



Formal Languages and Their Applications

An Introduction via PLY

— 2024 —

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These lecture notes, along with their \LaTeX source files and any associated programs, are available on the following GitHub repository:

<https://github.com/karlstroetmann/Formal-Languages>.

Given the ever-changing nature of the field of computer science, these notes are subject to periodic updates. If you have `git` installed on your computer, you can clone the repository using the following command:

```
git clone https://github.com/karlstroetmann/Formal-Languages.git.
```

To update to the most recent version of these notes, execute the command

```
git pull
```

from the repository's root directory.

Since these notes are frequently updated, they may contain typos or errors. If you find any, please consider submitting a pull request or contacting me via [Discord](#) or email at karl.stroetmann@dhbw-mannheim.de.

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Chapter 1

Introduction and Motivation

This lecture explores the theory of formal languages and delves into various applications, with a focus on the development of scanners, parsers, interpreters, and compilers. We also introduce a tool designed to aid in the creation of these components, specifically we will introduce [PLY](#), which can be used both as a scanner generator and as a parser generator.

1.1 Basic Definitions

The cornerstone of this lecture is the concept of a [formal language](#), essentially defined as a set of strings characterized by precise mathematical criteria. To establish this concept, we first introduce some foundational definitions.

Definition 1 (Alphabet) An [alphabet](#) Σ is a finite, non-empty set of [characters](#):

$$\Sigma = \{c_1, \dots, c_n\}.$$

The term [symbol](#) is occasionally used interchangeably with the term [character](#). □

Examples:

- (a) $\Sigma = \{0, 1\}$ is the alphabet for representing binary numbers.
- (b) $\Sigma = \{ \text{"a"}, \dots, \text{"z"}, \text{"A"}, \dots, \text{"Z"} \}$ is the standard alphabet for the English language.
- (c) $\Sigma_{\text{ASCII}} = \{0, 1, \dots, 127\}$ is known as the [ASCII-alphabet](#). Here, numbers correspond to letters, digits, punctuation marks, and control characters. For instance, the set of number $\{65, \dots, 90\}$ represents the letters $\{A, \dots, Z\}$: The number 65 represents the letter "A", the number 66 represents the letter "B", and the number 90 represents the letter "Z". Table 1.1 on the next page shows the complete ASCII-alphabet. Note that the first 32 codes are non-printable control codes, for example the number 10 encodes a line feed control symbol, while the number 13 encodes a carriage return control symbol.
- (d) $\Sigma_{\text{UNICODE}} = \{0, 1, \dots, 154\,998\}$ represents version 16.0 of the [UNICODE-alphabet](#). This extensive set accommodates a wide array of characters, including various scripts, symbols, and even emojis. Table 1.2 shows the Unicode encoding of the 120 most common Chinese characters. ◇

Definition 2 (Strings) For a given alphabet Σ , a [string](#) is a sequence of characters selected from Σ . In formal language theory, these sequences are written without brackets or separating commas. Thus, for $c_1, \dots, c_n \in \Sigma$, we write

$$w = c_1 \cdots c_n \quad \text{instead of} \quad w = [c_1, \dots, c_n].$$

The [empty string](#) is represented by λ , so $\lambda = ""$. The set of all possible strings formed from the alphabet Σ is denoted as Σ^* . Strings are often enclosed in quotation marks for emphasis. □

Character	Code	Character	Code	Character	Code
NUL	0	SOH	1	STX	2
ETX	3	EOT	4	ENQ	5
ACK	6	BEL	7	BS	8
TAB	9	LF	10	VT	11
FF	12	CR	13	SO	14
SI	15	DLE	16	DC1	17
DC2	18	DC3	19	DC4	20
NAK	21	SYN	22	ETB	23
CAN	24	EM	25	SUB	26
ESC	27	FS	28	GS	29
RS	30	US	31	Space	32
!	33	"	34	#	35
\$	36	%	37	&	38
'	39	(40)	41
*	42	+	43	,	44
-	45	.	46	/	47
0	48	1	49	2	50
3	51	4	52	5	53
6	54	7	55	8	56
9	57	:	58	;	59
<	60	=	61	>	62
?	63	@	64	A	65
B	66	C	67	D	68
E	69	F	70	G	71
H	72	I	73	J	74
K	75	L	76	M	77
N	78	O	79	P	80
Q	81	R	82	S	83
T	84	U	85	V	86
W	87	X	88	Y	89
Z	90	[91	\	92
]	93	^	94	-	95
`	96	a	97	b	98
c	99	d	100	e	101
f	102	g	103	h	104
i	105	j	106	k	107
l	108	m	109	n	110
o	111	p	112	q	113
r	114	s	115	t	116
u	117	v	118	w	119
x	120	y	121	z	122
{	123		124	}	125
~	126	DEL	127		

Table 1.1: ASCII Encoding Table

Glyph	Unicode	Glyph	Unicode	Glyph	Unicode	Glyph	Unicode
一	U+4E00	二	U+4E8C	三	U+4E09	四	U+56DB
五	U+4E94	六	U+516D	七	U+4E03	八	U+516B
九	U+4E5D	十	U+5341	人	U+4EBA	大	U+5927
天	U+5929	地	U+5730	日	U+65E5	月	U+6708
山	U+5C71	水	U+6C34	火	U+706B	木	U+6728
金	U+91D1	土	U+571F	子	U+5B50	女	U+5973
中	U+4E2D	国	U+56FD	王	U+738B	民	U+6C11
学	U+5B66	文	U+6587	本	U+672C	书	U+4E66
爱	U+7231	家	U+5BB6	友	U+53CB	生	U+751F
父	U+7236	母	U+6BCD	男	U+7537	女	U+5973
花	U+82B1	草	U+8349	狗	U+72D7	猫	U+732B
鸟	U+9E1F	鱼	U+9C7C	风	U+98CE	雨	U+96E8
云	U+4E91	雪	U+96EA	电	U+7535	光	U+5149
车	U+8F66	船	U+8239	米	U+7C73	茶	U+8336
饭	U+996D	面	U+9762	东	U+4E1C	西	U+897F
南	U+5357	北	U+5317	左	U+5DE6	右	U+53F3
高	U+9AD8	矮	U+77EE	长	U+957F	短	U+77ED
快	U+5FEB	慢	U+6162	早	U+65E9	晚	U+665A
你	U+4F60	好	U+597D	谢	U+8C22	是	U+662F
不	U+4E0D	我	U+6211	他	U+4ED6	她	U+5979
它	U+5B83	很	U+5F88	要	U+8981	有	U+6709
吃	U+5403	喝	U+559D	看	U+770B	听	U+542C
走	U+8D70	跑	U+8DD1	跳	U+8DF3	说	U+8BF4
问	U+95EE	答	U+7B54	学	U+5B66	校	U+6821
老	U+8001	师	U+5E08	生	U+751F	书	U+4E66

Table 1.2: UNICODE Encoding of some Chinese Characters

Examples:

(a) Let $\Sigma = \{0, 1\}$. If we define

$$w_1 := "01110" \text{ and } w_2 := "11001",$$

then both w_1 and w_2 are strings over Σ . Hence, we can state:

$$w_1 \in \Sigma^* \text{ and } w_2 \in \Sigma^*.$$

(b) Consider $\Sigma = \{a, \dots, z\}$. With

$$w := "example",$$

it follows that $w \in \Sigma^*$. ◇

The *length* of a string w is denoted by $|w|$ and represents the number of characters in w . We employ [array notation](#) for character extraction: given a string w and a natural number $i \leq |w|$, $w[i]$ specifies the i -th character in w . Character counting starts at 0, adhering to conventions in modern programming languages like C, Java, and Python.

Next, we introduce the concept of [concatenation](#) of two strings w_1 and w_2 . Concatenation results in a new string w formed by appending w_2 to w_1 . It is represented as $w_1 \cdot w_2$. In most books the concatenation of w_1 and w_2 is written without the dot as w_1w_2 .

Example: For $\Sigma = \{0, 1\}$ and $w_1 = "01"$ and $w_2 = "10"$, the concatenated strings are:

$$w_1 \cdot w_2 = "0110" \text{ and } w_2 \cdot w_1 = "1001".$$
◇

Definition 3 (Formal Language)

Given an alphabet Σ , a subset $L \subseteq \Sigma^*$ is called a **formal language** if it is *precisely defined*. □

You may wonder what it means for a subset to be *precisely defined*. A subset L is precisely defined if for every $w \in \Sigma^*$ the notion $w \in L$ is unambiguously defined. For instance, English doesn't qualify as a formal language because it lacks a precise rule set for valid sentences.

This initial definition is intentionally broad. As we progress through the lecture, we will explore specialized versions, most notably **regular languages** and **context-free languages**. These are particularly relevant in computer science.

Examples:

1. Assume that $\Sigma = \{0, 1\}$. Define

$$L_{\mathbb{N}} = \{1 \cdot w \mid w \in \Sigma^*\} \cup \{0\}$$

Then $L_{\mathbb{N}}$ is the language consisting of all strings that can be interpreted as natural numbers given in binary notation. The language contains all strings from Σ^* that start with the character 1 as well as the string 0, which only contains the character 0. For example, we have

$$"100" \in L_{\mathbb{N}}, \quad \text{but} \quad "010" \notin L_{\mathbb{N}}.$$

Let us define a function

$$\text{value} : L_{\mathbb{N}} \rightarrow \mathbb{N}$$

on the set $L_{\mathbb{N}}$. We define $\text{value}(w)$ by induction on the length of w . We call $\text{value}(w)$ the **interpretation** of w . The idea is that $\text{value}(w)$ computes the number represented by the string w :

- (a) $\text{value}(0) = 0$, $\text{value}(1) = 1$,
- (b) $|w| > 0 \rightarrow \text{value}(w0) = 2 \cdot \text{value}(w)$,
- (c) $|w| > 0 \rightarrow \text{value}(w1) = 2 \cdot \text{value}(w) + 1$.

