Summary of Chapter 3

The fact that a particle can be in several, well separated, quantum states, suggests the possibility of encoding data with those states. Using the vectorial representation of the quantum states (Chap.2, Sec.4), assume that the basic states of a particle are denoted by $|0\rangle$, $|1\rangle$, …, $|d\text{-}1\rangle$. Then, the quantum state of a set of m particles is (Chap.2, Sec.7)

$$|\psi\rangle = \sum_{ij\ldots k} c_{ij\ldots k} \left(|i\rangle\times|j\rangle\times \ldots \times|k\rangle\right) \quad (1)$$

where $|c_{ij\ldots k}|^2$ is the probability that a measure of the m individual particle states gives results equal to $i, j, \ldots, k$, respectively. A kind of d-valued logic is thus defined, with the peculiarity that the coefficients $c_{ij\ldots k}$ are not binary, so that superposition states can be defined.

In principle, any isolated physical system, observed at a small enough scale, behaves according to the quantum mechanics postulates. So, there must exist numerous ways to implement a data encoding system based on the quantum states of elementary particles such as electrons, atoms, photons and many others. Nevertheless, those real quantum objects are difficult to observe and to control. For that reason, electronic components, that behave according to the quantum mechanics postulates and are observable and controllable, have been developed. They might be called artificial atoms. In this chapter, a particular type of microwave circuit, with observable and controllable quantum states, is described.

Apart from common electronic components such as capacitors, inductors and sources, those artificial atoms include components whose quantum state is observable and controllable. In this chapter, one of those components is described. It consists of a superconducting island coupled by a Josephson junction to a superconducting reservoir. A major technological problem is that the Josephson effect only occurs at very low temperature, practically zero Kelvin grades.

1. Quantum model of an electrical circuit

In Chap.2, the main postulates of quantum mechanics have been presented and applied to the behavior description of an isolated particle. The particle energy is first expressed (Chap.2, relation 1) as the sum of the kinetic energy $p^2/2m$ and the potential energy $V(x)$. Then, the Hamiltonian operator is derived by replacing p and $V(x)$ by operators. The values of the particle energy are the eigenvalues of the Hamiltonian, and the eigenstates of the particle are the corresponding normalized eigenvectors. Finally, the particle state evolution is defined by a unitary operator derived from the Hamiltonian.

A similar theory can be developed in the case of an electrical circuit. Consider the circuit of Fig.1 made up of a capacitor C (farads) and an inductor L (henrys), and let $v(t)$ be the voltage across C and $i_L(t)$ the current across L.

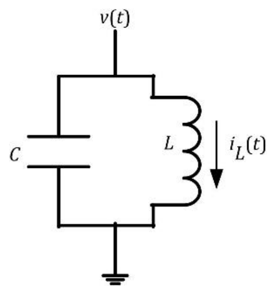

Figure 1 LC circuit

The energy stored in C, due to the electric field generated by the electric charges, is

$$E_C = C\frac{v^2}{2}, \tag{2}$$

and the energy stored in L, due to the magnetic field generated by the current of charges, is equal to

$$E_l = L\frac{{i_L}^2}{2}. \quad (3)$$

The process for quantizing electrical circuits has been described in several publications, for instance [2, 3]. First define the node flux

$$\Phi = \int_{-\infty}^{t} v(\tau)d\tau, \quad (4)$$

so that

$$v(t) = \frac{d\Phi}{dt} = \dot{\Phi}. \quad (5)$$

Then, as the voltage across the inductor terminals is proportional to the derivative of the current,

$$v(t) = -L\frac{di_L(t)}{dt}, \quad (6)$$

and, according to (4),

$$\Phi = -L\int_{-\infty}^{t} \frac{di_L(\tau)}{d\tau} d\tau = -Li_L, \quad (7)$$

assuming that $i_L(-\infty) = 0$. Relation (7) indicates that Φ is the magnetic flux generated by the inductor.

Using Φ and $\dot{\Phi}$ instead of i and v, the capacitive and inductive energies can be expressed as

$$E_c = C\frac{\dot{\Phi}^2}{2}, \quad E_l = \frac{C\omega^2\Phi^2}{2}, \text{ with } \omega = \frac{1}{\sqrt{LC}}. \quad (8)$$

Thus, the total energy is

$$E = \frac{Q^2}{2C} + V(\Phi), \text{ with } Q = C\dot{\Phi} \text{ and } V(\Phi) = \frac{C\omega^2\Phi^2}{2}. \quad (9)$$

It is the same expression as in the case of a particle (Chap.2, relation 1) substituting m by C, x by Φ, and $p = mv$ by $Q = C\dot{\Phi}$, being Q the electric charge (coulomb) stored in C.

The Hamiltonian associated to the LC circuit is deduced from (9) by replacing Q and Φ by operators:

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{C\omega^2\hat{\Phi}^2}{2}. \tag{10}$$

$\hat{Q}$ and $\hat{\Phi}$ are defined in the same way as in Chap.2, (3) and (5), that is

$$f(\Phi,t) \xrightarrow{\hat{Q}} -i\hbar\frac{\partial f(\Phi,t)}{\partial \Phi},\ f(\Phi,t) \xrightarrow{g(\hat{\Phi})} g(\Phi)f(\Phi,t). \tag{11}$$

The operators $\hat{Q}$ and $\hat{\Phi}$ are non-commutative and satisfy the relation (Chap.2, relation 4)

$$\hat{\Phi}\cdot\hat{Q} - \hat{Q}\cdot\hat{\Phi} = i\hbar. \tag{12}$$

Define two operators

$$\hat{a} = \sqrt{\frac{C\omega}{2\hbar}}\left(\hat{\Phi} + \frac{i}{C\omega}\hat{Q}\right) \text{ and } \hat{a}^+ = \sqrt{\frac{C\omega}{2\hbar}}\left(\hat{\Phi} - \frac{i}{C\omega}\hat{Q}\right). \tag{13}$$

From (12) and (13) the following properties can be derived:

$$\hat{a}^+ \cdot \hat{a} = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2},\ \hat{a}\cdot \hat{a}^+ = \frac{\hat{H}}{\hbar\omega} + \frac{1}{2},\ \hat{H} = \hbar\omega(\hat{a}^+ \cdot \hat{a} + \frac{1}{2}). \tag{14}$$

Assume that $e^{-i\omega_l t}\psi_l(\Phi)$ is a particular solution of the Schrödinger equation, that corresponds to the energy value $E_l = \hbar\omega_l$ (Chap.2, relation 9). Thus, E_l is an eigenvalue of $\hat{H}$:

$$\hat{H}e^{-i\omega_l t}\psi_l(\Phi) = E_l e^{-i\omega_l t}\psi_l(\Phi). \tag{15}$$

Then, using the definitions (13) and the properties (14), it can be demonstrated that

$$\hat{H}\cdot\hat{a}e^{-i\omega_l t}\psi_l(\Phi) = (E_l - \hbar\omega)\hat{a}e^{-i\omega_l t}\psi_l(\Phi),$$

$$\hat{H}\cdot\hat{a}^+e^{-i\omega_l t}\psi_l(\Phi) = (E_l + \hbar\omega)\hat{a}^+e^{-i\omega_l t}\psi_l(\Phi). \tag{16}$$

Thus, $\hat{a}e^{-i\omega_l t}\psi_l(\Phi)$ and $\hat{a}^+e^{-i\omega_l t}\psi_l(\Phi)$ are also eigenfunctions of $\hat{H}$, with eigenvalues equal to $E_l - \hbar\omega$ and $E_l + \hbar\omega$, respectively. For that reason, those operators are called annihilation and creation operators.

