$$

where ∇ is the Levi-Civita connection on (M^n, g). This result is known as the *Gauss Formula.* It has to be noted however that we are implicitly considering tangent vectors that are not in the same space. Indeed, making that more explicit, the formula should be:

$$\overline{\nabla}_{X_* v} X_* w = X_*(\nabla_v w) + \mathrm{I\!I}(v, w)$$

Proposition 1.2. (The Weingarten Equation) *If $v, w \in TM^n$ and $\nu \in NM^n$, if one considers the corresponding derivations in $T\overline{M}^{n+1}$ the following equation holds:*

$$\left\langle \overline{\nabla}_v \nu, w \right\rangle_{\overline{g}} = -\left\langle \nu, \mathrm{I\!I}(v, w) \right\rangle_{\overline{g}}$$

Proof. As $\langle \nu, w\rangle_{\overline{g}} \equiv 0$ on M,

$$\begin{aligned} 0 &= v\,\langle \nu, w\rangle_{\overline{g}} \\ &= \left\langle \overline{\nabla}_v \nu, w\right\rangle_{\overline{g}} + \left\langle \nu, \overline{\nabla}_v w\right\rangle_{\overline{g}} \\ &= \left\langle \overline{\nabla}_v \nu, w\right\rangle_{\overline{g}} + \langle \nu, \mathrm{I\!I}(v,w)\rangle_{\overline{g}} \end{aligned}$$

applying the Gauss Formula and the fact that $\nabla_v w \in TM^n$ in the last step □

It is also usual to define the associated Weingarten map, which is the linear map between sections of M $s : \Gamma(M) \to \Gamma(M)$ satisfying:

$$\langle s(v), w\rangle_g = \langle \nu, \mathrm{I\!I}(v,w)\rangle_{\overline{g}}$$

the linear map s is also known as the *shape operator of* M.

Combining this with the formula above, taking into account that it holds for a generic $w \in TM$:

$$s(v) = -(\overline{\nabla}_v \nu)^{\top}$$

We are going to use these equations in local coordinates, in the form shown below.

Proposition 1.3. *The above equations in local coordinates are equivalent to the following equations:*

$$\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial X^\alpha}{\partial x^k} + \overline{\Gamma}_{\beta\delta}^\alpha \frac{\partial X^\beta}{\partial x^i}\frac{\partial X^\delta}{\partial x^k} = -h_{ij}\nu^\alpha \tag{1.1}$$

$$\frac{\partial \nu^\alpha}{\partial x^i} + \overline{\Gamma}_{\beta\delta}^\alpha \frac{\partial X^\beta}{\partial x^i}\nu^\delta = h_{ij} g^{jl} \frac{\partial X^\alpha}{\partial x^l} \tag{1.2}$$

where ν is the normal unit vector at the point and $A = \{h_{ij}\}$ is the second fundamental form, thus $h_{ij} = \left\langle \nu, \overline{\nabla}_{\overline{\partial_i}} \overline{\partial_j}\right\rangle_{\overline{g}}$

Proof. For any connection ∇. and any derivations $v = v^i \partial_i$ and $w = w^j \partial_j$:

$$\nabla_v w = \nabla_{(v^i\partial_i)}(w^j \partial_j) = v(w^k)\partial_k + (v^i w^j \Gamma_{ij}^k)\partial_k$$

Let $\partial_1, \ldots \partial_n$ be a basis of TM^n at a point, and let $\overline{\partial_i} = X_* \partial_i$, $\overline{\partial_{n+1}} = \nu$. Let's consider the Gauss Formula for two generic ∂_i, ∂_j, using Roman letters for indices varying between 1 and n and Greek letters for indices varying between 1 and $n+1$:

$$\begin{aligned} \overline{\nabla}_{X_*\partial_i} X_* \partial_j &= X_*(\nabla_{\partial_i}\partial_j) + \mathrm{I\!I}(\partial_i, \partial_j) \\ (\overline{\partial_i}(X_*\partial_j)^\alpha)\partial_\alpha + ((X_*\partial_i)^\beta (X_*\partial_j)^\delta \overline{\Gamma}_{\beta\delta}^\alpha)\overline{\partial_\alpha} &= X_*(\Gamma_{ij}^k \partial_k) - h_{ij}\nu^\alpha \overline{\partial_\alpha} \\ \frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} + \overline{\Gamma}_{\beta\delta}^\alpha \frac{\partial X^\beta}{\partial x^i}\frac{\partial X^\delta}{\partial x^k} &= \Gamma_{ij}^k \frac{\partial X^\alpha}{\partial x^k} - h_{ij}\nu^\alpha \end{aligned}$$

Which is the formula (1.1). To get the second formula, first note that $s(v) = -(\overline{\nabla}_v \nu)^\top$. We then compute $-\langle s(\partial_i), \overline{\partial_\alpha}\rangle_{\overline{g}}$:

$$\left\langle \overline{\nabla}_{\overline{\partial_i}} \nu, \overline{\partial_\alpha} \right\rangle_{\overline{g}} = -\left\langle s(\partial_i), \overline{\partial_\alpha} \right\rangle_{\overline{g}}$$
$$\left\langle \left(\frac{\partial \nu^\alpha}{\partial x^i} + \overline{\Gamma}^\alpha_{\beta\delta} \frac{\partial X^\beta}{\partial x^i} \nu^\delta \right) \overline{\partial_\alpha}, \overline{\partial_\alpha} \right\rangle_{\overline{g}} = h_{ij} g^{jl} \frac{\partial X^\alpha}{\partial x^l}$$

leading to (1.2). □

1.2 Local representation as a graph

Consequence of the Inverse Function theorem is a powerful result that allows one to locally represent a submanifold of $\mathbb{R}^{n+1}$ as the graph of a smooth function. We provide a version of this theorem below:

Theorem 1.4 (Local representation as a graph)**.** *Let X^n be a submanifold $X^n \subset \mathbb{R}^{n+1}$ and let $x_0 \in X$. Then there exists a neighbourhood of x_0, $U \subset X^n$, such that U is the graph of a function. Moreover, this function can be of the form*

$$f : \pi(U) \subset \mathbb{R}^n \to U$$
$$U = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} | x_0 = f(x_1, ..., x_n)\}$$

for any of the possible orders of the usual basis for $\mathbb{R}^n$, $(e_0, \dots, e_n)$, as long as $e_0 \notin T_x M$, where $\pi(U)$ is the projection on the last n coordinates ($(x_0, \dots, x_n) \mapsto (x_1, \dots, x_n)$).

A proof of the 2D-case of the version of the theorem can be found in [6] which extends naturally to the n dimensional case, with almost no changes. This immediately extends to:

Corollary 1.5 (Local representation as a graph on the tangent)**.** *Let X^n be a submanifold $X^n \subset \mathbb{R}^{n+1}$ and let $x_0 \in X$. Then there exists a neighbourhood of x_0 $U \subset X^n$ and a smooth function $f : T_x X^n \to \mathbb{R}$ such that any $x_0 \in U$ can be expressed as*

$$x_0 = p + f(p)\nu$$

where ν is the vector normal to $T_{x_0} X^n$, for an appropriate point $p \in T_{x_0} X^n$. In other words, every submanifold $X^n \subset \mathbb{R}^{n+1}$ is locally expressible as a graph on its tangent space.

Proof. By rotation, we may assume $T_x X^n$ orthogonal to e_1. Then one can just apply the previous theorem. □

We will use this later to prove Theorem 3.4.

1.3 Some well established results from analysis

We now include some well known results from analysis which will be useful later. The first result we introduce is the maximum principle.

The maximum principle is a classical result of mathematical analysis, and it is usually introduced in a first course on partial differential equations. It is a fundamental tool in the theory of partial differential equations. It is a statement about the behaviour of solutions to certain types of PDEs and provides a method for obtaining upper and lower bounds on the solutions. The principle states that the maximum and minimum values of a solution to elliptic or parabolic PDE occur on the boundary of the domain unless the function is constant.

The maximum principle can be used to prove the existence, uniqueness, and regularity of solutions to elliptic and parabolic PDEs. It can also be used to obtain estimates on the behaviour of solutions and to study the asymptotic behaviour of solutions as the domain becomes large. The principle is widely used in many fields of mathematics and physics, such as geometric analysis, mathematical physics, and fluid dynamics. One of the many versions of this well know theorem is this:

Theorem 1.6 (Maximum principle for parabolic equations)**.** *Let Ω be an open, bounded, connected set. Assume $u \in C_1^2(\Omega \times [0,T]) \cap C^1(\overline{\Omega} \times [0,T])$. Suppose u satisfies:*

$$-\frac{\partial u}{\partial t} + \left(\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i}^{n} b_i \frac{\partial}{\partial x_i} + c \right) u = -u_t + Lu \geq 0 \tag{1.3}$$

where L is an elliptic differential operator, i.e. there exists $\theta > 0$ such that $\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2$ for all $\xi \in \mathbb{R}^n$ and $(x,t) \in \Omega \times [0,T]$. Suppose also that $c \equiv 0$ in Ω. Then:

- *if u attains its maximum in an interior point $(x_0, t_0) \in \Omega \times [0,T]$, then u is constant in $\Omega \times [0, t_0]$.*
- *If, instead, under the same conditions, $u_t - Lu \geq 0$ and attains its minimum in an interior point of $\Omega \times [0,T]$, then u is constant in $\Omega \times [0, t_0]$*

A proof of this result can be found, for example, in [7]. The theorem extends also to situations where the condition holds in a bounded connected region $R \subseteq \Omega \times [0,T]$: in that case, if u attains its maximum in an interior point then u has the same value at any point in R that can be connected to it through a segment going in the backwards direction of time and a "horizontal" line contained in Ω. This version of the theorem can be found for example in [18]:

Theorem 1.7. *Let u satisfy the uniformly parabolic differential inequality (1.3) with $c(x) \leq 0$ in a region $R_T = \{(x_1, x_2, \dots, x_n, t) \in R | t \leq T\}$ where R is a non-empty connected open set, and suppose that the coefficients of L are bounded. Suppose that the maximum of u in R_T is M and that it is attained at a point (x, t) of R_T. Thus if (y, s) is a point of R which can be connected to (x, t) by a path in R consisting only of horizontal segments and upward vertical segments, then $u(y, s) = M$.*

Hopf's boundary point lemma is another important classical tool in the study of PDEs that provides a criterion for determining the behaviour of solutions to certain types of elliptic or parabolic PDEs near the boundary of the domain. The lemma states that if one has a solution to some kinds of partial differential inequalities, then the normal derivative of the solution at that point is strictly positive.

It is often used to obtain estimates on the behaviour of solutions near the boundary, and to prove the existence and uniqueness of solutions to boundary value problems. The lemma is named after the German mathematician Eberhard Hopf, who first formulated it in the 1950s. In [18] we find the following version of the Hopf's boundary point lemma:

Theorem 1.8. *Let u be a solution to the parabolic inequality*

$$-u_t + Lu \geq 0$$

with L an elliptic linear differential operator with bounded coefficients such that $c(x) \leq 0$,in a domain E, and let $E_t = \{(x, s) \in E | s \leq t\}$. Suppose the maximum M of u is attained at a point $P = (x, t)$ on the boundary ∂E.
Assume that a sphere through P can be constructed which is in E such that

- *tangent to ∂E at P*
- *the set of point of its interior (y, s) such that $s \leq t$ lies in E_s,*
- *$u < M$ in its interior.*

Also, suppose that the radial direction from the centre of the sphere to P is not parallel to the t-axis.
Then, if $\frac{\partial}{\partial \nu}$ denotes any directional derivative in an outward direction from E_s , we have

$$\frac{\partial u}{\partial \nu} > 0$$

at P.

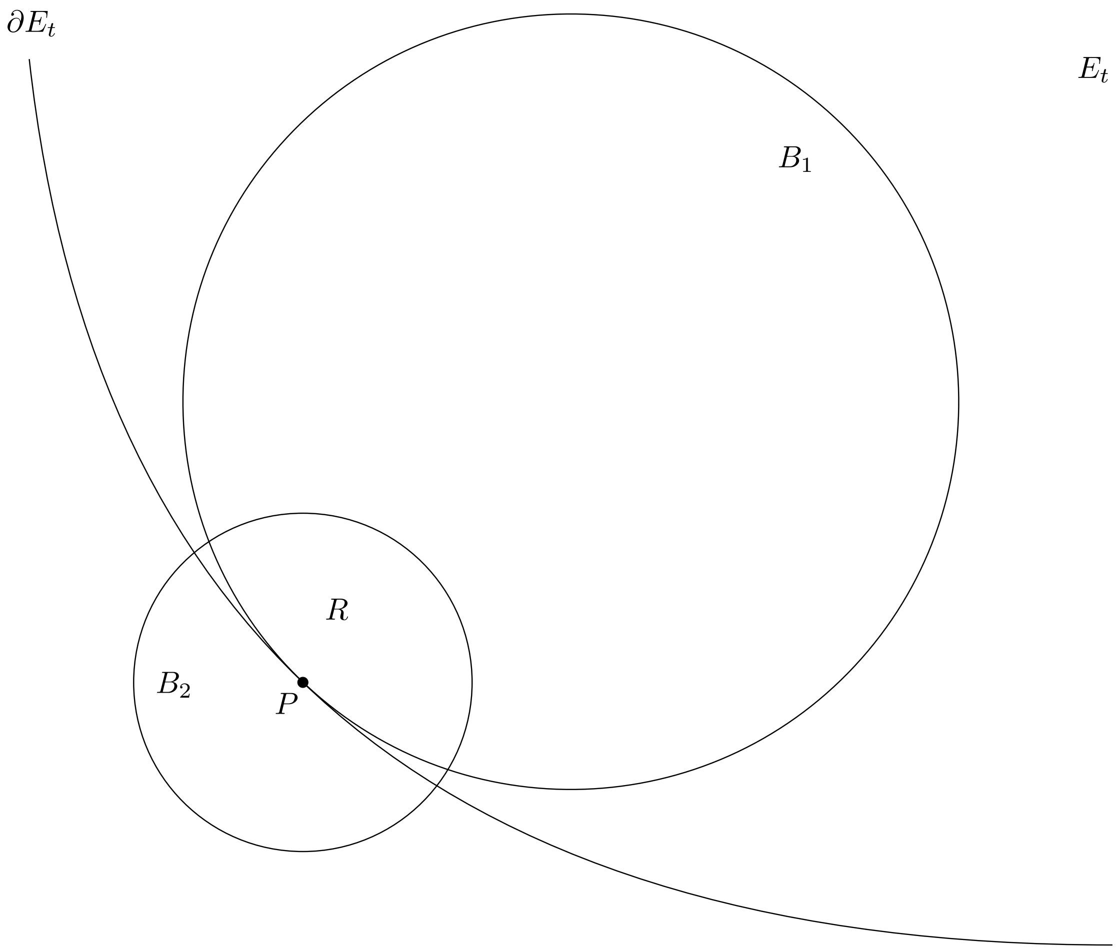

Figure 1.3: The setup in the proof at time t when there are two space dimensions. The region R also extends for times $s < t$ between the two spheres. The centre of B_1 can have time-coordinate different from t if the domain E is not "straight" in the time direction.

Proof. Let the sphere through P be B_1. We may construct a smaller sphere B_2 centred at P. Let now:

$$\begin{aligned} S_1 &= \partial B_1 \cap B_2 \cap E_t, \\ S_2 &= B_1 \cap \partial B_2 \cap E_t, \text{and} \\ S_3 &= B_1 \cap B_2 \cap \partial E_t = B_1 \cap B_2 \cap \{(x,s) \in E | s = t\}. \end{aligned}$$

The three sets satisfy $S_1 \cup S_2 \cup S_3 = \partial(B_1 \cap B_2 \cap E_t)$, we may call this region $R = B_1 \cap B_2 \cap E_t$. Without loss of generality, potentially taking a smaller sphere B_1, we may assume that $u < M$ on B_1 except at P. As $R \subset B_1$, we also get $u < M$ on R. We may thus conclude that:

- $u < M$ on R except at P
- $u \leq M - \delta$ on S_2 for a sufficiently small $\delta > 0$
- $u = M$ at P.

Now, let the centre of B_1 be $Q = (z, t_0)$ and let r be its radius. we can now introduce the function

$$v(y,s) = exp\left(-\alpha(s-t_0)^2 - \sum_{i=1}^{n} \alpha(y_i - z_i)^2\right) - exp\left(-\alpha r^2\right)$$

This function is such that $v(y,s) = 0$ if $(y,s) \in S_1$ - including $v(x,t) = 0$, as there the first term is $e^{-\alpha r^2}$, and $v(y,s) > 0$ in the interior of B_1.
Thus, in the region R, $v(y,s) \geq 0$ and has a minimum point at the boundary on (x,t), where $v(x,t) = 0$.

We can also compute Lv. After some calculation, we get that

$$Lv = 2\alpha e^{\left(-\alpha(s-t_0)^2 - \sum_{i=1}^{n} \alpha(y_i - z_i)^2\right)} [2\,\alpha\,(y-z)^t A(y-z) + \\ + \sum_{i}^{n} [b_i(y_i - z_i) + a_{i,i}] + (s-t)]$$

where A is the matrix of the $a_{i,j}$. In particular, one can choose an α large enough, so that $Lv > 0$ in $R \cup \partial R$.

