

# Three ways to Understand the Mixed Product of Vectors!

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**Hello!**

This property of vector operation bother me for a loooooong time:

**Mixed product**<sup>1</sup>:

$$\boxed{a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b).} \quad (1)$$

<sup>1</sup> Also known as “scalar triple product”.

This says nothing but the mixed product is unchanged under a circular shift.

You will understand it in this article!

## Three Approaches to the Mixed Product

Lemma

**Lemma 0.1.** *Let  $a, b, c \in \mathbb{R}^3$ , then*

$$a \cdot (b \times c) = \begin{vmatrix} | & | & | \\ a & b & c \\ | & | & | \end{vmatrix}$$

Once this is established, we can use the fact that:

$$\begin{vmatrix} | & | & | \\ a & b & c \\ | & | & | \end{vmatrix} = \begin{vmatrix} | & | & | \\ b & c & a \\ | & | & | \end{vmatrix} = \begin{vmatrix} | & | & | \\ c & a & b \\ | & | & | \end{vmatrix}$$

to prove Equation 1.

Now let's understand Lemma 0.1 in three different ways!

### Geometric Approach

See Figure 1, dot product make the slanted black box straight but maintains its volume.

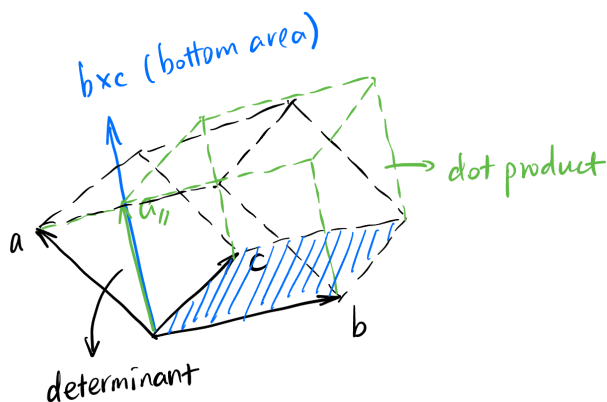


Figure 1: Geometric proof of Lemma 0.1

### 3b1b Approach

I called this “3b1b Approach” because this method is inspired by a great mathematician Grant Sanderson. “3b1b” is the name of [his Youtube channel](#).

In a [video](#) created by him, there is a very interesting function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ :

$$f(x) = \begin{vmatrix} | & | & | \\ x & v & w \\ | & | & | \end{vmatrix} \quad (2)$$

which takes in a vector in  $\mathbb{R}^3$  and output a number in  $\mathbb{R}$  ( $v$  and  $w$  are predefined and fixed).<sup>2</sup> Now I claim that:

Claim

**Theorem 0.1.** *The functional  $f$  in Equation 2 is linear, i.e.,*

$$f(x + y) = f(x) + f(y), \forall x, y \in \mathbb{R}^3$$

and

$$f(\alpha x) = \alpha f(x), \forall \alpha \in \mathbb{R}.$$

This is trivial.

There is another theorem<sup>3</sup>:

Riesz Representation Theorem (reduced version)

**Theorem 0.2.** *Every linear functional in  $\mathbb{R}^n$  induces a vector  $p \in \mathbb{R}^n$  such that*

$$f(x) = x \cdot p, \forall x \in \mathbb{R}^n.$$

Therefore, Equation 2 becomes

$$x \cdot p = \begin{vmatrix} | & | & | \\ x & v & w \\ | & | & | \end{vmatrix} \quad (3)$$

**i** Mnemonic device inspired by Equation 3

The Equation 3 gives us a way to write the cross product in a more “dot-product” way:

$$v \times w = \begin{vmatrix} \hat{i} & v_1 & w_1 \\ \hat{j} & v_2 & w_2 \\ \hat{k} & v_3 & w_3 \end{vmatrix}$$

because the result  $v \times w = (v_2 w_3 - v_3 w_2) \hat{i} + (v_3 w_1 - v_1 w_3) \hat{j} + (v_1 w_2 - v_2 w_1) \hat{k}$  is very much like a dot product! Remember  $v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1$  are just three numbers unless you associate each number with a direction.

<sup>2</sup> Functions from a vector space to a scalar is often referred to as a **functional**.

<sup>3</sup> There is a [video](#) created by myself that explains this in detail.

