Summary of Chapter 4

The main conclusion of the preceding chapters, in relation with the study of quantum computation systems, is that they can be modelled by space vectors over the complex field. Furthermore, given a particular quantum system, described by a space V, the following postulates are satisfied:

- The quantum state of the system is defined by a unitary vector of V.
- The evolution of the quantum state in function of t is defined by a unitary linear transformation.
- The measurement of the quantum state collapses the system state.

Those postulates constitute the starting point of the Quantum Computation theory. The vector spaces under consideration are the sets C^n of n-tuples of complex numbers, that is, dimension-n spaces. It is assumed that any unitary linear transformation over C^n can be implemented by a quantum system. This might seem a maximalist hypothesis. Nevertheless, decomposition techniques, that permit to express any unitary transformation as a product of simple implementable transformations, have been developed.

The method of design of a quantum system can be summarized as follows:

- Basic quantum systems, modelled by vector spaces C^{2^n}, with $n \leq 3$, to which can be associated actual implementations, are defined. Those basic quantum systems play the same role as the logic gates in the case of a digital circuit. This is the central topic of this chapter.
- Decomposition methods for expressing unitary operators as products of basic quantum systems are developed. This is the central topic of chapter 6.

1. Qubits

In a digital circuit, the basic information unit is a bit. They are stored within electronic components, for instance capacitors, flip-flops made up of transistors, floating-gate transistors, and so on. The important point is that those devices have two states, easily and reliably differentiable, to which are associated the values 0 or 1. The information stored by a digital circuit consists of the state of all the bits that it stores. This information must be inputted to and processed by the circuit. For that, predefined components are available, among others logic gates and look-up tables. By assembling those components with connections, digital circuits are created.

When quantum systems were considered as possible information processing hardware platforms, similar terms and concepts were used. The unit of information is the qubit (quantum bit), the predefined available operators are quantum gates, the action of an operator on a qubit, or on a set of qubits, is a quantum operation, and the set of operations performed on a set of qubits is a quantum circuit.

In order to define the state of a qubit, there are two commonly used methods: an algebraic description, that is, a linear combination of vectors of some base, and a geometric description, that is, a point on a unitary sphere.

1.1. Vectorial representation

Consider a quantum system such as the transmon cell of Chap.3. Assume that it can only be in two quantum states: the ground state and the first excited state. This system is a qubit. Let $|0\rangle$ and $|1\rangle$ be the vectorial representation of those states. They constitute a base of the space V associated to this qubit. So, the qubit state $|\psi\rangle$ can be expressed as a linear combination

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle, \tag{1}$$

where a_0 and a_1 are complex numbers. It is a superposition state whose physical meaning is the following: when measuring this qubit state, the result will be 0 with a probability equal to $|a_0|^2$ and 1 with a probability equal to $|a_1|^2$. Furthermore, after the measurement, the qubit state will be $|0\rangle$ or $|1\rangle$, depending on the result.

The following condition expresses that there are no other measurement results than 0 or 1:

$$|a_0|^2 + |a_1|^2 = 1. \tag{2}$$

To summarize, the state of a qubit can be expressed as a unitary vector belonging to a dimension-2 vector space over the field of complex numbers, generated by the orthonormal base $\{|0\rangle, |1\rangle\}$.

In matrix form

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}, |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}, |\psi\rangle = \begin{bmatrix}a_0\\a_1\end{bmatrix}. \tag{3}$$

1.2. Geometric representation

The coefficients a_0 and a_1 of (1) can be expressed in polar coordinates:

$$a_0 = |a_0|e^{i\alpha_0}, a_1 = |a_1|e^{i\alpha_1}. \tag{4}$$

Then, define an angle

$$\theta = 2\cos^{-1}|a_0|, \tag{5}$$

between 0 and π radians, so that

$$|a_0| = \cos\frac{\theta}{2}, |a_1| = \sqrt{1 - |a_0|^2} = \sin\frac{\theta}{2}. \tag{6}$$

With those definitions, the quantum state (1) is

$$|\psi\rangle = e^{i\alpha_0}(\cos(\theta/2)|0\rangle + \sin(\theta/2)\, e^{i(\alpha_1-\alpha_0)}|1\rangle). \tag{7}$$