We can thus introduce $w = u + \varepsilon v$. As both Lu and Lv are positive in R, $Lw > 0$ in R. We can also choose ε small enough so that $w < M$ on S_2. Also, as $v = 0$ on S_1, $w < M$ on S_1 except at P, and $w = M$ at P.

Therefore, we can apply the Strong Maximum Principle 1.7 to the region R to conclude that the maximum of w in R is attained at P. Therefore:

$$\frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu} + \varepsilon \frac{\partial v}{\partial \nu} \geq 0$$

But:

$$\frac{\partial v}{\partial \nu} = \nu \cdot n \frac{\partial v}{\partial R} = -2\nu \cdot n\alpha R e^{-\alpha R} < 0$$

Where n is the vector orthogonal to the sphere S_1. Therefore, one must have:

$$\frac{\partial u}{\partial \nu} > 0$$

as we wanted. □

Remark 1.9. If $c(x)$ is now just bounded, we can consider, instead of u, $v = ue^{-\lambda t}$, thus, by change of variables

$$-v_t + Lv - \lambda v \geq 0$$

whenever $-u_t + Lu \geq 0$, and we can chose λ large enough such that $c(x) - \lambda < 0$ and thus we can remove the hypothesis $c(x) \leq 0$ in both theorems when c is bounded.

1.4 Applying the maximum principle to non-linear PDEs

Following the approach in [18], it is easy to show that the maximum principle 1.7 and Hopf's boundary point lemma 1.8 can be applied - appropriately adapted - in some non-linear settings. Firstly, we must clarify what we mean by parabolic non-linear problem.

Definition 1.10. *A differential non-linear problem in the form*

$$Lu = F\left(t, x, v, \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_j}\right) - v_t = f(x,t) \tag{1.4}$$

given a smooth F is parabolic with respect to a function v *if for any real vector ξ*

$$\sum_{i,j=1}^{n} F_{ij}\xi_i\xi_j > 0$$

where F_{ij} are the derivatives of F with respect to $\frac{\partial^2 v}{\partial x_i \partial x_j}$.

Secondly, we remind the reader of the following generalized version of the theorem of the mean, a.k.a. Lagrange's theorem:

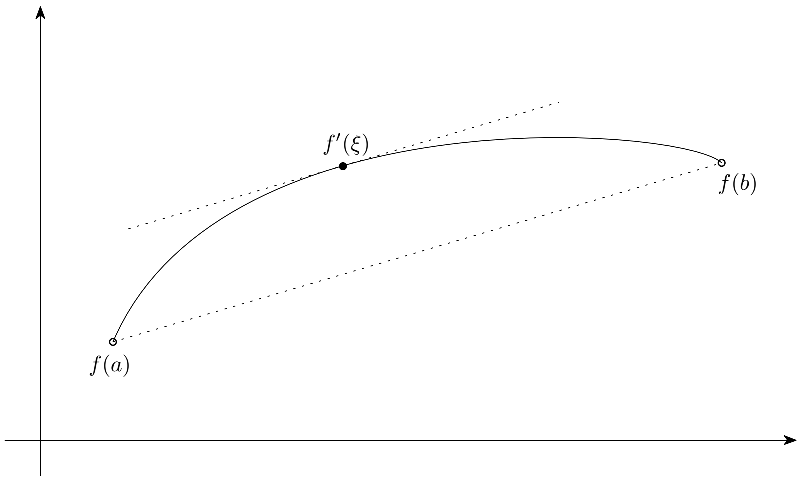

Figure 1.4: In one dimension, Lagrange's theorem states that given a smooth function f on an interval $[a,b]$, there exists a ξ such that $f(b)-f(a) = f'(\xi)(b-a)$.

Theorem 1.11 (Lagrange's theorem). *Given a convex open set $U \subseteq \mathbb{R}^n$ and a real function $F \in C^1(U)$, and given to points x, y in U, there exists a point z in the segment connecting x and y such that*

$$F(y) - F(x) = \langle \nabla F(z), (y-x) \rangle$$

Suppose that we have a solution to a non-linear parabolic problem v, i.e. v solves (1.4):

$$Lv = F\left(t, x, v, \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_j}\right) - v_t = f(t,x)$$

for a non-linear elliptic operator L in some region E, where we assume that $F(t, x, a, b_i, c_{i,j})$ is a given C^1 function. Suppose also that there is a w which is a solution of the corresponding differential inequality:

$$Lw = F\left(t, x, w, \frac{\partial w}{\partial x_i}, \frac{\partial^2 w}{\partial x_i \partial x_j}\right) - w_t \leq f(t,x)$$

One can then consider $u = v - w$, and by combining the above we get:

$$\left(F\left(t, x, v, \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_j}\right) - F\left(t, x, w, \frac{\partial w}{\partial x_i}, \frac{\partial^2 w}{\partial x_i \partial x_j}\right)\right) - u_t \leq 0$$

Now, we can apply Lagrange's theorem to F to get

$$\tilde{L}u = \left\langle \left(\frac{\partial F}{\partial a}, \frac{\partial F}{\partial b_i}, \frac{\partial F}{\partial c_{i,j}}\right)(\xi(t,x)), \left(u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}\right)\right\rangle - u_t \leq 0$$

for a fixed $\xi(t,x) = \theta(t,x)v(t,x) + (1-\theta(t,x))w(t,x)$, $\theta \in [0,1]$.

Thus, the difference u of two sub-solutions to a non-linear differential problem is a sub-solution to a (different) *linear* parabolic problem, as the derivatives of F and ξ do not depend on u (ξ can be chosen a-priori).

We can thus see that this new problem must be parabolic, because by definition 1.10 the matrix whose entries are the second order derivatives must be positive definite, and apply the maximum principle and the Hopf's boundary point lemma to u. This can allow us to state the following two results which we will be using later:

Proposition 1.12 (Maximum principle for parabolic non-linear differential equations)**.** *Suppose we have two solution v and w on the interval $[0,T]$ with different values at $t=0$ to the same non-linear differential equation (1.4) on an bounded open set Ω, parabolic at v, w and the functions between them. Suppose also that F is smooth on $\overline{\Omega}$. Then, if $v > w$ in the interior of Ω at $t=0$ and $v \geq w$ on $\partial\Omega$, $v > w$ for all $t \in [0,T]$ in the interior of Ω.*

Proof. $u = v - w \geq 0$ is a solution of a parabolic *linear* differential equation, where the term independent of u is bounded. Furthermore, if we take $c(x,t)$ it must be bounded by compactness. At $t=0$, $u > 0$ in the interior of Ω. If, at an interior point x, $v = w$ at a certain time $t = \tau$, $u(\tau, x) = 0$, and thus u is not constant. However, it attains minimum ($u = 0$) at an interior point, thus by Theorem 1.6 it must be constant, a contradiction. □

Proposition 1.13 (Hopf's boundary point lemma for parabolic non-linear differential equations)**.** *Suppose we have two solution v and w with different starting values to the same non-linear differential equation (1.4), parabolic at v, w and the functions between them, in a domain Ω. Suppose also that F is smooth on $\overline{\Omega}$. Let $u = v - w$ and suppose that the maximum of u is attained at the point P. Furthermore, assume that the conditions on the shape of the region Ω from theorem 1.8 hold. Then,*

$$\frac{\partial u}{\partial \nu}(P) > 0$$

where we take ν as the normal to $\partial\Omega$.

Proof. u is a solution of a parabolic *linear* differential equation, where $c(x,t)$ is bounded by compactness. We can then apply Theorem 1.8 to v (see also remark 1.9). □

Chapter 2

The Alexandrov Moving Planes Method

2.1 Reflections on spheres and hyperbolic spaces

In what follows, we will focus on manifolds embedded in spaces which have constant sectional curvature. Constant curvature manifolds are classified into three types based on the sign of the curvature:

- **Positive curvature**: *Spherical geometry*, where the curvature is positive and the manifold locally resembles a sphere (e.g., the standard sphere $\mathbb{S}^n$).
- **Zero curvature**: *Flat geometry*, where the curvature is zero and the manifold locally resembles Euclidean space $\mathbb{R}^n$.
- **Negative curvature**: *Hyperbolic geometry*, where the curvature is negative and the manifold locally resembles hyperbolic space $\mathbb{H}^n$.

As in [5], we will use the symbol $\mathbb{M}^n$ to indicate a Riemannian manifold that can be replaced by any one of $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{H}^n$: the n-dimensional sphere, Euclidean plane or Hyperbolic space respectively.

We will also use use the symbol $\mathbb{M}^n_+$ to indicate a Riemannian manifold that can be replaced by $\mathbb{S}^n_+$, $\mathbb{R}^n$ or $\mathbb{H}^n$: the n-dimensional hemisphere, Euclidean plane or Hyperbolic space, respectively.

We will not be considering other ambient spaces, which can be obtained through quotients from the three cases above.

Definition 2.1. $\mathbb{H}^n$ *is the* n-dimensional hyperbolic plane*. We can define it as the half space* $\{x \in \mathbb{R}^n | x_n > 0\}$ *with the Riemannian metric*

$$g_x = \frac{1}{x_n^2}\langle \cdot, \cdot \rangle$$

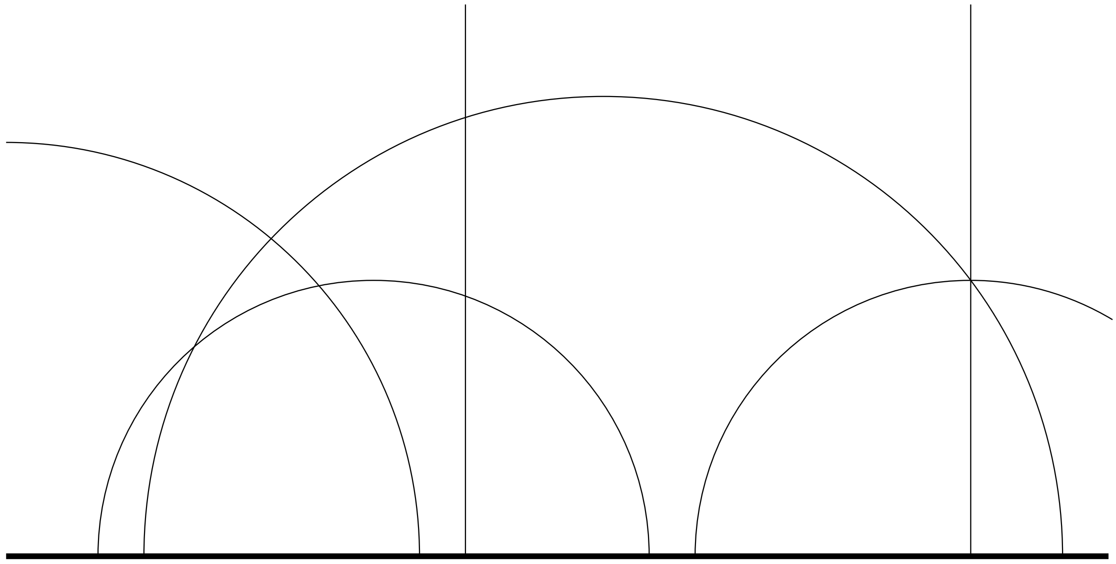

Figure 2.1: On the half-space model of $\mathbb{H}^n$, geodesics are half lines parallel to the x_n axis and circumferences with centre on the $x_n = 0$ hyperplane (see [15]).

where $\langle \cdot, \cdot \rangle$ *is the Euclidean dot product on* $\mathbb{R}^n$

Remark 2.2. This is not the only way we could define the hyperbolic space $\mathbb{H}^n$. Another alternative is taking the n-dimensional disc $D^n = \{x \in \mathbb{R}^n | \|x\| < 1\}$ with the Riemannian metric

$$g_x = \frac{4}{(1 - \|x\|^2)^2} \langle \cdot, \cdot \rangle$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean dot product on $\mathbb{R}^n$.

Remark 2.3. Let $\mathbb{S}^n \setminus \{P\}$ be the standard n-dimensional unitary sphere minus a point, with the standard induced Euclidean metric. Through stereographic projection it is isometric to $\mathbb{R}^n$ with metric:

$$g_x = \frac{4}{(1 + \|x\|^2)^2} \langle \cdot, \cdot \rangle$$

where $\langle \cdot, \cdot \rangle$ is again the Euclidean dot product on $\mathbb{R}^n$.

Remark 2.4. The models above have curvature ± 1. For other values of the curvature one has to consider the following Riemannian metrics

- for the half-space model of hyperbolic space:

$$g_x = \frac{R^2}{x_n^2} \langle \cdot, \cdot \rangle$$

- for the Poincaré ball model of hyperbolic space one has to take the disk with radius R and:

$$g_x = \frac{4R^4}{(R^2 - \|x\|^2)^2}\langle \cdot, \cdot \rangle$$

- For the stereographic projection of a sphere with radius R:

$$g_x = \frac{4R^4}{(R^2 + \|x\|^2)^2}\langle \cdot, \cdot \rangle$$

We will now define reflections in $\mathbb{M}^n$. This is relatively straightforward of S^n and a bit more involved for $\mathbb{H}^n$. The ingredients that we need are three: a geodesic, a family of hyperplanes associated to it, and reflections about those. The first one is easy, just take any point and any direction to get a geodesic. Then, at each point on the geodesic, there is an orthogonal totally geodesic hyperplane passing through that point: this is the family we consider. Finally, for each of these hyperplanes there is a reflection about it, which we can describe also mapping the hyperplanes into each other. While this could certainly be done in a more straightforward way compared to the approach we will follow, as we do not really need to know at this stage about what happens to the other planes in the family, and therefore we could just define a reflection about a plane and be satisfied. Using a heavier notation at this stage, however, will make it easier to introduce the moving planes method later.

We will now describe these choices for each ambient space in greater detail:

- On $\mathbb{R}^n$ we can choose any straight line as a geodesic, the hyperplanes orthogonal to it as the associated family, and use the usual reflections.

- On S^n, reflections are those induced by $\mathbb{R}^{n+1}$ when the fixed plane passes through the origin. Each one can be identified by vector orthogonal to the plane we chose. Each hyperplane through the origin in $\mathbb{R}^{n+1}$ defines a $(n-1)$-sphere through intersection with S^n. What we want to consider are reflections about n-planes through the origin in $\mathbb{R}^{n+1}$ orthogonal to a given geodesic of an immersed copy of S^n. To build an analogy between S^n and $\mathbb{H}^n$, we can parametrize each point in S^n_+ in the following way, which will be useful when dealing with both at the same time:

    - choose a point O and a direction $v \in T_O S^n$, and consider the geodesic γ_v such that $\gamma_v(0) = O$ and $\dot{\gamma_v}(0) = v$. Assume that γ is parametrised by arc-length.

  - Consider the $(n-1)$-sphere π_0 passing through $\gamma_v(0)$ and orthogonal to $\dot{\gamma_v}(0)$. One can rotate that sphere along the geodesic γ_v so that it touches each point in S^n_+. We will call the sphere passing through $\gamma_v(t)$ $\pi_{v,t}$. Each point x is in a unique $\pi_{v,t}$.
  - We can then assign to each point in S^n_+ a unique couple of coordinates (x,t), where x is the coordinate in $S^{n-1} \cap S^n_+$ when rotating it back to π_0, and t is the unique t such that the point is in $\pi_{v,t}$.

  Observe that a reflection about $\pi_{v,s}$ is $(x,t) \mapsto (x, 2s-t)$. This can be either taken as a definition (while being careful dealing with points in the hemisphere opposite to O having ambiguous coordinates) or as a consequence of considering the aforementioned reflection induced on S^n by $\mathbb{R}^{n+1}$ about a plane through the origin.

- On $\mathbb{H}^n$ we can also take a similar construction:
  - As a first step, choose a point a point $O \in \mathbb{H}^n$. $\mathbb{M}^n$ is a homogeneous space, so the construction does not depend on the choice we make.
  - Choose any direction in $v \in T_O\mathbb{H}^n$ and consider the geodesic $\gamma_v : \mathbb{R} \to \mathbb{H}^n$ satisfying $\gamma_v(0) = O$ and $\dot{\gamma_v}(0) = v$. Assume that γ is parametrised by arc-length.
  - Consider the hyperplane π_0 passing through $\gamma_v(0)$ and orthogonal to $\dot{\gamma_v}(0)$. Then consider the 1-parameter group of isometries of H^n such that $g_t(\gamma_v(0)) = \gamma_v(t)$ and such that the curves $t \mapsto g_t(x)$ are orthogonal to π_0 for each $x \in \pi_0$. This allows us to assign to each point in H^n coordinates (x,t) where $x \in \pi_0$ and $t \in \mathbb{R}$.
  - consider now any hyperplane π_t passing through $\gamma_v(t)$ and orthogonal to $\dot{\gamma_v}(t)$. The reflection fixing π_t will be the one given by the formula $(x,t) \mapsto (x, 2s-t)$.

  Through rotations and translations it is possible to assume without loss of generality that the point we chose is e_n and the direction is e_1 (assuming the half-space model in the definition above). We can do this because $\mathbb{H}^n$ is frame-homogenous (which means that we can move isometrically any point into any other and any direction into any other, therefore making any frame of reference identical to each other); for a proof that $\mathbb{H}^n$ is frame-homogenous, see proposition 3.9 in [15]. Then, the geodesic is the unit circle in the plane spanned by e_1 and e_n, and the (euclidean) hemispheres centred on the $x_n = 0$ plane and normal to the geodesic at one of its points are the totally geodesic hyperplanes (see figure 2.2). The reflection at t=0 coincides with the one for $\mathbb{R}^n$ about the same plane.