In conclusion, the possible values of the circuit energy are

$$E_0, E_1 = E_0 + \hbar\omega, E_2 = E_0 + 2\hbar\omega, \dots, E_l = E_0 + l\hbar\omega, \dots, \tag{17}$$

being E_0 the minimum energy.

To compute E_0, assume that the annihilation operator $\hat{a}$, applied to the first solution $e^{-i\omega_0 t}\psi_0(\Phi)$, generates the function 0, that is

$$\hat{a}e^{-i\omega_0 t}\psi_0(\Phi) = 0.$$

Then, according to (14),

$$\hat{H}e^{-i\omega_0 t}\psi_0(\Phi) = \frac{\hbar\omega}{2}e^{-i\omega_0 t}\psi_0(\Phi),$$

so that

$$E_0 = \frac{\hbar\omega}{2}. \tag{18}$$

Consider an orthonormal base of V:

$$B = \{|\varphi_0\rangle, |\varphi_1\rangle, \dots, |\varphi_{d-1}\rangle\} \tag{19}$$

Then, according to (17), (18), and to the spectral decomposition of a Hermitian operator, the Hamiltonian can be represented as follow:

$$\hat{H} = \hbar\omega\sum_{l=0}^{d-1}\left(\frac{1}{2} + l\right)|\varphi_l\rangle\langle\varphi_l|. \tag{20}$$

The energy levels are shown in Fig.2 (d = 5).

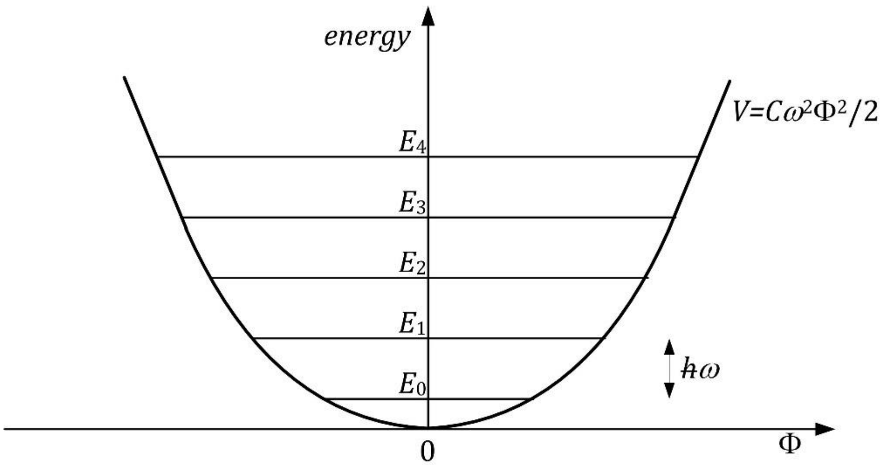

Figure 2 Energy levels

In a circuit made up of discrete capacitors and inductors (Fig.1), working at frequencies whose wave lengths are several magnitude orders smaller than the component sizes, the quantum effects are not observable. They could be observed in a microwave cavity, modelled by an LC circuit, working at very low temperature, when thermal noise does not affect quantum coherence. Even in this case, the LC circuit is not a good artificial atom. All energy differences $E_{l+1} - E_l$ are equal to $\hbar\omega$ - for that reason, it is called harmonic oscillator- so that the external energy necessary to excite the system from the ground state E_0 to E_1 is the same as the energy necessary to excite the system from the state E_1 to E_2, from the state E_2 to E_3, and so on. This means that a harmonic oscillator is not easily controllable. Thus, anharmonic oscillator should be considered, that is, oscillating circuits with unequal differences between eigenfrequencies.

2. Cooper pairs and Josephson effect

The electrical current within superconductors is produced by pairs of electrons called Cooper pairs. They behave as particles characterized by their wave function $\psi = |\psi|e^{i\phi}$. Furthermore, all pairs are simultaneously in the same quantum state. When two superconductors are separated by a thin ($\cong$ 1 nm) insulating barrier, Cooper pairs can cross the barrier. This is the Josephson effect.

Consider (Fig.3) two superconductors separated by an insulating barrier. Let $\phi_1(t)$ and $\phi_2(t)$ be the phases of the Cooper pairs wave functions in superconductor 1 and 2, respectively. The following equations describe the Josephson effect:

$$v(t) = \frac{\hbar}{2e}\frac{d\phi(t)}{dt}, \; i(t) = I_c \sin\phi(t), \tag{21}$$

where $v(t)$ is the difference of potential across the junction terminals, $i(t)$ is the current across the junction, $\phi(t) = \phi_1(t)$-$\phi_2(t)$, I_c is a constant value named critical current, and e is the elementary charge ($1.60217663 \times 10^{-19}$ Coulombs), so that $-2e$ is the charge of a Cooper pair.

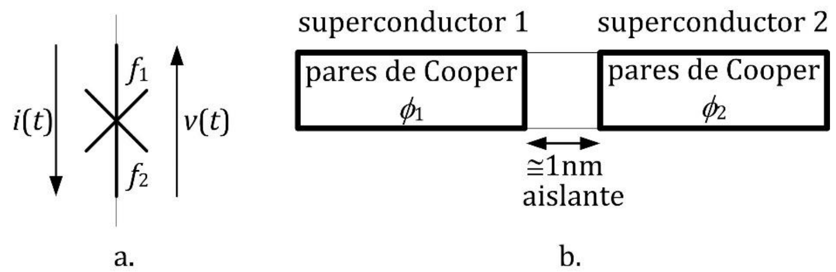

Figure 3 Josephson effect: a. symbol, b. structure

The model of a Josephson junction is shown in Fig.4. It consists of an insulating barrier, between two superconductors, characterized by its critical current I_C, and of a capacitor C_J that models the parasitic capacitance.