2. Again we have $\Sigma = \{0, 1\}$. Define the language $L_{\mathbb{P}}$ to be the set of all strings from $L_{\mathbb{N}}$ that are prime numbers:

$$L_{\mathbb{P}} := \{w \in L_{\mathbb{N}} \mid \text{value}(w) \in \mathbb{P}\}$$

Here, \mathbb{P} denotes the set of **prime numbers**, which is the set of all natural numbers p bigger than 1 that have no divisor other than 1 or p :

$$\mathbb{P} = \{p \in \mathbb{N} \mid \{t \in \mathbb{N} \mid \exists k \in \mathbb{N} : k \cdot t = p\} = \{1, p\}\}.$$

3. Define $\Sigma_{\text{ASCII}} = \{0, \dots, 127\}$. Furthermore, define L_C as the set of all strings of the form

$$\text{char}^* f(\text{char}^* x) \{ \dots \}$$

that are, furthermore, valid C function definitions. Therefore, L_C contains all those strings that can be interpreted as a C function f such that f takes a single argument which is a string and returns a value which is also a string.

4. Define $\Sigma := \Sigma_{\text{ASCII}} \cup \{\dagger\}$, where \dagger is some new symbol that is different from all symbols in Σ_{ASCII} . The **universal language** L_u is the set of all strings of the form

$$p\dagger x\dagger y$$

such that

- (a) $p \in L_C$,
- (b) $x, y \in \Sigma_{\text{ASCII}}^*$,
- (c) if f is the function that is defined by p , then $f(x)$ yields the result y . ◇

The above examples illustrate the expansive scope of what can be considered a formal language. While it's straightforward to identify strings in $L_{\mathbb{N}}$, it becomes substantially more challenging for languages like $L_{\mathbb{P}}$ or L_C . Moreover, since the [halting problem](#) is undecidable, no algorithm can definitively determine membership in L_u . Nevertheless, L_u is [semi-decidable](#): if a string w belongs to L_u , we can eventually establish this fact.

1.2 Overview

The aim of this lecture is to cover a series of interconnected topics. While this overview will introduce some terms that may not be immediately clear, rest assured that these terms will be elaborated upon as the lecture progresses. The lecture is structured into three main parts:

1. The first part delves into the theory of [regular languages](#).
 - (a) We begin with an exploration of [regular expressions](#). After formally defining this term, we will discuss its applications in *Python*.
 - (b) We then demonstrate how the [PLY](#) tool can be utilized to create [scanners](#).
 - (c) Subsequently, we examine the implementation of regular expressions via [finite state machines](#).
 - (d) A discussion on how to verify the [equivalence](#) of regular expressions follows.
 - (e) We conclude this section by discussing the limitations of regular languages, focusing particularly on the [Pumping Lemma](#).
2. The second part is centered on [context-free languages](#), which are instrumental in describing the syntax of programming languages.
 - (a) We first introduce [context-free grammars](#), which are employed to define context-free languages.
 - (b) Next, we talk about using [PLY](#) as a parser generator.
 - (c) Essential theory for understanding the parser generator [PLY](#) is then presented.
 - (d) Lastly, we discuss the limitations of context-free languages.
3. The final part of this lecture covers interpreters and compilers.
 - (a) We implement an interpreter for a toy language.
 - (b) Then we discuss how to implement a compiler that translates another toy language into [Java byte code](#).

1.3 Literature

Besides these lecture notes, there are three books I highly recommend:

- (a) *Introduction to Automata Theory, Languages, and Computation* [[HMU06](#)] — Often considered the definitive text on formal languages, this book encompasses all the theoretical concepts discussed in this lecture, although we will only touch on a subset of its content.
- (b) *Introduction to the Theory of Computation* [[Sip12](#)] — This book provides another accessible introduction to formal languages and extends to the theory of computability, which is beyond the scope of this lecture.
- (c) *Compilers — Principles, Techniques and Tools* [[ASUL06](#)] — A standard reference in compiler theory, this book also covers a substantial portion of formal languages.
- (d) The tool [ChatGPT o1](#) has reached a quality level that allows it to serve as a tutor for many concepts discussed in this course.

Nevertheless, I would like to warn you that quality of the free version of ChatGPT is rather poor and hence is useless when trying to understand scientific theories.

1.4 Check your Understanding

- (a) Define the notion of an [alphabet](#).
- (b) Based on a given alphabet, define the concept of a [string](#).
- (c) Explain the concept of a [formal language](#).

Chapter 2

Regular Expressions

Regular expressions are linguistic constructs designed to specify formal languages simple enough to be recognized by a *finite state machine*. The notion of finite state machines will be elaborated upon in Chapter 4. Regular expressions enable the specification of:

1. Alternatives among different options,
2. Concatenation of elements, and
3. Repetition of elements.

Regular expressions form the backbone of many modern scripting languages. For instance, the initial surge in the popularity of the programming language *Perl* can be largely attributed to its efficient handling of regular expressions. Nowadays, nearly all high-level programming languages, such as *Python*, *Java*, and *C#*, offer comprehensive libraries to support regular expressions.

Moreover, various UNIX utilities, including *grep*, *sed*, and *awk*, are fundamentally based on regular expressions. Consequently, proficiency in regular expressions is indispensable for budding computer scientists.

In this chapter, we will introduce a definition of regular expressions that is more concise than those typically found in programming languages. This streamlined definition will facilitate our theoretical examination of regular languages in Chapters 4 and 5. The syntax of regular expressions as used in *Python* will be covered in the subsequent chapter.

2.1 Preliminary Definitions

Before delving into the syntax and semantics of regular expressions, we need to introduce some preliminary definitions.

Definition 4 (Product of Languages) Given an alphabet Σ and formal languages $L_1, L_2 \subseteq \Sigma^*$, the *product* of L_1 and L_2 is denoted as $L_1 \cdot L_2$ and is defined as the set of all concatenations $w_1 \cdot w_2$ where $w_1 \in L_1$ and $w_2 \in L_2$. Formally,

$$L_1 \cdot L_2 := \{w_1 \cdot w_2 \mid w_1 \in L_1 \wedge w_2 \in L_2\}. \quad \diamond$$

Example: Suppose $\Sigma = \{a, b, c\}$ and L_1 and L_2 are given by

$$L_1 = \{ab, bc\} \quad \text{and} \quad L_2 = \{ac, cb\},$$

then $L_1 \cdot L_2 = \{abac, abcb, bcac, bccb\}$. \diamond

Definition 5 (Power of a Language) Let Σ be an alphabet, $L \subseteq \Sigma^*$ a formal language, and $n \in \mathbb{N}$. The n -th **power** of L , denoted as L^n , is defined inductively as follows:

B.C.: $n = 0$:

$$L^0 := \{\lambda\}, \quad \text{where } \lambda \text{ is the empty string.}$$

I.S.: $n \mapsto n + 1$:

$$L^{n+1} = L^n \cdot L. \quad \diamond$$

Example: For $\Sigma = \{a, b\}$ and $L = \{ab, ba\}$, we find:

(a) $L^0 = \{\lambda\},$

(b) $L^1 = L,$

(c) $L^2 = \{abab, abba, baab, baba\}.$ \diamond

Definition 6 (Kleene Closure) Given an alphabet Σ and a formal language $L \subseteq \Sigma^*$, the **Kleene closure** of L , denoted as L^* , is the union of all L^n for $n \in \mathbb{N}$:

$$L^* := \bigcup_{n \in \mathbb{N}} L^n = L^0 \cup L^1 \cup L^2 \cup \dots$$

Note that $\lambda \in L^*$, so L^* is never empty, even when $L = \{\}$. \square

Example: Let $\Sigma = \{a, b\}$ and $L = \{a\}$. Then

$$L^* = \{a^n \mid n \in \mathbb{N}\},$$

where a^n represents a string of n consecutive a 's. \diamond

Formally, for a string s and a non-negative integer $n \in \mathbb{N}$, the expression s^n is defined inductively:

B.C.: $n = 0$:

$$s^0 := \lambda.$$

I.S.: $n \mapsto n + 1$:

$$s^{n+1} := s^n \cdot s, \quad \text{where } s^n \cdot s \text{ represents the concatenation of } s^n \text{ and } s. \quad \diamond$$

This highlights that the Kleene closure of a finite language can be infinite. Specifically, L^* will be infinite whenever L contains a string different from λ .

2.2 The Formal Definition of Regular Expressions

We proceed to define the set of regular expressions for a given alphabet Σ . This set is denoted as RegExp_Σ , and is defined inductively. Simultaneously, we define the function

$$L : \text{RegExp}_\Sigma \rightarrow 2^{\Sigma^*},$$

which interprets each regular expression r as a formal language $L(r) \subseteq \Sigma^*$.¹

¹Given a set M , the **power set** of M , i.e., the set of all subsets of M , is denoted as 2^M .

Definition 7 (Regular Expressions) The set RegExp_Σ of **regular expressions** on the alphabet Σ is defined inductively as follows:

1. $\emptyset \in \text{RegExp}_\Sigma$

The regular expression \emptyset denotes the empty language, we have

$$L(\emptyset) := \{\}.$$

In order to avoid confusion we assume that the symbol \emptyset is not a member of the alphabet Σ , i.e. we have $\emptyset \notin \Sigma$.

2. $\varepsilon \in \text{RegExp}_\Sigma$

The regular expression ε denotes the language that only contains the empty string λ :

$$L(\varepsilon) := \{\lambda\}.$$

Furthermore, we assume that $\varepsilon \notin \Sigma$.

3. $c \in \Sigma \rightarrow c \in \text{RegExp}_\Sigma$.

Every character from the alphabet Σ is a regular expression. This expression denotes the language that contains only the string c :

$$L(c) := \{c\}.$$

Observe that we identify characters with strings of length one.

4. $r_1 \in \text{RegExp}_\Sigma \wedge r_2 \in \text{RegExp}_\Sigma \rightarrow r_1 + r_2 \in \text{RegExp}_\Sigma$

Starting from two regular expressions r_1 and r_2 we can use the infix operator “+” to build a new regular expression. This regular expression denotes the union of the languages described by r_1 and r_2 :

$$L(r_1 + r_2) := L(r_1) \cup L(r_2).$$

In order to avoid confusion we have to assume that the symbol “+” does not occur in the alphabet Σ , i.e. we have “+” $\notin \Sigma$.

5. $r_1 \in \text{RegExp}_\Sigma \wedge r_2 \in \text{RegExp}_\Sigma \rightarrow r_1 \cdot r_2 \in \text{RegExp}_\Sigma$

Starting from the regular expression r_1 and r_2 we can use the infix operator “.” to build a new regular expression. This regular expression denotes the product of the languages of r_1 and r_2 :

$$L(r_1 \cdot r_2) := L(r_1) \cdot L(r_2).$$

Again, in order to avoid confusion we have to assume that the symbol “.” does not occur in the alphabet Σ , i.e. we have “.” $\notin \Sigma$.

