EM Chapter I: Maxwell's First Equation

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Takeaway

1. Changing the mindset from *forces* to *fields*.

2. Maxwell's First Equation = Inverse Square + Superposition

Welcome!

This is the first chapter of my electromagnetics series. I will introduce to you the theory of electromagnetism through this series, not in the order of history, but in the order of a self-contained logic that satisfies people. The reason I did this is that history is too complicated, confusing and sometimes even wrong. One would not fully understand the trajactory of any subject (geometry, Newton's law, General relativity, etc) without *experiencing* it. So I would NOT recommend you go through the tedious history of *any* subject but develop **your own** logic-complete explanation of it after reading tons of wikipedia and stackexchange, in other words, that explanation should make sense in the history in another parallel universe!

You will understand the first one of the famous Maxwell's Equations in this blog:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \text{(Gauss's law)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Gauss's law for magnetism)}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's law)}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \text{(Ampère's law)}$$

Physics World to Math World

Physics originates from observations in the real world. If you keep an eye on it, you could discover the following phenomena yourself:

1. The force between two charges is proportional to their charge, and inversely proportional to their distance, squared.

2. No isolated "magnetic charges" found in nature.

3. Changing magnetic flux leads to current in a circuit!

4. Two parallel wires attract or repel each other when current is applied!

These four observations leads to the four equations by Maxwell.

Integral Form of Maxwell's First Equation

Inverse Square Law

According to observation 1, we could write[1]:

$$F \propto \frac{Qq}{r^2},$$

or in an equation:

$$\boxed{F = k\frac{Qq}{r^2},}$$

[1] Non-bold-face letters (F) are scalars, bold-face letters $(\mathbf{F})$ are vectors.

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where k is a constant, Q and q are the amount of ***charge***[2] on two small objects, r is their distance.

Conservative Property

Let's consider the work done when we fix Q and gradually move q in a certain path. Since the force from Q to q is always radially from Q, it turns out that the work done only depends on the *initial* and *final* position of q regardless of the moving process in between! We say a force like this ***Conservative***[3]. We would like to think that there is a number U attached at every spatial point around Q such that the the work done from point A and B is just that $U(A) - U(B)$[4], i.e.,

$$W_{AB} =: U(A) - U(B).$$

$U(r)$ is called the ***potential energy*** at r. We claim[5] that

$$\boxed{U(r) = -k\frac{Qq}{r},}$$

because

$$W_{AB} = \int_{A \to B} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{r_0}^{r_1} k\frac{Qq}{r^2}\mathrm{d}r = -k\frac{Qq}{r_1} - k\frac{Qq}{r_0} =: U(A) - U(B).$$

Therefore, the work done is just the difference of U with a negative sign:

$$W_{AB} = -\Delta U.$$

We also know that

$$\mathrm{d}W_{AB} = \mathbf{F} \cdot \mathrm{d}\mathbf{l} = F \cdot \mathrm{d}l_\parallel,$$

where $\mathrm{d}\mathbf{l}$ is a small displacement and $\mathrm{d}l_\parallel$ is the length of the projection of that small displacement onto the direction of $\mathbf{F}$, i.e., the direction of U, reversed. Hence,

$$F = \frac{\mathrm{d}W_{AB}}{\mathrm{d}l_\parallel} = -\frac{\mathrm{d}U}{\mathrm{d}l_\parallel}.$$

In vector notation,

$$\boxed{\mathbf{F} = -\,U.} \tag{1}$$

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[2] The unit of charge (Coulomb) is defined in the following way (The reason *Ampere* are defined first is because *Ampere* happens to be one of the *base unit* of SI): - **Current: Ampere** (I: A): First we define Ampere (surprise!) to be the intensity of current on two ideal wires that are 1 metre apart and produces a force of 2×10^{-7} N between them (1N is defined to be the force that makes an object of mass 1kg moves at the acceleration of 1m/s²). **Charge: Coulomb** (Q: C): Then we define Coulomb be the amount of charge the that passes through a point with a current of 1A over 1 second.

[3] It's worth to notice that conservativity has nothing to do with the inverse square property. The potential energies that corresponds to "inverse force" and "inverse cubic force" are:

