Q&A: Basis vectors are *exactly* the same as partial derivative operator?

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2025-03-04

Question

In differential geometry, we usually see a vector v is written as:

$$v = v^i \frac{\partial}{\partial x^i} \bigg|_p.$$

Why does a vector *naturally* relates to partial derivatives?

> One-line Solution
>
> $$T_p(\mathbb{R}^n) \cong \mathrm{Der}_p(C^\infty(\mathbb{R}^n))$$

Solution – From Derivative to Derivation

Directional derivative

We know from multivariable calculus that in high dimensions, we could not say the "derivative", but the *directional derivative* of a function[1]. The directional derivative is a measure of how quickly the function value vary when we step a tiny nudge along a vector v. Imagine we are at p in $\mathbb{R}^3$ and temperature is different everywhere. We are curiously about how this temperature field f changes in different directions. we move a tiny

[1] "Scalar field" in fancier term. A scalar field in $\mathbb{R}^n$ is a map from $\mathbb{R}^n$ to $\mathbb{R}$.

proportion[2] along v (say $\epsilon = 0.01\%$) and we feel the temperature changes by $\Delta f = f(p + \epsilon v) - f(p)$. So we define the directional derivative of f along v is

$$D_v f|_p := \lim_{\epsilon \to 0} \frac{\Delta f}{\epsilon}.$$

It turns out that there is an explicit formula for directional derivatives:

$$D_v f = \langle \nabla f, v \rangle,$$

i.e., the inner product between the gradient of f and v. The direction of the ∇f is the steepest ascend of f at p. In $\mathbb{R}^3$, this can be written as[3]

$$\begin{aligned}
D_v f &= \langle \frac{\partial f}{\partial x^1} e_1 + \frac{\partial f}{\partial x^2} e_2 + \frac{\partial f}{\partial x^3} e_3, v^1 e_1 + v^2 e_2 + v^3 e_3 \rangle \\
&= v^1 \frac{\partial f}{\partial x^1} + v^2 \frac{\partial f}{\partial x^2} + v^3 \frac{\partial f}{\partial x^3} \\
&= \sum_i v^i \frac{\partial f}{\partial x^i} \\
&=: v^i \frac{\partial f}{\partial x^i}.
\end{aligned} \tag{1}$$

The last step in Equation 1 where we drop the summation notation is a convention called Einstein notation.

We could view $D_v f$ as v acts on f. Some textbook uses $v[f]$ to represent this action, i.e.,

$$v[f] := D_v f.$$

Derivation

We know a normal derivative satisfy so-called chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}(fg) = \frac{\mathrm{d}f}{\mathrm{d}x} g + f \frac{\mathrm{d}g}{\mathrm{d}x}.$$

We extract this property and define abstractly the ***derivation*** operator on an algebra as follows:

[2] This is important! We are NOT moving a tiny *bit* but a tiny *proportion*, which means the length of v matters. Because if we move 0.01% on v and $2v$, f will vary Δf and $2\Delta f$ and therefore the directional derivative of f along $2v$ would be doubled! In some books, you will see we force v to be unit length, so we will not have this problem. But for me it's unnecessary.

[3] We use upper indices to represent coordinate components and lower indices to represent basis vectors, so Equation 1 in usually notation is just

$$D_v f = \langle \frac{\partial f}{\partial x} \hat{\imath} + \frac{\partial f}{\partial y} \hat{\jmath} + \frac{\partial f}{\partial z} \hat{k}, v_1 \hat{\imath} + v_2 \hat{\jmath} + v_3 \hat{k} \rangle.$$

> **Derivation on an Algebra**
>
> **Definition 0.1.** Let A be an algebra over field $\mathbb{F}$, a ***derivation*** is a linear map $D : A \to A$ s.t.,
>
> $$D(ab) = D(a)b + aD(b).$$

It's obvious that every v induces such a derivation on the algebra C_p^∞ by a map $\phi : v \mapsto D_v$. The question is: **Does every derivation necessarily induced by a vector?**

> **Vectors are Derivations**
>
> **Theorem 0.1.** *The space of all vectors emanating at p is isomorphic to the space of all derivations*
>
> $$T_p(\mathbb{R}^n) \cong Der_p(C^\infty(\mathbb{R}^n)).$$

In other words, every possible derivations on the algebra $C^\infty(\mathbb{R}^n)$ is some directional derivative along $v \in T_p(\mathbb{R}^n)$. Under this isomorphism, the basis vectors e_i is mapped to the partial derivative operator $\frac{\partial}{\partial x^i}$!

In a general manifold M, we actually use derivations to *define **tangent vectors*** on a manifold[4]. Because the concept of *derivations* are just functions that satisfy certain property, which is easy to define. While *vectors* seem exclusively belongs to Euclidean space. So :

> **Tangent Vector in a manifold**
>
> **Definition 0.2.** A tangent vector at a point p in a manifold M is a derivation at p.

This is common in mathematics. We call this "stereotyping", ah sorry, "abstraction". We find two similar concepts (e.g. vectors and directional derivatives) on some object (euclidean space). But one of them (directional derivative) can be easily generalized to another objects ("manifold"). So then Mathematicians use some of its properties back to *define* itself axiomatically and called it the same name just to confuse people ("tangent vectors")[5]. Or invent another name (e.g. topological space) just

[4] Tu's book is a very good book of differential geometry for beginners, check it out!

[5] Other examples include *topological spaces*, *groups*, σ-*algebra*, "*measurable spaces*", etc. These are just abstraction of open sets, closed stuff, events, volumes, etc.

to be intimidating. Anyway, you will feel comfortable once you get used to them.