In this expression, α_0 is a global phase, while α_1-α_0 is a relative phase. A measurement of the state $|\psi\rangle$ would give either 0 or 1 with probabilities equal to

$$p_0 = |e^{i\alpha_0}(\cos(\theta/2)|^2 = \cos^2(\theta/2), p_1 = |e^{i\alpha_1}(\sin(\theta/2)|^2 = \sin^2(\theta/2). \quad (8)$$

A wrong conclusion could be that global and relative phase do not matter. Assume that a linear operation A is applied to the qubit state $|\psi\rangle$. Then, using the vectorial representation (1),

$$A|\psi\rangle = a_0A|0\rangle + a_1A|1\rangle = b_0|0\rangle + b_1|1\rangle. \quad (9)$$

If a_0 and a_1 have a common factor $e^{i\varphi}$, then b_0 and b_1 will have the same common factor, so that the global phase φ remains unobservable. Nevertheless, the next example shows that relative phases can be observed. Assume that the qubit is in state

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \quad (10)$$

and execute the unitary operation

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \quad (11)$$

The result is

$$H|\psi_0\rangle = |0\rangle. \quad (12)$$

Instead, if the qubit is in state

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle + e^{i\pi}\frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle, \quad (13)$$

then

$$H|\psi_1\rangle = |1\rangle. \quad (14)$$

Thus, the linear unitary transformation H (Hadamard operator) makes observable the relative phase difference between states (10) and (13).

In conclusion, all quantum states (7) with the same values of θ and $\varphi = \alpha_1 - \alpha_0$ can be considered as equivalent: they only differentiate by the global phase. Choose the equivalence class element whose global phase is equal to 0. Then, the distinguishable qubit states are

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)\, e^{i\varphi}|1\rangle, \tag{15}$$

with $0 \le \theta \le \pi$ and $0 \le \varphi < 2\pi$. Thus, a quantum state is defined by two real numbers θ y φ, that is, two angles expressed in radians.

In Fig.1, a graphic representation of (15) is shown. The qubit state is a point of a unitary sphere. It is the Bloch sphere.

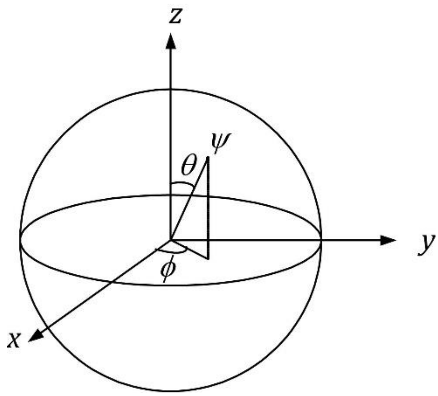

Figura 1 Bloch sphere

In terms of geographic coordinates, θ is the latitude, between 0 radians (North Pole) and π radians (South Pole), and φ is the longitude, between 0 radians (reference meridian) and 2π radians.

In Cartesian coordinates (x, y, z), the sphere equation is

$$x^2 + y^2 + z^2 = 1 \tag{16}$$

and the relation between (1) and the graphical representation of Fig.1 is the following:

- The coordinates that correspond to state (1) are

$$x = \sin\theta\cdot\cos\varphi, y = \sin\theta\cdot\sin\varphi, z = \cos\theta \tag{17}$$

where $\theta = 2\cos^{-1}|a_0|$ and φ is the phase difference between a_1 and a_0.

- The angles θ and φ that correspond to the coordinates (x, y, z) are

$$\theta = \cos^{-1}z,\ 0 \le \theta \le \pi,$$

$$\varphi = \cos^{-1}\frac{x}{\sqrt{x^2+y^2}},\ 0 \le \varphi < 2\pi,\ \mathrm{sign}(\sin\varphi) = \mathrm{sign}(y). \tag{18}$$

Examples of spherical and cartesian representations are given in Table 1 and Fig.2.