6. $r \in \text{RegExp}_\Sigma \rightarrow r^* \in \text{RegExp}_\Sigma$

Given a regular expression r , the postfix operator “*” can be used to create a new regular expression. This new regular expression denotes the Kleene closure of the language described by r :

$$L(r^*) := (L(r))^*.$$

We have to assume that “*” $\notin \Sigma$.

7. $r \in \text{RegExp}_\Sigma \rightarrow (r) \in \text{RegExp}_\Sigma$

Regular expressions can be surrounded by parentheses. This does not change the language denoted by the regular expression:

$$L((r)) := L(r).$$

We have to assume that the parentheses “(” and “)” do not occur in the alphabet Σ , i.e. we have “(” $\notin \Sigma$ and “)” $\notin \Sigma$. \diamond

Given the preceding definition, it is unclear whether to interpret the regular expression

$$a + b \cdot c,$$

as either

$$(a + b) \cdot c \quad \text{or} \quad a + (b \cdot c).$$

To remove this ambiguity, we assign **operator precedences** as follows:

1. The postfix operator $*$ has the highest precedence.
2. The infix operator \cdot has lower precedence than $*$ but higher precedence than $+$.
3. The operator $+$ has the lowest precedence.

With these conventions, the expression

$$a + b \cdot c^*$$

is interpreted as

$$a + (b \cdot (c^*)).$$

Examples: In the following examples, the alphabet Σ is defined as

$$\Sigma := \{a, b, c\}.$$

1. $r_1 := (a + b + c) \cdot (a + b + c)$

This expression r_1 denotes the set of all strings from Σ^* that have length 2:

$$L(r_1) = \{w \in \Sigma^* \mid |w| = 2\}.$$

2. $r_2 := (a + b + c) \cdot (a + b + c)^*$

This expression r_2 denotes the set of all strings from Σ^* that have at least length 1:

$$L(r_2) = \{w \in \Sigma^* \mid |w| \geq 1\}.$$

3. $r_3 := (b + c)^* \cdot a \cdot (b + c)^*$

This expression r_3 denotes the set of all strings from Σ^* that have exactly one occurrence of the letter a:

$$L(r_3) = \{w \in \Sigma^* \mid \#\{i \in \mathbb{N} \mid w[i] = a\} = 1\}.$$

4. $r_4 := (b + c)^* \cdot a \cdot (b + c)^* + (a + c)^* \cdot b \cdot (a + c)^*$

This expression r_4 denotes the set of all strings from Σ^* that either contain exactly one occurrence of the letter a or exactly one occurrence of the letter b:

$$L(r_4) = \{w \in \Sigma^* \mid \#\{i \in \mathbb{N} \mid w[i] = a\} = 1\} \cup \{w \in \Sigma^* \mid \#\{i \in \mathbb{N} \mid w[i] = b\} = 1\}. \quad \diamond$$

Remark: The syntax of regular expressions given here is the same as the syntax used in [HMU06]. However, the syntax used for regular expressions in programming languages like *Python* differs. These differences will be discussed later. \diamond

Exercise 1:

- (a) Assume $\Sigma = \{a, b, c\}$. Define a regular expression for the language $L \subseteq \Sigma^*$ that consists of those strings that contain at least one occurrence of the letter “a” and one occurrence of the letter “b”.
- (b) Assume $\Sigma = \{0, 1\}$. Specify a regular expression for the language $L \subseteq \Sigma^*$ that consists of those strings s such that the [antepenultimate](#) character is the symbol “1”.
- (c) Again, assume $\Sigma = \{0, 1\}$. Define a regular expression for the language $L \subseteq \Sigma^*$ containing all those strings that do not contain the substring 110.
- (d) Again, assume $\Sigma = \{0, 1\}$. What is the language L generated by the regular expression

$$(1 + \varepsilon) \cdot (0 \cdot 0^* \cdot 1)^* \cdot 0^*$$

◇

2.3 Algebraic Simplification of Regular Expressions

Given two regular expressions r_1 and r_2 , we use the notation

$$r_1 \doteq r_2 \quad \text{iff} \quad L(r_1) = L(r_2),$$

to indicate that r_1 and r_2 describe the same language. If the equation $r_1 \doteq r_2$ holds, then we refer to r_1 and r_2 as [equivalent](#). The following algebraic laws are applicable:

- (a) $r_1 + r_2 \doteq r_2 + r_1$

This equation is true because the union operator is commutative for sets:

$$L(r_1 + r_2) = L(r_1) \cup L(r_2) = L(r_2) \cup L(r_1) = L(r_2 + r_1).$$

- (b) $(r_1 + r_2) + r_3 \doteq r_1 + (r_2 + r_3)$

This equation is true because the union operator is associative for sets.

- (c) $(r_1 \cdot r_2) \cdot r_3 \doteq r_1 \cdot (r_2 \cdot r_3)$

This equation is true because the concatenation of strings is associative, for any strings u , v , and w we have

$$(uv)w = u(vw).$$

This implies

$$\begin{aligned} L((r_1 \cdot r_2) \cdot r_3) &= \{xw \mid x \in L(r_1 \cdot r_2) \wedge w \in L(r_3)\} \\ &= \{(uv)w \mid u \in L(r_1) \wedge v \in L(r_2) \wedge w \in L(r_3)\} \\ &= \{u(vw) \mid u \in L(r_1) \wedge v \in L(r_2) \wedge w \in L(r_3)\} \\ &= \{uy \mid u \in L(r_1) \wedge y \in L(r_2 \cdot r_3)\} \\ &= L(r_1 \cdot (r_2 \cdot r_3)). \end{aligned}$$

The following equations are more or less obvious.

- (d) $\emptyset \cdot r \doteq r \cdot \emptyset \doteq \emptyset$

- (e) $\varepsilon \cdot r \doteq r \cdot \varepsilon \doteq r$

- (f) $\emptyset + r \doteq r + \emptyset \doteq r$

- (g) $(r_1 + r_2) \cdot r_3 \doteq r_1 \cdot r_3 + r_2 \cdot r_3$

- (h) $r_1 \cdot (r_2 + r_3) \doteq r_1 \cdot r_2 + r_1 \cdot r_3$

(i) $r + r \doteq r$, because

$$L(r + r) = L(r) \cup L(r) = L(r).$$

(j) $(r^*)^* \doteq r^*$

We have

$$L(r^*) = \bigcup_{n \in \mathbb{N}} L(r)^n$$

and that implies $L(r) \subseteq L(r^*)$. This holds true if we replace r by r^* . Therefore

$$L(r^*) \subseteq L((r^*)^*)$$

holds. In order to prove the inclusion

$$L((r^*)^*) \subseteq L(r^*),$$

we consider the structure of the strings $w \in L((r^*)^*)$. Because of

$$L((r^*)^*) = \bigcup_{n \in \mathbb{N}} L(r^*)^n$$

we have $w \in L((r^*)^*)$ if and only if there is an $n \in \mathbb{N}$ such that there are strings $u_1, \dots, u_n \in L(r^*)$ satisfying

$$w = u_1 \cdots u_n.$$

Because of $u_i \in L(r^*)$ we find a number $m(i) \in \mathbb{N}$ for every $i \in \{1, \dots, n\}$ such that for $j = 1, \dots, m(i)$ there are strings $v_{i,j} \in L(r)$ satisfying

$$u_i = v_{1,i} \cdots v_{m(i),i}.$$

Combining these equations yields

$$w = v_{1,1} \cdots v_{m(1),1} v_{1,2} \cdots v_{m(2),2} \cdots v_{1,n} \cdots v_{m(n),n}.$$

Hence w is a concatenation of strings from the language $L(r)$ and hence we have

$$w \in L(r^*).$$

This shows the inclusion $L((r^*)^*) \subseteq L(r^*)$.

(k) $\emptyset^* \doteq \varepsilon$

(l) $\varepsilon^* \doteq \varepsilon$

(m) $r^* \doteq \varepsilon + r^* \cdot r$

(n) $r^* \doteq (\varepsilon + r)^*$

2.4 Check your Understanding

- (a) What is the definition of L^* for a given formal language L ?
- (b) How is the set RegExp_Σ formally defined?
- (c) How is the function $L : \text{RegExp}_\Sigma \rightarrow 2^{\Sigma^*}$ formally defined?
- (d) For $r_1, r_2 \in \text{RegExp}_\Sigma$, how is the equivalence $r_1 \doteq r_2$ defined?

Chapter 3

Building Scanners with Ply

After introducing the concept of regular expressions, we will now explore their practical applications. We examine the parser generator PLY, which is capable of generating both *Python scanners* and *Python parsers*. In this chapter, we will only discuss the generation of scanners via PLY. A [scanner](#) is a program designed to segment a given string into a sequence of *tokens*, where a [token](#) is a contiguous string of characters that logically group together. To illustrate, consider the input for a C-compiler, which is an ASCII-string representing a valid C program. The compiler initially organizes these characters into various tokens, including:

1. [Keywords](#) or reserved words, such as, for example, “if”, “while”, “for”, and “switch”.
2. [Operator symbols](#), for example “+”, “+=”, “<”, and “<=”.
3. [Parentheses](#), which include “(”, “[”, “{”, and their corresponding closing symbols.
4. [Constants](#), with C recognizing three types:
 - (a) Numerical constants, such as the integer “42” or the floating-point number “1.23e2”.
 - (b) String constants enclosed in double quotes, like “hello”.
 - (c) Single-character constants wrapped in single quotes, as in ‘a’.Note that the double quote character “” is part of the string constant.
Note that the single quote character ‘’ is part of the character constant.
5. [Identifiers](#), which may serve as variable names, function names, or type names.
6. [Comments](#), available in two forms: [Single-line comments](#) start with “//” and go to the end of the line, while [multi-line comments](#) begin with “/*” and end with “*/”.
7. [Whitespace characters](#), such as the [blank character](#) ‘ ’, the [tab](#) ‘\t’, the [newline](#) ‘\n’, and the [carriage return](#) ‘\r’.

Both whitespace characters and comments are generally discarded by the scanner.

For a more tangible example, consider Figure 3.1 on page 17, which presents a C program that outputs the string “Hello World!” along with a newline character. The scanner processes this program into the subsequent list of tokens:

```
[ "#", "include", "<", "stdio.h", ">", "int", "main", "(", ")", "{",  
  "printf", "(", "\"Hello World!\"", ")", ";", "return", "1", ";", "}" ]
```

Note that the scanner omits all white space characters, except those that occur inside string constants, as they are only used to demarcate tokens. Furthermore, the comment has been removed by the scanner.