The energy stored in C_J is equal to (2) with $C = C_J$. As regards the insulating barrier, the energy is equal to

$$E = \int_{-\infty}^{t} v(\tau) i(\tau) d\tau. \quad (22)$$

According to (21), and assuming that $\phi(-\infty) = 0$,

$$E = \frac{\hbar I_C}{2e} \int_0^{\phi} \sin\varphi \, d\varphi = E_J(1 - \cos\phi), \quad (23)$$

where

$$E_J = \frac{\hbar I_C}{2e} \quad (24)$$

is the Josephson junction energy.

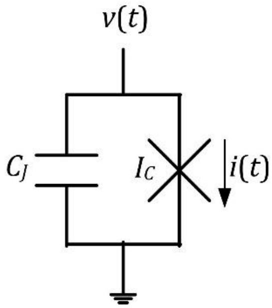

Figure 4 Josephson junction

As before, the energies are expressed in function of the node flux Φ (4) and of $\dot{\Phi}$, instead of i and v. According to (5) and (21)

$$\frac{d\phi(t)}{dt} = \frac{2e}{\hbar} v(t) = \frac{2e}{\hbar} \frac{d\Phi(t)}{dt}, \tag{25}$$

So that, assuming that $\phi(-\infty) = \Phi(-\infty) = 0$,

$$\phi = \frac{2e}{\hbar} \Phi. \tag{26}$$

The capacitive energy is (8) with $C = C_J$, that is,

$$E_C = C_J \frac{\dot{\Phi}^2}{2}, \tag{27}$$

and, according to (23) and (26), the insulating barrier energy is equal to

$$E_I = E_J (1 - \cos \frac{2e}{\hbar} \Phi). \tag{28}$$

The latter is a function of Φ, just like the inductive energy E_l in (8), but it is not a quadratic function; it is a cosine function. For that reason, the Josephson junction is considered as a non-linear inductor.

Using the same method as in the case of the LC circuit, define the momentum

$$C_J \dot{\Phi} = C_J v(t) = Q, \tag{29}$$

being Q the charge stored in C_J. Assume that (Fig.3) the superconductor 1 contains n_1 Cooper pairs and the superconductor 2 contains n_2 Cooper pairs. Then, the electrical charge stored in the first superconductor is equal to $Q_1 = -2n_1e$, and the electrical charge stored in the second superconductor is equal to $Q_2 = -2n_2e$. Thus, the electrical charge stored by the parasitic capacitor C_J is

$$Q = -2ne,\ n = n_1 - n_2. \tag{30}$$

Using (27), (29) and (30) the following expressions of the capacitive energy are obtained:

$$E_C = \frac{Q^2}{2C_J} = \frac{4n^2e^2}{2C_J}. \tag{31}$$

The total energy, in function of Φ and Q, is equal to

$$E = \frac{Q^2}{2C_J} + (1 - \cos\frac{2e}{\hbar}\Phi). \tag{32}$$

The Hamiltonian operator is derived from E by replacing Φ and Q by operators that satisfy (12):

$$\hat{H} = \frac{\hat{Q}^2}{2C_J} + E_J\,(1 - \cos\frac{2e}{\hbar}\hat{\Phi}). \tag{33}$$

Instead of $\hat{\Phi}$ and $\hat{Q}$, the following operators can be used:

$$\hat{\phi} = \frac{2e}{\hbar}\hat{\Phi} \text{ and } \hat{n} = \frac{\hat{Q}}{2e}. \tag{34}$$

Finally, the following expression is obtained:

$$\hat{H} = 4E_C\hat{n}^2 - E_J\cos\hat{\phi}, \text{ with } E_C = \frac{e^2}{2C_J}. \tag{35}$$

The constant term E_J has been omitted, so that the energy levels will be expressed with respect to the ground level.

The action of operators $\hat{n}$ and $\hat{\phi}$ on complex functions of two real variables is easily deduced from (34) and (11):

$$f(\phi,t) \xrightarrow{\hat{n}} -i\frac{\partial f(\phi,t)}{\partial \phi},\ f(\phi,t) \xrightarrow{g(\hat{\phi})} g(\phi)f(\phi,t). \tag{36}$$

The non-commutativity property is derived from (34) and (12):

$$\hat{\phi}\cdot\hat{n} - \hat{n}\cdot\hat{\phi} = i. \tag{37}$$

3. Charge states

The conclusion of the preceding section is that a Josephson junction can be considered as a quantum system whose Hamiltonian is (35). Thus (Chap.2), the quantum states can be represented by unitary vectors of a vector space V over the complex field. Let $|n\rangle$ be the unitary vector that represents the junction quantum state when the number of Cooper pairs in C_J is equal to n, so that the capacitive energy is equal to (31). Take into account that the Cooper pairs can cross the insulating barrier in both directions, so that n can be a positive or negative integer. If the system only consisted of the capacitance C_J, the Hamiltonian would reduce to the first term of (35) and the states $|n\rangle$ would be eigenstates of $\hat{H}$, with eigenvalues defined by (31), and thus

$$\hat{n}^2|n\rangle = n^2|n\rangle. \tag{38}$$

In the complete system, the states $|n\rangle$ are no longer eigenvectors, but they still constitute an orthonormal base.

Relation (37) suggests the following definition of $\hat{n}$:

$$\hat{n}|n\rangle = -n|n\rangle, \tag{39}$$

which is compatible with (38) and takes into account that the momentum $\hat{n} = \frac{\hat{Q}}{2e}$ has been derived from (30), that is $\frac{Q}{2e} = -n$.

In order to get a spectral decomposition of $\hat{H}$ (Chap.2, 17), using the orthonormal base

$$B = \{\dots, |-2\rangle, |-1\rangle, |0\rangle, |1\rangle, |2\rangle, \dots\}, \tag{40}$$

first consider the Hamiltonian $\hat{H}_C$ of the system consisting of the capacitor C_J. According to (31) and to the definition (35) of E_C, the energy that corresponds to the state $|n\rangle$ is equal to $4E_Cn^2$, so that the spectral decomposition is

$$\widehat{H}_C = \sum_{l\in B} 4E_C l^2 |l\rangle\langle l| . \tag{41}$$

On the other hand, according to (35), the Hamiltoniano $\widehat{H}_J$ of the Josephson junction, without the parasitic capacitance C_J, is

$$\widehat{H}_J = -E_J \cos\widehat{\phi},$$

that, in turn, can be expressed as

$$\widehat{H}_J = -\frac{E_J}{2}\left(e^{i\widehat{\phi}} + e^{-i\widehat{\phi}}\right). \tag{42}$$

The next property shows that $e^{i\widehat{\phi}}$ and $e^{-i\widehat{\phi}}$ are the annihilation and creation operators of the charge states.