$\lvert\psi\rangle$	θ, φ	x, y, z
$\lvert 0\rangle$	0, 0	0, 0, 1
$\lvert 1\rangle$	π, 0	0, 0,-1
$\lvert\boldsymbol{\psi_0}\rangle = \frac{1}{\sqrt{2}}\lvert 0\rangle + \frac{1}{\sqrt{2}}\lvert 1\rangle$	$\pi/2$, 0	1, 0, 0
$\lvert\boldsymbol{\psi_1}\rangle = \frac{1}{\sqrt{2}}\lvert 0\rangle - \frac{1}{\sqrt{2}}\lvert 1\rangle$	$\pi/2, \pi$	-1, 0, 0
$\lvert\boldsymbol{\psi_2}\rangle = \frac{1}{\sqrt{2}}\lvert 0\rangle + \frac{i}{\sqrt{2}}\lvert 1\rangle$	$\pi/2, \pi/2$	0, 1, 0
$\lvert\boldsymbol{\psi_3}\rangle = \frac{1}{\sqrt{2}}\lvert 0\rangle - \frac{i}{\sqrt{2}}\lvert 1\rangle$	$\pi/2, 3\pi/2$	0,-1, 0

Table 1 Spherical and Cartesian coordinates

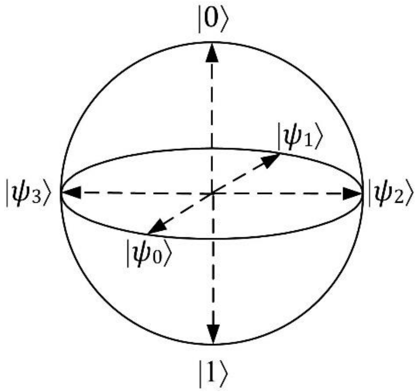

Figure 2 Graphic representation of $|0\rangle$, $|1\rangle$, $|\psi_0\rangle$, $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$

2. *n*-qubit register

A quantum computation system consists of a set $\{q_0, q_1, \ldots, q_{n-1}\}$ of qubits, that is, an n-qubit register. Its quantum state is defined by a linear combination of base vectors:

$$|\psi\rangle = a_0|00...00\rangle + a_1|00...01\rangle + a_2|00...10\rangle + ... + a_{2^n-1}|11...11\rangle. \quad (19)$$

The coefficients a_j are complex numbers and $|a_j|^2$ is the probability of observing the corresponding base state $|j_{n-1}\, j_{n-2} \ldots j_0\rangle$ when measuring the register state. Thus

$$|a_0|^2 + |a_1|^2 + ... + |a_{2^n-1}|^2 = 1. \quad (20)$$

In algebraic terms, the register quantum state is represented by a unitary vector (condition 20) over a dimension-2^n space generated by an orthonormal base $\{|00...00\rangle, |00...01\rangle, \ldots, |11...11\rangle\}$. In matrix form,

$$|00...00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, |00...01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \ldots, |11...11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 1 \end{bmatrix}, \ |\psi\rangle = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdots \\ a_{2^n-1} \end{bmatrix}. \quad (21)$$

3. Quantum gates

Quantum gates are operators able to modify the quantum state of a reduced number of qubits. As an example, consider a set of a few (1, 2 or 3) transmon cells (Chap.3). The application of sequences of microwave signals, with the convenient frequencies, amplitudes, phases and durations, permits to modify their quantum state in a previsible way. It is important to note that, in spite of the fact that the same word "gate", as in the case of digital circuits, is used, a quantum gate is not a component; it makes reference to the possibility of controlling the quantum state of a small register in a predefined way.

3.1. Unary gates

A unary gate (n = 1) is defined by a 2×2 matrix over the complex field:

$$U = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}. \tag{22}$$

Consider a qubit initially in state (1). The operator U executes the following transformation of the qubit state:

$$|\psi\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \xrightarrow{U} U|\psi\rangle = \begin{bmatrix} u_{00}a_0 + u_{01}a_1 \\ u_{10}a_0 + u_{11}a_1 \end{bmatrix} = |\psi'\rangle. \tag{23}$$

By definition of a unitary matrix, $UU^{+} = U^{+}U = I$, where U^{+} is the adjoint (conjugate transpose) of U and I the unitary matrix, so that the operation U is reversible:

$$|\psi'\rangle = U|\psi\rangle \xrightarrow{U^{+}} U^{+}U|\psi\rangle = |\psi\rangle. \tag{24}$$

Furthermore, if $|\psi\rangle$ is unitary, then $|\psi'\rangle$ is also unitary.