While it is feasible to manually create a scanner, using a [scanner generator](#) makes the task considerably simpler. We will explore one such utility in the following section.

```

1  /* Hello World program */
2  #include <stdio.h>
3
4  int main() {
5      printf("Hello World!");
6      return 1;
7  }

```

Figure 3.1: A straightforward C program.

3.1 The Structure of a Ply Scanner Specification

In this section, we explore [Python Lex-Yacc](#), commonly known as PLY. According to its [official webpage](#),

“PLY serves as a lex and yacc parsing toolkit for *Python*.”

The toolkit was developed by [David Beazley](#). In this section, we focus solely on its capabilities as a scanner generator. A more comprehensive discussion on its parser generation features will be given in [Chapter 9](#).

To illustrate the core components of a PLY scanner specification, we consider a simple example that tokenizes arithmetic expressions. This example is adapted from the official PLY [documentation](#). A PLY scanner specification generally consists of three key parts, as demonstrated in [Figure 3.2](#):

1. The module `ply.lex` contains the definition of the function `ply.lex.lex()` that is able to generate a scanner. Therefore, this module is imported in line 1.
2. The first part of a scanner specification is the [token declaration section](#). Syntactically, this is just a list containing the names of all tokens. Note that all token names have to start with a capital letter.

In [Figure 3.2](#) the token declaration section extends from line 3 to line 11.

3. The second part contains the [token definitions](#). There are two kinds of token definitions:

- (a) [Immediate token definitions](#) have the following form:

```
t_NAME = r'regexp'
```

Here *NAME* has to be one of the names declared in the declaration section and *regexp* is a regular expression using the syntax that is specified in the *Python re* module.

- (b) [Functional token definitions](#) are syntactically *Python function definitions* and have the following form:

```
def t_NAME(t):
    r'regexp'
    :
```

Here, the vertical dots “`:`” denote any *Python* code, while *NAME* has to be one of the token names declared in the declaration section and *regexp* is a regular expression.

The functional token definition shown in line 20–23 takes a token *t* as its argument. This token has the attribute `t.value`, which refers to the string that has been recognized as this token. In this case, this string is a sequence of digits that can be interpreted as a number. In line 22 the function `t.NUMBER` converts this string into a number and stores this number as the attribute `t.value`. Finally, the token *t* itself is returned. This is a typical case where we need a functional token definition since we want to modify the token that is returned.

In [Figure 3.2](#) the token definitions start in line 13 and end in line 23.

4. The third part deals with the handling of newlines, ignored characters, and scanner errors.

- (a) A PLY input file may contain the definition of the function `t_newline`. This function is supposed to deal with newlines contained in the input. The main purpose of this function is to set the counter `t.lexer.lineno`. Every token `t` has the attribute `t.lexer`, which is a reference to the scanner object. In turn, the scanner object has the attribute `lineno`, which is supposed to be an integer containing the number of the line currently scanned. This integer starts at the value 1. Every time a newline is read it should be incremented.

In line 26 the regular expression `r'\n+'` matches any positive number of newlines. Hence the counter `lineno` has to be incremented by the length of the string `t.value`.

Note that the function `t_newline` does not return a token.

- (b) Line 29 specifies that both blanks and tabs should be ignored by the scanner. Note that the string

```
"' \t'"
```

is not interpreted as a regular expression but rather as a list of its characters. Furthermore, this is not a raw string and must not be prefixed with the character `"r"`, for otherwise the character sequence `"\t"` would not be interpreted as a tab symbol.

- (c) The function `t_error` deals with characters that can not be recognized. An error message is printed and the call `t.lexer.skip(1)` discards the character that could not be matched.

5. In line 35 the function `lex.lex` creates the scanner that has been specified.

6. Line 38 shows how data can be fed into this scanner.

7. In order to use this scanner we can just iterate over it as shown in line 40. This iteration scans the input string using the generated scanner and produces the tokens that are recognized by the scanner one by one.

Executing the program depicted in Figure 3.2 yields the following output:

```
LexToken(NUMBER,3,1,0)
LexToken(PLUS,'+',1,2)
LexToken(NUMBER,4,1,4)
% ... (rest of the output)
LexToken(NUMBER,2,1,27)
```

The scanner returns tokens as instances of the `LexToken` class, which possess four distinct attributes:

1. The `type` attribute indicates the type of the token. It holds a string value that corresponds to one of the declared tokens.
2. The `value` attribute usually contains the recognized string. However, this attribute is mutable, allowing for transformations. For instance, the function `t_NUMBER` converts this string into an integer.
3. The `lineno` attribute specifies the line number where the token was identified.
4. The `lexpos` attribute serves as a counter, incremented for each character read.

Homework: Install PLY and verify that the previously presented example operates as expected. The program PLY can be installed via the following command provided the appropriate conda environment has already been activated:

```
pip install ply
```

```

1  import ply.lex as lex
2
3  tokens = [
4      'NUMBER',
5      'PLUS',
6      'MINUS',
7      'TIMES',
8      'DIVIDE',
9      'LPAREN',
10     'RPAREN'
11 ]
12
13 t_PLUS    = r'\+'
14 t_MINUS   = r'\-'
15 t_TIMES   = r'\*'
16 t_DIVIDE  = r'\/'
17 t_LPAREN  = r'\('
18 t_RPAREN  = r'\)'
19
20 def t_NUMBER(t):
21     r'0|[1-9][0-9]*'
22     t.value = int(t.value)
23     return t
24
25 def t_newline(t):
26     r'\n+'
27     t.lexer.lineno += len(t.value)
28
29 t_ignore  = ' \t'
30
31 def t_error(t):
32     print("Illegal character '%s'" % t.value[0])
33     t.lexer.skip(1)
34
35 lexer = lex.lex()
36
37 data = '3 + 4 * 10 + 007 + (-20) * 2'
38 lexer.input(data)
39
40 for tok in lexer:
41     print(tok)

```

Figure 3.2: A simple scanner Specification for *PLY*.

3.2 The Syntax of Regular Expressions in *Python*

In the preceding chapter, we introduced regular expressions using a minimalistic syntax. While this simplicity is advantageous for the theoretical discussions in the upcoming chapter—where we explore the implementation of regular expressions via finite state machines—it can be limiting in practical applications. To address this, the *Python* module [re](#) offers various syntactic abbreviations that facilitate the expression of complex regular expressions in a more concise manner. For an in-depth look at these features, I have authored a brief tutorial at

<https://github.com/karlstroetmann/Formal-Languages/blob/master/Python/Chapter-3/Regex-Tutorial.ipynb>

that focuses on the key aspects of the `re` module's regular expressions. For an interactive experience, it is advisable to consult this tutorial directly.

3.3 A Complex Example: Evaluating an Exam

This section presents a more complex example that shows some of the power of `PLY`. The task at hand is the evaluation of an exam. When I mark an exam I create a file that has a format similar to the example shown in Figure 3.3.

```

1  Class: Algorithms and Complexity
2  Group: TIT09AID
3  MaxPoints = 60
4
5  Exercise:      1. 2. 3. 4. 5. 6.
6  Jim Smith:    9 12 10 6 6 0
7  John Slow1:   4 4 2 0 - -
8  Susi Sorglos: 9 12 12 9 9 6

```

Figure 3.3: Results of an Exam

1. The first line contains the keyword “Class”, a colon “:”, and then the name of the lecture.
2. The second line specifies the group that has taken the exam.
3. The third line specifies the number of points that are necessary to obtain the best mark.
4. The fourth line is empty.
5. The fifth line numbers the exercises.
6. After that, there is a table. Every row in this table lists the scores achieved by a student for each of the exercises. The name of each student is at the beginning of each row. The name is followed by a colon and after that there is a list of the scores achieved for each exercise. If an exercise has not been attempted at all, the corresponding column contains a hyphen “-”.

I have written a *Jupyter notebook* that is able to evaluate data of this kind. You can find the notebook here:

<https://github.com/karlstroetmann/Formal-Languages/blob/master/Python/Chapter-3/Exam-Evaluation.ipynb>

3.4 Scanner States

Sometimes, regular expressions are not quite enough and it is beneficial for the scanner to have different states. The following example illustrates this. We will develop a program that is able to convert an HTML file into a pure text file. This program is actually quite useful: Some years ago I had a student that was blind. If he read a web page, he would use his Braille display. For him, the HTML markup was of no use so if the markup was removed, he could read web pages faster. In order to develop the program to remove HTML tags, we have to use [scanner states](#). The idea behind scanner states is that the scanner can use different regular expressions for different parts of the input. For example, the header of an HTML file, i.e. the part that is between the `<head>` and `</head>` tags, can just be skipped. On the other hand, the text inside the `<body>` and `</body>` tags needs to be echoed after any remaining tags have been removed. The easiest way to achieve this is by using scanner states and switching between them.

¹You know nothing, John Slow.

The following notebook shows how this can be done:

<https://github.com/karlstroetmann/Formal-Languages/blob/master/Python/Chapter-3/Html2Text.ipynb>

Exercise 2: The purpose of the following exercise is to transform \LaTeX into MATHML . \LaTeX is a document markup language that is especially well suited to present text that contains mathematical formulas. In fact, these lecture notes have all been typeset using \LaTeX . MATHML is the part of HTML that deals with the representation of mathematical formulas. As \LaTeX provides a very rich document markup language and we can only afford to spend a few hours on this exercise, we confine ourselves to a small subset of \LaTeX . Figure 3.4 on page 21 shows the example input file that we want to transform in HTML . If this example file is typeset using \LaTeX , it is displayed as shown in Figure 3.5 on page 21. The program that you are going to develop should transform the \LaTeX input file into an HTML file. For your convenience, all these files are available in the github directory

[Exercises/LaTeX2HTML](#).

This directory contains also the *Jupyter* notebook [LaTeX2HTML.ipynb](#). This notebook contains lots of predefined functions that are useful in order to solve the given task.

```
\documentclass{article}
\begin{document}
The sum of the squares of the first  $n$  natural numbers is given as:

$$\sum_{i=1}^n i^2 = \frac{1}{6} \cdot n \cdot (n+1) \cdot (2n+1).$$

According to Pythagoras, the length of the hypotenuse of a right triangle is
the square root of the squares of the length of the two catheti:

$$c = \sqrt{a^2 + b^2}.$$

The area of a circle is given as

$$A = \pi \cdot r^2,$$

while its circumference satisfies

$$C = 2 \cdot \pi \cdot r.$$

\end{document}
```

Figure 3.4: An example \LaTeX input file.

