Three ways to Understand the Mixed Product of Vectors!

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Hello!

This property of vector operation bother me for a loooooong time:

Mixed product[1]:

$$\boxed{a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b).} \tag{1}$$

[1] Also known as "scalar triple product".

This says nothing but the mixed product is unchanged under a circular shift.

You will understand it in this article!

Three Approaches to the Mixed Product

Lemma

Lemma 0.1. *Let $a, b, c \in \mathbb{R}^3$, then*

$$a \cdot (b \times c) = \begin{vmatrix} | & | & | \\ a & b & c \\ | & | & | \end{vmatrix}$$

Once this is established, we can use the fact that:

$$\begin{vmatrix} | & | & | \\ a & b & c \\ | & | & | \end{vmatrix} = \begin{vmatrix} | & | & | \\ b & c & a \\ | & | & | \end{vmatrix} = \begin{vmatrix} | & | & | \\ c & a & b \\ | & | & | \end{vmatrix}$$

to prove Equation 1.

Now let's understand Lemma 0.1 in three different ways!

Geometric Approach

See Figure 1, dot product make the slanted black box straight but maintains its volume.

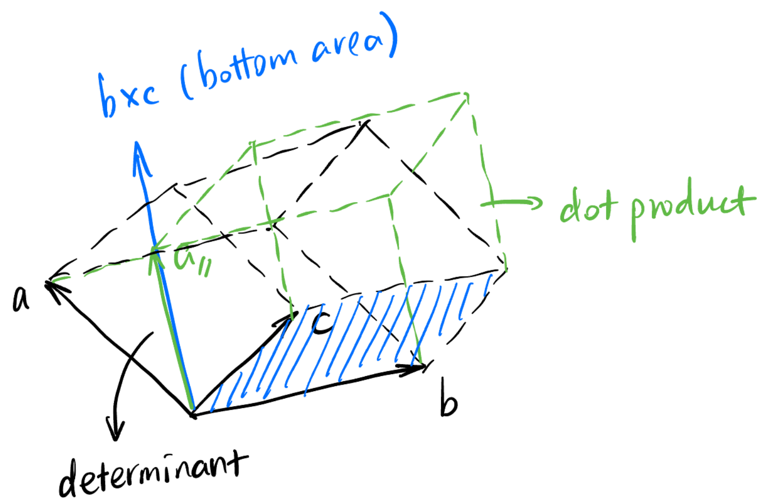

Figure 1: Geometric proof of Lemma 0.1

3b1b Approach

I called this "3b1b Approach" because this method is inspired by a great mathematician Grant Sanderson. "3b1b" is the name of his Youtube channel.

In a video created by him, there is a very interesting function $f : \mathbb{R}^3 \to \mathbb{R}$:

$$f(x) = \begin{vmatrix} | & | & | \\ x & v & w \\ | & | & | \end{vmatrix} \tag{2}$$

which takes in a vector in $\mathbb{R}^3$ and output a number in $\mathbb{R}$ (v and w are predefined and fixed).[2] Now I claim that:

Claim

Theorem 0.1. *The functional f in Equation 2 is* linear, *i.e.,*

$$f(x + y) = f(x) + f(y), \forall x, y \in \mathbb{R}^3$$

and

$$f(\alpha x) = \alpha f(x), \forall \alpha \in \mathbb{R}.$$

This is trivial.

There is another theorem[3]:

Riesz Representation Theorem (reduced version)

Theorem 0.2. *Every linear functional in $\mathbb{R}^n$ induces a vector $p \in \mathbb{R}^n$ such that*

$$f(x) = x \cdot p, \forall x \in \mathbb{R}^n.$$

Therefore, Equation 2 becomes

$$\boxed{x \cdot p = \begin{vmatrix} | & | & | \\ x & v & w \\ | & | & | \end{vmatrix}} \tag{3}$$

Mnemonic device inspired by Equation 3

The Equation 3 gives us a way to write the cross product in a more "dot-product" way:

$$v \times w = \begin{vmatrix} \hat{\imath} & v_1 & w_1 \\ \hat{\jmath} & v_2 & w_2 \\ \hat{k} & v_3 & w_3 \end{vmatrix}$$

[2] Functions from a vector space to a scalar is often referred to as a ***functional***.