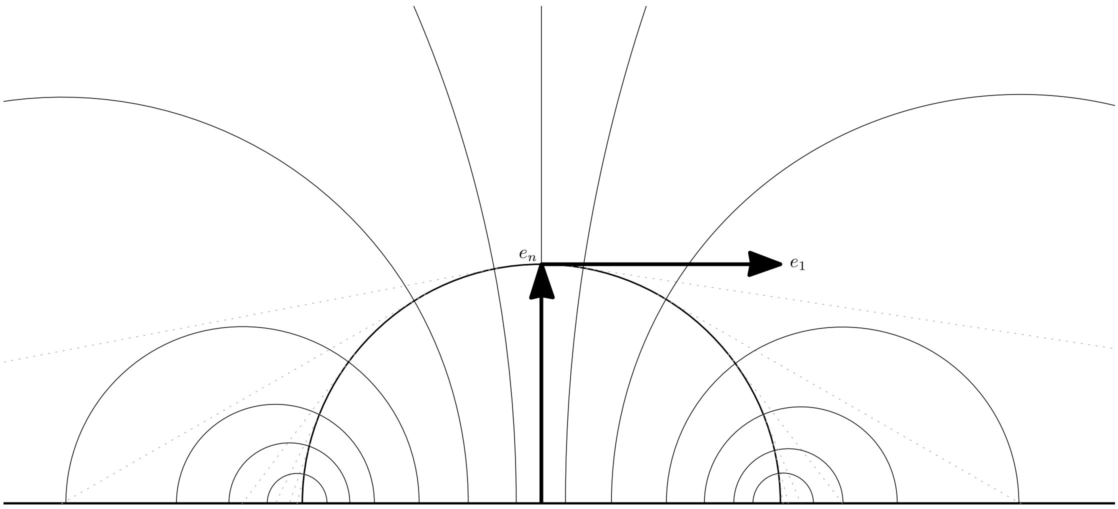

Figure 2.2: The moving hyperplanes in $\mathbb{H}^n$ when the point we chose is e_n and the direction is e_1, projected onto the plane spanned by these two vectors. The centres of the hemispheres are the intersection of the $x_n = 0$ plane with the line spanned by the tangent vector.

Remark 2.5. Another choice of direction on the hyperbolic plane allows one to write an explicit formula for the reflection about each hyperplanes, which may be more enlightening than the construction above on another important matter. One can repeat the construction, taking the point e_n and direction e_n. The geodesic is then the vertical x_n axis and the totally geodesic hyperplanes are therefore the euclidean spheres centred at the origin. Assume without loss of generality that curvature is ± 1, and take the sphere of radius r. We claim that the reflection about the sphere of radius r is the spherical inversion about the sphere. We remind the reader that a spherical inversion about a sphere of centre O is a transformation mapping a point P to the point P' on the (euclidean) line through O and P such that $\overline{OP} \cdot \overline{OP'} = r^2$. Therefore the point $(\underline{0}, a)$ on the geodesic is mapped to $(\underline{0}, \frac{r^2}{a})$ by the inversion. Computing the geodesic distance of these two points to $(\underline{0}, r)$:

$$\int_r^a \frac{1}{x} dx = [\ln x]_r^a = \ln a - \ln r$$
$$\int_{\frac{r^2}{a}}^r \frac{1}{x} dx = [\ln x]_{\frac{r^2}{a}}^r = \ln r - 2 \ln r + \ln a - \ln r = \ln a - \ln r$$

we see that the points are the equidistant from $(\underline{0}, r)$ and therefore the corresponding spheres/hyperplanes have to map to each other. On the other hand, it is an inversion, and inversions are orientation-reversing isometries in hyperbolic space. Please note however that, if we consider $I_r(P)$, for a fixed a point $P \in \mathbb{H}^n$, and let

r vary, it does not move along a geodesic unless it is on the geodesic generating the motion. Therefore, as the geodesic is the shortest path, taking r to be varying between R_0 and R_1

$$\mathrm{dist}(I_{R_0}(P), I_{R_1}(P)) \leq \mathrm{length}(I_r(P))$$

in general, with a strict inequality unless P is on the vertical geodesic. This is unlike the situation on the plane, where equality holds everywhere. The hyperplanes in the euclidean case are equidistant at all points from one another: this cannot be achieved in a curved setting.

Remark 2.6. The inequality

$$\mathrm{dist}(I_{R_0}(P), I_{R_1}(P)) \leq \mathrm{length}(I_r(P)) \tag{2.1}$$

where $I_{\cdot}(P) : t \mapsto (P, t)$ is also the path formed by points of the same space coordinate on the hyperplanes, holds also on S^n, because said paths are not geodesics in this case either.

2.2 The Moving Planes Method

The Moving Planes Method is a technique that can be used to prove radial symmetry of certain solutions to some differential equations. The method was originally introduced by Alexandrov to characterize the sphere as the only hypersurfaces with constant curvature (see for example [1]), and then used by Serrin on elliptic PDEs (see [20]) and Gidas-Ni-Nirenberg (see [10]).

To provide some justification and context to the next chapters, we here describe the method in $\mathbb{M}^{n+1}_+$.

Let $X : M^n \to \mathbb{M}^{n+1}_+$ be a hypersurface in a constant curvature ambient space. If the ambient space is a sphere, X must be contained in a hemisphere to avoid issues with multiple self-intersections. Assume also that $X = \partial\Omega$ for a bounded domain Ω in $\mathbb{M}^{n+1}_+$.

- As a first step, choose a point a point $O \in \mathbb{M}^n$. As $\mathbb{M}^n$ is a homogeneous space, so the construction does not depend on the choice we make. Without loss of generality, we choose the origin in $\mathbb{R}^n$, e_n in $\mathbb{H}^n$ and the north pole in S^n.

- Choose any direction in $v \in T_O\mathbb{M}^n$ and consider the geodesic $\gamma_v : I \to \mathbb{M}^n$ satisfying $\gamma_v(0) = O$ and $\dot{\gamma_v}(0) = v$. Assume that γ is parametrised by arclength. Here $I = \mathbb{R}$ if $\mathbb{M}^n_+$ is flat or hyperbolic, and $I = (-\frac{\pi}{2}, \frac{\pi}{2})$ if $\mathbb{M}^n_+$ is a hemisphere.

- Consider the hyperplanes $\pi_{v,s}$ passing through $\gamma_v(s)$ and orthogonal to $\dot{\gamma_v}(s)$.

The method consists of reflecting the part of X "below" the hyperplane into the top part, and using properties of both copies of the hypersurface together to prove some statement about the non-reflected hypersurface.

To make this more precise, we can define:

$$X_{v,s} = \{p \in X \mid p \in \pi_{v,t} \text{ for some } t < s\}$$

We will use the notation $X^{\pi}_{v,s}$ to indicate the reflection of $X_{v,s}$ about $\pi_{v,s}$.

Finally, we define:

$$\begin{aligned} m_v &= \sup\left\{s \in I \mid X^{\pi}_{v,s} \subset \Omega \text{ for every } t < s\right\} \\ &= \sup\left\{s \in I \mid X \cap X^{\pi}_{v,s} = \emptyset \text{ for every } t < s\right\} \end{aligned}$$

the last time at which the hypersurface and its reflection do not touch internally. Please note that at m_v the two surfaces are tangent at some point, which can be either in the interior of X, or on the boundary. The hyperplane π_{v,m_v} is the critical hyperplane.

2.3 The Alexandrov soap-bubble theorem

To give some justification to the method we described, we will outline the proof an important result that uses this method, the so-called Alexandrov soap bubble theorem. This section will gloss over some technical matters which can be made more precise with some effort. For more details, see [5].

Theorem 2.7. *The only C^2-regular connected hypersurfaces embedded in $\mathbb{M}^{n+1}_+$ and such that the mean curvature is constant are the distance spheres.*

In [5] this theorem is proved more generally: let H_X be a C^2 function of the ordered principal curvatures $H_X = f(\kappa_1, \ldots, \kappa_n)$, and

$$f : \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \leq x_2 \leq \cdots \leq x_n\} \to \mathbb{R}$$

is such that

$$f(x) > 0 \text{ if } x_i > 0 \text{ for every } i = 1, \ldots, n$$

and it is concave on the component of $\{x \in \mathbb{R}^n \mid f(x) > 0\}$ containing $\{x \in \mathbb{R}^n \mid x_i > 0\}$. Then the following more general theorem holds:

Theorem 2.8. *The only C^2-regular connected hypersurfaces embedded in $\mathbb{M}^{n+1}_+$ and such that H_X is constant are the distance spheres.*

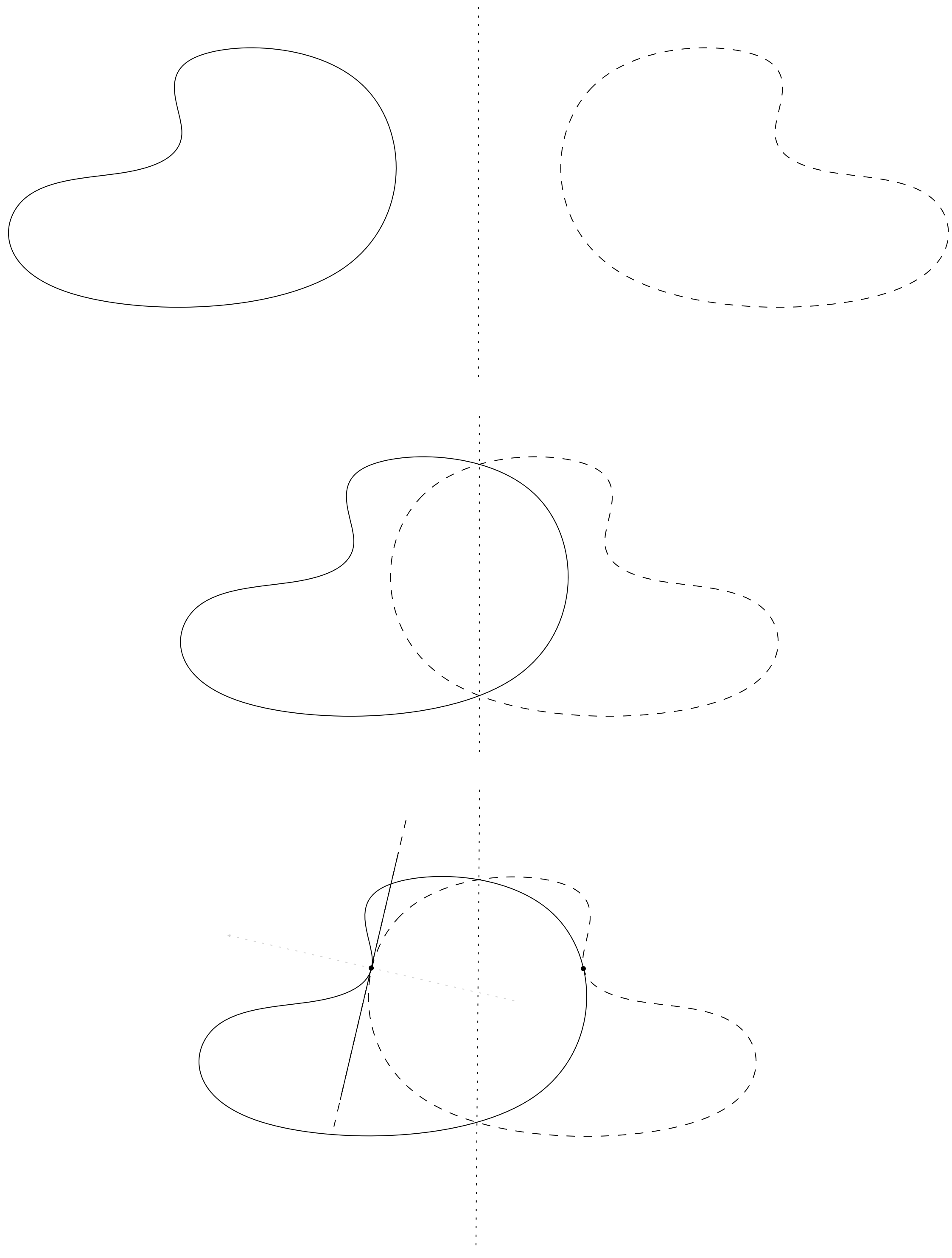

Figure 2.3: The method of the moving planes in $\mathbb{R}^2$. In the figure, the "plane" is centred for easier pagination, but the hypersurface is fixed and the plane varies. The third figure shows the critical time m_v when the hypersurface and its reflection touch internally.

The following proposition holds:

Proposition 2.9. *Let $X = \partial\Omega$ be a C^2-regular, connected, closed hypersurface embedded in $\mathbb{M}_+^{n+1}$, where Ω is a bounded domain. Assume that for every geodesic $\gamma : \mathbb{R} \to \mathbb{M}^{n+1}$ there exists a hyperplane orthogonal to γ such that X is symmetric about π. Then X is a distance sphere about its centre of mass O, i.e. the unique minimum of*

$$P_\Omega(x) = \int_\Omega d(x,a)^2 da$$

Its proof is in [5] and is omitted here. We now move on to the proof of Theorem 2.7:

Sketch of proof. Assume that X is a manifold with constant mean curvature. We want to show that, for any point O and for every direction $v \in T_O\mathbb{M}^{n+1}$, X is symmetric about a plane perpendicular to the geodesic $\exp_O(tv)$. We put ourselves in the hypothesis of the method of the moving planes described in section 2.2. At m_v, $X \cap X^\pi_{v,m_v}$ is non-empty by definition of m_v and therefore it is closed in X^π_{v,m_v}. We want to show that $X \cap X^\pi_{v,m_v}$ is an open set in X^π_{v,m_v}.

Let $p \in X \cap X^\pi_{v,m_v}$. As the two manifolds are tangent

$$T_pX = T_pX^\pi_{v,m_v}.$$

By Theorem 1.5, we can represent both manifold as the Euclidean graph of two functions C^2 functions u and $\tilde{u}$ defined in a neighbourhood of p inside the tangent space. Consider now the differential equation:

$$K(u(x)) = K(\tilde{u}(x)) = \text{ constant}$$

where K is the mean-curvature operator. As X is a manifold with constant mean curvature, it holds everywhere. Looking at equation (1.1), we see that, given that the principal curvature are a function of the h_{ij}, the operator K is an elliptic operator. Reasoning exactly like we did in section 1.4 for parabolic differential equations, we see that $u - \tilde{u}$ is the solution of a linear elliptic differential equation of the form $L(u - \tilde{u}) = 0$, with $u(p) = \tilde{u}(p) = 0$.

Without loss of generality, we can assume that, in the neighbourhood where we defined u and $\tilde{u}$, $u - \tilde{u} \geq 0$.

If p is an interior point in X^π_{v,m_v}, we can then apply the maximum principle for elliptic equations (see [7] and [18]) and obtain that $u = \tilde{u}$ in the neighbourhood.

Otherwise, if $p \in \pi_{v,s}$, $\nabla u(p) = \nabla\tilde{u}(p) = 0$ and one can apply Hopf's boundary point lemma for elliptic equations (see [7] and [18]) to conclude again that $u = \tilde{u}$ in the neighbourhood.

Therefore, the whole neighbourhood is in $X \cap X^{\pi}_{v,m_v}$, hence the intersection is open, as every point p is contained in an open ball. This proves that X is symmetric about π_{v,m_v}. The theorem is then consequence of Proposition 2.9 □

Remark 2.10. To prove the theorem in the aforementioned more general setting, one would need to only prove that the differential equation $H_X(u(x)) = \text{constant}$ is an elliptic equation. Its ellipticity is a standard result in Geometric Analysis, and can be found in many other works. It is also equivalent to equation (3.2) being parabolic (see section 3.2).

Chapter 3

The Chow-Gulliver Critical Planes Result

The main result we want to establish in this chapter is theorem 3.8, a result about critical hyperplanes when applying the method of the moving planes to solutions of a large class of non-linear parabolic partial differential equations, and whose proof is somewhat similar to Theorem 2.7.

We will first describe the differential equations we are analysing, then prove that they are parabolic, and finally prove the theorem.

After this, we will include some corollaries of the result and an application to find some estimates for the gradient of the support function and the gradient of the radial function.

Finally, another application of the result is presented, showing that the condition of *coming out of a point* is particularly rigid on ancient expansive flows.

3.1 Class of Equations we analyze

We consider manifolds M^n embedded in $\mathbb{R}^{n+1}$, i.e. there is an embedding $X_0 : M^n \to \mathbb{R}^{n+1}$ parametrizing the hypersurface $X_0(M^n)$.

Let $F : \{(\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n | \kappa_1 \leq \cdots \leq \kappa_n\} \to \mathbb{R}$ be a C^1 function satisfying:

$$\frac{\partial F}{\partial \kappa_i} > 0 \text{ for all } i = 1, \ldots, n \tag{3.1}$$

and consider the evolution equation

$$\begin{cases} \dfrac{\partial X_t}{\partial t} = -F(\kappa_1(x), \ldots, \kappa_n(x))\nu \\ X(0) = X_0 \end{cases} \tag{3.2}$$

where ν is the outward normal to $X_t(M^n)$ at the point $X_t(x)$ and $\kappa_1 \leq \cdots \leq \kappa_n$ are the principal curvatures at $X_t(x)$.