$$U_{\frac{1}{r}} = -kQq \ln r,$$

and

$$U_{\frac{1}{r^3}} = \frac{kQq}{2}\frac{1}{r^2}.$$

[5] Actually, should be $U(r) = -k\frac{Q}{r} + $ const. But when $r \to \infty$, we expect U to be 0 because q do not have the ability to do work.
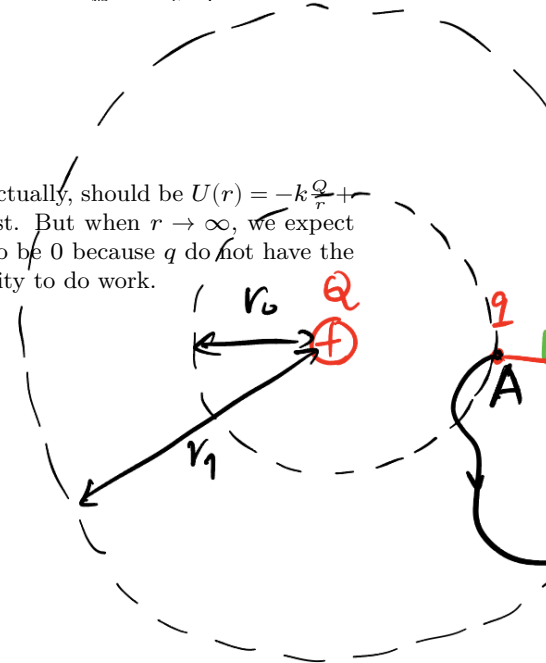
Figure 1: $U(A)$ is expected to be larger t

Get Rid of Test-charge

When q is far smaller than Q, it is called a test charge, which is used to "test" the effect of Q to its surroundings and minimize other interference. It is naturally to get rid of q and define a quantity E that only depends on Q, we expect that E satisfies:

$$\mathbf{F} =: \mathbf{E}q,$$

where E obviously equal to:

$$\boxed{E = k\frac{Q}{r^2}.}$$

We call E the **electric field** generated by Q[6].

Therefore, Equation 1 could be written as:

$$\mathbf{E}q = -U. \tag{2}$$

We could also get rid of q in Equation 2 by defining a quantity V called the *(Electric) potential* generated by Q:

$$\boxed{U := Vq,}$$

so we have

$$\mathbf{E}q = -Vq$$
$$\implies \boxed{\mathbf{E} = -V.} \tag{3}$$

You can compare Equation 1 and Equation 3, the latter is test-charge-free version of the former!

> 💡 Tip 1: The spirit of *fields*
>
> This mindset from *force* to *field* is extremely important! "Field" originates from "force" but later evolves independently from it, as you will see. Thinking in terms of "fields" rather than "forces" is a key factor that distinguishes beginners from experts. Now you have evolved to the second level – make "fields" be your **second nature**!

[6] Since $\mathbf{E}$ is equivalent to force (just up to a constant), all properties of $\mathbf{E}$ is inherited from $\mathbf{F}$, such as *vector property*, *superposition*, *conservativity*, etc.

An Interesting Question

How to know how many charges inside some closed region?

Proposition 0.1. *The charge inside some closed surface S can be calculated by only looking at the field sitting on its* surface[7]*:*

$$Flux \propto Q_{in},$$

where the flux Φ is defined:

$$\Phi := \oiint_S \mathbf{E} \cdot \mathrm{d}A.$$

Solution 0.1.

1. We will consider the case then there is only one point charge q inside S.

   Consider a sphere R of radius r around q in Figure 3, it's obviously that the flux through R does not related to r, because the surface area increases at the rate of r^2 and the field decays at the rate of $1/r^2$. Just to be intimidating,

   $$\Phi_R = \oiint_R k\frac{q}{r^2}dA = k\frac{q}{r^2} \cdot 4\pi r^2 = 4\pi kq \propto q.$$

   We commonly let

   $$\boxed{k = \frac{1}{4\pi\epsilon_0}} \tag{4}$$

   to simpify[8] Equation 4 to be

   $$\Phi_R = \frac{q}{\epsilon_0}.$$

2. Now we claim that

   $$\Phi_R = \Phi_S,$$

   where S is an arbitrary closed surface outside R.

   Again we use the *inverse square property*, the flux through $\mathrm{d}R$ should be same as the flux through the blue circle in Figure 3. Plus, the flux through the blue circle is exactly the same as the flux through $\mathrm{d}S$ as shown in Figure 4 (since their "perpendicular" surface area are the same)

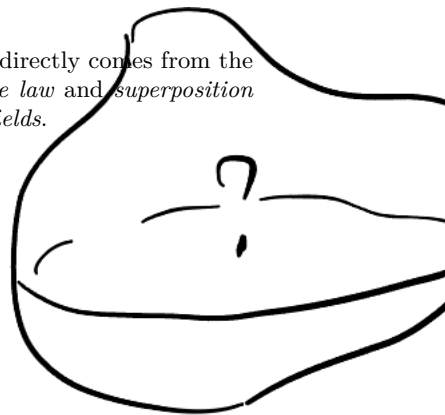[7] This claim directly comes from the *inverse square law* and *superposition principle of fields.*

Figure 2: The question

[8] We introduce the symbol ϵ_0 by the motivation to eliminate the "4π" Equation 4. But the meaning of ϵ_0 would be clear later until we introduce the electric fields in matter. Don't worry.

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3. We then use *superposition property* of fields to obtain the equation of multiple charges enclosed.