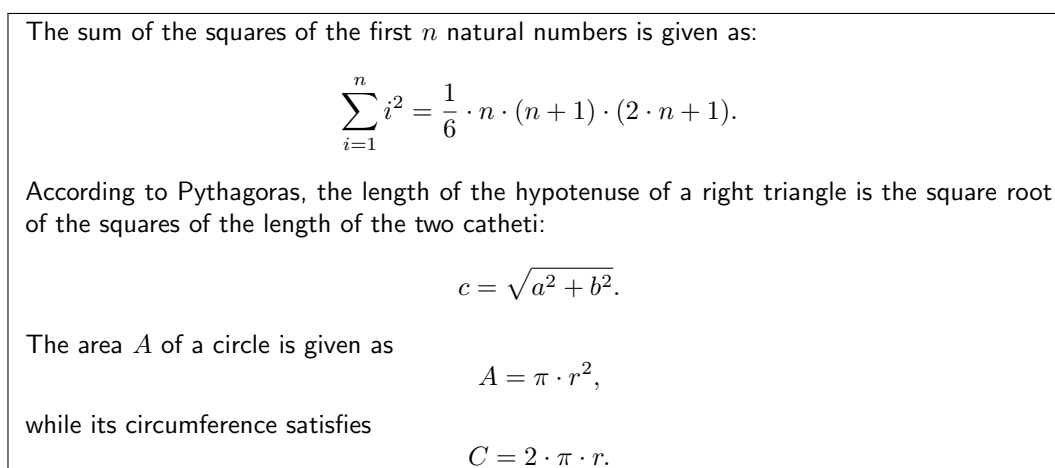


Figure 3.5: Output produced by the \LaTeX file shown in Figure 3.4

In order to do this exercise, you have to understand a little bit about \LaTeX and about MATHML . In the following, we discuss those features of these two language that are needed in order to solve the given problem.

1. A \LaTeX input file has the following structure:

- (a) The first line list the type of the document. In our example, it reads

```
\documentclass{article}.
```

This line will be transformed into the following HTML:

```
<html>
<head>
<script type="text/javascript"
src="http://cdn.mathjax.org/mathjax/latest/MathJax.js?config=TeX-AMS-MML_HTMLorMML">
</script>
```

Here, the `<script>` tag is necessary in order for the MATHML to be displayed correctly.

- (b) The next line has the form:

```
\begin{document}
```

This line precedes the content and should be translated into the tag

```
<body>.
```

- (c) After that, the \LaTeX file consists of text that contains mathematical formula.

- (d) The \LaTeX input file finishes with a line of the form

```
\end{document}.
```

This line should be translated into the tags

```
</body></html>.
```

2. In \LaTeX , an inline formula is started and ended with a single dollar symbol “\$”. In MATHML, an inline formula is written as

```
<math xmlns="http://www.w3.org/1998/Math/MathML" display='inline'>...</math>.
```

Here, I have used “...” to represent the mathematical content of the formula.

3. In \LaTeX , a formula that is displayed in its own line is started and ended with the string “\$\$. In MATHML, these formulas are called **block formulas** and are written as

```
<math xmlns="http://www.w3.org/1998/Math/MathML" display='block'>...</math>.
```

Again, I have used “...” to represent the mathematical content of the formula.

4. While in \LaTeX a mathematical variable does not need any special markup, in MATHML a mathematical variable is written using the tags `<mi>` and `</mi>`. For example, the mathematical variable n is written as

```
<mi>n</mi>.
```

5. While in \LaTeX a number does not need any special markup, in MATHML a number is written using the tags `<mn>` and `</mn>`. For example, the number 3.14149 is written as

```
<mn>3.14159</mn>.
```

6. In \LaTeX the mathematical constant π is written using the command “`\pi`”. In MATHML, we have to make use of the HTML entity “`π`” and hence we would write π as

```
<mn>&pi;</mn>.
```

7. In \LaTeX the multiplication operator “.” is written using the command “`\cdot`”. In MATHML, we have to make use of the HTML entity “`⋅`” and hence we would write “.” as

```
<mop>&sdot;</mop>.
```

8. While in \LaTeX most operator symbols stand for themselves, in MATHML an operator is surrounded by the tags `<mop>` and `</mop>`. For example, the operator $+$ is written as

`<mop>+</mop>`.

9. In \LaTeX , raising an expression e to the n th power is done using the operator `^`. Furthermore, the exponent should be enclosed in the curly braces `"{"` and `"}"`. For example, the code to produce the term x^2 is

`x^{2}`.

In MATHML , raising an expression to a power is achieved using the tags `<msup>` and `</msup>`. For example, in order to display the term x^2 , we have to write

`<msup><mi>x</mi><mn>2</mn></msup>`.

10. In \LaTeX , taking the square root of an expression is done using the command `"\sqrt"`. The argument has to be enclosed in curly braces. For example, in order to produce the output $\sqrt{a+b}$, we have to write

`\sqrt{a+b}`.

In MATHML , taking the square root makes use of the tags `<msqrt>` and `</msqrt>`. The example shown above can be written as

`<msqrt><mi>a</mi><mop>+</mop><mi>b</mi></msqrt>`.

11. In \LaTeX , writing a fraction is done using the command `"\frac"`. This command takes two arguments, the numerator and the denominator. Both of these have to be enclosed in curly braces. For example, in order to produce the output $\frac{a+b}{2}$, we have to write

`\frac{a+b}{2}`.

In MATHML , a fraction is created via the tags `<mfrac>` and `</mfrac>`. Additionally, if the arguments contain more than a single element, each of them has to be enclosed in the tags `<mrow>` and `</mrow>`. The example shown above can be written as

`<mfrac><mrow><mi>a</mi><mop>+</mop><mi>b</mi></mrow><mn>2</mn></mfrac>`.

12. In \LaTeX , writing a sum is done using the command `"\sum\limits"`. This command takes two arguments: The first argument gives the indexing variable together with its lower bound, while the second argument gives the upper bound. The first argument is started using the string `"_{"` and ended using the string `"}"`, while the second argument is started using the string `"^{"` and ended using the string `"}"`. For example, in order to produce the output

$$\sum_{i=1}^n i,$$

we have to write

`\sum\limits_{i=1}^n i`.

In MATHML , a sum with lower and upper limits is created via the tags `<munderover>` and `</munderover>` and the HTML entity `"&sum"`. The tag `munderover` takes three arguments:

- The first argument is the operator, so in this case it is the entity `"&sum"`.
- The second argument initializes the indexing variable of the sum.
- The third argument provides the upper bound.

The second argument usually contains more than a single item and therefore has to be enclosed in the tags `<mrow>` and `</mrow>`. Hence, the example shown above would be written as follows:


```

<munderover>
  <mo>&sum;</mo>
  <mrow>
    <mi>i</mi> <mo>=</mo> <mn>1</mn>
  </mrow>
  <mi>n</mi>
</munderover>

```

Remark: The most important problem that you have to solve is the following: Once you encounter a closing brace “}” you have to know whether this brace closes the argument of a square root, a fraction, a sum, or an exponent. You should be aware that, for example, square roots and fractions can be nested. Hence, it is not enough to have a single variable that remembers whether you are parsing, say, a square root or a fraction. Instead, every time you encounter a string like, e.g.

`\sqrt{` or `\frac{`,

you should store the current state on a stack and set the new state according to whether you have just seen the keyword “`\frac`” or “`\sqrt`” or whatever caused the curly brace to be opened. When you encounter a closing brace “}”, you should restore the state to its previous value by looking up this value from the stack. ◇

3.5 Check your Understanding

- (a) How are regular expressions used in *Python*?
- (b) Do you understand the structure of a *PLY* scanner specification?
- (c) Are you able to use *PLY* to write a program that reads a given text, finds all numbers inside this text that are followed by a \$ symbol and that then converts these numbers to their equivalent value in €?

Chapter 4

Finite State Machines

In the previous chapter we have seen how to generate a scanner using PLY. In this chapter we learn how regular expressions can be implemented using [finite state machines](#), abbreviated as FSMs. There are two kinds of FSMs: The deterministic ones and non-deterministic ones. Although non-deterministic FSMs seem to be more powerful than deterministic FSMs, we will see that every non-deterministic FSM can be transformed into an equivalent deterministic FSM. After proving this result, we show how a regular expression can be translated into an equivalent non-deterministic FSM. Finally, we show that the language recognized by any FSM can be described by an equivalent regular expression. Therefore, the central result of this chapter is the equivalence of finite state machines and regular expressions. Hence, the results proved in this chapter are as follows:

1. The language described by a regular expression can be defined by a non-deterministic FSM.
2. Every non-deterministic FSM can be transformed into an equivalent deterministic FSM.
3. For every deterministic FSM there is a regular expression specifying the language recognized by the FSM.

4.1 Deterministic Finite State Machines

The FSMs that we are going to discuss in this chapter are used to read a string and to decide whether this string is an element of a given language. Hence, the input of these FSMs is a string, while the output is either the value `True` or the value `False`. The name giving feature of an FSM is the fact that an FSM only has a [finite](#) number of possible states. Reading a character causes the FSM to change its state. An FSM accepts its input if it has reached a so called [accepting state](#) after reading all characters of the input string. Let me explain this idea more precisely:

1. Initially, the FSM is in a state that is known as the [start state](#).
2. In every step of its computation, the FSM reads one character of the input string s . Every time a character is processed, the state of the FSM might change.
3. Some states of the FSM are designated as [accepting states](#). If the FSM has consumed all characters of the given input string s and the FSM has reached an accepting state, then the input string s is accepted and the FSM returns `True`. Otherwise the FSM returns `False` and the string s is rejected.



Figure 4.1: An FSM to recognize the language $L(a^* \cdot b \cdot a^*)$.

Finite state machines are best presented graphically. Figure 4.1 depicts a simple FSM that recognizes those strings that are specified by the regular expression

$$a^* \cdot b \cdot a^*.$$

This FSM has but two states. These states are called 0 and 1.

1. State 0 is the start state. In Figure 4.1, the start state is indicated by an arrow coming from the left that points to it.