Remark 3.1. Condition (3.1) does not need to hold for every possible choice of κ_i, it just needs to hold at all points of the solution to (3.2).

3.2 Parabolicity of the differential equation (3.2)

The condition (3.1) will guarantee that equation (3.2) is a parabolic equation. This may be confusing, as (3.2) does not make it obvious how to apply definition 1.10.

In order to justify that this is a parabolic non-linear partial differential equation we can also try to understand how it behaves "close to a solution" in the solutions space. We want to prove that very close to any solution, "moving in any direction", the change in the equation is always a parabolic PDE. This will then tell us that our equation is parabolic, and that the theorems that apply to solutions of parabolic partial differential equations apply to our equation as well. To do so, we are going to "linearise" the differential equation about a solution.

On a more technical note, what one finds is that the equation is degenerate, as there is a scalar product that "erases" what happens in the tangential direction. It is indeed evident that diffeomorphisms tangential to the manifold do not influence the flow. This issue can be solved using a result from Hamilton (see [11] and related papers) or reformulating the problem as a scalar equation. We will not delve into the details and keep the discussion at an intuitive level. A more systematic approach proving that the differential equation is parabolic can be found in [9].

Following the approach in [12], as F is a symmetric function in the principal curvatures, we may interchangeably take F to be a function of the Weingarten map tensor or of the second fundamental form, and thus we get:

$$\frac{\partial X_t}{\partial t} = -F(h_{ij}(X_t))\nu$$

To understand the behaviour close to a solution, we can substitute in our equation X_t with a $X_t + \varepsilon u_t$ to get:

$$\frac{\partial X_t}{\partial t} + \varepsilon \frac{\partial u_t}{\partial t} = -F(h_{ij}(X_t + \varepsilon u_t))\nu_{(X_t+\varepsilon u_t)} \tag{3.3}$$

where we mean that $\nu_{(X_t+\varepsilon u_t)}$ is the normal to the perturbed immersion. We are interested in the behaviour of this equation for a small ε. This equation when taking the limit for $\varepsilon \to 0$ is the so-called linearisation of the PDE; we want this PDE to be a parabolic equation to apply our results.

We can use the Weingarten equation (1.1) to write the RHS explicitly:

$$
\begin{aligned}
h_{ij}(X_t + \varepsilon u_t) &= -\left\langle v, \nu_{(X_t+\varepsilon u_t)} \right\rangle \text{ where} \\
v^\alpha &= \frac{\partial^2 X_t^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial X_t^\alpha}{\partial x^k} + \overline{\Gamma}_{\beta\delta}^\alpha \frac{\partial X_t^\beta}{\partial x^i}\frac{\partial X_t^\delta}{\partial x^k} + \\
&\quad + \varepsilon \left(\frac{\partial^2 u_t^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u_t^\alpha}{\partial x^k} + \overline{\Gamma}_{\beta\delta}^\alpha \left(\frac{\partial X_t^\beta}{\partial x^i}\frac{\partial u_t^\delta}{\partial x^k} + \frac{\partial u_t^\beta}{\partial x^i}\frac{\partial X_t^\delta}{\partial x^k} \right) \right) + \\
&\quad + \varepsilon^2 \left(\overline{\Gamma}_{\beta\delta}^\alpha \frac{\partial u_t^\beta}{\partial x^i}\frac{\partial u_t^\delta}{\partial x^k} \right) \\
v^\alpha &= w + \varepsilon \left(\frac{\partial^2 u_t^\alpha}{\partial x^i \partial x^j} + \text{lower order terms} \right) + o(\varepsilon)
\end{aligned}
$$

where $h_{ij}(X_t) = -\langle w, \nu_{X_t} \rangle$.

Putting it all together in the first line:

$$
\begin{aligned}
h_{ij}(X_t + \varepsilon u_t) &= -\left\langle w + \varepsilon \left(\frac{\partial^2 u_t}{\partial x^i \partial x^j} + \text{lower order terms} \right) + o(\varepsilon), \nu_{(X_t+\varepsilon u_t)} \right\rangle \\
&= \left\langle w, \nu_{(X_t+\varepsilon u_t)} \right\rangle - \varepsilon \left\langle \left(\frac{\partial^2 u_t}{\partial x^i \partial x^j} + \text{lower order terms} \right), \nu_{(X_t+\varepsilon u_t)} \right\rangle + o(\varepsilon) \\
&= h_{ij}(X_t) + \left\langle w, \nu_{(X_t+\varepsilon u_t)} - \nu_{X_t} \right\rangle - \varepsilon H_{ij} + o(\varepsilon) \\
&= h_{ij}(X_t) - \varepsilon H_{ij} + o(\varepsilon)
\end{aligned}
$$

Were on the last step we are using the fact that $\nu_{(X_t+\varepsilon u_t)} - \nu_{X_t} = O(\varepsilon)$, and as this gets smaller the component of w parallel to the difference also is $O(\varepsilon)$, as w is parallel to ν_{X_t}. We can then expand F in the RHS of the equation (3.3) to the first order, as it is a C^1 function:

$$
\begin{aligned}
\cancel{\frac{\partial X_t}{\partial t}} + \varepsilon \frac{\partial u_t}{\partial t} &= -\left(\cancel{F(h_{ij}(X_t))} + \varepsilon \langle DF, H \rangle + o(\varepsilon) \right) \nu_{(X_t+\varepsilon u_t)} \\
\varepsilon \frac{\partial u_t}{\partial t} &= \varepsilon \left(\frac{\partial F}{\partial h_{ij}} g^{ik} g^{jl} \left\langle \frac{\partial^2 u_t}{\partial x^k \partial x^l}, \nu_{(X_t+\varepsilon u_t)} \right\rangle + \text{lower order terms} \right) \nu_{(X_t+\varepsilon u_t)} + o(\varepsilon) \\
\frac{\partial u_t}{\partial t} &= \frac{\partial F}{\partial h_{ij}} g^{ik} g^{jl} \left\langle \frac{\partial^2 u_t}{\partial x^k \partial x^l}, \nu_{(X_t+\varepsilon u_t)} \right\rangle \nu_{(X_t+\varepsilon u_t)} + \text{lower order terms} + o(1)
\end{aligned}
$$

Letting $\varepsilon \to 0$, we get to:

$$
\frac{\partial u_t}{\partial t} = \frac{\partial F}{\partial h_{ij}} g^{ik} g^{jl} \left\langle \frac{\partial^2 u_t}{\partial x_k \partial x_l}, \nu \right\rangle \nu + \text{lower order terms}
$$

For the second order term to be positive definite, then,

$$\left(\frac{\partial F}{\partial h_{ij}}\right)_{i,j}$$

must be positive definite. Or equivalently, as the principal curvatures are the eigenvalues of the matrix $(h_{ij})_{i,j}$,

$$\frac{\partial F}{\partial \kappa_i} > 0$$

Remark 3.2. While in this chapter we are using the standard metric of $\mathbb{R}^n$, the calculation above is valid using any other metric. We will take advantage of this fact in the next chapter.

3.3 An existence result

In [12] one finds the following comforting existence result for the class of equations we are analysing under very broad hypothesis. The same paper also includes a proof of the result with some more restrictive hypotheses. A complete proof is quite more involved. A much more extended analysis of the problem of the existence of solution to the equation can be found in [9].

Theorem 3.3 (Short term existence of a solution for (3.2))**.** *Suppose $X_0 : M^n \to \mathbb{R}^{n+1}$ is a smooth, closed hypersurface in $\mathbb{R}^{n+1}$, such that (3.1) holds at all points in X_0, i.e. for all the values of the principal curvatures κ_i realized at some point on X_0. Then, (3.2) has a smooth solution, at least on some short time interval $[0, T)$, $T > 0$.*

3.4 Local representation as a graph of a solution of (3.2)

As a first step, from the Corollary 1.5, we can establish the following:

Theorem 3.4 (Local representation as a graph of a solution of (3.2))**.** *Let X^n be a submanifold $X^n \subset \mathbb{R}^{n+1}$ and let $F : X^n \times (0, T) \to \mathbb{R}^{n+1}$ be a solution of (3.2). Also let $t \in (0, T)$ and $x \in F(X^n, t)$. Then there exists a neighbourhood of (x, t), $U \subset F(X^n \times (0, T))$, and a smooth function $f : T_xF(X^n, t) \times (0, T) \to \mathbb{R}$ such that any $(x_0, t_0) \in U$ can be expressed as*

$$x_0 = p + f(p, t_0)\nu$$

where ν is a vector normal to $T_xF(X^n, t)$, for an appropriate point $p \in T_xF(X^n, t)$.

Proof. We can consider the image of $F : X^n \times (0,T) \to \mathbb{R}^{n+1}$ as a manifold in $\mathbb{R}^{n+2}$ by considering $G := (F(x,t),t)$. Moreover $\frac{\partial G_t}{\partial e_j} \equiv 0$ for all possible vectors of the canonical basis of $\mathbb{R}^{n+1} \times \mathbb{R}$ except for the one corresponding to the time coordinate, where it is 1. Also, $\frac{\partial G_i}{\partial e_j} \equiv \frac{\partial F_i}{\partial e_j}$ and the first n coordinates of $\frac{\partial G}{\partial t}$ form a vector normal to T_xX^n by (3.2). Thus, the tangent space of $\mathrm{Im}(G)$ is $T_xX^n \times \{0\} \oplus \mathrm{span}\langle(\nu,1)\rangle$ and we can apply corollary 1.5 to $\mathrm{Im}(G)$ to get a function $\tilde{f}$ such that

$$(x_0,t) = [(p,0) + (s\nu,s)] + \tilde{f}(p,s)(\nu, -\|\nu\|)$$

as $(\nu,-\|\nu\|)$ is the vector orthogonal to $T_xX^n \times \{0\} \oplus \mathrm{span}\langle(\nu,1)\rangle$. Let $\sigma_p : I \subset \mathbb{R} \to \mathbb{R}$ be the function that associates to t the appropriate s in the expression above. Projecting to the first $n+1$ coordinates and calling $f(p,t) = \sigma_p(t) + \tilde{f}(p,\sigma_p(t))$ one gets:

$$x_0 = p + f(p,t)\nu$$

which is our thesis, as long as $\sigma_p(t)$ is smooth. This is indeed the case, as the graph function $\Gamma_f : x \mapsto (x, f(x))$ is smooth for any smooth function f and has a smooth inverse (and thus, the inverse $(x_0,t) \mapsto ((p,0) + (s\nu,s))$ is smooth). $\square$

Remark 3.5. One can show through direct calculation (see [16], Exercise 1.1.2) that, if an immersed hypersurface $\phi : M \to \mathbb{R}^{n+1}$ is locally the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$ (i.e., locally, $(x,f(x)) = \phi$), then:

$$\begin{aligned} g_{ij} &= \delta_{ij} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} \\ \nu &= -\frac{(\nabla f, -1)}{\sqrt{1+|\nabla f|^2}} \\ h_{ij} &= -\frac{H_{ij}}{\sqrt{1+|\nabla f|^2}} \end{aligned}$$

Where the matrix $(H_{ij})_{ij}$ is the hessian of f. Thus, if one considers the principal curvatures, they are closely related to the eigenvalues of the hessian of f. This could also be used to do the calculation in section 2.2 but it may be harder to extend it to constant curvature spaces.

3.5 The Moving Planes Method and the Chow result

We will now present the Chow-Gulliver result, roughly following paragraph 2 of the original paper [4]. Suppose that we have a hypersurface embedded in $\mathbb{R}^{n+1}$

evolving according to equation (3.2). For a fixed time t, we can apply the method of the moving planes as described in section 2.2: we can take parallel hyperplanes $\pi_{v,s}$ orthogonal to v intersecting X, and consider the reflection $X^{\pi}_{v,s}$. There will be a hyperplane π_{v,m_v} where X and $X^{\pi}_{v,s}$ are tangent. As the hypersurface evolves, we may wonder how this critical threshold changes over time. If it behaves in a predictable way, we can hope to use the Moving Planes Method on the evolving manifold, otherwise, if the critical plane moves back and forth multiple times, it may be a hopeless endeavour. In the next section, we are going to prove a result in this general direction, to show a form of "regularity" in this sense. First, however, we need to introduce a marginally stricter definition for the concept of "reflecting inside itself".

Let π be a hyperplane in $\mathbb{R}^{n+1}$. We may assume π orthogonal to a unit vector $v \in \mathbb{R}^{n+1}$, i.e. $\langle x, v\rangle = C$ for all $x \in \pi$ for some constant C. In our notation for the method of moving planes, assuming we pick the origin as a starting point, this means that $\pi = \pi_{v,C}$.

Then, $\mathbb{R}^{n+1}$ is divided by π into two half-spaces, which we will name

$$H^+(\pi) = \{x \in \mathbb{R}^{n+1} : \langle x, v\rangle > C\} = \bigcup_{s>C} \pi_{v,s} \text{ and}$$
$$H^-(\pi) = \{x \in \mathbb{R}^{n+1} : \langle x, v\rangle < C\} = \bigcup_{s<C} \pi_{v,s}.$$

Definition 3.6. *We say* we can reflect $X : M^n \to \mathbb{R}^{n+1}$ strictly with respect to π *if both:*

- $X^\pi \cap H^-(\pi) \subset \operatorname{int}(X) \cap H^-(\pi)$ *where* X^π *is the reflection of* X *about* π *and* $\operatorname{int}(X)$ *is the region inside* X*.*
- $V \notin T_xM$ *for all* $x \in M^n \cap \pi$

This fundamentally means that the reflection of one of the halves of X on the other side of π is contained in the region inside M^n and the tangent spaces of X and of the half-reflection do not form a ninety degree angle with π, at all points on $\pi \cap X$. As the two tangent spaces are one the reflection of the other, this means that they do not coincide.

Definition 3.7. *We say* we can reflect $X : M^n \to \mathbb{R}^{n+1}$ strictly up to (π, v) *if we can reflect* M^n *strictly with respect to* $\pi_{v,s}$ *for all hyperplanes* $\pi_{v,s}$ *such that* $s < C$*.*

The key idea of the main result in the next section is as follows: suppose we have an embedded smooth hypersurface X evolving according to (3.2) and a fixed hyperplane π, intersecting X. Suppose that, at some time t, X and X_π touch

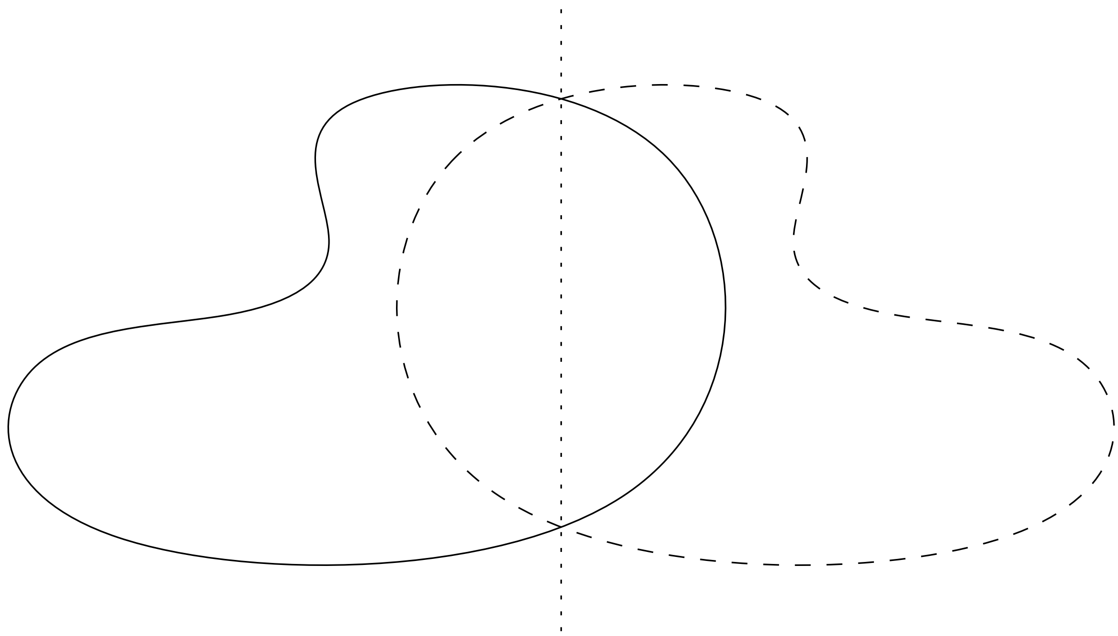

Figure 3.1: Example: We can reflect $X : M^n \to \mathbb{R}^{n+1}$ strictly with respect to π

outside of π. We can consider X and X_π as local graphs over the same hyperplane π, and we can show that these function evolve according to the same differential equation. Using the strong maximum principle and the Hopf boundary point lemma, then, one can conclude that the two functions coincide, and have been coinciding up until that point. We can then conclude that if X and X_π only touch in $X \cap \pi$ at the beginning of the evolution, then they will never touch elsewhere.

