What is symmetry?   ?

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Discover Symmetry

You may have noticed these concepts:

(Additive) Even/Odd Functions

Even/odd complex-valued function

Definition 0.1. A function $f : \mathbb{R}^n \to \mathbb{C}$ is called

- ***conjugate symmetric*** $:\iff f(-\mathbf{v}) = \overline{f(\mathbf{v})}, \forall \mathbf{v} \in \mathbb{R}^n$,
- ***conjugate anti-symmetric*** $:\iff -f(-\mathbf{v}) = \overline{f(\mathbf{v})}, \forall \mathbf{v} \in \mathbb{R}^n$

Special cases of Definition 0.1

Even/odd real function

Definition 0.2. A function $f : \mathbb{R} \to \mathbb{R}$ is called

- ***even*** $:\iff f(-x) = f(x), \forall x \in \mathbb{R}$,
- ***odd*** $:\iff -f(-x) = f(x), \forall x \in \mathbb{R}$

Even/odd multivariate real function

Definition 0.3. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called

- ***even*** $:\iff f(-\mathbf{v}) = f(\mathbf{v}), \forall \mathbf{v} \in \mathbb{R}^n$,
- ***odd*** $:\iff -f(-\mathbf{v}) = f(\mathbf{v}), \forall \mathbf{v} \in \mathbb{R}^n$

Decomposition Property

Theorem 0.1. *Any function $f : \mathbb{R}^n \to \mathbb{C}$ can be decomposed*[1] *into a symmetric part Sf and a anti-symmetric part Af:*

$$f = \frac{Sf + Af}{2},$$

$$Sf := f(\mathbf{v}) + \overline{f(-\mathbf{v})},$$

$$Af := f(\mathbf{v}) - \overline{f(-\mathbf{v})}.$$

In fancier language,

$$\mathbb{C}^{\mathbb{R}^n} = S\mathbb{C}^{\mathbb{R}^n} \oplus A\mathbb{C}^{\mathbb{R}^n}.$$

Note

As a special case of Theorem 0.1, any function $f : \mathbb{R} \to \mathbb{R}$ is a sum of an even and an odd function:

$$f = \frac{(f(x) + f(-x)) + (f(x) - f(-x))}{2}.$$

(Multiplicative) Even/Odd Functions

There are also multiplicative version of Definition 0.1 and Theorem 0.1:

[1]The reason why I do NOT define $Sf = (f(\mathbf{v}) + \overline{f(-\mathbf{v})})/2$ will be clear later.

Multiplicative version of Definition 0.1

Definition 0.4. A function $f : (\mathbb{R}^\times)^n \to \mathbb{C}$ is called[2]

- ***Multiplicative conjugate symmetric*** $:\iff f(\frac{1}{\mathbf{v}}) = \overline{f(\mathbf{v})}, \forall \mathbf{v} \in \mathbb{R}^n$,
- ***conjugate anti-symmetric*** $:\iff \frac{1}{f(\frac{1}{\mathbf{v}})} = \overline{f(\mathbf{v})}, \forall \mathbf{v} \in \mathbb{R}^n$,

where $\frac{1}{\mathbf{v}}$ is another vector in $(\mathbb{R}^\times)^n$ whose components are the reciprocal of those of $\mathbf{v}$.

Multiplicative version of Decomposition Property

Theorem 0.2. *Any function $f : (\mathbb{R}^\times)^n \to \mathbb{C}$ can be decomposed into a symmetric part $S^\bullet f$ and a anti-symmetric part $A^\bullet f$:*

$$f = \sqrt{S^\bullet f \cdot A^\bullet f},$$
$$S^\bullet f := f(\mathbf{v}) \cdot \overline{f(\mathbf{v}^{-1})},$$
$$A^\bullet f := \frac{f(\mathbf{v})}{\overline{f(\mathbf{v}^{-1})}}.$$

Symmetric/Alternating Tensor

[2] $\mathbb{R}^\times := \mathbb{R}\backslash\{0\}$.

Symmetric/Alternating Tensor

Definition 0.5. A symmetric rank-k tensor $f : V^k \to \mathbb{R}$ is ***symmetric*** iff

$$f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = f(v_1, \dots, v_k)$$

for all permutations $\sigma \in S_k$.
It is ***alternating*** iff

$$f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = (\operatorname{sgn} \sigma) f(v_1, \dots, v_k)$$

for all permutations $\sigma \in S_k$.

Though generally we cannot decompose an arbitrary tensor into a symmetric and alternating part, we could build them by introducing two operators:

Symmetric/Alternating Operator for Tensors

Definition 0.6. Given $\forall f : V^k \to \mathbb{R}$, the operator S and A defined below always give a symmetric and alternating tensor[3]:

$$Sf := \sum_{\sigma \in S_k} \sigma f,$$

$$Af := \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sigma f.$$

Matrix

Self-adjoint and Skew-adjoint Matrices

Definition 0.7. A linear operator $\phi \in \operatorname{Hom}(V)$ is called ***self-adjoint*** iff

$$\phi^H = \phi,$$

and ***skew-adjoint*** iff

$$\phi^H = -\phi.$$

[3] σf is defined by $(\sigma f)(v_1, v_2, \dots, v_k) := f(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k)})$.