3.1.1. Rotations and Pauli operators

Consider a qubit q whose quantum state $|\psi\rangle$ is described by the spheric coordinates φ and θ on the Bloch sphere (Fig.1). Thus

$$|\psi\rangle = e^{i\alpha_0}(\cos(\theta/2)|0\rangle + \sin(\theta/2)\, e^{i\varphi}|1\rangle), \tag{25}$$

whatever the global phase α_0 is. Rotations about a radius of the sphere are reversible operations, so that they might be considered as possible quantum gates.

Let $R_x(\pi)$, $R_y(\pi)$ and $R_z(\pi)$ denote rotations by π radians about the coordinate axes $0x$, $0y$ and $0z$. In Cartesian coordinates, they execute the following transformations:

$$(x, y, z) \xrightarrow{R_x(\pi)} (x, -y, -z),\ (x, y, z) \xrightarrow{R_y(\pi)} (-x, y, -z),\ (x, y, z) \xrightarrow{R_z(\pi)} (-x, -y, z). \tag{26}$$

In spheric coordinates, according to (18), the operations are the following:

$$(\theta, \varphi) \xrightarrow{R_x(\pi)} (\pi-\theta, 2\pi-\varphi),\ (\theta, \varphi) \xrightarrow{R_y(\pi)} (\pi-\theta, \pi-\varphi),\ (\theta, \varphi) \xrightarrow{R_z(\pi)} (\theta, \pi+\varphi). \tag{27}$$

The corresponding transformations of the qubit quantum state are

$$\cos(\theta/2)|0\rangle + \sin(\theta/2)\, e^{i\varphi}|1\rangle \xrightarrow{R_x(\pi)} \sin(\theta/2)\,|0\rangle + \cos(\theta/2)e^{-i\varphi}\,|1\rangle, \quad (28)$$

$$\cos(\theta/2)|0\rangle + \sin(\theta/2)\, e^{i\varphi}|1\rangle \xrightarrow{R_y(\pi)} \sin(\theta/2)\,|0\rangle - \cos(\theta/2)e^{-i\varphi}\,|1\rangle, \quad (29)$$

$$\cos(\theta/2)|0\rangle + \sin(\theta/2)\, e^{i\varphi}|1\rangle \xrightarrow{R_z(\pi)} \cos(\theta/2)\,|0\rangle - \sin(\theta/2)e^{i\varphi}\,|1\rangle. \quad (30)$$

In matrix form, the operations (28), (29) and (30) are

$$A = \begin{bmatrix} 0 & e^{-i\varphi} \\ e^{-i\varphi} & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & e^{-i\varphi} \\ -e^{-i\varphi} & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (31)$$

Taking into account that the global phases are not observable, the same transformations are defined by the following matrices:-

$$X = e^{i\varphi}A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = e^{i(\varphi-\pi/2)}B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (32)$$

Matrices X, Y and Z define the Pauli operators. They are unitary ($X^{-1} = X^{+}$, $Y^{-1} = Y^{+}$, $Z^{-1} = Z^{+}$) and Hermitian ($X = X^{+}$, $Y = Y^{+}$, $Z = Z^{+}$). They are basic quantum gates that execute the following transformations of the quantum state (1):

- The gate X exchanges the coefficients of $|0\rangle$ and $|1\rangle$:

$$a_0|0\rangle + a_1|1\rangle \xrightarrow{X} a_1|0\rangle + a_0|1\rangle. \quad (33)$$

  In particular, it transforms the basic state $|0\rangle$ into $|1\rangle$, and inversely.

- The gate Y exchanges the coefficients of $|0\rangle$ and $|1\rangle$ modifies the relative phase by π radians:

$$a_0|0\rangle + a_1|1\rangle \xrightarrow{Y} -i\cdot a_1|0\rangle + i\cdot a_0|1\rangle \equiv a_1|0\rangle - a_0|1\rangle. \quad (34)$$

  In particular, it transforms the basic state $|0\rangle$ into $i|1\rangle \equiv |1\rangle$ and the basic state $|1\rangle$ into $-i|0\rangle \equiv |0\rangle$.