3.6 The Chow-Gulliver result

The main theorem is the following:

Theorem 3.8 (Chow-Gulliver). *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a C^2 solution to equation (3.2). Then, if we can reflect $X(M^n, 0) = X_0$ strictly with respect to π, then for all $t \in [0,T)$ we can reflect $X(M^n, t) = X_t$ strictly with respect to π.*

Proof. By contradiction, suppose that there is a time t such that the thesis is false, and that it is the smallest such t. Then, for all $\tau \in [0,t)$, $X_{\tau,\pi} \cap H^-(\pi) \subset \mathrm{int}(X_\tau) \cap H^-(\pi)$; the unit vector orthogonal to π, V, is such that $V \notin T_x X_\tau$ for all $x \in X_\tau \cap \pi$ and $\tau \in [0,t)$; and either of the conditions fails at t, i.e. either:

(i) $X_{t,\pi} \cap H^-(\pi) \cap X_t \neq \emptyset$

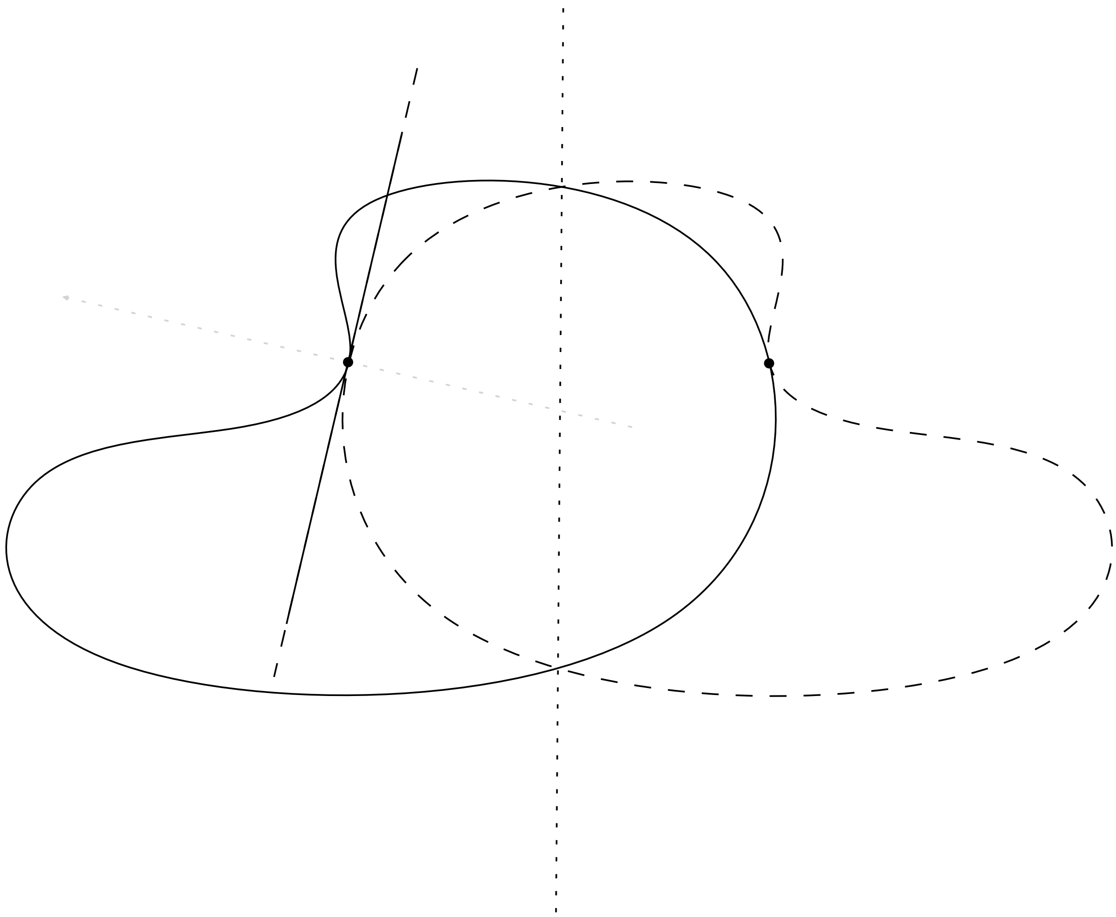

Figure 3.2: Example: We cannot reflect $X : M^n \to \mathbb{R}^{n+1}$ strictly with respect to π, because there is an interior contact

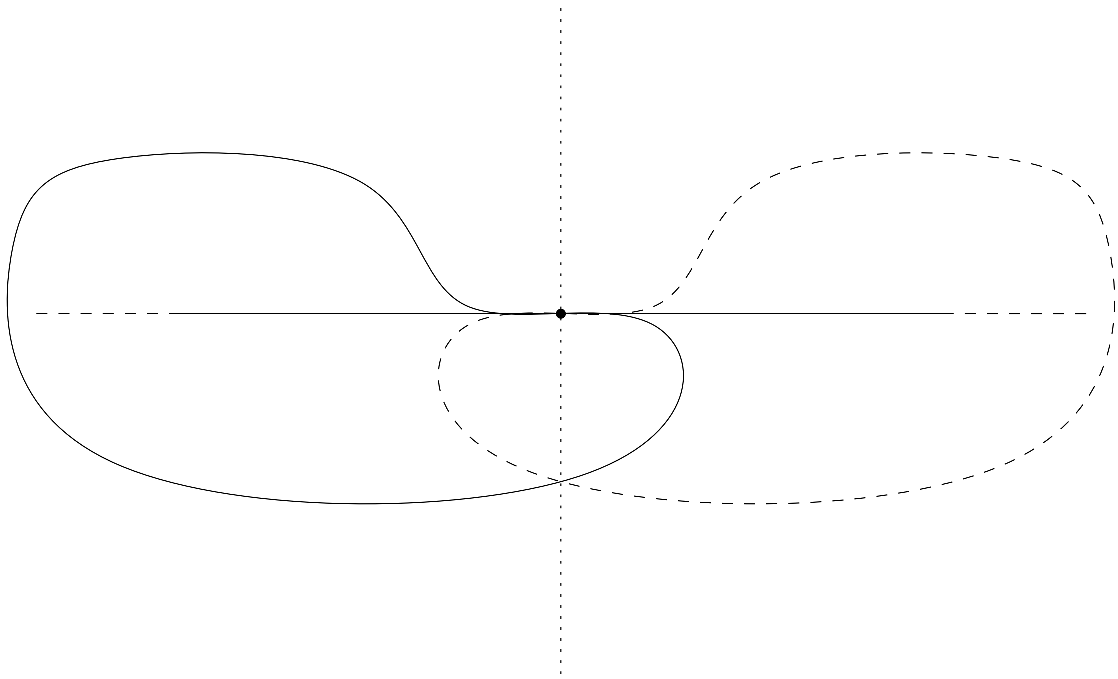

Figure 3.3: Example: We cannot reflect $X : M^n \to \mathbb{R}^{n+1}$ strictly with respect to π, because the tangent spaces coincide at a point about which we are reflecting

(ii) $V \in T_x X_t$ for some $x \in \pi$.

(i) Suppose the first case is true. Then, there exists $x_0 \in X_{t,\pi} \cap H^-(\pi) \cap X_t$ such that at x_0 the two manifolds are tangent.
We can take a neighbourhood of $(x_0, t) \in X_t \times \mathbb{R}$ such that both $X_{t,\pi}$ and X_t are graphs over $T_{x_0} X_t$ by 3.4.
We can explicitly write the functions $f : U \times (t-\varepsilon, t+\varepsilon) \to X_t$, where $U \subset T_{x_0} X_t$, and the corresponding f_π for $X_{t,\pi}$. We can also write

$$
\begin{aligned}
f &: (x,t) \mapsto x + \tilde{f}(x,t)\nu \\
f_\pi &: (x,t) \mapsto x + \tilde{f}_\pi(x,t)\nu
\end{aligned}
$$

for appropriate functions $\tilde{f} : U \times (t-\varepsilon, t+\varepsilon) \to \mathbb{R}$ and $\tilde{f}_\pi : U \times (t-\varepsilon, t+\varepsilon) \to \mathbb{R}$, where ν is a fixed unit vector normal to $T_{x_0} X_t$. $\tilde{f}$ and $\tilde{f}_\pi$ are solutions to the same second order PDE, which is parabolic by what was discussed in paragraph 3.2, hence we can apply Proposition 1.12 to conclude that $\tilde{f} \equiv \tilde{f}_\pi$, and thus $X_{t,\pi}$ and X_t coincide in a neighbourhood of (x,t), a contradiction as we assumed that t is the first t where the flows touch.

(ii) Suppose instead that $V \in T_x X_t$ for some $t \in [0,t)$ and some $x \in X_t \cap \pi$. Then $T_x X_t = T_x X_{t,\pi}$ and in a neighbourhood of (x,t) both X_t and $X_{t,\pi}$ are graphs of two smooth functions over T_xX_t by 3.4, i.e. again

$$\begin{aligned} f &: (x,t) \mapsto x + \tilde{f}(x,t)\nu \\ f_\pi &: (x,t) \mapsto x + \tilde{f}_\pi(x,t)\nu \end{aligned}$$

Moreover, in $\overline{H^-(\pi)}$, $f_\pi \geq f$, because $M^n_\pi \cap H^-(\pi) \subset \mathrm{int}(M^n) \cap H^-(\pi)$. Finally, $f(x,t) = f_\pi(x,t)$, hence $f_\pi - f(x,t) = 0$, and thus (x,t) is a minimum point on the boundary for $f_\pi - f$. Also, we must have

$$\frac{\partial f}{\partial V}(x,t) = \frac{\partial f_\pi}{\partial V}(x,t)$$

because the graphs are both tangent to T_xX_t, and V here is the outward pointing normal to the boundary by definition of the reflection. Thus,

$$\frac{\partial(f - f_\pi)}{\partial V}(x,t) = 0$$

But we must have

$$\frac{\partial(f - f_\pi)}{\partial V}(x,t) > 0$$

at a minimum on the boundary by Proposition 1.13, a contradiction. □

3.7 Some corollaries of the result

In this section we collect a number of corollaries to the main result above. This first one is an immediate consequence of the main result:

Corollary 3.9. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a C^2 embedded solution to equation (3.2). Then, if we can reflect X_0 strictly up to $(\pi_{v,C}, v)$, for all $t \in [0,T)$ $v \notin T_xX_t$ for all $x \in X_t \cap \overline{H^+(\pi)}$. In particular, $X_t \cap \overline{H^+(\pi)}$ is a graph over π for all $t \in [0,T)$.*

Another consequence of theorem 3.8 is the following:

Corollary 3.10. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a C^2 solution to equation (3.2). Then, if we can reflect X_0 strictly up to $(\pi_{v,C}, v)$, for all $t \in [0,T)$ we can reflect $X(M,t)$ strictly up to $(\pi_{v,C}, v)$.*

Proof. The hypothesis of the theorem are true for each π_K in the definition, thus we can reflect strictly with respect to each π_K for all $t \in [0,T)$, and thus we can reflect $X(M,t)$ strictly up to $(\pi_{v,C}, v)$. □

Furthermore, it is clear that, for every direction v, there exists a hyperplane Π, perpendicular to v, such that we can reflect X_0 up to (Π, v) and Π intersects the interior of X_0. To be more precise, for every direction v, there exists a hyperplane Π_0^v tangent to X_0 and such that $X_0 \cap H^+(\Pi_0^v) = \emptyset$. Suppose that for every $\varepsilon > 0$ we can find a plane π_ε such $H^+(\pi_\varepsilon) \cap B_{R-\varepsilon}(C) = \emptyset$ and we cannot reflect X_0 strictly at π_ε, where R is such that $X_0 \subset B_R(C)$. We can take the corresponding plane Π_0^ε parallel to π_ε and tangent to X_0 at the point p_ε. Taking a sequence $\varepsilon_n \to 0$ we find a corresponding limited sequence p_{ε_n} which, by compactness, has a subsequence converging to a point $p \in X_0$. By construction, this point p is such that arbitrarily close to it there is a point in X_0 such that we cannot reflect by more than any chosen $\varepsilon > 0$ in the direction of its normal. As we can always represent X_0 as the graph of a function in a neighbourhood of p, we get a contradiction, because strict reflection in the direction of the normal by at least a fixed uniform amount ε at each point is always possible locally for any graph of a smooth function. Thus, we obtain the following:

Corollary 3.11. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a C^2 embedded solution to equation (3.2). There exists $\varepsilon > 0$ depending only on X_0 such that for all $t \in [0,T)$ we can reflect X_t up to $(\Pi_0^v + \epsilon v, v)$ for every $v \in S^n$. In particular, if $X_0 \subset B_R(C)$, then we can always reflect X_t up to (Π, v) whenever $H^+(\Pi) \cap B_{R-\varepsilon}(C) = \emptyset$.*

In other words, we can always reflect a little ε in any direction, uniformly, from X_0. We can use the fact that we can always reflect about a plane outside a sphere containing X_0 to prove the following estimate:

Corollary 3.12. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a C^2 embedded solution to equation (3.2). There exists $C > 0$ depending only on X_0 such that for all $t \in [0,T)$:*

$$\max_{x \in X_t} |x| - \min_{x \in X_t} |x| < C$$

Proof. We can reflect $X_0 \subset B_R(0)$ up to any plane tangent to $B_R(0)$, i.e. any plane $\pi_{v,K}$ for any $K \geq R$ and v unit vector, i.e. $\pi_{v,K} = \{p \in \mathbb{R}^{n+1} : \langle p, v \rangle = K\}$. Let $x_1, x_2 \in X_t$ such that $|x_1| = \min_{x \in X_t} |x|$ and $|x_2| = \max_{x \in X_t} |x|$. Let $v = \frac{x_2 - x_1}{|x_2 - x_1|}$: we can reflect X_t up to $(\pi_{v,R}, v)$ by theorem 3.8, therefore $\mathrm{dist}(x_2, \pi_{v,R}) <$

$\mathrm{dist}(x_1, \pi_{v,R})$, or in other words:

$$
\begin{aligned}
\left\langle x_2, \frac{x_2 - x_1}{|x_2 - x_1|} \right\rangle - R &< \left\langle x_1, \frac{x_1 - x_2}{|x_2 - x_1|} \right\rangle + R \\
\frac{|x_2|^2}{|x_2 - x_1|} - \cancel{\frac{\langle x_2, x_1\rangle}{|x_2 - x_1|}} - R &< \frac{|x_1|^2}{|x_2 - x_1|} - \cancel{\frac{\langle x_2, x_1\rangle}{|x_2 - x_1|}} + R \\
\frac{|x_2|^2 - |x_1|^2}{|x_2 - x_1|} &< 2R \\
|x_2|^2 - |x_1|^2 &< 2R|x_2 - x_1| \leq 4R|x_2| \\
|x_2|^2 &< |x_1|^2 + 4R|x_2| \\
|x_2| &< |x_1|\frac{|x_1|}{|x_2|} + 4R < |x_1| + 4R \\
|x_2| - |x_1| &< 4R
\end{aligned}
$$

□

Remark 3.13. This result has an important meaning if the hypersurface uniformly expands to infinity, i.e. $\lim_{t\to T} \min_{x\in X_t} |x| = \infty$. We can then consider the rescaled hypersurfaces

$$\widetilde{X_t} = \frac{1}{\min_{x\in X_t} |x|} X_t$$

Immediately, we find that the $\widetilde{X_t}$ must converge uniformly to a sphere, because $\frac{C}{\min_{x\in X_t} |x|} \to 0$, and therefore

$$\max_{x\in\widetilde{X_t}} |x| - \min_{x\in\widetilde{X_t}} |x| \to 0$$

The following result about the part of X_t outside a ball is also a surprisingly powerful tool:

Corollary 3.14. *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be an embedded solution to equation (3.2). Then, if, for a sphere B, $X_0 \subset B$, at all times $t \in [0, T)$ $X_t \backslash B$ is star-shaped with respect to the centre of B.*

Proof. $X_0 \subset B$ therefore we can reflect X_t about any hyperplane tangent to B by Corollary 3.11. By Corollary 3.9 $X_t \cap \overline{H^+(\pi)}$ is a graph over π, therefore, it is not possible for a normal line coming out of B and orthogonal to π to intersect $X_t \cap \overline{H^+(\pi)}$ more than once. This implies that $X_t \setminus B$ is star-shaped with respect to the centre of B □

Lastly:

Corollary 3.15. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a C^2 embedded solution to equation (3.2). Let $s_v : [0,T) \to I$ be such that*

$$s_v(t) = \sup \{ s \in I \mid \text{ we can reflect } X_t \text{ strictly up to } (\pi_{v,s}, v) \} .$$

Then $s_v(t)$ is a non-decreasing function. Also, if X_0 is compact, the limit

$$\lim_{t \to T^-} s_v(t)$$

exists and is finite.