If the FSM is in the state 0 and reads the character “a”, then the FSM stays in the state 0. This is specified in the figure by an arrow labeled with the character “a” that both starts and ends in the state 0. On the other hand, if the character “b” is read while the FSM is in state 0, then the FSM switches into the state 1. This is depicted by an arrow labeled with the character “b” that originates from the state 0 and points to the state 1.

2. State 1 is an accepting state. In Figure 4.1 this is specified by the fact that the state 1 is decorated by a double circle.

If the character “a” is read while the FSM is in state 1, then the FSM does not change its state. On the other hand, if the FSM reads the character “b” while in state 1, then the next state is undefined since there is no arrow originating from state 1 that is labeled with the character “b”.

In general, a FSM *dies* if it reads a character c in a state s such that there is no transition from s when c is read. In this case, the FSM returns the value `False` to signal that it does not accept the given input string.

Formally, a *finite state machine* is defined as a 5-tuple.

Definition 8 (Fsm) A *finite state machine* (abbreviated as FSM) is a 5-tuple

$$F = \langle Q, \Sigma, \delta, q_0, A \rangle$$

where the components Q , Σ , δ , q_0 , and A have the following properties:

1. Q is the *finite set of states*.
2. Σ is the *input alphabet*. Therefore, Σ is a set of characters and the strings read by the FSM F are strings from the set Σ^* .
3. $\delta : Q \times \Sigma \rightarrow Q \cup \{\Omega\}$
is the *transition function*. For every state $q \in Q$ and for all characters $c \in \Sigma$ the expression $\delta(q, c)$ computes the new state of the FSM F that is reached if F reads the character c while in state q . If $\delta(q, c) = \Omega$, then F *dies* when it is in state q and the next character is c .
In the figures depicting FSMs transitions of the form $\delta(q, c) = \Omega$ are not shown.
4. $q_0 \in Q$ is the *start state*.
5. $A \subseteq Q$ is the set of *accepting states*. □

Example: The FSM that is shown in Figure 4.1 is formally defined as follows:

$$F = \langle Q, \Sigma, \delta, q_0, A \rangle,$$

where we have:

1. $Q = \{0, 1\}$,
2. $\Sigma = \{a, b\}$,
3. $\delta = \{ \langle 0, a \rangle \mapsto 0, \langle 0, b \rangle \mapsto 1, \langle 1, a \rangle \mapsto 1, \langle 1, b \rangle \mapsto \Omega \}$,
4. $q_0 = 0$,
5. $A = \{1\}$.

In order to formally define the language $L(F)$ that is accepted by an FSM F we generalize the transition function δ to a new function

$$\delta^* : Q \times \Sigma^* \rightarrow Q \cup \{\Omega\}$$

that, instead of a single character, accepts a string as its second argument. The definition of $\delta^*(q, w)$ is given by induction on the string w .

I.A. $w = \lambda$: We define

$$\delta^*(q, \lambda) := q,$$

because if a deterministic FSM does not read any character, it cannot change its state.

I.S. $w = cv$ where $c \in \Sigma$ and $v \in \Sigma^*$: We define

$$\delta^*(q, cv) := \begin{cases} \delta^*(\delta(q, c), v) & \text{provided } \delta(q, c) \neq \Omega; \\ \Omega & \text{otherwise.} \end{cases}$$

If F reads the string cv , it first reads the character c . Now if this causes F to change into the state $\delta(q, c)$, then F has to read the string v in the state $\delta(q, c)$. However, if $\delta(q, c)$ is undefined, then $\delta^*(q, cv)$ is undefined too.

Definition 9 (Accepted Language, $L(F)$) For an FSM $F = \langle Q, \Sigma, \delta, q_0, A \rangle$ the **language accepted by F** is called $L(F)$ and is defined as

$$L(F) := \{s \in \Sigma^* \mid \delta^*(q_0, s) \in A\}.$$

Hence, the accepted language of F is the set of all those strings that take F from its start state into an accepting state. \diamond

Exercise 3: Specify an FSM F such that $L(F)$ is the set of all those strings $s \in \{a, b\}^*$, such that s contains the substring “aba”. \diamond

Complete Finite State Machines Occasionally it is beneficial for an FSM F to be **complete**: An FSM

$$F = \langle Q, \Sigma, \delta, q_0, A \rangle,$$

is **complete** if the transition function δ never returns the undefined value Ω , i.e. we have

$$\delta(q, c) \neq \Omega \quad \text{for all } q \in Q \text{ and } c \in \Sigma.$$

Proposition 10 For every FSM F there exists a complete FSM \hat{F} that accepts the same language as the FSM F , i.e. we have:

$$L(\hat{F}) = L(F).$$

Proof: Assume F is given as

$$F = \langle Q, \Sigma, \delta, q_0, A \rangle.$$

The idea is to define \hat{F} by adding a new state to the set of states Q . This new state is called the **dead state**. If there is no next state for a given state $q \in Q$ when a character c is processed, i.e. if we have

$$\delta(q, c) = \Omega,$$

then F changes into the dead state. Once F has reached a dead state, it will never leave this state.

The formal definition of the FSM \hat{F} is done as follows: We introduce a new state Ω which serves as the **dead state**. The only requirement is that $\Omega \notin Q$.

$$1. \hat{Q} := Q \cup \{\Omega\},$$

the dead state is added to the set Q .

$$2. \hat{\delta} : \hat{Q} \times \Sigma \rightarrow \hat{Q},$$

where the function $\hat{\delta}$ is defined as follows:

$$(a) \delta(q, c) \neq \Omega \rightarrow \hat{\delta}(q, c) = \delta(q, c) \quad \text{for all } q \in Q \text{ and } c \in \Sigma.$$

If the state transition function is defined for the state q and the character c , then $\hat{\delta}(q, c)$ is the same as $\delta(q, c)$.

$$(b) \delta(q, c) = \Omega \rightarrow \hat{\delta}(q, c) = \text{☠} \quad \text{for all } q \in Q \text{ and } c \in \Sigma.$$

If the state transition function δ is undefined for the state q and the character c , then $\hat{\delta}(q, c)$ returns the dead state ☠.

$$(c) \hat{\delta}(\text{☠}, c) = \text{☠} \quad \text{for all } c \in \Sigma,$$

because there is no escape from death¹.

Hence the FSM \hat{F} is given as follows:

$$\hat{F} = \langle \hat{Q}, \Sigma, \hat{\delta}, q_0, A \rangle.$$

If F reads a string s without reaching an undefined state, then the behavior of F and \hat{F} is the same. However, if F reaches an undefined state, then \hat{F} instead switches into the dead state ☠ and remains in this state regardless of the rest of the input string. As the dead state ☠ is not an accepting state, the languages accepted by F and \hat{F} are identical. \square

Exercise 4: Define an FSM that accepts the language specified by the regular expression

$$r := (a + b)^* \cdot b \cdot (a + b) \cdot (a + b).$$

◇

Solution: The regular expression r specifies those strings s from the alphabet $\Sigma = \{a, b\}$ such that the antepenultimate character of s is the character “b”. In order to recognize this fact, the FSM has to remember the last three characters. As there are eight different possible combinations for the last three characters, the FSM needs to have eight states. Let us number these states 0, 1, 2, \dots , 7. We describe the purpose of these states in the following:

State 0: In this state, the character “b” has not yet been seen. Depending on how many characters have been read, there are four cases:

- (a) At least three characters have been read. In this case, the last three characters are “aaa”.
- (b) Two characters have been read. In this case, the string that has been read so far is the string “aa”.
- (c) Only one character has been read so far. In this case, the string that has been read is “a”.
- (d) Nothing has yet been read and therefore the string that has been read is λ .

For the remaining states we list the last three characters that have been read without further comment.

State 1: “aab”.

This case also covers the cases where the strings “ab” and “b” have been read.

State 2: “aba”.

This case also cover the case where the string “ba” has been read.

State 3: “abb”.

This case also cover the case where the string “bb” has been read.

State 4: “bab”.

State 5: “bba”.

State 6: “bbb”.

¹Or, as the disciples of the [Drowned God](#) say: “What is dead may never die”.

State 7: “baa”.

Obviously, the states 4, 5, 6 and 7 are the accepting states because here the antepenultimate character is the character “b”. Next, we construct the transition function δ .

0. First, let us consider the state 0. If the last three characters that have been read are “aaa” and if we read the character “a” next, then the last three characters read will again be “aaa”. Hence, we must have

$$\delta(0, a) = 0.$$

However, if instead we read the character “b” in state 0, then the last three characters that have been read are “aab”, which is exactly the last three characters that have been read in state 1. Hence we have

$$\delta(0, b) = 1.$$

1. Next we consider state 1. If the last three characters are “aab” and we read the character “a” next, then the last three characters are “aba”. This corresponds to the state 2. Therefore, we must have

$$\delta(1, a) = 2.$$

If instead we read the character “b” while in state 1, then the last three characters will be “abb”, which corresponds to the state number 3. Hence we have

$$\delta(1, b) = 3.$$

The remaining transitions are found in a similar way. Figure 4.2 on page 29 shows the resulting FSM. We still have to explain how we have chosen the start state. When the computation starts, the finite state machine has not read any character. In particular, this implies that neither of the last three characters is the character “b”. Hence we can use the state 0 as the start state of our FSM.

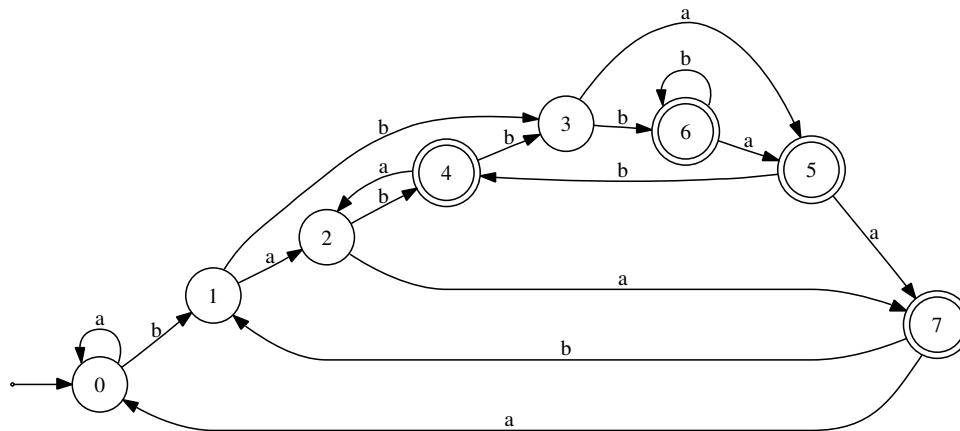


Figure 4.2: An FSM accepting $L((a + b)^* \cdot b \cdot (a + b) \cdot (a + b))$.

