$$\boxed{\begin{array}{l} y_t = \beta_0 + \beta_1 t + \varepsilon_t \\ y_{t-1} = \beta_0 + \beta_1(t-1) + \varepsilon_{t-1} \end{array}}$$

$Lets\ state\ a\ new\ function\ composed\ from\ y\ and\ y\ shifted:$

$z_t = y_t - y_{t-1}$

$z_t = \beta_0 + \beta_1 t + \varepsilon_t - (\beta_0 + \beta_1(t-1) + \varepsilon_{t-1})$

$z_t = \beta_0 + \beta_1 t + \varepsilon_t - \beta_0 - \beta_1(t-1) - \varepsilon_{t-1}$

$z_t = \beta_1 t - \beta_1(t-1) + \varepsilon_{t-1} - \varepsilon_{t-1}$

$z_t = \beta_1 t - \beta_1 t + \beta_1 + \varepsilon_{t-1} - \varepsilon_{t-1}$

$z_t = \beta_1 + (\varepsilon_t - \varepsilon_{t-1})$

$Lets\ find\ out\ its$ $\boldsymbol{variance}$ and $\boldsymbol{expected\ value}$ $, remember\ expected\ value\ of\ noise\ is\ 0:$

$Expected\ Value\ (mean):$

$E[Z_t] = E[\beta_1 + (\varepsilon_t - \varepsilon_{t-1})]$

$E[Z_t] = E[\beta_1] + E[\varepsilon_t - \varepsilon_{t-1}]$

$E[Z_t] = E[\beta_1] + E[\varepsilon_t - \varepsilon_{t-1}]$

$E[Z_t] = E[\beta_1] + E[\varepsilon_t] - E[\varepsilon_{t-1}]$

$E[Z_t] = E[\beta_1]$

$But\ \beta_1 is\ a\ constant, then:$

$$\boxed{E[Z_t] = \boldsymbol{\beta_1}}$$

$Variance:$

$Var_{Z_t} = Var[\beta_1 + (\varepsilon_t - \varepsilon_{t-1})]$

$But\ terms:\ \beta_1\ and\ (\varepsilon_t - \varepsilon_{t-1})\ are\ independent\ , then:$

$Var_{Z_t} = Var[\beta_1] + Var[(\varepsilon_t - \varepsilon_{t-1})]$

$Var_{Z_t} = E[(\beta_1 - \boldsymbol{E}[\boldsymbol{\beta_1}])^2] + Var[(\varepsilon_t - \varepsilon_{t-1})]$

$But\ \beta_1\ coeficient\ is\ constant, then\ E[\beta_1] = \beta_1$

$$Var_{Z_t} = E[(\beta_1 - \beta_1)^2] + Var[(\varepsilon_t - \varepsilon_{t-1})]$$

$$Var_{Z_t} = Var[(\varepsilon_t - \varepsilon_{t-1})]$$

$$Var_{Z_t} = E\left[\left((\varepsilon_t - \varepsilon_{t-1}) - E[\varepsilon_t - \varepsilon_{t-1}]\right)^2\right]$$

$$Var_{Z_t} = E[(\varepsilon_t - \varepsilon_{t-1})^2 - 2(\varepsilon_t - \varepsilon_{t-1})(E[\varepsilon_t - \varepsilon_{t-1}]) + E[\varepsilon_t - \varepsilon_{t-1}]^2]$$

But $E[\varepsilon_t]$ *and* $E[\varepsilon_{t-1}]$ *is* 0 *since both terms represent noise.*

$$Var_{Z_t} = E[(\varepsilon_t - \varepsilon_{t-1})^2 - 2(\varepsilon_t - \varepsilon_{t-1})(E[\varepsilon_t] - E[\varepsilon_{t-1}]) + E[\varepsilon_t - \varepsilon_{t-1}]^2]$$

$$Var_{Z_t} = E[(\varepsilon_t - \varepsilon_{t-1})^2 + E[\varepsilon_t - \varepsilon_{t-1}]^2]$$

The expected value of an expected value can be colapsed into one expectation since first calculation brings out a constant.

$$Var_{Z_t} = E[(\varepsilon_t - \varepsilon_{t-1})^2] + \boldsymbol{E}[\boldsymbol{E}[\varepsilon_t - \varepsilon_{t-1}]^2]$$

$$Var_{Z_t} = E[(\varepsilon_t - \varepsilon_{t-1})^2] + E[\varepsilon_t - \varepsilon_{t-1}]^2$$

$$Var_{Z_t} = E[(\varepsilon_t - \varepsilon_{t-1})^2] + E[\varepsilon_t - \varepsilon_{t-1}]^2$$

Let's approach the quadratic term $(\varepsilon_t - \varepsilon_{t-1})^2$ *first*

$$Var_{Z_t} = E[(\varepsilon_t)^2 - 2(\varepsilon_t)(\varepsilon_{t-1}) + (\varepsilon_{t-1})^2] + E[\varepsilon_t - \varepsilon_{t-1}]^2$$

But ε_t *and* ε_{t-1} *are independent given its randomness* :

$$Var_{Z_t} = E[(\varepsilon_t)^2] - E[2(\varepsilon_t)(\varepsilon_{t-1})] + E[(\varepsilon_{t-1})^2] + E[\varepsilon_t - \varepsilon_{t-1}]^2$$

Again , $E[\varepsilon_t]$ *and* $E[\varepsilon_{t-1}]$ *is* 0 *since both terms represent noise.*

$$Var_{Z_t} = E[(\varepsilon_t)^2] - 2E[(\varepsilon_{t-1})]E[(\varepsilon_t)] + E[(\varepsilon_{t-1})^2] + E[\varepsilon_t - \varepsilon_{t-1}]^2$$

$$Var_{Z_t} = E[(\varepsilon_t)^2] + E[(\varepsilon_{t-1})^2] + E[\varepsilon_t - \varepsilon_{t-1}]^2$$

$$Var_{Z_t} = E[(\varepsilon_t)^2] + E[(\varepsilon_{t-1})^2] + (E[\varepsilon_t] - E[\varepsilon_{t-1}])^2$$

$$Var_{Z_t} = E[(\varepsilon_t)^2] + E[(\varepsilon_{t-1})^2] + (E[\varepsilon_t] - E[\varepsilon_{t-1}])^2$$

$$Var_{Z_t} = E[(\varepsilon_t)^2] + E[(\varepsilon_{t-1})^2]$$

$But\ we\ know\ that\ Var_{E_t} = E[(\varepsilon_t)^2] - (E[\varepsilon_t])^2\ ,so:$

$$Var_{Z_t} = Var_{E_t} + (E[\varepsilon_t])^2 + Var_{E_{t-1}} + (E[\varepsilon_{t-1}])^2$$

$$Var_{Z_t} = Var_{E_t} + Var_{E_{t-1}}$$

$$Var_{Z_t} = \sigma_\varepsilon{}^2 + \sigma_\varepsilon{}^2$$

$$\boxed{Var_{Z_t} = 2\sigma_\varepsilon{}^2}$$