Proof. When taking X_t as the starting manifold, the hypothesis of the theorem are still true, therefore we can reflect about $\pi_{c,v}$ for all $c < s_v(t)$ at all subsequent times. Thus $s_v(t)$ is non-decreasing. $\lim_{t \to T^-} s_v(t) = \sup s_v(t)$, therefore the limit exists. Also, if X_0 is bounded, there exists $R > 0$ such that $X_0 \subset B_R(0)$, therefore we can reflect X_0 strictly about any hyperplane non intersecting $B_R(0)$, as it does not touch X_0, and therefore we can also reflect X_t strictly about the same hyperplanes by theorem 3.8. At the same time, there exists a hyperplane such that X_t cannot be reflected strictly about it, because there will always be a straight line parallel to v intersecting X_t at more than one point, letting us consider the hyperplane orthogonal to v passing through their midpoint, about which X_t cannot be reflected strictly, implying $s_v(t) \neq +\infty$. Therefore $s_v(t) \in [-R, R]$, and the limit above is finite. $\square$

3.8 Applying the result to find gradient estimates

In this section we collect some applications of theorem 3.8 to gradient estimates for the support function and the radial function, originally included in [4] (a more in-depth definition these two functions can be found in the introduction of [8]). Central in what will follow is this corollary providing an estimate for the tangent component of the position vector x of a point on the hypersurface in its own tangent space:

Corollary 3.16. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be an embedded solution to equation (3.2). There exists a constant C, depending only on the initial hypersurface X_0, such that for all points $x \in X_t$ and $t \in [0,T)$, the following inequality holds:*

$$|x - \langle x \cdot \nu \rangle \nu| \leq C,$$

where ν is the unit normal to X_t at the point x.

Proof. : Choose $C > 0$ such that $X_0 \subset B_C(0)$. By Theorem 3.8, we can reflect X_t strictly up to any plane tangent to the ball, like in the previous corollaries. Thus, for any point $x \in X_t$ and outside the ball, we know that whenever $(x, V) > C$, then $V \notin T_x X_t$. This is equivalent to saying that for all $W \in T_x X_t$, we have:

$$(x, W) \leq C.$$

If we take now the projection of x on $T_x X_t$ and rescale it to be a unit vector, $W = \frac{x-(x\cdot\nu)\nu}{|x-(x\cdot\nu)\nu|} \in T_x X_t$, then we obtain:

$$C \geq (x, W) = \frac{|x - (x \cdot \nu)\nu|}{|W|} = |x - (x \cdot \nu)\nu|.$$

Thus, the corollary is proved. □

Let's now assume that the hypersurfaces X_t in the solution to the equation are convex. We are going to show a gradient estimate for the support function of the hypersurface. Support functions are one of the most important concepts stemming from the study of convex sets.

Definition 3.17. *Let K be a non-empty compact convex set in $\mathbb{R}^n$. We define the* support function *$u_K : S^n \to \mathbb{R}$ as*

$$u_K(\nu) = \sup\{\langle x, \nu\rangle : x \in K\}$$

Remark 3.18. Many authors define the support function on the whole euclidean space. As its value (thus ∇u_K) scales linearly with the distance from the origin, we will not be doing so out of simplicity. Notice that given two convex sets K_1 and K_2, $K_1 \subseteq K_2$ if and only if $u_{K_1} \leq u_{K_2}$. In this sense, a convex set is determined by its support function. Also, the tangent plane at the point in ∂K which has ν as a normal vector is $h_\nu = \{x \in \mathbb{R}^n : x \cdot \nu = u_K(\nu)\}$.

Corollary 3.19. *Let $u : S^n \times [0, T) \to \mathbb{R}$ be the support function of convex hypersurfaces X_t, solving the equation (3.2). There exists a constant C, depending only on $u(0)$, such that:*

$$|\nabla u(\nu, t)| \leq C,$$

for all $(\nu, t) \in S^n \times [0, T)$.

Proof. For each unit normal vector $\nu \in S^n$, let $x_t \in X_t$ be the unique point such that ν is the outward unit normal to X_t at x_t. We compute $\nabla u(\nu)$: let R_θ be a

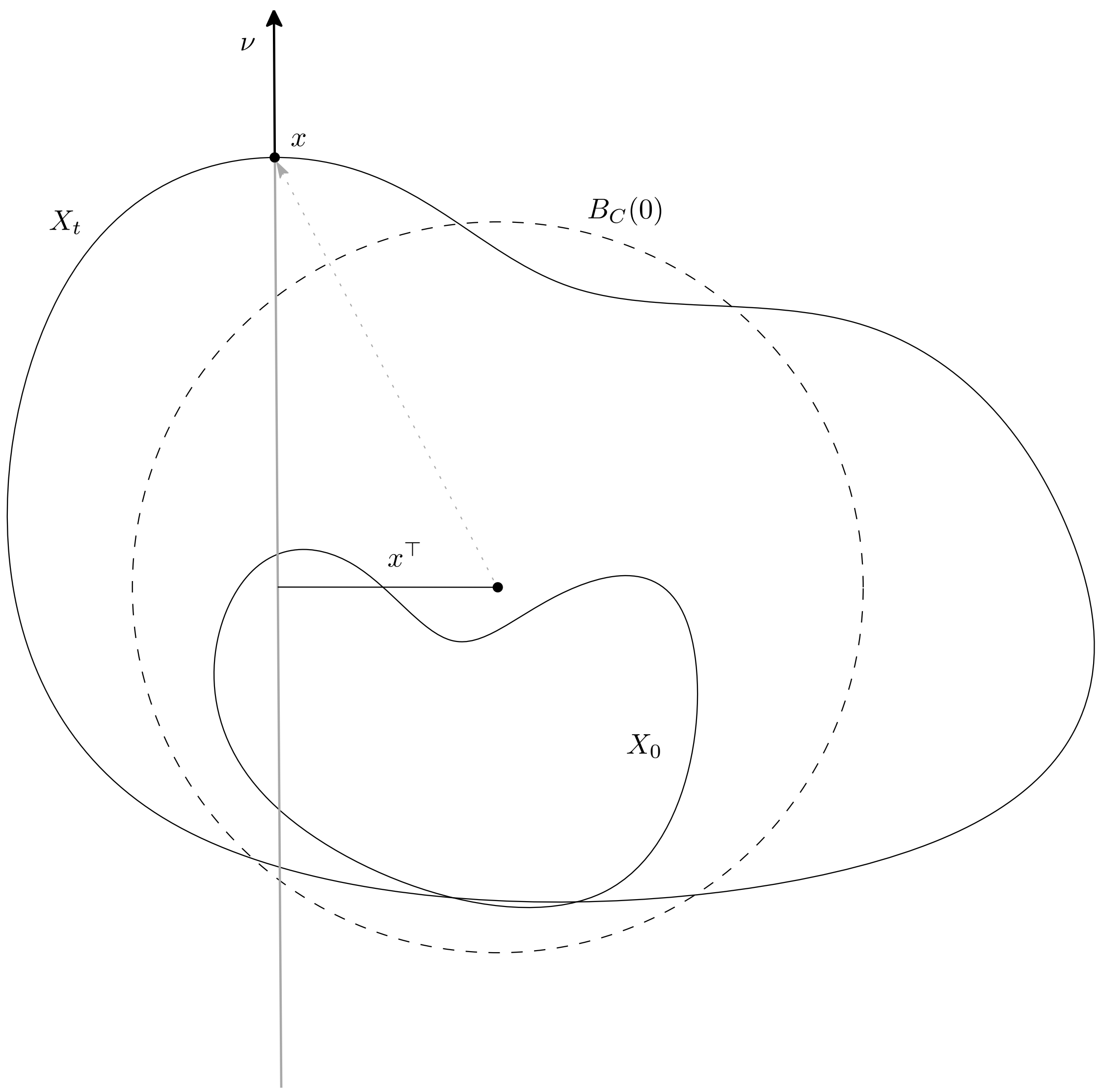

Figure 3.4: A geometric interpretation of corollary 3.16: the normal line always intersects $B_C(0)$. $x^\top = x - (x \cdot \nu)\nu$.

rotation of an angle θ in the direction $\partial_i \in T_\nu S^n$:

$$\begin{aligned}
\partial_i u(\nu) &= \lim_{\theta\to 0} \left[\langle x_t(R_\theta\nu), R_\theta\nu\rangle - \langle x_t(\nu), \nu\rangle\right]/\theta \\
&= \lim_{\theta\to 0} \left[\langle x_t(R_\theta\nu), \nu - \nu + R_\theta\nu\rangle - \langle x_t(\nu), \nu\rangle\right]/\theta \\
&= \lim_{\theta\to 0} \left[\langle x_t(R_\theta\nu), R_\theta\nu - \nu\rangle + \langle x_t(R_\theta\nu) - x_t(\nu), \nu\rangle\right]/\theta \\
&= \lim_{\theta\to 0} \left[\left\langle x_t(R_\theta\nu), \frac{R_\theta\nu - \nu}{\theta}\right\rangle + \left\langle \frac{x_t(R_\theta\nu) - x_t(\nu)}{\theta}, \nu\right\rangle\right] \\
&= \langle x_t(\nu), \partial_i\rangle + \cancel{\langle (dx_t)_\nu(\partial_i), \nu\rangle}
\end{aligned}$$

where $\langle (dx_t)_\nu(\partial_i), \nu\rangle = 0$ because $(dx_t)_\nu(\partial_i) \in T_{x_t}X_t$. Therefore

$$\begin{aligned}
\nabla u(\nu) &= (x_t)^\top = x_t - (x_t)^\perp \\
\nabla u(\nu) &= x_t - \langle x_t, \nu\rangle\nu = x_t - u(\nu)\nu \\
|\nabla u(\nu, t)| &= |x_t - \langle x_t, \nu\rangle\nu| \leq C
\end{aligned}$$

by applying Corollary 3.16, which completes the proof. □

Now, let's consider the case where the hypersurfaces X_t are starshaped for all $t \in [0, T)$. We obtain a gradient estimate for the radial function at points outside a certain compact starshaped region associated with the initial hypersurface X_0.

Definition 3.20. *Suppose that $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ parametrizes starshaped hypersurfaces X_t with respect to the origin. The* radial function $r : S^n \times [0, T) \to \mathbb{R}^+$ *is defined so that for each $(z, t) \in S^n \times [0, T)$, the point $r(z, t)z$ belongs to X_t.*

We will need the following lemma:

Lemma 3.21. *In the hypothesis of the definition above, there exists a constant C, depending only on X_0, such that for all points $(v, t) \in S^n \times [0, T)$, we have:*

$$r^2|\nabla r|^2 \leq C(r^2 + |\nabla r|^2).$$

In particular, if $r^2 > C$, then:

$$|\nabla r|^2 \leq \frac{Cr^2}{r^2 - C}.$$

Proof. As $x = r(z)z$, let $\partial_i \in T_z S^n$ and $\overline{\partial_i} = \partial_i x \in T_z X_t$ for $i = 1 \ldots n$ be corresponding bases in the tangent spaces. Computing this explicitly in $\mathbb{R}^{n+1}$ yields:

$$\begin{aligned}
\overline{\partial_i} &= \partial_i x \\
&= \partial_i(r(z)z) \\
&= (\partial_i r(z))z + r(z)\partial_i
\end{aligned}$$

Notice here that z and the ∂_i are orthogonal, therefore $|az + b^i\partial_i|^2 = a^2 + \sum_i (b^i)^2$. Computing this vector's scalar product with $r(z)z - \nabla r(z)$ yields:

$$\begin{aligned}
\langle r(z)z - \nabla r(z), \overline{\partial_i}\rangle &= \langle r(z)z - \sum_j \partial_j r(z)\partial_j, \overline{\partial_i}\rangle \\
&= \langle r(z)z, \overline{\partial_i}\rangle - \langle \sum_j \partial_j r(z)\partial_j, \overline{\partial_i}\rangle \\
&= r(z)\langle z, \overline{\partial_i}\rangle - \sum_j \partial_j r(z)\langle \partial_j, \overline{\partial_i}\rangle \\
&= r(z)(\partial_i r(z)) - \partial_i r(z) r(z) \\
&= 0
\end{aligned}$$

Moreover, notice that $\nabla r(z) \in T_z X_t$, which is orthogonal to z. Therefore, the following formula holds for the normal at a point with respect to the radial function:

$$\nu = \frac{r(z)z - \nabla r(z)}{\sqrt{r(z)^2 + |\nabla r(z)|^2}}$$

when taking $x = r(z)z$. Substituting the expressions above into the inequality from Corollary 3.16, we obtain:

$$\begin{aligned}
C &\geq \left| r(z)z - \langle r(z)z, r(z)z - \nabla r(z)\rangle \frac{r(z)z - \nabla r(z)}{r(z)^2 + |\nabla r(z)|^2} \right| \\
&\geq \left| r(z)z - \frac{r(z)^2(r(z)z - \nabla r(z))}{r(z)^2 + |\nabla r(z)|^2} \right| \\
&\geq \left| \frac{r(z)z(r(z)^2 + |\nabla r(z)|^2) - r(z)^2(r(z)z - \nabla r(z))}{r(z)^2 + |\nabla r(z)|^2} \right| \\
&\geq \left| \frac{r(z)|\nabla r(z)|^2 z + r(z)^2 \nabla r(z)}{r(z)^2 + |\nabla r(z)|^2} \right|
\end{aligned}$$

Multiplying by $r(z)^2 + |\nabla r(z)|^2$ and squaring (still using the fact that z and $\nabla r(z)$ are orthogonal):

$$\begin{aligned}
\left| r(z)|\nabla r(z)|^2 z + r(z)^2 \nabla r(z) \right| &\leq C(r(z)^2 + |\nabla r(z)|^2) \\
r(z)^2|\nabla r(z)|^4 + r(z)^4|\nabla r(z)|^2 &\leq C^2(r(z)^2 + |\nabla r(z)|^2)^2 \\
r^2|\nabla r|^4 + r^4|\nabla r|^2 &\leq C^2(r^2 + |\nabla r|^2)^2.
\end{aligned}$$

From here, the lemma follows. □

The estimate is:

Proposition 3.22. *With the same hypotheses as the lemma above, there exists a constant C, depending only on X_0, such that for all $(v,t) \in S^n \times [0,T)$, taking Π_0^v as in corollary 3.11, if $r(v,t)v \in \overline{H^+(\Pi_0^v)}$, then:*

$$|\nabla r(v,t)| \leq K.$$

Proof. By Corollary 3.11, there exists a constant $\epsilon > 0$, depending only on X_0, such that for all $(z,t) \in S^n \times [0,T)$ with $r(z,t)z \in H^+(\Pi_0^v)$, and for all $w \in S^n$ with $\langle W, z\rangle > 1-\epsilon$, we can reflect X_t up to the hyperplane $\Pi_w = \{r(z,t)z + p : \langle p, w\rangle = 0\}$. This means that $w \notin T_{r(z,t)z}X_t$, i.e., W is not tangent to X_t. In the proof of the previous lemma we showed that $r(z)z - \nabla r(z)$ is normal to the manifold's tangent plane at $r(z)z$, so, letting w_z be the projection of w in the z direction:

$$\begin{aligned}
0 &< \langle w, r(z)z - \nabla r(z)\rangle \\
\langle w, \nabla r(z)\rangle &< \langle w, r(z)z\rangle \\
\langle w - w_z, \nabla r(z)\rangle &< r(z)\langle w, z\rangle \\
|w - w_z||\nabla r(z)| &< r(z)\langle w, z\rangle \\
|\nabla r(z)| &< \frac{r(z)\langle w, z\rangle}{|w - w_z|}
\end{aligned}$$

by the fact that w is arbitrary, and estimating $|w - w_z|$ as $\sqrt{1^2 - (1-\epsilon)^2}$:

$$\begin{aligned}
|\nabla r(z)| &< \frac{r(z)(1-\epsilon)}{\sqrt{1^2 - (1-\epsilon)^2}} \\
|\nabla r(z)| &< r(z)\frac{(1-\epsilon)}{\sqrt{\epsilon(2-\epsilon)}}
\end{aligned}$$

By the preceding lemma, if $r^2 > C$, then:

$$\begin{aligned}
|\nabla r|^2 &\leq \frac{Cr^2}{r^2 - C} < C \\
|\nabla r| &\leq \sqrt{C}
\end{aligned}$$

otherwise,

$$|\nabla r(z)| < C\frac{(1-\epsilon)}{\sqrt{\epsilon(2-\epsilon)}}$$

Hence:

$$|\nabla r(z)| < \max\left(\sqrt{C}, C\frac{(1-\epsilon)}{\sqrt{\epsilon(2-\epsilon)}}\right)$$

∎

3.9 Expansive flows and ancient solutions

In this section, roughly following [19], we consider solutions to (3.2) which are defined on a larger interval (T_0, T_1), with $-\infty \leq T_0 \leq 0 < T_1 \leq \infty$.

We will limit our analysis to a sub-class of solutions to (3.2):

Definition 3.23. *We say that a solution to equation (3.2) is* expansive *if* $F < 0$.

An example of such a flow is the inverse mean curvature flow ($F = -\frac{1}{\kappa_1 + \cdots + \kappa_n}$), extensively studied in the literature.

Remark 3.24. This assumption on F implies that whenever $t < s$, $X_t \subset \mathrm{int}(X_s)$. This is implied by the fact that, in the equation, the time derivative is always an outward pointing non-zero vector. In this sense, the hypersurface is expanding in the ambient space.

Definition 3.25. *Let* $X : M^n \times (T_0, T_1) \to \mathbb{R}^{n+1}$ *be an embedded expansive solution to equation (3.2). We say that* X comes out of a point *if there exists a point* y_∞ *such that for every* $\varepsilon > 0$*, there exists a time* $\tau \in (T_0, T_1)$ *such that* $X_\tau \subset B_\varepsilon(y_\infty)$.

Remark 3.26. If we take the optimal τ in the definition above, Remark 3.24 implies that the function $\tau(\varepsilon)$ mapping ε to the last time where X_τ is contained in the ball is non-increasing.