Property 1

$$e^{i\widehat{\phi}}|n\rangle = |n\text{-}1\rangle,\ e^{-i\widehat{\phi}}\,|n\rangle = |n+1\rangle$$

Proof

According to (36),

$$e^{i\widehat{\phi}}\cdot \hat{n}\,\psi(\phi,t) = -i\,e^{i\phi}\frac{\partial\psi(\phi,t)}{\partial\phi},\ \hat{n}\cdot e^{i\widehat{\phi}}\,\psi(\phi,t) = e^{i\phi}\psi(\phi,t)\ \text{-}i\,e^{i\phi}\frac{\partial\psi(\phi,t)}{\partial\phi},$$

so that

$$e^{i\widehat{\phi}}\cdot \hat{n}\ \text{-}\ \hat{n}\cdot e^{i\widehat{\phi}} = \text{-}\ e^{i\widehat{\phi}}. \tag{43}$$

Apply this non-commutativity property to state $|n\rangle$. From (43) and (39)

$$\hat{n}\cdot e^{i\widehat{\phi}}|n\rangle = e^{i\widehat{\phi}}\cdot \hat{n}|n\rangle + e^{i\widehat{\phi}}|n\rangle = \text{-}(n\text{-}1)e^{i\widehat{\phi}}|n\rangle.$$

Thus, $e^{i\widehat{\phi}}|n\rangle$ is an eigenvector of $\hat{n}$, with eigenvalue $-(n\text{-}1)$. According to (39), $e^{i\widehat{\phi}}|n\rangle = |n\text{-}1\rangle$. The second property is deduced from the first:

$$|n+1\rangle = e^{-i\widehat{\phi}}\cdot e^{i\widehat{\phi}}|n+1\rangle = e^{-i\widehat{\phi}}|n\rangle.$$

From property 1 the following expressions of $e^{i\widehat{\phi}}$ and $e^{-i\widehat{\phi}}$are deduced:

$$e^{i\widehat{\phi}} = \sum_{l\in B}|l-1\rangle\langle l|,\ e^{-i\widehat{\phi}} = \sum_{l\in B}|l+1\rangle\langle l|. \tag{44}$$

Actually, computing $e^{i\hat{\phi}}|n\rangle$ and $e^{-i\hat{\phi}}|n\rangle$ with (44), then, by orthonormality of B, the results are $|n\text{-}1\rangle$ and $|n+1\rangle$, respectively.

Finally, from (41), (42) and (44), the following expression of the Hamiltonian operator $\hat{H} = \hat{H}_C + \hat{H}_J$ is obtained:

$$\hat{H} = \sum_{l \in B} \left[4E_C l^2 |l\rangle\langle l| - \frac{E_J}{2} (|l+1\rangle\langle l| + |l-1\rangle\langle l|) \right]. \qquad (45)$$

4. Energy values

According to (45), the charge states $|n\rangle$ are not the eigenvectors of the Halmitonian operator, unless $E_J \cong 0$; in this case, the system reduces to a capacitor, and the energy (31) is proportional to the square of the electric charge Q. If the value of E_J is not negligible, then Cooper pairs that cross the insulating barrier modify the energy value in a non-negligible way.

The calculus of the eigenvalues of the Hamiltonian is not an easy task. A rigorous method consists of solving the Schrödinger equation, that is a non-linear grade-two differential equation. A not so precise but easier option is to truncate the expression (45) and to use algebraic or numerical methods to get the eigenvalues:

$$\hat{H}_{\text{trunc}} = \sum_{l=-N}^{N} \left[4E_C l^2 |l\rangle\langle l| - \frac{E_J}{2} (|l\rangle\langle l+1| + |l\rangle\langle l-1|) \right]. \qquad (46)$$

Assume that $|\psi\rangle$ is an eigenvector of $\hat{H}_{\text{trunc}}$ represented in the truncated base

$$B_{\text{trunc}} = \{|\text{-}N\rangle, \ldots, |\text{-}2\rangle, |\text{-}1\rangle, |0\rangle, |1\rangle, |2\rangle, \ldots, |N\rangle\},$$

that is

$$|\psi\rangle = \sum_{l=-N}^{N} c_l |l\rangle. \qquad (47)$$

If $|\psi\rangle$ is an eigenvalue of $\hat{H}_{\text{trunc}}$, with eigenvalue E, then $\hat{H}_{\text{trunc}}|\psi\rangle = E|\psi\rangle$. Thus, according to (45) and (47), assuming that B_{trunc} is orthonormal, in spite of the truncation:

$$\left(4E_C N^2 c_{-N} - \frac{E_J}{2} c_{-N+1}\right)|-N\rangle + \sum_{l=-N+1}^{N-1} \left(4E_C l^2 c_l - \frac{E_J}{2}(c_{l-1} + c_{l+1})\right)|l\rangle$$

$$+ \left(4E_C N^2 c_N - \frac{E_J}{2} c_{N-1}\right)|N\rangle = \sum_{l=-N}^{N} c_l E|l\rangle. \qquad (48)$$

From (48) a homogeneous system of $2N+1$ linear equations, with $2N+1$ unknowns c_l, is derived:

$$4E_C N^2 c_{-N} - \frac{E_J}{2} c_{-N+1} - Ec_{-N} = 0,$$

$$4E_C l^2 c_l - \frac{E_J}{2}(c_{l-1} + c_{l+1}) - Ec_l = 0, \quad l = -N+1, \dots, N-1,$$

$$4E_C N^2 c_N - \frac{E_J}{2} c_{N-1} - Ec_N = 0. \qquad (49)$$

The characteristic equation, deduced from the determinant of the matrix associated to (49), is a grade $2N+1$ algebraic equation whose $2N+1$ solutions are the eigenvalues, that is, the energy values.

As an example, with $N = 1$, system (49) in matrix form is

$$\begin{bmatrix} 4E_C & -\frac{E_J}{2} & 0 \\ -\frac{E_J}{2} & 0 & -\frac{E_J}{2} \\ 0 & -\frac{E_J}{2} & 4E_C \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = E \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix}.$$

This system has solutions if

$$\det \begin{bmatrix} 4E_C - E & -\frac{E_J}{2} & 0 \\ -\frac{E_J}{2} & -E & -\frac{E_J}{2} \\ 0 & -\frac{E_J}{2} & 4E_C - E \end{bmatrix} = 0.$$

Thus, the characteristic equation is

$$(4E_C - E)(E^2 - 4E_C E - \frac{E_J^2}{2}) = 0,$$

whose solutions are

$$E_0 = 2E_C - \sqrt{4E_C^2 + \frac{E_J^2}{2}},\ E_1 = 4E_C,\ E_2 = 2E_C + \sqrt{4E_C^2 + \frac{E_J^2}{2}}.$$

Remember that the Hamiltonian has been calculated without taking into account constant values. For that reason, a negative E_0 has been obtained. In fact, all energy levels must be expressed with respect to this ground energy level.

Observe that the energy differences $E_1 - E_0$ and $E_2 - E_1$ are not equal, as was the case in the harmonic oscillator (Sec.1, Fig.2). Thus, the Josephson junction is an anharmonic oscillator and it can be considered as a candidate to implement an artificial atom.