- The gate Z modifies the relative phase by π radians:

$$a_0|0\rangle + a_1|1\rangle \xrightarrow{Z} a_0|0\rangle - a_1|1\rangle. \quad (35)$$

Rotations by angles γ different from π can also be defined. Relatively simple calculus, based on the spectral decomposition of Hermitian matrices, give the following results, in matrix form:

$$R_x(\gamma) = \begin{bmatrix} \cos\frac{\gamma}{2} & -i\cdot\sin\frac{\gamma}{2} \\ -i\cdot\sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{bmatrix},\ R_y(\gamma) = \begin{bmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{bmatrix},$$

$$R_z(\gamma) = \begin{bmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{bmatrix}. \tag{36}$$

Other commonly used unary gates are the following

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix},\ T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}, \tag{37}$$

that, in turn, are particular cases of the relative phase change gate

$$R_\varphi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix}. \tag{38}$$

In fact, gates (37) and (38) are rotations about the coordinate axis $0z$:

$$R_\varphi = e^{i\varphi/2}R_z(\varphi),\ S = R_{\pi/2} = e^{i\pi/4}R_z(\pi/2),\ T = R_{\pi/4} = e^{i\pi/8}R_z(\pi/4). \tag{39}$$

Observe also that

$$Z = S^2,\ \ S = T^2. \tag{40}$$

3.1.2. Hadamard operator

The Hadamard operator transforms basic states into superposition states. For that reason, it is an essential component of practically all quantum algorithms. It is a unitary and Hermitian operator defined by the matrix

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{41}$$

It executes the following transformation of the quantum state (1):

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle \xrightarrow{H} H|\psi\rangle = \frac{(a_0+a_1)}{\sqrt{2}}|0\rangle + \frac{(a_0-a_1)}{\sqrt{2}}|1\rangle. \tag{42}$$

In particular,

$$|0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, |1\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle. \tag{43}$$

Using the following notations to represent the two preceding superpositions states

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle, \tag{44}$$

relation (43) becomes

$$|0\rangle \xrightarrow{H} |+\rangle, |1\rangle \xrightarrow{H} |-\rangle, \tag{45}$$

and, inversely,

$$|+\rangle \xrightarrow{H} |0\rangle, |-\rangle \xrightarrow{H} |1\rangle. \tag{46}$$

Relations (45) and (46) can be interpreted as encoding operations: they associate superposition states $|+\rangle$ and $|-\rangle$ to the basic states $|0\rangle$ and $|1\rangle$. Observe that if a qubit is in state $|+\rangle$ or $|-\rangle$, as defined by (44), the information is not the probability of observing 0 or 1, when measuring the quantum state - the probabilities are the same, whatever the state - ; the information is the relative phase, 0 in the case of $|+\rangle$ or π in the case of $|-\rangle$. Many quantum algorithms use this type of encoding: encode data with Hadamard operators (more generally, with Fourier transform operators, Chap.7); execute operations that take profit of the superposition states; decode the result with Hadamard operators (with inverse Fourier transform operators).

3.2. Binary gates

A quantum binary gate is an operator able to modify the state of a 2-qubit register. For example, it may consist of two coupled adjacent transmon cells.

The quantum state of a 2-qubit register can be expressed as follows:

$$|\psi\rangle = a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle, \tag{47}$$

with

$$|a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2 = 1. \qquad (48)$$

A binary gate is defined by a 4×4 matrix

$$U = \begin{bmatrix} u_{00} & u_{01} & u_{02} & u_{03} \\ u_{10} & u_{11} & u_{12} & u_{13} \\ u_{20} & u_{21} & u_{22} & u_{23} \\ u_{30} & u_{31} & u_{32} & u_{33} \end{bmatrix}. \qquad (49)$$

over the complex field. It executes the following transformation of the register state:

$$|\psi\rangle = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \xrightarrow{U} U|\psi\rangle = \begin{bmatrix} u_{00}a_0 + u_{01}a_1 + u_{02}a_2 + u_{03}a_3 \\ u_{10}a_0 + u_{11}a_1 + u_{12}a_2 + u_{13}a_3 \\ u_{20}a_0 + u_{21}a_1 + u_{22}a_2 + u_{23}a_3 \\ u_{30}a_0 + u_{31}a_1 + u_{32}a_2 + u_{33}a_3 \end{bmatrix} = |\psi'\rangle. \qquad (50)$$

By definition of a unitary matrix, $UU^+ = U^+U = I$, so that the operation U is reversible, and if $|\psi\rangle$ is unitary, then $|\psi'\rangle$ is also unitary.