Remark: There is a nice tool available that can be used to better understand finite state machines. This tool is available at

https://ivanzuzak.info/noam/webapps/fsm_simulator/.

4.2 Non-Deterministic Finite State Machines

For many applications, the finite state machines introduced in the previous section are unwieldy because they have a large numbers of states. For example, the regular expression to recognize the language

$$L((a + b)^* \cdot b \cdot (a + b) \cdot (a + b))$$

needs 8 different states since the FSM needs to remember the last three characters that have been read and there are $2^3 = 8$ combinations of these characters. It would be possible to simplify this FSM if the FSM would be permitted to *choose* its next state from a given set of states.

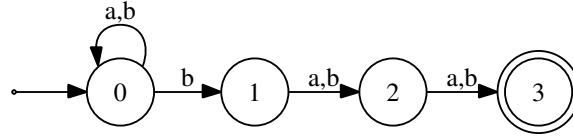


Figure 4.3: A non-deterministic finite state machine to recognize $L((a + b)^* \cdot b \cdot (a + b) \cdot (a + b))$.

Figure 4.3 presents a **non-deterministic finite state machine** that accepts the language specified by the regular expression

$$(a + b)^* \cdot b \cdot (a + b) \cdot (a + b).$$

This finite state machine has only 4 different states that are named 0, 1, 2 and 3.

1. 0 is the start state. If the FSM reads the letter a while it is in this state, the FSM will stay in state 0. However, if the FSM reads the character b, then the finite state machine has a *choice*: It can either stay in state 0, or it might switch to the state 1.
2. In state 1 the finite state machine switches to state 2 if it reads either the character a or the character b.
3. In state 2 the FSM switches to state 3 if it reads either the character a or the character b.
4. State 3 is the accepting state. There is no transition from this state. Hence, if the FSM is in state 3 and there are still characters to read, then the FSM dies.

The finite state machine in Figure 4.3 is non-deterministic because it has to guess the next state if it is in state 0 and reads the character “b”. Let us consider a possible *computation* of the FSM when it reads the input “abab”:

$$0 \xrightarrow{a} 0 \xrightarrow{b} 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

In this computation, the FSM has chosen the correct transition when reading the first occurrence of the character “b”. If the FSM had stayed in the state 0 instead of switching into the state 1, it would have been impossible to reach the accepting state 3 later because then the computation would have worked out as follows:

$$0 \xrightarrow{a} 0 \xrightarrow{b} 0 \xrightarrow{a} 0 \xrightarrow{b} 1$$

Here, the FSM is in state 1 after consuming the input string “abab” and as state 1 is not an accepting state, the FSM would have falsely rejected the string “abab”. Let us consider a different example where the input is the string “bbbb”:

$$0 \xrightarrow{b} 0 \xrightarrow{b} 1 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} \Omega$$

Here, the FSM has switched to early into the state 1. In this case, the FSM dies when reading the last character “b”. If the FSM has stayed in state 0 when reading the second occurrence of the character “b”, then it would have correctly accepted the string “bbbb” since then the computation could have been as follows:

$$0 \xrightarrow{b} 0 \xrightarrow{b} 0 \xrightarrow{b} 1 \xrightarrow{b} 2 \xrightarrow{b} 3.$$

The previous examples show that in order to avoid premature death, the given non-deterministic FSM has to choose its successor state **wisely**. If F is a non-deterministic FSM and s is a string such that F can, when reading s , choose its successor so that it reaches an accepting state after having read s , then the string s is an element of the language $L(F)$.

It seems that the concept of a non-deterministic FSM is far more powerful than the concept of a deterministic FSM. After all, a non-deterministic FSM appears to have some form of clairvoyance for else it could not guess which

states to choose. However, we will prove in the next section that both deterministic and non-deterministic FSMs have the same power to recognize languages: Every language recognized by a non-deterministic FSM is also recognized by a deterministic FSM. In order to prove this claim, we have to formalize the notion of a non-deterministic FSM. The definition that follows is more general than the informal description of non-deterministic FSMs given so far, as we will allow the FSM to also have ε -transitions. A ε transition allows the FSM to switch its state without reading any character. For example, if there is an ε -transition from the state 1 into the state 2, we write

$$1 \xrightarrow{\varepsilon} 2.$$

Definition 11 (NFA) A **non-deterministic FSM** (abbreviated as **NFA** for non-deterministic automaton) is a 5-tuple

$$\langle Q, \Sigma, \delta, q_0, A \rangle,$$

such that the following holds:

1. Q is the finite **set of states**.
2. Σ is the **input alphabet**.
3. δ is a function from $Q \times (\Sigma \cup \{\varepsilon\})$ that assigns a set of states $\delta(q, a) \subseteq Q$ to every pair $\langle q, a \rangle$ from $Q \times (\Sigma \cup \{\varepsilon\})$:

$$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow 2^Q.$$

If $a \in \Sigma$, then $\delta(q, a)$ is the set of states the FSM can switch to after reading the character a in state q . The set $\delta(q, \varepsilon)$ is the set of states that can be reached from the state q without reading a character.

As in the deterministic case, δ is called the **transition function**.

4. $q_0 \in Q$ is the start state.
5. $A \subseteq Q$ is the set of accepting states.

If we have $q_2 \in \delta(q_1, \varepsilon)$, then the FSM has an ε -transition from the state q_1 into the state q_2 . This is written as

$$q_1 \xrightarrow{\varepsilon} q_2.$$

If $c \in \Sigma$ and $q_2 \in \delta(q_1, c)$, we write

$$q_1 \xrightarrow{c} q_2.$$

□

In order to distinguish a deterministic FSM from a non-deterministic FSM, deterministic FSMs are also called DFA which is short for **deterministic finite automaton**.

Example: For the FSM F shown in Figure 4.3 on page 30 we have

$$F = \langle Q, \Sigma, \delta, 0, A \rangle \quad \text{where}$$

1. $Q = \{0, 1, 2, 3\}$.
2. $\Sigma = \{a, b\}$.
3. $\delta = \{ \langle 0, a \rangle \mapsto \{0\}, \langle 0, b \rangle \mapsto \{0, 1\}, \langle 0, \varepsilon \rangle \mapsto \{\}, \langle 1, a \rangle \mapsto \{2\}, \langle 1, b \rangle \mapsto \{2\}, \langle 1, \varepsilon \rangle \mapsto \{\}, \langle 2, a \rangle \mapsto \{3\}, \langle 2, b \rangle \mapsto \{3\}, \langle 2, \varepsilon \rangle \mapsto \{\}, \langle 3, a \rangle \mapsto \{\}, \langle 3, b \rangle \mapsto \{\}, \langle 3, \varepsilon \rangle \mapsto \{\} \}$.

It is more convenient to specify the transition function δ as follows:

$$\delta := \{ 0 \xrightarrow{a} 0, 0 \xrightarrow{b} 0, 0 \xrightarrow{b} 1, 1 \xrightarrow{a} 2, 1 \xrightarrow{b} 2, 2 \xrightarrow{a} 3, 2 \xrightarrow{b} 3 \}.$$

4. The start state is 0.
5. $A = \{3\}$, hence the only accepting state is 3.

◇

In order to formally define how a non-deterministic FSM processes its input we introduce the notion of a **configuration** of a non-deterministic FSM. A configuration is defined as a pair

$$\langle q, s \rangle$$

where q is a state and s is a string. Here, q is the current state of the FSM and s is the part of the input that has not yet been consumed. We define a binary relation \rightsquigarrow on configurations as follows:

$$\langle q_1, cs \rangle \rightsquigarrow \langle q_2, s \rangle \quad \text{iff} \quad q_1 \xrightarrow{c} q_2, \quad \text{i.e. if } q_2 \in \delta(q_1, c).$$

Therefore, we have $\langle q_1, cs \rangle \rightsquigarrow \langle q_2, s \rangle$ if and only if the FSM transitions from the state q_1 into the state q_2 when the character c is consumed. Furthermore, we have

$$\langle q_1, s \rangle \rightsquigarrow \langle q_2, s \rangle \quad \text{iff} \quad q_1 \xrightarrow{\varepsilon} q_2, \quad \text{i.e. if } q_2 \in \delta(q_1, \varepsilon).$$

This accounts for the ε transitions. The **reflexive-transitive closure** of the relation \rightsquigarrow is written as \rightsquigarrow^* . The language accepted by a non-deterministic FSM F is denoted as $L(F)$ and is defined as

$$L(F) := \{s \in \Sigma^* \mid \exists p \in A : \langle q_0, s \rangle \rightsquigarrow^* \langle p, \lambda \rangle\}.$$

Here, q_0 is the start state and A is the set of accepting states. Hence, a string s is an element of the language $L(F)$, iff there is an accepting state p such that the configuration $\langle p, \varepsilon \rangle$ is reachable from the configuration $\langle q_0, s \rangle$.

Example: The FSM F shown in Figure 4.3 accepts those strings $w \in \{a, b\}^*$ such that the antepenultimate character of w is the character “b”:

$$L(F) = \{w \in \{a, b\}^* \mid |w| \geq 3 \wedge w[-3] = b\} \quad \diamond$$

Exercise 5: Specify a non-deterministic FSM F such that $L(F)$ is the set of those strings from the language $\{a, b\}^*$ that contain the substring “aba”. \(\diamond\)

4.3 Equivalence of Deterministic and Non-Deterministic FSMs

In this section we show how a non-deterministic FSM

$$F = \langle Q, \Sigma, \delta, q_0, A \rangle$$

can be transformed into a deterministic FSM $\text{det}(F)$ such that both FSMs accept the same language, i.e. we have

$$L(F) = L(\text{det}(F))$$

The idea behind this transformation is that the FSM $\text{det}(F)$ has to compute the **set** of all states that the FSM F could be in after reading a string in the start state. Hence the states of the deterministic FSM $\text{det}(F)$ are **sets** of states of the non-deterministic FSM F . A set of these states contains all those states that the non-deterministic FSM F could have reached when reading a character. Furthermore, a set M of states of the FSM F is an accepting state of the FSM $\text{det}(F)$ if the set M contains an accepting state of the FSM F .

