

# Q&A: Basis vectors are *exactly* the same as partial derivative operator?

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## Question

In differential geometry, we usually see a vector  $v$  is written as:

$$v = v^i \frac{\partial}{\partial x^i} \Big|_p.$$

Why does a vector *naturally* relates to partial derivatives?

One-line Solution

$$T_p(\mathbb{R}^n) \cong \text{Der}_p(C^\infty(\mathbb{R}^n))$$

## Solution – From Derivative to Derivation

### Directional derivative

We know from multivariable calculus that in high dimensions, we could not say the “derivative”, but the *directional derivative* of a function<sup>1</sup>. The directional derivative is a measure of how quickly the function value vary when we step a tiny nudge along a vector  $v$ . Imagine we are at  $p$  in  $\mathbb{R}^3$  and temperature is different everywhere. We are curious about how this temperature field  $f$  changes in different directions. we move a tiny

<sup>1</sup> “Scalar field” in fancier term. A scalar field in  $\mathbb{R}^n$  is a map from  $\mathbb{R}^n$  to  $\mathbb{R}$ .

proportion<sup>2</sup> along  $v$  (say  $\epsilon = 0.01\%$ ) and we feel the temperature changes by  $\Delta f = f(p + \epsilon v) - f(p)$ . So we define the directional derivative of  $f$  along  $v$  is

$$D_v f|_p := \lim_{\epsilon \rightarrow 0} \frac{\Delta f}{\epsilon}.$$

It turns out that there is an explicit formula for directional derivatives:

$$D_v f = \langle \nabla f, v \rangle,$$

i.e., the inner product between the gradient of  $f$  and  $v$ . The direction of the  $\nabla f$  is the steepest ascend of  $f$  at  $p$ . In  $\mathbb{R}^3$ , this can be written as<sup>3</sup>

$$\begin{aligned} D_v f &= \left\langle \frac{\partial f}{\partial x^1} e_1 + \frac{\partial f}{\partial x^2} e_2 + \frac{\partial f}{\partial x^3} e_3, v^1 e_1 + v^2 e_2 + v^3 e_3 \right\rangle \\ &= v^1 \frac{\partial f}{\partial x^1} + v^2 \frac{\partial f}{\partial x^2} + v^3 \frac{\partial f}{\partial x^3} \\ &= \sum_i v^i \frac{\partial f}{\partial x^i} \\ &=: v^i \frac{\partial f}{\partial x^i}. \end{aligned} \tag{1}$$

The last step in Equation 1 where we drop the summation notation is a convention called [Einstein notation](#).

**We could view  $D_v f$  as  $v$  acts on  $f$ .** Some textbook uses  $v[f]$  to represent this action, i.e.,

$$v[f] := D_v f.$$

## Derivation

We know a normal derivative satisfy so-called chain rule:

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f \frac{dg}{dx}.$$

We extract this property and define abstractly the **derivation** operator on an algebra as follows:

<sup>2</sup> This is important! We are NOT moving a tiny *bit* but a tiny *proportion*, which means the length of  $v$  matters. Because if we move 0.01% on  $v$  and  $2v$ ,  $f$  will vary  $\Delta f$  and  $2\Delta f$  and therefore the directional derivative of  $f$  along  $2v$  would be doubled! In some books, you will see we force  $v$  to be unit length, so we will not have this problem. But for me it's unnecessary.

<sup>3</sup> We use upper indices to represent coordinate components and lower indices to represent basis vectors, so Equation 1 in usually notation is just

$$D_v f = \left\langle \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}, v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k} \right\rangle.$$

### Derivation on an Algebra

**Definition 0.1.** Let  $A$  be an algebra over field  $\mathbb{F}$ , a **derivation** is a linear map  $D : A \rightarrow A$  s.t.,

$$D(ab) = D(a)b + aD(b).$$

It's obvious that every  $v$  induces such a derivation on the algebra  $C_p^\infty$  by a map  $\phi : v \mapsto D_v$ . The question is: **Does every derivation necessarily induced by a vector?**

### Vectors are Derivations

**Theorem 0.1.** *The space of all vectors emanating at  $p$  is isomorphic to the space of all derivations*

$$T_p(\mathbb{R}^n) \cong \text{Der}_p(C^\infty(\mathbb{R}^n)).$$

In other words, every possible derivations on the algebra  $C^\infty(\mathbb{R}^n)$  is some directional derivative along  $v \in T_p(\mathbb{R}^n)$ . Under this isomorphism, the basis vectors  $e_i$  is mapped to the partial derivative operator  $\frac{\partial}{\partial x^i}$ !

In a general manifold  $M$ , the concept of *derivations* remain while *vectors* is hard to define. So we will actually use derivations to define **tangent vector** in a manifold<sup>4</sup>:

### Tangent Vector in a manifold

**Definition 0.2.** A tangent vector at a point  $p$  in a manifold  $M$  is a derivation at  $p$ .

<sup>4</sup> [Tu's book](#) is a very good book of differential geometry for beginners, check it out!