[3] There is a video created by myself that explains this in detail.

because the result $v \times w = (v_2w_3 - v_3w_2)\hat{\imath} + (v_3w_1 - v_1w_3)\hat{\jmath} + (v_1w_2 - v_2w_1)\hat{k}$ is very much like a dot product! Remember $v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1$ are just three numbers unless you you associate each number with a direction.
If you feel uncomfortable with this notation, there is more:

$$\operatorname{curl} F \equiv \nabla \times F = \begin{vmatrix} \hat{\imath} & \frac{\partial}{\partial x} & F_x \\ \hat{\jmath} & \frac{\partial}{\partial y} & F_y \\ \hat{k} & \frac{\partial}{\partial z} & F_z \end{vmatrix}$$

where we treat ∇ like a vector! The first column is just an indicator that the result should be interpreted as a vector not a dot product. I took long time to think ∇ as a kind of special vector, but I failed. Feel free to ignore the notation $\nabla \times F$ if you don't like it!

What is p then? Well, p **is a special vector in $\mathbb{R}^3$ such that the dot product with *any* vector x gives a number that is equal to the volume of a box spaned by x, v, and w.** This is a little bit mouthful, but we can see immediately from Figure 1 that p is just the blue vector, which is $v \times w$ in this case! So Equation 3 becomes:

$$x \cdot (v \times w) = \begin{vmatrix} | & | & | \\ x & v & w \\ | & | & | \end{vmatrix}$$

We are done!

Tensor Approach

Now this explanation requires some knowledge about tensors[4]. But once you understand it, **you will completely change your view on *determinants***! (If you did not respect them at all in the past anyway.)

[4] Just a quick joke, tensors are exactly linear functionals but we allow multiple vectors to be the input (instead of one).

Proposition

Proposition 0.1. *The mapping* $g(a, b, c) := a \cdot (b \times c)$ *and* $\det$ *are both alternating 3-tensor, i.e.,*

$$g, \det \in \bigwedge^3 (\mathbb{R}^3) . \tag{4}$$

We also have this theorem:

Uniqueness of volume form

Theorem 0.3. *The dimension of the vector space of* k*-tensors on* $\mathbb{R}^n$ *is* $\binom{n}{k}$*:*

$$\dim \bigwedge^k (\mathbb{R}^n) = \binom{n}{k}.$$

Demonstration of Theorem 0.3

Pick $n = 3$ and $k = 2$. This is because in order to get the result of a 2-tensor[1] B acting on *any* two vectors of dimension 3, we would only need to specify the values of B acting on any 2 of the three basis vectors (e_x, e_y, e_z). How many numbers do we need? Well, only $\binom{3}{2} = 3$ numbers:

$$B(e_x, e_y) = a_{12}, B(e_y, e_z) = a_{23}, B(e_z, e_x) = a_{31}.$$

These are the "basis" of the space $\bigwedge^2 (\mathbb{R}^3)$. So its dimension is 3. Then we just use the bilinear and alternating property of B to calculate the result of *any* two input vectors say $x = 2e_x + e_y - e_z$ and $y = -e_x + 3e_z$:

$$\begin{aligned}
B(x, y) &= B(2e_x + e_y - e_z, -e_x + 3e_z) \\
&= B(2e_x + e_y - e_z, -e_x) + B(2e_x + e_y - e_z, 3e_z) \\
&= -2B(e_x, e_x) - B(e_y, e_x) + B(e_z, e_x) \\
&+ 6B(e_x, e_z) + 3B(e_y, e_z) - 3B(e_z, e_z) \\
&= 0 - (-a_{12}) + a_{31} \\
&+ 6(-a_{31}) + 3a_{23} - 0 \\
&= a_{12} + 3a_{23} - 5a_{31}.
\end{aligned}$$

As a special case[5] when $n = k = 3$, we have:

$$\boxed{\dim \bigwedge^3 (\mathbb{R}^3) = \binom{3}{3} = 1.}$$

[5] This type of tensor is also called "volume form".

Let's think about what this means. **The determinant is a very special object that *every* volume form in n dimension is just a scalar multiple of it! In other words, *every* alternating n-tensor in $\mathbb{R}^n$ must be the determinant (up to a scalar)!** So with Equation 4, both g and det are volume forms! So g must be a scalar multiple of det:

$$g = k \det .$$

We can evaluate k by choosing a very special set of $a, b, c \in \mathbb{R}^3$, say $a = e_x, b = e_y, c = e_z$:

$$g(e_x, e_y, e_z) = e_x \cdot (e_y \times e_z) = e_x \cdot e_x = 1,$$

$$\det(e_x, e_y, e_z) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Therefore, $k = 1$ and $g = \det$. We are done!

[1]This is also called an alternating bilinear form.