3.2.1. Controlled unary operators

Consider two qubits c and t, and a unary unitary transformation U. A controlled CU operator can be defined. It executes the following transformation of the state of qubits c and t: if the control qubit c is in the basic state $|0\rangle$, then the state of the target qubit t doesn't change, but if c is in the basic state $|1\rangle$, then the state transformation U is executed on t. By linearity, the effect of CU on any register state can be defined. Assume that initially the state of r is $|\psi\rangle$ defined by (1). Then

$$|\psi\rangle \xrightarrow{CU} a_0|00\rangle + a_1|01\rangle + a_2(|1\rangle\times U|0\rangle) + a_3(|1\rangle\times U|1\rangle). \qquad (51)$$

If U is defined by (22), so that $U|0\rangle = u_{00}|0\rangle + u_{10}|1\rangle$ y $U|1\rangle = u_{01}|0\rangle + u_{11}|1\rangle$, then

$$|\psi\rangle \xrightarrow{CU} a_0|00\rangle + a_1|01\rangle + (a_2u_{00}+a_3u_{01})\,|10\rangle + (a_2u_{10}+a_3u_{11})\,|11\rangle. \qquad (52)$$

In matrix form,

$$CU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{00} & u_{01} \\ 0 & 0 & u_{10} & u_{11} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix}, \tag{53}$$

and

$$CU^{+} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{00}^{*} & u_{10}^{*} \\ 0 & 0 & u_{01}^{*} & u_{11}^{*} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & U^{+} \end{bmatrix}. \tag{54}$$

The operator CU is unitary, given that U is unitary, and is Hermitian only if U is Hermitian.

Several particular cases of operators CU deserve a special attention. The most important is when U is the Pauli operator X. The CX gate, also called $CNOT$ gate, executes the following transformation:

$$|\psi\rangle \xrightarrow{CX} a_0|00\rangle + a_1|01\rangle + a_3|10\rangle + a_2|11\rangle, \tag{55}$$

to which corresponds the matrix

$$CX = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix}. \tag{56}$$

Observe that if only the basic quantum states $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$ are considered, then the operator CX executes the transformation

$$|xy\rangle \xrightarrow{CX} |x\, x\oplus y\rangle, \tag{57}$$

where x and y are binary values, that is, the same function as a digital XOR gate.

Other CU gates can be defined, based on the unary gates of Sec.3.1.1: CY, CZ, $CR_x(\gamma)$, $CR_y(\gamma)$, $CR_z(\gamma)$, CR_φ.

3.2.2. *SWAP* operator

Given a 2-qubit register (t,u), this operators swaps the quantum states of qubits t and u, that is

$$(t_0|0\rangle + t_1|1\rangle) \times (u_0|0\rangle + u_1|1\rangle) \xrightarrow{SWAP} (u_0|0\rangle + u_1|1\rangle) \times (t_0|0\rangle + t_1|1\rangle). \quad (58)$$

From (58), defining $a_0 = t_0u_0$, $a_1 = t_0u_1$, $a_2 = t_1u_0$, $a_3 = t_1u_1$, the following operation definition is derived:

$$a_0|00\rangle+a_1|01\rangle+a_2|10\rangle+a_3|11\rangle \xrightarrow{SWAP} a_0|00\rangle+a_2|01\rangle+a_1|10\rangle+a_3|11\rangle. \quad (59)$$

In matrix form

$$SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (60)$$

It is unitary and Hermitian.

3.3. Ternary gates

In spite of the fact that their implementation is more difficult, sometimes impossible, depending on the used technology, ternary gates can be defined.

3.3.1. *C^2U* operators

Consider a 3-qubit register (c, d, t) and a unary unitary transformation U. Then, the ternary operator C^2U executes the following transformation of the qubit states: if the control qubit c is in the basic state $|0\rangle$, or if the control qubit d is in the basic state $|0\rangle$, then the state of the target qubit t doesn't change. On the contrary, if both control qubits c and d are in the basic state $|1\rangle$, then the state transformation U is executed on t. By linearity, the effect of C^2U on any register state can be defined.