The energy levels, with respect to the ground level E_0, expressed in multiples of E_C, are equal to

$$e_1 = (E_1 - E_0)/E_C = 2 + \sqrt{4 + \frac{E_J^2/E_C^2}{2}},\ e_2 = (E_2 - E_0)/E_C = 2\cdot\sqrt{4 + \frac{E_J^2/E_C^2}{2}}.$$

With $E_J = 10E_C$: $e_1 = 9.34...$ and $e_2 = 14.69...$ (Fig.5).

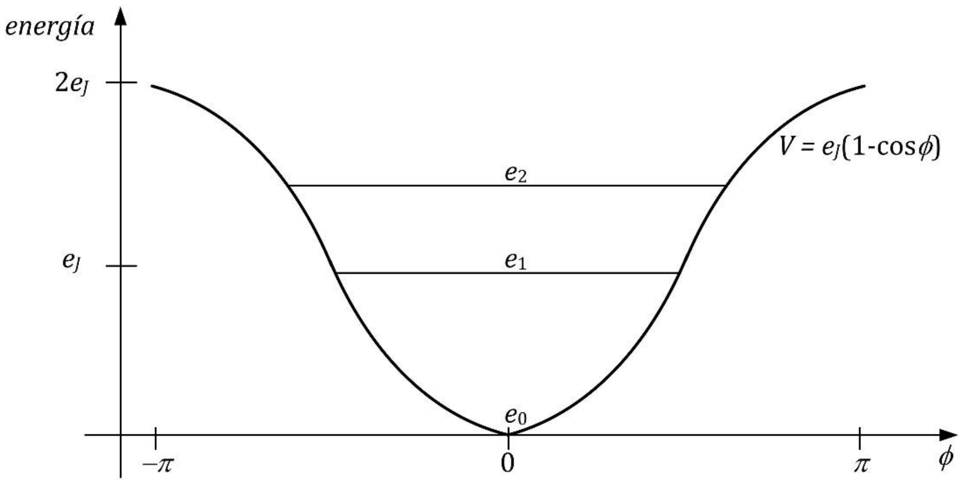

Figure 5 Energy levels, with respect to E_0, in multiples of E_C

5. Charge qubit

The charge qubit (quantum bit) structure is shown in Fig.6. It consists of a Josephson junction whose operating point can be controlled with a voltage source V_g coupled to the junction by a capacitor C_g. This device is made up of a superconductor island separated from a superconductor electrode by an insulating barrier (Fig.6.a). The electrode serves as a Cooper pair reservoir. An electrical model is shown in Fig.6.b.

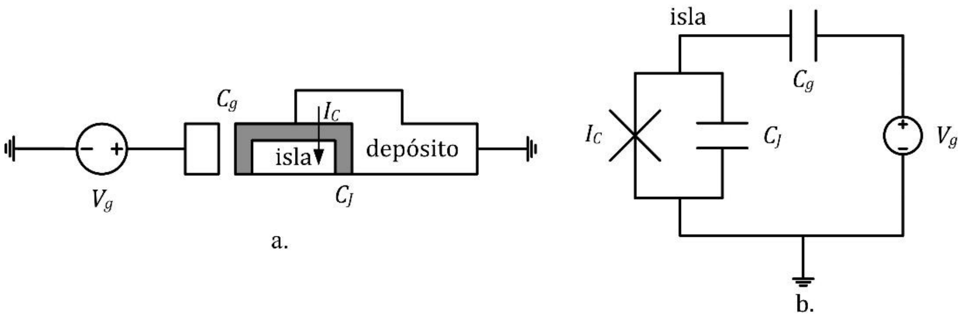

Figure 6 Charge qubit

The capacitive energy of the charge qubit is derived from (31) by replacing C_J by $C_J + C_g$ and n by $n - n_g$, where

$$n_g = C_g V_g / 2e, \tag{50}$$

so that the capacitive energy is

$$E_c = \frac{4(n-n_g)^2 e^2}{2(C_J+C_g)} = 4E_C(n-n_g)^2, \tag{51}$$

with

$$E_C = \frac{e^2}{2(C_J+C_g)} \cdot \tag{52}$$

The Hamiltonian operator is

$$\hat{H} = 4E_C(\hat{n} - n_g)^2 - E_J \cos\hat{\phi}. \tag{53}$$

Using the same orthonormal base B as before, namely, the charge states (Sec.3), the Hamiltonian can be expressed as follows:

$$\widehat{H} = \sum_{l \in B} \left[4E_C (l - n_g)^2 |l\rangle\langle l| - \frac{E_J}{2} (|l+1\rangle\langle l| + |l-1\rangle\langle l|) \right]. \quad (54)$$

The eigenvalues of the Hamiltonian can be calculated with a truncated expression

$$\widehat{H}_{\text{trunc}} = \sum_{l=-N}^{N} \left[4E_C (l - n_g)^2 |l\rangle\langle l| - \frac{E_J}{2} (|l\rangle\langle l+1| + |l\rangle\langle l-1|) \right]. \quad (55)$$

Consider an eigenvector of $\widehat{H}_{\text{trunc}}$, with eigenvalue E, represented in the same base B by (47). Then, defining $e = E/E_C$ and $e_J = E_J/E_C$, the following equations can be derived in the same way as equations (49):

$$4(-N - n_g)^2 c_{-N} - \frac{e_J}{2} c_{-N+1} = e c_{-N},$$

$$4(l - n_g)^2 c_l - \frac{e_J}{2} (c_{l-1} + c_{l+1}) = e c_l, \quad l = \; ...., -1, 0, 1, ... ,$$

$$4(N - n_g)^2 c_N - \frac{e_J}{2} c_{N-1} = e c_N. \quad (56)$$

This linear system can be expressed in matrix form:

$$H[c_{-N} \; c_{-N+1} \ldots c_{-1} \; c_0 \; c_1 \ldots c_{N-1} \; c_N]^{\mathrm{T}} = e[c_{-N} \; c_{-N+1} \ldots c_{-1} \; c_0 \; c_1 \ldots c_{N-1} \; c_N]^{\mathrm{T}}.$$

The eigenvalues of H are the energy values. They can be calculated with numerical methods. In [1], Python libraries are used for that purpose. The energy values have been calculated with $N = 5$, so that H is an 11×11 matrix. With $E_J/E_C = 1$, the four first eigenvalues, in function of n_g, are

```
ng = -1:
[-0.12176554  3.97918922  4.1009006  16.00831046]

ng = -0.5:
[0.47065435 1.46676684 9.01371984 9.01760693]

ng = 0:
[-0.12176554  3.97918922  4.1009006  16.00831046]

ng = 0.5:
[0.47065435 1.46676684 9.01371984 9.01760693]

ng = 1:
[-0.12176554  3.97918922  4.1009006  16.00831046]
```

and with $E_J/E_C = 50$, the four first eigenvalues are

```
ng = -1:
[-40.25444748 -21.28756866  -3.37847899  13.44009709]

ng = -0.5:
[-40.25636872 -21.30927963  -3.48560099  13.09829506]

ng = 0:
[-40.25665949 -21.3130015   -3.50880969  13.0453763 ]

ng = 0.5:
[-40.25636872 -21.30927963  -3.48560099  13.09829506]

ng = 1:
[-40.25444748 -21.28756866  -3.37847899  13.44009709]
```

A graphical representation of those results is shown in Fig.7 and 8. The energy levels are expressed with respect to the ground level E_0.