Suppose there are $N = 3$ point charges inside S as shown in Figure 5, the total flux is

$$\Phi = \oiint_S \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \sum_{i=1}^{3} \left(\oiint_S \mathbf{E}_i \cdot \mathrm{d}\mathbf{A} \right) = \sum_{i=1}^{3} \frac{q_i}{\epsilon_0} = \frac{Q_{\text{in}}}{\epsilon_0}.$$
(5)

Of course Equation 5 can be generalized when N is arbitrary. And ura! We have just proof the integral version of Maxwell's first equation!

$$\boxed{\oiint_S \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{Q_{\text{in}}}{\epsilon_0}.}$$
(6)

This is also known as **Gauss's Law**. Equation 6 holds for *any* closed surface S.

Differential Form of Maxwell's First Equation

The charges in the real world are not commonly appears like an infinitesimal point. They distributed evenly through a body instead of concentrate on a point of no size. Therefore, Q_{in} on the RHS[9] of Equation 6 could be write as an integral:

$$\frac{Q_{\text{in}}}{\epsilon_0} = \iiint_V \frac{\rho}{\epsilon_0} \mathrm{d}V,$$

where V is the region enclosed by S, ρ is the **density**[10] of charges at some place inside S.

What about the LHS of Equation 6? We *expect* it to be also written in a kind of volume integral to cancel it out with the RHS:

$$\oiint_S \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \iiint_V \boxed{???} \, \mathrm{d}V.$$

And then

$$\iiint_V \boxed{???} \, \mathrm{d}V = \iiint_V \frac{\rho}{\epsilon_0} \mathrm{d}V$$
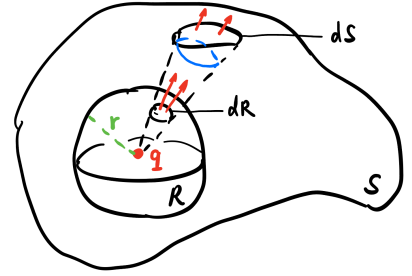(7)

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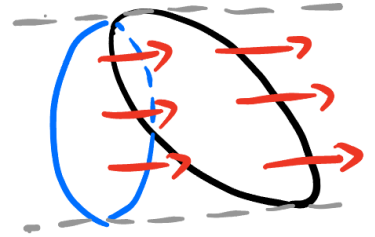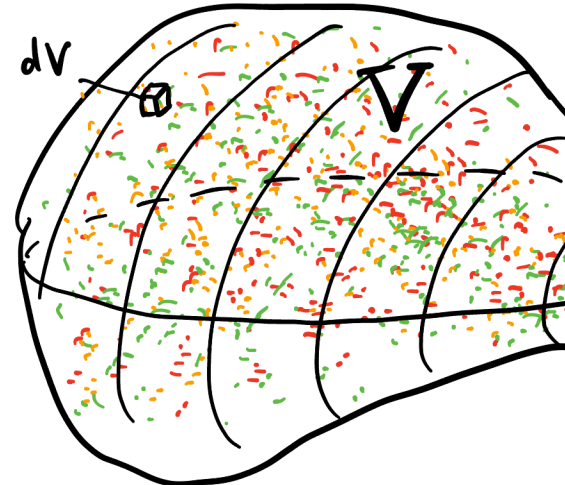Figure 3: Single point charge case

Figure 4: The flux through the blue circle is exactly the same as the flux through $\mathrm{d}S$

the behaviour of this kind of "degenerated" density. It is not a func-
Figure 6: Real world charges are not po
tion, just a symbol following some
rules that deals with the Mathemati-

holds for any^{11} volume V, so we can claim that

$$\boxed{???} = \frac{\rho}{\epsilon_0}. \tag{8}$$

Luckily! **Stoke's Theorem** tells us

$$\boxed{???} = \operatorname{div} \mathbf{E}.$$

For those of you not familiar with multivariable calculus, $\operatorname{div} \mathbf{E}$ is called the **divergence**[12] of $\mathbf{E}$, which is a scalar-valued function purely derived from $\mathbf{E}$.

Finally! We got the differential form:

$$\boxed{\operatorname{div} \mathbf{E} = \frac{\rho}{\epsilon_0}}. \tag{9}$$

[12] Also denoted $\cdot \mathbf{E}$.

[11] *Any* is very important! Without *any*, we cannot derive Equation 7 from Equation 8.

Thinking Problem

Exercise 0.1. Think about what properties of $\mathbf{E}$ ensures that Equation 9 holds true?

Solution 0.2. There are only two properties of electric fields that are used to obtain Equation 9:

- Inverse square
- Superposition

Stoke's Theorem don't count because it holds for any vector fields.

Superposition is trivial. The non-trivial part is inverse square. This property ensures that we can extend the sphere to an arbitrary surface in Figure 3. What is the nature of inverse square?

Well, we live in 3-dimensional space. Everything that spreads should somehow decay at the rate of $1/r^2$, like light, gravity, sound, etc. Otherwise it will against the conservation of energy. In general, if we live in a N-dim world, fields should naturally decay at the rate of $1/r^{N-1}$. In other words, everything in N-dim world should decay at the rate of "area" decay. This paper by myself delved a little deeper inside this inspiration.