In particular, we will show that expansive solutions "coming out of a point" must be expanding spheres. It is easy to check that homothetically expanding spheres do satisfy the equation: Any spherical solution is completely determined by its radius at time t, because the equation is invariant at each point, being the principal curvatures constant: thus, solving the ordinary differential equation $r'(t) = \varphi(r(t))$ completely determines the flow, where $\varphi(r) = -F(\frac{1}{r}, \ldots, \frac{1}{r})$.

We say that a solution to the equation is ancient if $T_0 = -\infty$. In the case of the spherical solution coming out of a point, there exists an ancient solution if and only if

$$+\infty = (T_1 - T_0) = \int_{T_0}^{T_1} 1 dt = \int_{T_0}^{T_1} \frac{r'(t)}{\varphi(r(t))} dt = \int_{r(T_0)}^{r(T_1)} \frac{1}{\varphi(r)} dr = \int_0^c \frac{1}{\varphi(r)} dr$$

$$\int_0^c \frac{1}{\varphi(r)} dr = +\infty$$

This equation allows us to determine, given F, if an ancient solution can exist (even if not coming out of a point). Solutions to (3.2) obey an avoidance principle: if a solution is inside another one, it stays inside it at all times. This can be proven similarly to theorem 3.8: if at some (x, t) this fails and they are tangent, one can apply the maximum principle to show that they coincide in a neighbourhood, which is a contradiction if one takes the first or the last t where this happens. A generic solution, then, will be sandwiched between two expanding sphere solutions, one inside and one outside, and therefore cannot be ancient if no ancient expanding sphere solution exists.

The idea of ancient solutions was introduced by Richard Hamilton in his work on the Ricci flow. It has since been applied to other geometric flows as well as to other partial differential equations. Ancient solutions are significant because they capture key asymptotic features of the flow and often have unique, rigid properties that distinguish them from other solutions. For instance, in contractive flows, ancient solutions can form complex, non-spherical shapes, whereas in expansive flows, they tend to exhibit strong rigidity, commonly resulting in spherical shapes under certain conditions. These solutions thus help in understanding the geometry and topology of hypersurfaces as they evolve, especially in applications like mean curvature flow and inverse mean curvature flow in mathematics and physics.

Indeed, the property of coming out a point is particularly rigid, as shown in the following result:

Theorem 3.27. *Let $X : M^n \times (T_0, T_1) \to \mathbb{R}^{n+1}$ be a smooth, closed, embedded expansive solution to equation (3.2) coming out of a point. Then it is a family of expanding spheres.*

Previous results in geometric flows have shown multiple non-trivial examples of contracting flows sweeping the whole space. This could therefore be somewhat surprising, because it shows an opposite result, when running the equation in the opposite direction. It can however be explained intuitively by the idea that parabolic flows tend to *smooth things out*: it is thus possible to arrive to a point from a more irregular hypersurface, however the only thing that can "come out" of a point is something just as symmetric as a point, i.e. a sphere.

The proof is a relatively simple application of the reflection technique. The outline of the proof in [19] is the same as the one below, but using Corollary 3.11 makes it a bit simpler:

Proof. Fix any hyperplane π not passing through y_∞. There is $R > 0$ such that $B_{2R}(y_\infty)$ does not intersect it. By definition, there is also a time $\tau \in (T_0, T_1)$ such that $X_\tau \subset B_R(y_\infty)$. By Corollary 3.11, then, we can reflect strictly X_t up to π for any $t > \tau$.

Now consider a sequence $\epsilon_n \to 0$. Up to a subsequence, we can then find a corresponding converging non-increasing sequence $\tau_n \to \bar{t} \in [T_0, T_1)$ (here note that $\bar{t}$ can be $-\infty$) such that $X_{\tau_n} \subseteq B_{\epsilon_n}(y_\infty)$, therefore in (τ_n, T_1) we can reflect up to any hyperplane tangent to $B_{\epsilon_n}(y_\infty)$ in any direction, reasoning like we just did. At time $\bar{t}$, $X_{\bar{t}} \subseteq \cap_r B_r(y_\infty) = \{y_\infty\}$, thus we would have a singularity at $\bar{t}$ if $\bar{t} \in (T_0, T_1)$ and therefore $\bar{t} = T_0$.

On the other hand, by construction, we can reflect X_t strictly about any hyperplane not intersecting $\cap_r B_r(y_\infty) = \{y_\infty\}$ at any time $t > \bar{t} = T_0$, hence we can reflect X_t strictly up to any hyperplane passing through y_∞, in both directions, at any time $t \in (T_0, T_1)$. We observe that in the limit, the reflection property becomes non-strict, in the sense that we have to replace the interior of X in definition 3.6 with its closure, therefore X_t may touch its reflection at the limit plane, i.e. the one passing through y_∞. Similarly, it cannot be that the other condition is the one causing the strict reflection definition to fail, as the other condition stays instead strict.

This implies that, taking any hyperplane passing through y_∞ and considering opposite directions for the reflection, X_t is symmetric about said hyperplane for any time $t \in (T_0, T_1)$. By Proposition 2.9, then, we conclude that X_t must be a ball. $\square$

Chapter 4

Extension to constant curvature spaces

In this chapter we want to extend the Chow-Gulliver result (theorem 3.8) to constant curvature spaces. We will use the same notation as in section 2.1 and 2.2. Throughout the chapter, by hyperplane we mean a totally geodesic hypersurface.

4.1 The equation in constant curvature spaces

As shown in [12], the equation we analysed in the previous chapter can be also considered in non-flat ambient spaces. In particular, we will analyse the case where the ambient space is one of those described in section 2.1: $\mathbb{R}^{n+1}$, $\mathbb{H}^{n+1}$, or $\mathbb{S}^n_+$. We again use the symbol $\mathbb{M}^{n+1}_+$ to indicate any of these spaces. Let $X_0 : M^n \to \mathbb{M}^{n+1}_+$ be a manifold embedded in $\mathbb{M}^{n+1}_+$. Let $F : \{(\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n | \kappa_1 \leq \cdots \leq \kappa_n\} \to \mathbb{R}$ be a C^1 function satisfying:

$$\frac{\partial F}{\partial \kappa_i} > 0 \text{ for all } i = 1, \ldots, n \tag{4.1}$$

and consider the evolution equation

$$\begin{cases} \dfrac{\partial X_t}{\partial t} = -F(\kappa_1(x), \ldots, \kappa_n(x))\nu \\ X(0) = X_0 \end{cases} \tag{4.2}$$

where ν is the outward normal to $X_t(M^n)$ at the point $X_t(x)$ and $\kappa_1 \leq \cdots \leq \kappa_n$ are the principal curvatures at $X_t(x)$.

As we saw in the previous chapter, it is a non-linear parabolic differential equation, as the calculation in chapter 3.2 is valid for any metric on $\mathbb{R}^{n+1}$, and in particular, for the metrics in our models for $\mathbb{M}^{n+1}_+$ in section 2.1. The existence result in [12] also holds as well in this case. Finally, the result in section 3.4 is not using the metric tensor, and therefore is valid in this setting as well, again because there are models on $\mathbb{R}^{n+1}$.

Remark 4.1. The hemisphere has a finite diameter in each direction. It is therefore possible that the flow stops in a finite time when it touches the equator. One could extend it a little bit by rotating the hypersurface in S^{n+1} so that it only touches the equator at the last possible moment. It is however possible that the hypersurface touches its reflection *on the wrong side* if we let it expand past the equator, (a sphere is, after all, round). It is easier to avoid this possibility by considering only flows defined in the hemisphere.

4.2 Extension of theorem 3.8

Assume the hypothesis in section 2.1: $X : M^n \to \mathbb{M}_+^{n+1}$ is a hypersurface in a constant curvature ambient space, and we choose a point and a direction v to foliate the ambient space. Consider a hyperplane in the foliation, $\pi = \pi_{v,C}$. As in section 2.1 we can define the reflection about π. As in the previous chapter, let X^π be the reflection of X about π.

Then, $\mathbb{M}_+^{n+1}$ is divided by π into two half-spaces:

$$H^+(\pi) = \bigcup_{s>C} \pi_{v,s} \text{ and } H^-(\pi) = \bigcup_{s<C} \pi_{v,s}.$$

Definition 4.2. *We say* we can reflect $X : M^n \to \mathbb{M}_+^{n+1}$ strictly with respect to π *if both:*

- $X^\pi \cap H^-(\pi) \subset \mathrm{int}(X) \cap H^-(\pi)$
- *At all $x \in X(M^n) \cap \pi$ the normal vectors to the X and X^π do not coincide (which would also imply that $T_xX = T_xX^\pi$).*

This fundamentally means that the reflection of one of the halves of X on the other side of π is contained in the region inside M^n and the tangent spaces of X and of the half-reflection do not coincide unless X and the half-reflection are tangent at that point.

Definition 4.3. *We say* we can reflect $X : M^n \to \mathbb{R}^{n+1}$ strictly up to (π, v) *if we can reflect M^n strictly with respect to $\pi_{v,s}$ for all hyperplanes $\pi_{v,s}$ such that $s < C$.*

The result then becomes:

Theorem 4.4 (Extended Chow-Gulliver). *Let $X : M^n \times [0,T) \to \mathbb{M}_+^{n+1}$ be a C^2 solution to equation (4.2). Then, if we can reflect $X(M^n,0) = M_0$ strictly with respect to π, then for all $t \in [0,T)$ we can reflect $X(M^n,t) = M_t$ strictly with respect to π.*

Proof. As before, by contradiction, suppose that there is a time t such that the thesis is false, and that it is the smallest such t. Then, for all $\tau \in [0,t)$, $M_{\tau,\pi} \cap H^-(\pi) \subset \mathrm{int}(M_\tau) \cap H^-(\pi)$; the unit vector orthogonal to π, V, is such that $V \notin T_xM_\tau$ for all $x \in M_\tau \cap \pi$ and $\tau \in [0,t)$; and either of the conditions fails at t, i.e. either:

(i) $M_{t,\pi} \cap H^-(\pi) \cap M_t \neq \emptyset$

(ii) The tangent spaces T_xX and T_xX^π coincide for some $x \in \pi$.

(i) Suppose the first case is true. Then, there exists $x_0 \in M_{t,\pi} \cap H^-(\pi) \cap M_t$ such that at x_0 the two manifolds are tangent.
We can then reason as in the proof of theorem 3.8.

(ii) Suppose instead that $T_xM_t = T_xM_t^\pi$ and in a neighbourhood of (x,t) both M_t and M_t^π are graphs of two smooth functions over T_xM_t by 3.4, i.e. again

$$\begin{aligned} f \ &: \ (x,t) \mapsto x + \tilde{f}(x,t)\nu \\ f_\pi \ &: \ (x,t) \mapsto x + \tilde{f}_\pi(x,t)\nu \end{aligned}$$

We can do this because the theorem allowing us to do so is a theorem on smooth manifolds, and requires nothing on the metric, therefore the existence of models in $\mathbb{R}^{n+1}$ of the ambient spaces allows us to do the same procedure. Reasoning again as in the proof of theorem 3.8, in $\overline{H^-(\pi)}$, $f_\pi \geq f$, because $M_\pi^n \cap H^-(\pi) \subset \mathrm{int}(M^n) \cap H^-(\pi)$. Finally, $f(x,t) = f_\pi(x,t)$, hence $f_\pi - f(x,t) = 0$, and thus (x,t) is a minimum point on the boundary for $f_\pi - f$. Also, for the outward pointing normal to the boundary V

$$\frac{\partial f}{\partial V}(x,t) = \frac{\partial f_\pi}{\partial V}(x,t)$$

because the graphs are both tangent to T_xM_t. We note also that the models in section 2.1 are conformal, so a vector is orthogonal to the boundary if and only if it is orthogonal to the corresponding region when seen as a region in $\mathbb{R}^{n+1}$. Thus,

$$\frac{\partial(f - f_\pi)}{\partial V}(x,t) = 0$$

But we must have

$$\frac{\partial(f - f_\pi)}{\partial V}(x,t) > 0$$

at a minimum on the boundary by Proposition 1.13, a contradiction. □

4.3 Extending Corollaries

We now shift the focus to extending the corollaries of theorem 3.8 in section 3.7.

The author is not aware of a standard definition of a support and a radial function in a curved setting, so we do not attempt to extend the results in section 3.8.

Clearly, corollary 3.10 has a direct equivalent in this setting:

Corollary 4.5. *Let $X : M^n \times [0,T) \to \mathbb{M}_+^{n+1}$ be a C^2 solution to equation (3.2). Then, if we can reflect X_0 strictly up to $(\pi_{v,C}, v)$, for all $t \in [0,T)$ we can reflect X_t strictly up to $(\pi_{v,C}, v)$.*

The second result that we can extend is Corollary 3.9, although the meaning of *graph* in curved spaces is ambiguous. We adapt the corollary as follows:

Corollary 4.6. *Let $X : M^n \times [0,T) \to \mathbb{M}_+^{n+1}$ be a C^2 embedded solution to equation (4.2). Then, if we can reflect X_0 strictly up to $(\pi_{v,C}, v)$, then for all $t \in [0,T)$, $X_t \cap H^+(\pi_{v,C})$ is such that the projection of the coordinates of its points onto $\pi_{v,C}$, $(p,\tau) \mapsto p$, is injective.*

This condition guarantees that we can build a map from $s : \overline{\mathrm{int}(X_t \cap \pi_{v,C})} \to \mathbb{R}$ such that $(p, s(p)) \in X_t$, making $p \mapsto (p, s(p))$ act as a sort of curved graph.

Proof. By theorem 4.4 we can reflect up to $(\pi_{v,C}, v)$ at all times in $[0,T)$. Let $\gamma_p(s) = (p, s + C)$ be the path followed by $p \in \pi_{v,C}$ as the planes sweep through the ambient space. If two points exist with the same p coordinate, say (p, C_1) and (p, C_2), the reflection about the hyperplane $\pi_{v,\frac{C_1+C_2}{2}}$ would map one onto the other, but as both C_1 and C_2 are greater than C, then we should also be able to reflect strictly about it by theorem 4.4, as we would have $\frac{C_1+C_2}{2} > C$. □

Let $\tilde{B}_r(y) = \{p \in \mathbb{M}_+^{n+1} : \mathrm{dist}(p,y) < r\}$. Reasoning exactly like in corollary 3.11, we also can extend it to this setting:

Corollary 4.7. *Let $X : M^n \times [0,T) \to \mathbb{M}_+^{n+1}$ be a C^2 embedded solution to equation (4.2). There exists $\varepsilon > 0$ depending only on X_0 such that for all $t \in [0,T)$ we can reflect X_t up to $(\Pi_0^v + \epsilon v, v)$ for every $v \in S^n$. In particular, if $X_0 \subset \tilde{B}_R(C)$, then we can always reflect X_t up to (Π, v) whenever $H^+(\Pi) \cap \tilde{B}_{R-\varepsilon}(C) = \emptyset$.*

Other corollaries of the section, like Corollary 3.16 and Corollary 3.12 are harder to extend, as they rely on the points at each plane moving along geodesics on the plane, which is not true in the curved case. (see remarks 2.5 and 2.6)

4.4 Extending theorem 3.27

It is also possible to extend the result in section 3.9 to $\mathbb{M}_+^{n+1}$. We consider *expansive* solutions to 4.2 which are defined on a larger interval (T_0, T_1), with $-\infty \le T_0 \le 0 < T_1 \le \infty$. Like before:

Definition 4.8. *We say that a solution to equation (4.2) is* expansive *if* $F < 0$*.*

Like in the euclidean case, expansive solutions are those where $\frac{\partial X_t}{\partial t}$ is an outward pointing vector.

Definition 4.9. *Let* $X : M^n \times (T_0, T_1) \to \mathbb{M}_+^{n+1}$ *be an embedded expansive solution to equation (4.2). We say that* X comes out of a point *if there exists a point* y_∞ *such that for every* $\varepsilon > 0$*, there exists a time* $\tau \in (T_0, T_1)$ *such that* $X_\tau \subset \tilde{B}_\varepsilon(y_\infty)$*, where* $\tilde{B}_r(y) = \{p \in \mathbb{M}_+^{n+1} : \mathrm{dist}(p, y) < r\}$*.*

Like in the euclidean case, homothetically expanding distance spheres satisfy the equation: distance spheres have constant principal curvatures in $\mathbb{M}_+^{n+1}$ as well; thus, any spherical solution is completely determined by its radius at time t, because the equation is invariant at each point. Again, the ordinary differential equation $r'(t) = \varphi(r(t))$ completely determines the flow, where $\varphi(r) = -F(\kappa(r), \dots, \kappa(r))$. Like before, we can find an ancient solution if and only if

$$\int_0^c \frac{1}{\varphi(r)} dr = +\infty.$$

It is no surprise that in constant curvature spaces we can prove a statement equivalent to theorem 3.27:

Theorem 4.10. *Let* $X : M^n \times (T_0, T_1) \to \mathbb{M}_+^{n+1}$ *be a smooth, closed, embedded expansive solution to equation (4.2) coming out of a point. Then it is a family of expanding distance spheres.*

Remark 4.11. At a glance, it could be confusing why the shape changes, after all it is not immediately apparent what changes in equation 4.2 from the previous chapter, as we do not find the metric in it, and one might wonder if it could be seen as a version of equation 3.2 with a different F. This is not the case, as computing the principal curvatures interacts with the metric at a fundamental level, making it so that the two equations are not comparable.