If you feel uncomfortable with this notation, there is more:

$$\operatorname{curl} F \equiv \nabla \times F = \begin{vmatrix} \hat{i} & \frac{\partial}{\partial x} & F_x \\ \hat{j} & \frac{\partial}{\partial y} & F_y \\ \hat{k} & \frac{\partial}{\partial z} & F_z \end{vmatrix}$$

where we treat  $\nabla$  like a vector! The first column is just an indicator that the result should be interpreted as a vector not a dot product. I took long time to think  $\nabla$  as a kind of special vector, but I failed. Feel free to ignore the notation  $\nabla \times F$  if you don't like it!

What is  $p$  then? Well,  $p$  is a special vector in  $\mathbb{R}^3$  such that the dot product with *any* vector  $x$  gives a number that is equal to the volume of a box spanned by  $x$ ,  $v$ , and  $w$ . This is a little bit mouthful, but we can see immediately from Figure 1 that  $p$  is just the blue vector, which is  $v \times w$  in this case! So Equation 3 becomes:

$$x \cdot (v \times w) = \begin{vmatrix} | & | & | \\ x & v & w \\ | & | & | \end{vmatrix}$$

We are done!

## Tensor Approach

Now this explanation requires some knowledge about tensors<sup>4</sup>. But once you understand it, **you will completely change your view on determinants!** (If you did not respect them at all in the past anyway.)

<sup>4</sup> Just a quick joke, tensors are exactly linear functionals but we allow multiple vectors to be the input (instead of one).

### Proposition

**Proposition 0.1.** *The mapping  $g(a, b, c) := a \cdot (b \times c)$  and  $\det$  are both alternating 3-tensor, i.e.,*

$$g, \det \in \bigwedge^3(\mathbb{R}^3). \quad (4)$$

We also have this theorem:

### Uniqueness of volume form

**Theorem 0.3.** *The dimension of the vector space of  $k$ -tensors on  $\mathbb{R}^n$  is  $\binom{n}{k}$ :*

$$\dim \bigwedge^k (\mathbb{R}^n) = \binom{n}{k}.$$

#### **i** Demonstration of Theorem 0.3

Pick  $n = 3$  and  $k = 2$ . This is because in order to get the result of a 2-tensor<sup>1</sup>  $B$  acting on *any* two vectors of dimension 3, we would only need to specify the values of  $B$  acting on any 2 of the three basis vectors  $(e_x, e_y, e_z)$ . How many numbers do we need? Well, only  $\binom{3}{2} = 3$  numbers:

$$B(e_x, e_y) = a_{12}, B(e_y, e_z) = a_{23}, B(e_z, e_x) = a_{31}.$$

These are the “basis” of the space  $\bigwedge^2 (\mathbb{R}^3)$ . So its dimension is 3. Then we just use the bilinear and alternating property of  $B$  to calculate the result of *any* two input vectors say  $x = 2e_x + e_y - e_z$  and  $y = -e_x + 3e_z$ :

$$\begin{aligned} B(x, y) &= B(2e_x + e_y - e_z, -e_x + 3e_z) \\ &= B(2e_x + e_y - e_z, -e_x) + B(2e_x + e_y - e_z, 3e_z) \\ &= -2B(e_x, e_x) - B(e_y, e_x) + B(e_z, e_x) \\ &\quad + 6B(e_x, e_z) + 3B(e_y, e_z) - 3B(e_z, e_z) \\ &= 0 - (-a_{12}) + a_{31} \\ &\quad + 6(-a_{31}) + 3a_{23} - 0 \\ &= a_{12} + 3a_{23} - 5a_{31}. \end{aligned}$$

As a special case<sup>5</sup> when  $n = k = 3$ , we have:

$$\dim \bigwedge^3 (\mathbb{R}^3) = \binom{3}{3} = 1.$$

Let’s think about what this means. **The determinant is a very special object that *every* volume form in  $n$  dimen-**

<sup>5</sup> This type of tensor is also called “volume form”.

<sup>1</sup>This is also called an alternating bilinear form.

sion is just a scalar multiple of it! In other words, *every* alternating  $n$ -tensor in  $\mathbb{R}^n$  must be the determinant (up to a scalar)! So with Equation 4, both  $g$  and  $\det$  are volume forms! So  $g$  must be a scalar multiple of  $\det$ :

$$g = k \det.$$

We can evaluate  $k$  by choosing a very special set of  $a, b, c \in \mathbb{R}^3$ , say  $a = e_x, b = e_y, c = e_z$ :

$$g(e_x, e_y, e_z) = e_x \cdot (e_y \times e_z) = e_x \cdot e_x = 1,$$

$$\det(e_x, e_y, e_z) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Therefore,  $k = 1$  and  $g = \det$ . We are done!