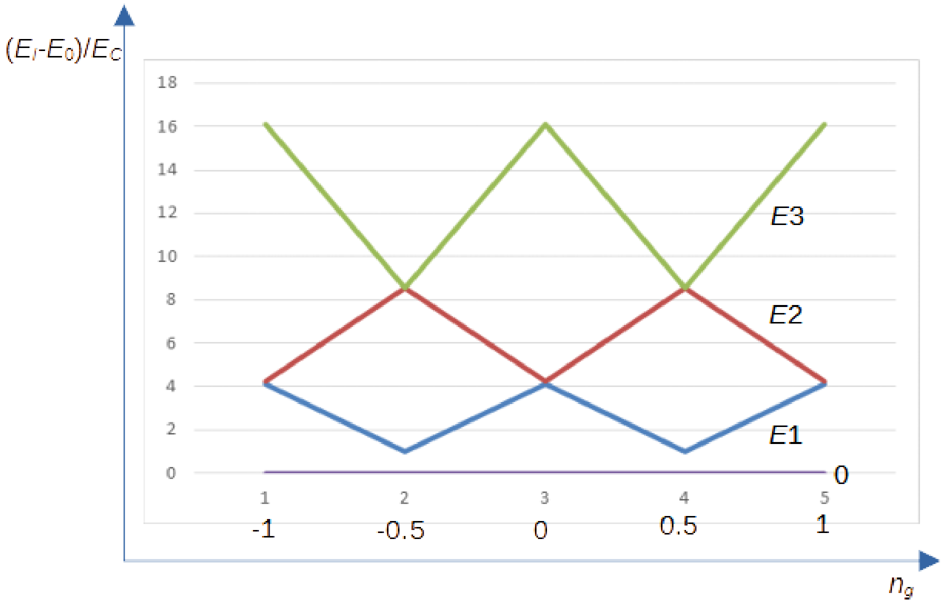

Figure 7 $(E_l - E_0)\,/\,E_C$ with $E_J/\,E_C = 1$

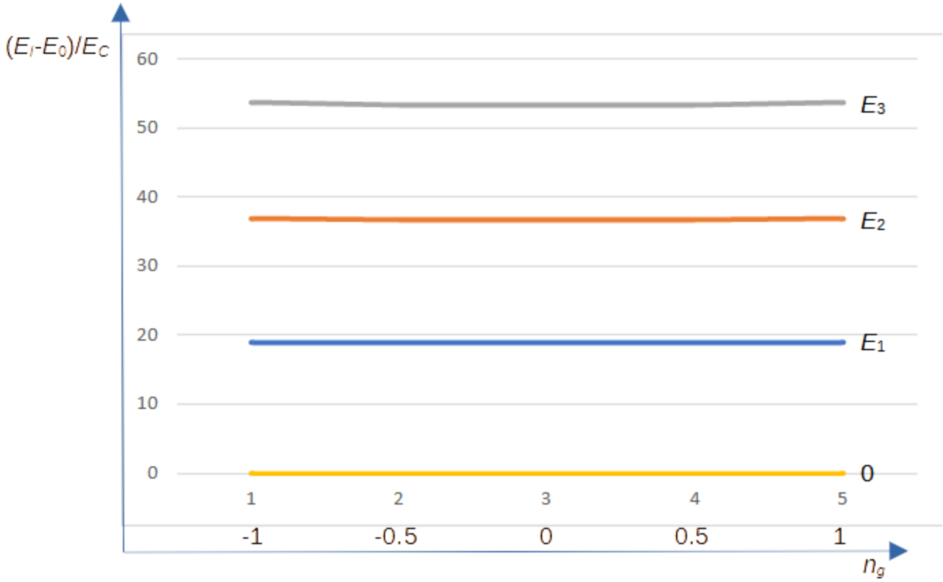

Figure 8 $(E_l - E_0)\,/\,E_C$ with $E_J/\,E_C = 50$

Several conclusions can be drawn.

- When the capacitive and potential energies are comparable ($E_J \cong E_C$, Fig.7), the energy levels are very sensitive to the value of n_g, and thus to the electromagnetic noise. This characteristic is known as "charge dispersion". On the other hand, the difference between adjacent energy levels is high, a characteristic known as "anharmonicity". For example, with $n_g = 0$, E_1 - E_0 and E_2 - E_1 are approximately equal to $4E_C$ and $0.12E_C$; with $n_g = \pm\,0{,}5$, E_1 - E_0 and E_2 - E_1 are approximately equal to E_C and $7.5E_C$.
- When the potential energy is much greater than the capacitive energy ($E_J >> E_C$, Fig.8), the differences between energy levels are much smaller. For example, with $n_g = 0$ and $E_J = 50E_C$, E_1 - E_0 and E_2 - E_1 are approximately equal to $18.943E_C$ and $17.805E_C$; with $n_g = \pm\,0{,}5$, E_1 - E_0 and E_2 - E_1 are approximately equal to $18.947E_C$ and $17.824E_C$. So, the anharmonicity is smaller, but the charge dispersion is also smaller.

It has been calculated that the decrease in sensitivity to charge noise is exponential in $(E_J/E_C)^{1/2}$, while the anharmonicity only decreases linearly. For that reason, relatively high values of E_J/E_C are preferred, and thus, according to (52), relatively high values of C_g should be considered. As regards the value of n_g, the curves of Fig.7 indicate that the derivatives of the energy levels with respect to n_g are equal to zero when $n_g = k + 0.5$, being k an integer. With those values of n_g the charge dispersion is reduced. According to (50), the value of n_g can be set with V_g. It will be seen below that the value of E_J is also adjustable.