Symmetry as Group Action

Problem

Is there any way to unify these seemingly "symmetric" concepts? What kind of mathematical object can be symmetrize and and alternate? When does the object itself expressible by only its symmetrized and and alternated ones?

Important Observation

The common thing of the above examples in Section is that the domain of the objects (functions, tensors, matrices[1]) could be manipulated by some kind of actions:

- $f : \mathbb{R}^n \to \mathbb{C}$: additive inversion,
- $f : (\mathbb{R}^\times)^n \to \mathbb{C}$: multiplicative inversion,
- $f : V^k \to \mathbb{R}$: permutation.

[1] This is left as an exercise.

The first two can be viewed as the 2-element *group* S_2 *acts* on the domain of f, where S_2 is the group generated by the operation of "taking inverse":

$$S_2 := \langle \cdot^{-1} \rangle = \{e, \cdot^{-1}\},$$

or equivalently, the permutation group on two letters:

$$S_2 = \{e, (12)\}.$$

Therefore, in the first two cases, we could define a S_2-action:

$$(\sigma f)(\mathbf{v}) := \overline{f(\mathbf{v}^{-1})},$$

where $\mathbf{v}^{-1}$ is either $-\mathbf{v}$ (additive inverse) or $1/\mathbf{v}$ (multiplicative inverse).

Therefore, the definition of the operator S and A in Definition 0.6 also applies for the first two cases:

$$Sf := \sum_{\sigma \in S_2} \sigma f = f(\mathbf{v}) + \overline{f(-\mathbf{v})} \quad (\text{or } f(\mathbf{v}) \cdot \overline{f(-\mathbf{v})}),$$

$$Af := \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sigma f = f(\mathbf{v}) - \overline{f(-\mathbf{v})} \quad (\text{or } \frac{f(\mathbf{v})}{\overline{f(\mathbf{v}^{-1})}}).$$

When Decomposable?

In the first two cases, f can be expressed purely by Sf and Af:

$$f = \frac{Sf + Af}{2} \quad (\text{or } \sqrt{Sf \cdot Af}),$$

which is just the *average* of them! (Arithmetic average and geometric average respectively)

But we don't have this relationship for tensors, i.e., not every rank k tensor can be purely expressed using Sf and Af – apart from the case when $k = 2$:

$$f(v_1, v_2) = \frac{(f(v_1, v_2) + f(v_2, v_1)) + (f(v_1, v_2) - f(v_2, v_1))}{2} = \frac{Sf + Af}{2}.$$

What happened when $k \geq 3$?

Let $f : V^3 \to \mathbb{R}$, we have

$$Sf = f(v_1, v_2, v_3) + f(v_2, v_3, v_1) + f(v_3, v_1, v_2) + f(v_2, v_1, v_3) + f(v_1, v_3, v_2) + f(v_3, v_2, v_1),$$

$$Af = f(v_1, v_2, v_3) + f(v_2, v_3, v_1) + f(v_3, v_1, v_2) - f(v_2, v_1, v_3) - f(v_1, v_3, v_2) - f(v_3, v_2, v_1).$$

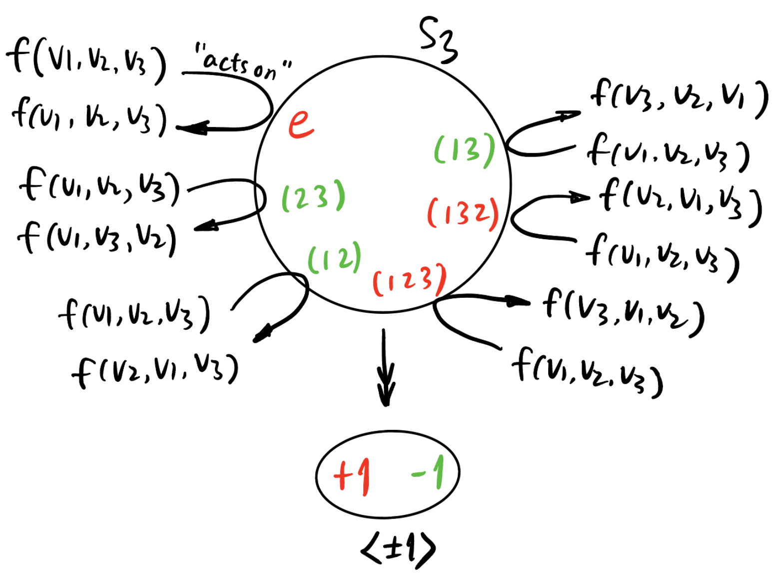

Figure 1: Visualize group action

The result

$$\frac{Sf + Af}{2} = f(v_1, v_2, v_3) + f(v_2, v_3, v_1) + f(v_3, v_1, v_2) = \sum_{\sigma \in A_3} \sigma f \neq f,$$

where A_3 is the alternating group (the group of even permutations) on three letters.

Try Yourself!

Exercise 0.1 **(S and A operator for matrices ϕ).** Let $\phi \in \mathrm{End}(\mathbb{C}^n)$, derive the definition of $S\phi$ and $A\phi$.

Solution

$$S\phi := \frac{\phi + \phi^H}{2},$$

$$A\phi := \frac{\phi - \phi^H}{2}.$$

We also have

$$\phi = \frac{S\phi + A\phi}{2}.$$