Assume that initially the state $|\psi\rangle$ of the 3-qubit register is

$$|\psi\rangle = a_0|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle +$$

$$a_4|100\rangle + a_5|101\rangle + a_6|110\rangle + a_7|111\rangle. \quad (61)$$

Then

$$|\psi\rangle \xrightarrow{C^2U} a_0|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle +$$

$$a_4|100\rangle + a_5|101\rangle + a_6(|11\rangle \times U|0\rangle) + a_7(|11\rangle \times U|1\rangle). \quad (62)$$

In matrix form,

$$C^2U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_{00} & u_{01} \\ 0 & 0 & 0 & 0 & 0 & 0 & u_{10} & u_{11} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix}. \quad (63)$$

This operator is unitary, given that U is unitary, and is Hermitian only if U is Hermitian.

The most important particular case is the Toffoli operator C^2X. It executes the following transformation of the initial state $|\psi\rangle$ defined by (61):

$$|\psi\rangle \xrightarrow{C^2X} a_0|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle +$$

$$a_4|100\rangle + a_5|101\rangle + a_7|110\rangle + a_6|111\rangle. \quad (64)$$

In matrix form:

$$C^2X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix}. \quad (65)$$

Observe that if only the basic quantum states $|xyz\rangle$, where x, y and z are binary values, are considered, then the C^2X gate can executes the same functions as digital *AND* or *NAND* gates:

$$|xy0\rangle \xrightarrow{C^2X} |x\,y\,AND(x,y)\rangle,\ |xy1\rangle \xrightarrow{C^2X} |x\,y\,NAND(x,y)\rangle. \qquad (66)$$

3.3.2. Controlled *SWAP* operator

Given a 3-qubit register (c,t,u), this operator swaps the quantum states of qubits t and u, under the control of qubit c, that is

$$(c_0|0\rangle + c_1|1\rangle) \times (t_0|0\rangle + t_1|1\rangle) \times (u_0|0\rangle + u_1|1\rangle) \xrightarrow{CSWAP}$$

$$[c_0|0\rangle\times(t_0|0\rangle+t_1|1\rangle)\times(u_0|0\rangle+u_1|1\rangle)]+$$

$$[c_1|1\rangle\times(u_0|0\rangle+u_1|1\rangle)\times(t_0|0\rangle+t_1|1\rangle)]. \qquad (67)$$

From (67), defining $a_0 = c_0t_0u_0$, $a_1 = c_0t_0u_1$, $a_2 = c_0t_1u_0$, $a_3 = c_0t_1u_1$, $a_4 = c_1t_0u_0$, $a_5 = c_1t_0u_1$, $a_6 = c_1t_1u_0$, $a_7 = c_1t_1u_1$, the following operation definition is deduced:

$$a_0|000\rangle+a_1|001\rangle+a_2|010\rangle+a_3|011\rangle+a_4|100\rangle+a_5|101\rangle+a_6|110\rangle+a_7|111\rangle$$

$$\xrightarrow{CSWAP}$$

$$a_0|000\rangle+a_1|001\rangle+a_2|010\rangle+a_3|011\rangle+a_4|100\rangle+a_6|101\rangle+a_5|110\rangle+a_7|111\rangle. \quad (68)$$

In matrix form

$$CSWAP = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & SWAP \end{bmatrix}. \qquad (69)$$

The *CSWAP* operator, also known as Fedkin operator, is unitary and Hermitian.

4. Quantum circuits

Once quantum gates have been defined, quantum circuits can be generated. Nevertheless, it is essential to understand that those gates are not components that must be placed and interconnected. A quantum circuit is a fixed predefined structure, for example an array of transmon cells, to which can be applied control signals that correspond to the predefined quantum gate operations. By sending control signals to individual qubits, or to pairs of qubits, or even to more qubits, the quantum state of those qubits is modified, accordingly to the successively applied operations.

The symbols used to represent quantum gates are shown in Fig.3 (unary gates, Fig.4 (binary gates) and Fig.5 (ternary gates).