As already mentioned, the charge states $|n\rangle$ are not eigenstates of the Hamiltonian operator. The eigenstates $|\psi_l\rangle$ of the charge qubit are superpositions of charge states. Using the truncated Hamiltonian (55) with $N = 2$, $n_g = 0$ and $e_J = E_J/E_C = 50$, and a Python library of linear algebra functions, the following eigenstates have been calculated [1]:

$$|\psi_0\rangle = -0.225\,|-2\rangle - 0.498\,|-1\rangle - 0.635\,|0\rangle - 0.498\,|1\rangle - 0.225\,|2\rangle,$$
$$|\psi_1\rangle = 0.438|-2\rangle + 0.555|-1\rangle \qquad - 0.555|1\rangle - 0.438\,|2\rangle.$$

The physical meaning of the first relation is the following: when the charge qubit is in ground state $|\psi_0\rangle$, the observation of the difference in number of Cooper pairs, between both sides of the barrier, would give a result equal to 0 with a probability equal to $(0.635)^2$, equal to 1 (o -1) with a probability equal to $(0.498)^2$, and equal to 2 (o -2) with a probability equal to $(0.225)^2$. It behaves as an oscillator, with an angular frequency equal to $\omega_0 = E_0/\hbar$. When it is in quantum state $|\psi_1\rangle$, the probabilities are 0, $(0.555)^2$ and $(0.438)^2$, and the angular frequency is $\omega_1 = E_1/\hbar$.

6. State evolution

Assuming that the only possible quantum states are superpositions of $|\psi_0\rangle$ and $|\psi_1\rangle$, the Hamiltonian operator can be expressed as (Chap.2,17)

$$\hat{H} = E_0|\psi_0\rangle\langle\psi_0| + E_1|\psi_1\rangle\langle\psi_1|.$$

The evolution of the charge qubit in function of t is defined by the unitary operator (Chap.2, 20)

$$\hat{U} = e^{-i\frac{E_0}{\hbar}\Delta t}|\psi_0\rangle\langle\psi_0| + e^{-i\frac{E_1}{\hbar}\Delta t}|\psi_1\rangle\langle\psi_1|.$$

Discarding the global phase and using the eigenstate base $\{|\psi_0\rangle, |\psi_1\rangle\}$, the system evolution is defined by the following unitary matrix

$$U = \begin{bmatrix} e^{-\frac{i\omega\Delta t}{2}} & 0 \\ 0 & e^{\frac{i\omega\Delta t}{2}} \end{bmatrix}, \ \omega = \frac{E_0 - E_1}{2\hbar}. \tag{57}$$

Assuming that the system is in a superposition state

$$|\psi\rangle = a|\psi_0\rangle + b|\psi_1\rangle, \tag{58}$$

where a and b are complex numbers such that $|a|^2 + |b|^2 = 1$, then

$$U|\psi\rangle = e^{-\frac{i\omega\Delta t}{2}} a|\psi_0\rangle + e^{i\omega\Delta t} b|\psi_1\rangle. \tag{59}$$

This relation shows that the probabilities

$$\left|e^{-\frac{i\omega\Delta t}{2}} a\right|^2 = |a|^2 \text{ and } \left|e^{\frac{i\omega\Delta t}{2}} b\right|^2 = |b|^2$$

of observing either the state $|\psi_0\rangle$ or the state $|\psi_1\rangle$ are unchanged. Relation (59) shows that the system evolution is a modification of the relative phase between two components of the qubit wave function (Chap.2, relation 14, with $d = 2$, $x = \Phi$). This evolution doesn't need any interchange of energy with the environment.

7. Transmon

A conclusion of the previous section is that, to minimize the charge dispersion, the potential energy E_J should be greater than the capacitive energy E_C. This can be achieved by connecting an additional capacitor C_B in parallel with C_J. Furthermore, a structure known as "SQUID" can be used to tune the value of E_J. Apart from giving the possibility to increase the value of the quotient E_J/E_C, it permits to modify the difference between the ground state energy $E_0 = \hbar\omega_0$ and the first excited state energy $E_1 = \hbar\omega_1$.

7.1 Structure

The electric model of a transmon is shown in Fig.9. It contains two Josephson junctions, connected in parallel, and a capacitor C_B whose capacitance is much greater than the parasitic capacitance C_J of the junctions. It also includes a current source that generates a magnetics flux Φ that crosses the loop made up of the two junctions. This SQUID (*Superconducting QUantum Interference Device*) configuration is equivalent to a single junction whose potential energy E_J depends on the magnetic flux Φ:

$$E_J = (E_{J1} + E_{J2})\cos\left(\frac{\pi\Phi}{\Phi_0}\right)\sqrt{1 + d^2\tan^2\left(\frac{\pi\Phi}{\Phi_0}\right)} \tag{60}$$

where

$$d = \frac{E_{J2} - E_{J1}}{E_{J2} + E_{J1}},\ \Phi_0 = \frac{\hbar}{2e}.$$

Assuming that $C_B >> C_J$, the value of E_C is deduced from (52):

$$E_C = \frac{e^2}{2(C_B + C_g)}. \tag{61}$$

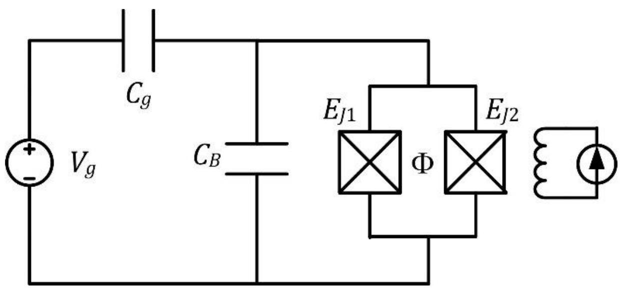

Figure 9 Transmon

7.2 Control

To control the state of a transmon cell, microwave circuit techniques are used. They consist of coupling the circuit with transmission lines modelled by resonant circuits. The necessary basic operations are: the generation of an external magnetic flux to set the value of E_J, the initialization and modification of the quantum state, and the measurement of the quantum state. Furthermore, in a system made up of several qubits, the entanglement of adjacent circuits must be made possible.

From now on, the symbols $|0\rangle$ and $|1\rangle$ will stand for the first eigenstates $|\psi_0\rangle$ and $|\psi_1\rangle$ and no longer for charge states.

An electrical model of a transmon, with external signals and coupling circuits, is shown in Fig.10.