The proof is effectively the same as before:

Proof. Fix any hyperplane π not passing through y_∞. There is $R > 0$ such that $\tilde{B}_{2R}(y_\infty)$ does not intersect it. By definition, there is also a time $\tau \in (T_0, T_1)$ such that $X_\tau \subset \tilde{B}_R(y_\infty)$. By Corollary 4.7, then, we can reflect strictly X_t up to π for any $t > \tau$.

Now consider a sequence $\epsilon_n \to 0$. Up to a subsequence, we can then find a corresponding converging non-increasing sequence $\tau_n \to \bar{t} \in [T_0, T_1)$ (here note that $\bar{t}$ can be $-\infty$) such that $X_{\tau_n} \subseteq B_{\epsilon_n}(y_\infty)$, therefore in (τ_n, T_1) we can reflect up to any hyperplane outside $\tilde{B}_{\epsilon_n}(y_\infty)$ in any direction, reasoning like we just did. At time $\bar{t}$, $X_{\bar{t}} \subseteq \cap_r \tilde{B}_r(y_\infty) = \{y_\infty\}$, thus we would have a singularity at $\bar{t}$ if $\bar{t} \in (T_0, T_1)$ and therefore $\bar{t} = T_0$.

On the other hand, by construction, we can reflect X_t strictly about any hyperplane not intersecting $\cap_r B_r(y_\infty) = \{y_\infty\}$ at any time $t > \bar{t} = T_0$, hence we can reflect X_t strictly up to any hyperplane passing through y_∞, in both directions, at any time $t \in (T_0, T_1)$. We observe that in the limit, the reflection property becomes non-strict, in the sense that we have to replace the interior of X in definition 3.6 with its closure, therefore X_t may touch its reflection at the limit plane, i.e. the one passing through y_∞. Similarly, it cannot be that the other condition is the one causing the strict reflection definition to fail, as the other condition stays instead strict.

This implies that, taking any hyperplane passing through y_∞ and considering opposite directions for the reflection, X_t is symmetric about said hyperplane for any time $t \in (T_0, T_1)$. By Proposition 2.9, then, we conclude that X_t must be a ball. $\square$

4.5 Shrinking flows on the sphere

As noted in [3], something can be said about the opposite problem for a shrinking flow on the sphere S^{n+1}. In fact, shrinking flows of convex manifolds on a sphere are also quite rigid, and are limited under certain condition to being shrinking spheres. This was shown in [2].

Definition 4.12. *We say that a solution to equation (4.2) is* shrinking *if $F \geq 0$, and $F(0) = 0$.*

The condition $F(0) = 0$ ensures that equators on the sphere are static solutions to equation (4.2). On the whole sphere one must choose a conventional direction for the unit normal.

Theorem 4.13. *Let $X : M^n \times (T_0, T_1) \to S^{n+1}_+$ be a smooth, convex, embedded shrinking solution to equation (4.2), and (T_0, T_1) be the maximum interval where the solution is defined. Suppose also that:*

$$\limsup_{t \to T_0} \max_{X_t} \sum_i \kappa_i(x) < \infty$$

Then X_t is a distance sphere for all $t \in (T_0, T_1)$.

The proof of this result is long and involved, and the main result in [2]. It also uses a reflection argument, but not precisely the Alexandrov Moving Planes method.

Remark 4.14. This result acts as a sort of converse for theorem 4.10 on the hemisphere, showing that all ancient convex shrinking solutions on the sphere are the trivial shrinking spheres.

Chapter 5

Area- and volume-preserving flows

In this chapter area-preserving and volume-preserving flows are discussed. These are flows similar to the equation (3.2), with a non-local term added. It can be shown that the theorems in chapter 2 also apply to these flows. Finally, we discuss an application of these flows.

5.1 The flows we consider in this chapter

Volume-preserving flows were introduced by Gerhard Huisken in 1987 (see [13]). Unlike the mean curvature flow, which contracts hypersurfaces to a point, volume-preserving flows maintain a constant enclosed volume while reducing the hypersurface area. These flows are governed by the modified evolution equation:

$$\frac{\partial X}{\partial t} = [h(t) - H(x,t)]\,\nu,$$

where $H(x,t)$ is the is the mean curvature at a point $x \in X_t$ and $h(t)$ is the average mean curvature at time t, a non-local term defined as

$$h(t) = \frac{\int_{X_t} H(x,t)\,d\mu}{\int_{X_t} 1\,d\mu} = \frac{\int_{M_t} H(x,t)\,d\mu}{|M_t|},$$

where $d\mu$ is the volume element on the hypersurface X_t. Notice that mean-curvature flows ($\frac{\partial X}{\partial t} = -H(t)\nu$) is a possible choice of F in (3.2), therefore compared to the previous chapters we are just adding the global term $h(t)$. Huisken's results demonstrate that such flows lead to the convergence of the hypersurfaces to round spheres, provided the initial hypersurface is uniformly convex.

A more general family of flows is in [21]. Consider flows:

$$\begin{cases} \dfrac{\partial X_t}{\partial t} = \left[-H^k(x,t) + \phi(t)\right] \nu \\ X(0) = X_0 \end{cases} \tag{5.1}$$

where $\phi(t)$ is an appropriate non-local term and $k \in (0, \infty)$. If $k \neq 1$, we will assume that $H > 0$ everywhere on X_t for all t (see remark below).

Remark 5.1. When considering $\frac{a+b}{2}$, the derivatives with respect to a and b are $\frac{1}{2} > 0$. However, when one considers $\left(\frac{a+b}{2}\right)^2 = \frac{a^2}{4} + \frac{ab}{2} + \frac{b^2}{4}$, the derivatives depend on the values of a and b, and in particular it may be negative depending on their values. a and b here can be our mean curvatures, and therefore this simple observation shows that there is no guarantee that the flow

$$\begin{cases} \dfrac{\partial X_t}{\partial t} = -H^k(x,t)\nu \\ X(0) = X_0 \end{cases}$$

is parabolic. One needs stronger hypothesis to guarantee this. By the formula for the derivative of the composite function, one gets easily

$$\frac{\partial}{\partial \kappa_i} H^k = kH^{k-1} \frac{\partial}{\partial \kappa_i} H = \frac{k}{n} H^{k-1}$$

To guarantee that $\frac{\partial F}{\partial \kappa_i} > 0$ we need to assume that $H > 0$. We will assume this on the hypersurface at all times whenever $k \neq 1$.

A possible choice for ϕ is:

$$\phi(t) = \frac{\int_{X_t} H^k(x,t)\, d\mu}{|X_t|}, \tag{5.2}$$

corresponding to volume-preserving flows. Another possible choice is

$$\phi(t) = \frac{\int_{X_t} H^{k+1}(x,t)\, d\mu}{\int_{X_t} H(x,t)\, d\mu}, \tag{5.3}$$

which corresponds to area-preserving flows. We know that, indeed, if we choose ϕ as in (5.2), the volume of the domain enclosed by X_t remains constant, while choosing ϕ as in (5.3) keeps the area $|X_t|$ constant.

The fact that choosing ϕ as in (5.2) preserves the volume is immediately apparent, as the change of the volume inside the hypersurface is the integral over X_t of $\frac{\partial X_t}{\partial t} \cdot \nu$ (by Reynolds transport theorem for a constant function):

$$\int_{X_t} \frac{\partial X_t}{\partial t} \cdot \nu \, d\mu = \int_{X_t} \left[-H^k(x,t) + \phi(t)\right] d\mu = -\int_{X_t} H^k(x,t) d\mu + \phi(t)|X_t| = 0$$

On the other hand, choosing ϕ as in (5.3), the formula for the first variation of area says that

$$\frac{d}{dt}\int_{X_t} d\mu = \int_{X_t} \left\langle \frac{\partial X_t}{\partial t}, H\nu \right\rangle d\mu + \cancel{\int_{\partial X_t} \nu \frac{\partial X_t}{\partial t} d\mu}$$

where the second term is cancelled because $\frac{\partial X_t}{\partial t}$ is orthogonal to the surface. Therefore

$$\begin{aligned}\frac{d}{dt}\int_{X_t} d\mu &= \int_{X_t} \left[-H^k + \phi(t)\right] H d\mu \\ &= \phi(t)\int_{X_t} H d\mu - \int_{X_t} H^{k+1} d\mu = 0\end{aligned}$$

Remark 5.2. Another possible choice for an area-preserving flow is in [17] where he considers a flow

$$\begin{cases} \dfrac{\partial X_t}{\partial t} = \left[1 - H(x,t)\psi(t)\right]\nu \\ X(0) = X_0 \end{cases} \tag{5.4}$$

where

$$\psi(t) = \frac{\int_{X_t} H(x,t)\, d\mu}{\int_{X_t} H^2(x,t)\, d\mu}$$

Again, computing the first variation of the area:

$$\begin{aligned}\frac{d}{dt}\int_{X_t} d\mu &= \int_{X_t} \left[1 - H\phi(t)\right] H d\mu \\ &= \int_{X_t} H d\mu - \psi(t)\int_{X_t} H^2 d\mu = 0\end{aligned}$$

Notice, however, that it is almost the same flow as (5.1) with $k = 1$, where we divided the RHS by $\phi(t)$.

5.2 Theorem 3.8 and corollaries

Because of the non-local term, equation (5.1) does not belong to the class considered in the previous chapters. However, it can be shown that $\phi(t)$ behaves like a lower-order term and that the flow is still parabolic. In particular, small time existence holds as in Theorem 3.3. Given a solution X_t, we can consider the evolution equation

$$\begin{cases} \dfrac{\partial Y_t}{\partial t} = \left[-H^k(x,t) + \varphi_{X_t}(t)\right]\nu \\ Y(0) = Y_0 \end{cases} \tag{5.5}$$

where φ_{X_t} is the constant function independent of Y given by ϕ when computed on the specific solution X_t. Clearly, X_t is a solution also to the second equation, as the values of ϕ and φ_{X_t} coincide when one considers the solution X_t. At the same time, equation (5.5) is a parabolic equation: the only difference with (3.2) is the addition of a constant function φ_{X_t} independent of the solution multiplied by the normal vector, which cannot affect the second order terms (the normal vector only depends on first-order derivatives). Therefore, following this line of reasoning, one can apply theorems like the maximum principle (Proposition 1.12) and the boundary point lemma (Proposition 1.13) also to solutions of (5.1) as long as they share the same φ_{X_t} (which is the case for reflections), as they must hold for solutions of the associated parabolic equation (5.5).

Immediate consequence of the above is the fact that theorem 3.8 extends to solutions of equation (5.1):

Theorem 5.3. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a C^2 solution to equation (5.1). Then, if we can reflect $X(M^n,0) = X_0$ strictly with respect to π, then for all $t \in [0,T)$ we can reflect $X(M^n,t) = X_t$ strictly with respect to π.*

The proof is word for word identical to that of theorem 3.8, once one notices that it is possible to apply the maximum principle and Hopf's boundary point lemma.

Remark 5.4. There is also a simpler approach to apply the maximum principle in this specific case: in the proof of the maximum principle one considers the difference of two solutions; in principle the extra term $\phi(t)\nu$ could be different, providing an obstruction in the proof. However, reflections have the same $\phi(t)$, as the two surfaces in this case are just one the reflection of the other, and therefore this non-local term cancels out completely when considering the equation of the difference, leaving only the same terms as we had when dealing with equation (3.2). Thus, the maximum principle 1.12 can be extended to this situation. Similarly, this also applies to Hopf's boundary point lemma 1.13.

As a direct consequence, all the corollaries of theorem 3.8 can be extended. In particular:

Corollary 5.5. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a C^2 solution to equation (5.1). Then, if we can reflect X_0 strictly up to $(\pi_{v,C}, v)$, for all $t \in [0,T)$ we can reflect $X(M,t)$ strictly up to $(\pi_{v,C}, v)$.*

Corollary 5.6. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a C^2 embedded solution to equation (5.1). Then, if we can reflect X_0 strictly up to $(\pi_{v,C}, v)$, for all $t \in [0,T)$ $v \notin T_x X_t$ for all $x \in X_t \cap \overline{H^+(\pi)}$. In particular, $X_t \cap \overline{H^+(\pi)}$ is a graph over π for all $t \in [0,T)$.*

Corollary 5.7. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a C^2 embedded solution to equation (5.1). There exists $C > 0$ depending only on X_0 such that for all $t \in [0,T)$:*

$$\max_{x \in X_t} |x| - \min_{x \in X_t} |x| < C$$

Corollary 5.8. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be an embedded solution to equation (5.1). Then, if, for a sphere B, $X_0 \subset B$, at all times $t \in [0,T)$ $X_t \backslash B$ is star-shaped with respect to the centre of B.*

These correspond to corollaries 3.10, 3.9, 3.12 and 3.14, respectively.

5.3 The solution stays inside a compact

One of the results that is proven in [17] is that the solution to (5.4) remains inside a bounded region of euclidean space. As a final application of the technique, a simple proof of this fact for solutions to (5.1) is presented. The only other result we need do prove this is the well known:

Theorem 5.9 (Isoperimetric inequality)**.** *Among all measurable sets in $\mathbb{R}^n$ with a given volume and C^1 boundary, the sphere has the smallest possible surface area.*

We show that the solution does not leave a compact ball if it has a sufficiently large radius.

Proposition 5.10. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a C^2 embedded solution to equation (5.1). There exists $C > 0$ depending only on X_0 such that for all $t \in [0,T)$ such that*

$$X_t \subset B_C(0)$$

Proof. The thesis can be written as

$$\max_{x\in X_t} |x| < C$$

In light of corollary 5.7, it suffices to prove that

$$\min_{x\in X_t} |x| < K$$

as in that case

$$\max_{x\in X_t} |x| < C + \min_{x\in X_t} |x| < C + K$$

Like in the proof of corollary 5.7, assume $X_0 \subset B_K(0)$. We may take, without loss of generality, K such that the surface area of $\partial B_K(0)$ is greater than the surface area of X_0. Suppose that, at some time t, $\min_{x\in X_t} |x| > K$. Then, all the points in X_t are outside $B_K(0)$. By corollary 5.8, therefore, the hypersurface is star-shaped with respect to the origin and $B_K(0) \subset \operatorname{int}(X_t)$. If we chose $\phi(t)$ such that it is an area-preserving flow, this is a contradiction, because in this case we would have found an X_t with a bigger volume than the sphere $B_K(0)$ but smaller surface area. For the volume-preserving flows, instead, we can compute the derivative of the total surface-area:

$$\begin{aligned}
\frac{d}{dt}\int_{X_t} d\mu &= \int_{X_t} \left[-H^k + \phi(t)\right] H d\mu \\
&= -\int_{X_t} H^{k+1} d\mu + \phi(t)\int_{X_t} H d\mu \\
&= -\int_{X_t} H^{k+1} d\mu + \frac{1}{|X_t|}\left(\int_{X_t} H^k(x,t)\, d\mu\right)\left(\int_{X_t} H d\mu\right)
\end{aligned}$$

We find that

$$\begin{aligned}
\frac{1}{|X_t|}\int_{X_t} H^{k+1} d\mu &= \left(\frac{1}{|X_t|}\int_{X_t} H^{k+1} d\mu\right)^{\frac{k}{k+1}} \left(\frac{1}{|X_t|}\int_{X_t} H^{k+1} d\mu\right)^{\frac{1}{k+1}} \\
&= \left(\frac{1}{|X_t|}\int_{X_t} \left(H^k\right)^{\frac{k+1}{k}} d\mu\right)^{\frac{k}{k+1}} \left(\frac{1}{|X_t|}\int_{X_t} H^{k+1} d\mu\right)^{\frac{1}{k+1}} \\
&\geq \left(\frac{1}{|X_t|}\int_{X_t} H^k d\mu\right)\left(\frac{1}{|X_t|}\int_{X_t} H d\mu\right)
\end{aligned}$$

by Jensen's inequality. Therefore,

$$\frac{1}{|X_t|}\left(\int_{X_t} H^k d\mu\right)\left(\int_{X_t} H d\mu\right) \leq \int_{X_t} H^{k+1} d\mu$$

$$\frac{d}{dt}\int_{X_t} d\mu \leq 0$$

which implies that the surface area is non-increasing, and we have a contradiction similar to the one for the area-preserving case: X_t is a hypersurface containing the sphere $B_K(0)$, while having a smaller surface area than said sphere. This implies that

$$\min_{x \in X_t} |x| < K$$

and the final theorem is proven. $\square$

Remark 5.11. A careful reader might remember that in the previous chapters we cited a result showing that solutions of (3.2) satisfy an *avoidance principle*: when a solution is contained in the interior of another solution, without touching each other, they will never touch and stay one inside the other. As the spheres are constant solutions to (5.1), one may wonder if a solution inside a sphere stays inside the sphere thanks to a similar result, therefore proving that the solution stays inside a compact in a much simpler way and making this whole discussion trivial. This line of reasoning is unfortunately not possible in this case: the aforementioned theorem is proven using the maximum principle at the first contact point of the two solutions, but in this case the extra term $\phi(t)$ in (5.1) is different between the two, therefore its impact on the equation cannot be ignored, and, thus, it is not possible to apply an avoidance principle in this case.

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