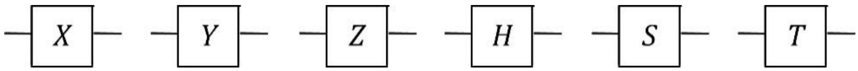

Figure 3 Unary quantum gates

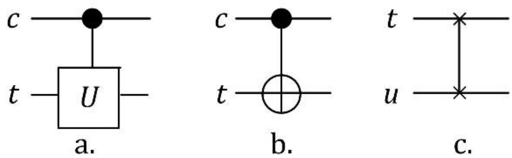

Figure 4 a. Generic *CU* gate, b. *CX* gate, c. *SWAP* gate

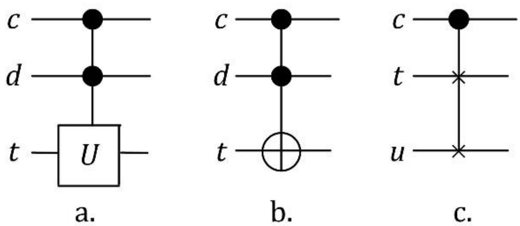

Figure 5 a. Generic C^2U gate, b. C^2X gate, c. *CSWAP* gate

With those symbols, quantum circuits can be created. An example is given in Fig.6. The circuit is made up of two qubits, q_0 and q_1. Assume that initially q_0 is in state $a_0|0\rangle + a_1|1\rangle$ and q_1 in state $b_0|0\rangle + b_1|1\rangle$. Then apply a Hadamard operator H on q_0. As a result, the following transformation is executed:

$$a_0|0\rangle + a_1|1\rangle \xrightarrow{H} \frac{a_0+a_1}{\sqrt{2}}|0\rangle + \frac{a_0-a_1}{\sqrt{2}}|1\rangle. \tag{70}$$

After that, apply a CX operator on q_1 under the control of q_0. The following transformation is executed:

$$\left(\frac{a_0+a_1}{\sqrt{2}}|0\rangle + \frac{a_0-a_1}{\sqrt{2}}|1\rangle\right) \times (b_0|0\rangle + b_1|1\rangle)$$

$$\xrightarrow{CX} \frac{b_0(a_0+a_1)}{\sqrt{2}}|00\rangle + \frac{b_1(a_0+a_1)}{\sqrt{2}}|01\rangle + \frac{b_1(a_0-a_1)}{\sqrt{2}}|10\rangle + \frac{b_0(a_0-a_1)}{\sqrt{2}}|11\rangle. \tag{71}$$

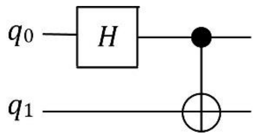

Figure 6 Bell states generation

In particular, if the initial state of qubits q_0 and q_1 are basic states, $|0\rangle$ o $|1\rangle$, the circuit of Fig.6 generates entangled states, known as Bell states:

$$|00\rangle \xrightarrow{H,CX} \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = |\Psi^+\rangle, \tag{72}$$

$$|01\rangle \xrightarrow{H,CX} \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle = |\Phi^+\rangle, \tag{73}$$

$$|10\rangle \xrightarrow{H,CX} \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle = |\Psi^-\rangle, \tag{74}$$

$$|11\rangle \xrightarrow{H,CX} \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle = |\Phi^-\rangle. \tag{75}$$

Observe that, in all four Bell states, the measure of the state of one of the qubits permit to foresee which will be the result of measuring the state of the other qubit. They are entangled.

5. Measurement operators

Measurement operators are unary operators, very different from those defined above. In particular, they are not unitary. Consider a qubit whose quantum state is defined by relation (1). A measurement operator collapses (a non-reversible operation) the qubit in state $|0\rangle$ with a probability equal to $|a_0|^2$ and in state $|1\rangle$ with a probability equal to $|a_1|^2$. Furthermore, it generates binary values, 0 or 1, under some readable form, with the same probabilities $|a_0|^2$ and $|a_1|^2$ as before.

The symbol of a measurement operator is shown in Fig.7.

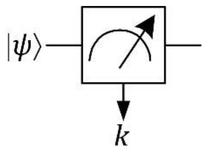

Figure 7 Measurement operator, k = 0 or 1