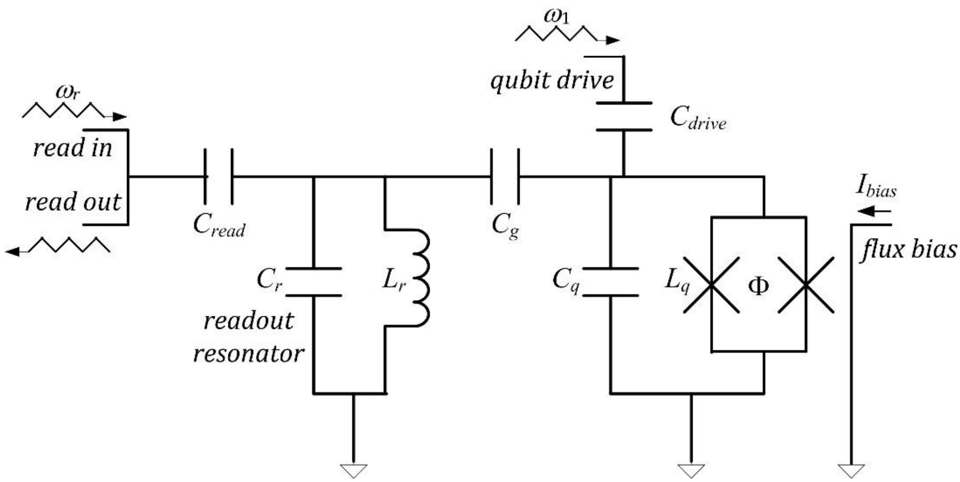

Figure 10 Transmon: electrical model

The operations are performed as follows:

- The flux bias line permits to control the magnetic flux Φ, and thus to define the potential energy E_J.
- In Sec.6 (relation 57), it was seen that the natural evolution of a two-level charge qubit amounts to a modification of the phase difference

between the basic wave functions, without energy interchange with the environment. Nevertheless, not any unitary transformation could be executed in this way. An operation that modifies the measurement result probabilities implies an energy interchange with the control circuitry. For example, if the charge qubit is in the ground state, to which corresponds the energy level $E_0 = \hbar\omega_0$, then to take the qubit to the first excited state, to which corresponds the energy level $E_1 = \hbar\omega_1$, a microwave signal, at frequency $\omega/2\pi = (\omega_1 - \omega_0)/2\pi$, the so-called “transition frequency”, must be inputted through the qubit drive line. More generally, if $|0\rangle$ and $|1\rangle$ are the ground state and the first excited state, the application of sequences of microwave signals, at the transition frequency $(\omega_1 - \omega_0)/2\pi$, with the convenient amplitudes, phases and durations, permits to take the qubit to any superposition of states $|0\rangle$ and $|1\rangle$.

- As regards the state measurement, a microwave resonator can be used. Its effective resonance frequency depends on the qubit state (Fig. 11). If the charge qubit is in state and $|0\rangle$, the resonance frequency is equal to $(\omega_r+\chi)/2\pi$, where $\omega_r/2\pi$ is the central frequency and $\chi/2\pi$ a frequency shift due to the interference with the charge qubit state, and if the qubit is in state $|1\rangle$, the resonance frequency is equal to $(\omega_r - \chi)/2\pi$. This variation of the resonance frequency is detected by inputting (read in line) a microwave signal at frequency $\omega_r/2\pi$ and by measuring the reflected signal frequency (read out line).

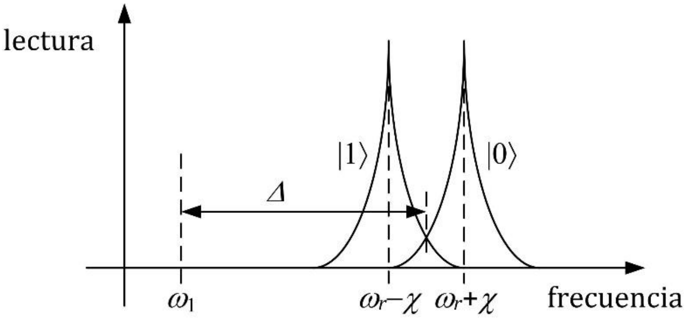

Figure 11 State measurement

- Another necessary operation is the initialization of the charge qubit state. According to the quantum mechanics postulates, a measurement operation collapses the quantum state. Thus, in order to set a charge qubit in e.g. the ground state $|0\rangle$, the following operations are executed: measure the quantum state; if the result is the value that corresponds to state $|0\rangle$, the initialization is done; if the result is the value that corresponds to state $|1\rangle$, apply the control signals that takes the state from $|1\rangle$ to $|0\rangle$. Another method consists of controlling the qubit environment to induce relaxation to the ground state.

The coupling of adjacent transmon cells is necessary to make possible the execution of unitary operations over a pair of cells. For that purpose, several methods have been developed. The transition frequencies can be tuned using the flux bias lines. If two adjacent transmon cells have the same transition frequency, they are coupled. If their transition frequencies are different, they are not. Another technique consists of connecting two adjacent transmon cells, having the same transition frequency, though a tunable resonant circuit. The transmon cells are coupled only if the resonant circuit frequency is equal to the common transition frequency.

In conclusion, thanks to the possibility of coupling adjacent transmon cells, quantum systems made up of several charge qubits can be generated. They are characterized by their Hamiltonian operator and by their eigenstates. The application of sequences of microwave signals, with the convenient frequencies, amplitudes, phases and durations, permits to control their quantum state. Obviously, this is not an easy task; the development of physical configurations that implement unitary binary, or even ternary, operations, with an acceptable reliability level, is still a research topic.

7.3 Final comments

Transmon cells are anharmonic oscillating circuits, with frequencies within the range 1 ·· 10 GHz. Their transition frequencies (difference beween $\omega_1/2\pi$ and $\omega_0/2\pi$) are within the range 100 ·· 300 MHz. The execution time of an operation that involves two cells is within the range 10 ·· 20 ns, and the corresponding control signals are microwave pulses whose duration is within the same range. Those are performances achievable with the current microwave circuit technology. Techniques that reduce the charge dispersion permit to develop quantum circuits with a relatively long coherence time, up to hundredths of microseconds. Thus, with those values, thousands of operations could be executed, with a high success probability.

Apart from the charge qubit, other artificial atoms have been developed, many of them based on the Josephson junction, for example the flux qubit and the phase qubit. Whatever the chosen option, there are several conditions that qubits must satisfy to be basic components of quantum computers. Those conditions are known as DiVicenzo criteria [4]. They could be summarized as follows: robustness, controllability and measurability.

- Robustness makes reference to the reliability of the qubit over time. Qubits must maintain their quantum characteristics during time intervals much longer than the execution time of simple operations. For that, the reduction of the charge dispersion is essential. The system must be isolated, as much as possible, from electromagnetic noise.
- Controllability means ability to initialize and to transform the qubit quantum state, under the control of external signals. This is the way quantum algorithms can be executed.
- Measurability is the ability to read out the quantum state of a set of qubits. The reading of the quantum state of a set of qubits is the final step of all quantum algorithms.

Observe that the preceding conditions are, up to a certain point, contradictory. To keep its quantum characteristics, a quantum system should be as isolated as possible (robustness condition). Nevertheless, to be controllable, it must be connected to external signals, for example to a digital computer. Furthermore, the basic components are Josephson junctions that work at practically zero Kelvin grades. So, the building of a quantum computer raises numerous problems of electronic and electrical engineering, computer engineering, manufacturing, and others. As a result, the performances of the quantum computers currently available are still relatively limited. Several quantum processors have already been developed by big electronics and computing companies, such as Google and IBM, but with still limited capacities, namely hundredths of qubits.

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