Q&A: Basis vectors are *exactly* the same as partial derivative operator?

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Question

In differential geometry, we usually see a vector v is written as:

$$v = v^i \frac{\partial}{\partial x^i}\bigg|_p .$$

Why does a vector *naturally* relates to partial derivatives?

> One-line Solution
>
> $$T_p(\mathbb{R}^n) \cong \mathrm{Der}_p(C^\infty(\mathbb{R}^n))$$

Solution – From Derivative to Derivation

Directional derivative

We know from multivariable calculus that in high dimensions, we could not say the "derivative", but the *directional derivative* of a function[1]. The directional derivative is a measure of how quickly the function value vary when we step a tiny nudge along a vector v. Imagine we are at p in $\mathbb{R}^3$ and temperature is different everywhere. We are curiously about how this temperature field f changes in different directions. we move a tiny

[1] "Scalar field" in fancier term. A scalar field in $\mathbb{R}^n$ is a map from $\mathbb{R}^n$ to $\mathbb{R}$.

proportion[2] along v (say $\epsilon = 0.01\%$) and we feel the temperature changes by $\Delta f = f(p + \epsilon v) - f(p)$. So we define the directional derivative of f along v is

$$D_v f\big|_p := \lim_{\epsilon \to 0} \frac{\Delta f}{\epsilon}.$$

It turns out that there is an explicit formula for directional derivatives:

$$D_v f = \langle \nabla f, v \rangle,$$

i.e., the inner product between the gradient of f and v. The direction of the ∇f is the steepest ascend of f at p. In $\mathbb{R}^3$, this can be written as[3]

$$\begin{aligned} D_v f &= \langle \frac{\partial f}{\partial x^1} e_1 + \frac{\partial f}{\partial x^2} e_2 + \frac{\partial f}{\partial x^3} e_3, v^1 e_1 + v^2 e_2 + v^3 e_3 \rangle \\ &= v^1 \frac{\partial f}{\partial x^1} + v^2 \frac{\partial f}{\partial x^2} + v^3 \frac{\partial f}{\partial x^3} \\ &= \sum_i v^i \frac{\partial f}{\partial x^i} \\ &=: v^i \frac{\partial f}{\partial x^i}. \end{aligned} \tag{1}$$

The last step in Equation 1 where we drop the summation notation is a convention called Einstein notation.

We could view $D_v f$ as v acts on f. Some textbook uses $v[f]$ to represent this action, i.e.,

$$v[f] := D_v f.$$

Derivation

We know a normal derivative satisfy so-called chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}(fg) = \frac{\mathrm{d}f}{\mathrm{d}x} g + f \frac{\mathrm{d}g}{\mathrm{d}x}.$$

We extract this property and define abstractly the ***derivation*** operator on an algebra as follows:

[2] This is important! We are NOT moving a tiny *bit* but a tiny *proportion*, which means the length of v matters. Because if we move 0.01% on v and $2v$, f will vary Δf and $2\Delta f$ and therefore the directional derivative of f along $2v$ would be doubled! In some books, you will see we force v to be unit length, so we will not have this problem. But for me it's unnecessary.

[3] We use upper indices to represent coordinate components and lower indices to represent basis vectors, so Equation 1 in usually notation is just

$$D_v f = \langle \frac{\partial f}{\partial x} \hat{\imath} + \frac{\partial f}{\partial y} \hat{\jmath} + \frac{\partial f}{\partial z} \hat{k}, v_1 \hat{\imath} + v_2 \hat{\jmath} + v_3 \hat{k} \rangle.$$

Derivation on an Algebra

Definition 0.1. Let A be an algebra over field $\mathbb{F}$, a ***derivation*** is a linear map $D : A \to A$ s.t.,

$$D(ab) = D(a)b + aD(b).$$

It's obvious that every v induces such a derivation on the algebra C_p^∞ by a map $\phi : v \mapsto D_v$. The question is: **Does every derivation necessarily induced by a vector?**

Vectors are Derivations

Theorem 0.1. *The space of all vectors emanating at p is isomorphic to the space of all derivations*

$$T_p(\mathbb{R}^n) \cong Der_p(C^\infty(\mathbb{R}^n)).$$

In other words, every possible derivations on the algebra $C^\infty(\mathbb{R}^n)$ is some directional derivative along $v \in T_p(\mathbb{R}^n)$. Under this isomorphism, the basis vectors e_i is mapped to the partial derivative operator $\frac{\partial}{\partial x^i}$!

In a general manifold M, the concept of *derivations* remain while *vectors* is hard to define. So we will actually use derivations to define ***tangent vector*** in a manifold[4]:

[4] Tu's book is a very good book of differential geometry for beginners, check it out!

Tangent Vector in a manifold

Definition 0.2. A tangent vector at a point p in a manifold M is a derivation at p.