In order to present the construction of $\text{det}(F)$ we first have to define two auxiliary functions. We start with the **ε -closure** of a given state. For every state q of the non-deterministic FSM F the function

$$ec : Q \rightarrow 2^Q$$

computes the set $ec(q)$ of all those states that the FSM F can reach by ε transitions from the state q . Formally, the set $ec(q)$ is computed inductively:

B.C.: $q \in ec(q)$.

I.S.: $p \in ec(q) \wedge r \in \delta(p, \varepsilon) \rightarrow r \in ec(q)$.

If the state p is an element of the ε -closure of the state q and there is an ε -transition from p to some state r , then r is also an element of the ε -closure of q .

Figure 4.4: A non-deterministic FSM with ε -transitions.

Example: Figure 4.4 shows a non-deterministic FSM with ε -transitions. In this figure, the ε -transitions are shown as unlabelled arrows. We compute the ε -closure for all states:

1. $ec(q_0) = \{q_0, q_1, q_2\}$,
2. $ec(q_1) = \{q_1\}$,
3. $ec(q_2) = \{q_2\}$,
4. $ec(q_3) = \{q_3\}$,
5. $ec(q_4) = \{q_4\}$,
6. $ec(q_5) = \{q_5, q_7, q_0, q_1, q_2\}$,
7. $ec(q_6) = \{q_6, q_7, q_0, q_1, q_2\}$,
8. $ec(q_7) = \{q_7, q_0, q_1, q_2\}$.

□

In order to transform a non-deterministic FSM into a deterministic FSM $det(F)$ we have to extend the function $\delta : Q \cup \{\varepsilon\} \times \Sigma \rightarrow 2^Q$ into the function

$$\hat{\delta} : Q \times \Sigma \rightarrow 2^Q.$$

The idea is that given a state q and a character c , $\hat{\delta}(q, c)$ is the set of all states that the FSM F could reach when it reads the character c in state q and then performs an arbitrary number of ε -transitions. Formally, the definition of $\hat{\delta}$ is as follows:

$$\hat{\delta}(q_1, c) := \bigcup \{ec(q_2) \mid q_2 \in \delta(q_1, c)\}.$$

This formula is to be read as follows:

- (a) For every state $q_2 \in Q$ that can be reached from the state q_1 by reading the character c we compute the ε -closure $ec(q_2)$.
- (b) Then we take the union of all the sets $ec(q_2)$ where $q_2 \in \delta(q_1, c)$.

Example: In continuation of the previous example (shown in Figure 4.4) we have:

1. $\hat{\delta}(q_0, a) = \{\}$,
because in state q_0 there is no transition on reading the character a . Note that in our definition of the function $\hat{\delta}$ the ε -transitions are done only after the character has been read.
2. $\hat{\delta}(q_1, b) = \{q_3\}$,
because when the letter 'b' is read in the state q_1 the FSM switches into the state q_3 and the state q_3 has no ε -transitions.

$$3. \hat{\delta}(q_3, a) = \{q_5, q_7, q_0, q_1, q_2\},$$

because when the letter 'a' is read in the state q_3 the FSM switches into the state q_5 . From q_5 the states q_7 , q_0 , q_1 and q_2 are reachable by ε -transitions. \diamond

The function $\hat{\delta}$ maps a state into a set of states. Since the FSM $\det(F)$ uses sets of states of the FSM F as its states we need a function that maps sets of states of the FSM F into sets of states. Hence we generalize the function $\hat{\delta}$ to the function

$$\Delta : 2^Q \times \Sigma \rightarrow 2^Q$$

such that for a set M of states and a character c the expression $\Delta(M, c)$ computes the set of all those states that the FSM F could be in if it is in a state from M , then reads the character c , and finally makes some ε -transitions. The formal definition is as follows:

$$\Delta(M, c) := \bigcup \{ \hat{\delta}(q, c) \mid q \in M \}.$$

This formula is easy to understand: For every state $q \in M$ we compute the set of states that the FSM could be in after reading the character c and doing some ε -transitions. Then we take the union of these sets.

Example: Continuing our previous example (shown in Figure 4.4) we have:

1. $\Delta(\{q_0, q_1, q_2\}, a) = \{q_4\},$
2. $\Delta(\{q_0, q_1, q_2\}, b) = \{q_3\},$
3. $\Delta(\{q_3\}, a) = \{q_5, q_7, q_0, q_1, q_2\},$
4. $\Delta(\{q_3\}, b) = \{\},$
5. $\Delta(\{q_4\}, a) = \{\},$
6. $\Delta(\{q_4\}, b) = \{q_6, q_7, q_0, q_1, q_2\}.$

\diamond

Now we are ready to formally define how the deterministic FSM $\det(F)$ is constructed from the non-deterministic FSM $F := \langle Q, \Sigma, \delta, q_0, A \rangle$. We define:

$$\det(F) = \langle 2^Q, \Sigma, \Delta, ec(q_0), \hat{A} \rangle$$

where the components of this tuple are defined as follows:

1. The set of states of $\det(F)$ is the set of all subsets of Q and therefore it is equal to the power set 2^Q .
Later we will see that we do not need all of these subsets. The reason is that the states are those subsets that could be reached from the start state q_0 when some string has been read. In most cases there are some combinations of states that can not be reached and the corresponding sets are not really needed as states.
2. The input alphabet Σ does not change when going from F to $\det(F)$. After all, the deterministic FSM $\det(F)$ has to recognize the same language as the non-deterministic FSM F .
3. The previously defined function Δ specifies how the set of states changes when a character is read.
4. The start state $ec(q_0)$ of the non-deterministic FSM $\det(F)$ is the set of all states that can be reached from the start state q_0 of the non-deterministic FSM F via ε -transitions.
5. The set of accepting states \hat{A} is the set of those subsets of Q that contain an accepting state of the FSM F :

$$\hat{A} := \{M \in 2^Q \mid M \cap A \neq \{\}\}.$$

Exercise 6: Transform the non-deterministic FSM F that is shown in Figure 4.3 on page 30 into the deterministic FSM $\det(F)$. \diamond

Solution: We start by computing the set of states.

1. As we have $ec(0) = \{0\}$, the start state of $det(F)$ is the set containing 0.

$$S_0 := ec(0) = \{0\}.$$

2. As we have $\delta(0, a) = \{0\}$ and there are no ε -transitions we have

$$\Delta(S_0, a) = \Delta(\{0\}, a) = \{0\} = S_0.$$

3. As we have $\delta(0, b) = \{0, 1\}$ we conclude

$$S_1 := \Delta(S_0, b) = \Delta(\{0\}, b) = \{0, 1\}.$$

4. We have that $\delta(0, a) = \{0\}$ and $\delta(1, a) = \{2\}$. Hence

$$S_2 := \Delta(S_1, a) = \Delta(\{0, 1\}, a) = \{0, 2\}.$$

5. We have $\delta(0, b) \in \{0, 1\}$ and $\delta(1, b) = \{2\}$. Therefore

$$S_4 := \Delta(S_1, b) = \Delta(\{0, 1\}, b) = \{0, 1, 2\}$$

Similarly we derive the following:

$$6. S_3 := \Delta(S_2, a) = \Delta(\{0, 2\}, a) = \{0, 3\}.$$

$$7. S_5 := \Delta(S_2, b) = \Delta(\{0, 2\}, b) = \{0, 1, 3\}.$$

$$8. S_6 := \Delta(S_4, a) = \Delta(\{0, 1, 2\}, a) = \{0, 2, 3\}.$$

$$9. S_7 := \Delta(S_4, b) = \Delta(\{0, 1, 2\}, b) = \{0, 1, 2, 3\}.$$

$$10. \Delta(S_3, a) = \Delta(\{0, 3\}, a) = \{0\} = S_0.$$

$$11. \Delta(S_3, b) = \Delta(\{0, 3\}, b) = \{0, 1\} = S_1.$$

$$12. \Delta(S_5, a) = \Delta(\{0, 1, 3\}, a) = \{0, 2\} = S_2.$$

$$13. \Delta(S_5, b) = \Delta(\{0, 1, 3\}, b) = \{0, 1, 2\} = S_4.$$

$$14. \Delta(S_6, a) = \Delta(\{0, 2, 3\}, a) = \{0, 3\} = S_3.$$

$$15. \Delta(S_6, b) = \Delta(\{0, 2, 3\}, b) = \{0, 1, 3\} = S_5.$$

$$16. \Delta(S_7, a) = \Delta(\{0, 1, 2, 3\}, a) = \{0, 2, 3\} = S_6.$$

$$17. \Delta(S_7, b) = \Delta(\{0, 1, 2, 3\}, b) = \{0, 1, 2, 3\} = S_7.$$

These are all possible sets of states that the deterministic FSM $det(F)$ can reach. For a better overview let us summarize the definitions of the individual states of the deterministic FSM:

$$S_0 = \{0\}, S_1 = \{0, 1\}, S_2 = \{0, 2\}, S_3 = \{0, 3\}, S_4 = \{0, 1, 2\},$$

$$S_5 = \{0, 1, 3\}, S_6 = \{0, 2, 3\}, S_7 = \{0, 1, 2, 3\}$$

Therefore the set \hat{Q} of the deterministic FSM $det(F)$ is given as follows:

$$\hat{Q} := \{S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7\}.$$

The transition function Δ is shown as a table:

Δ	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7
a	S_0	S_2	S_3	S_0	S_6	S_2	S_3	S_6
b	S_1	S_4	S_5	S_1	S_7	S_4	S_5	S_7

Finally we recognize that only the sets S_3 , S_5 , S_6 and S_7 contain the accepting state 3. Therefore we have

$$\hat{A} := \{S_3, S_5, S_6, S_7\}.$$

Therefore we have now found the deterministic FSM $\det(F)$. We have

$$\det(F) := \langle \hat{Q}, \Sigma, \Delta, S_0, \hat{A} \rangle.$$

This FSM is shown in Figure 4.5 on page 37.

We realize that this deterministic FSM $\det(F)$ has 8 different states. The non-deterministic FSM F has 4 different states $Q = \{0, 1, 2, 3\}$. Hence the power set 2^Q has 16 elements. Why then has the FSM $\det(F)$ only 8 and not $2^4 = 16$ states? The reason is that we can only reach those sets of states from the start 0 that contain the state 0 because no matter whether we read an a or a b the FSM F can always choose to switch to the state 0. Therefore, every set of states that is reachable from the state 0 has to contain the state 0. Therefore, sets that do not contain 0 are not needed as states of the deterministic FSM $\det(F)$.

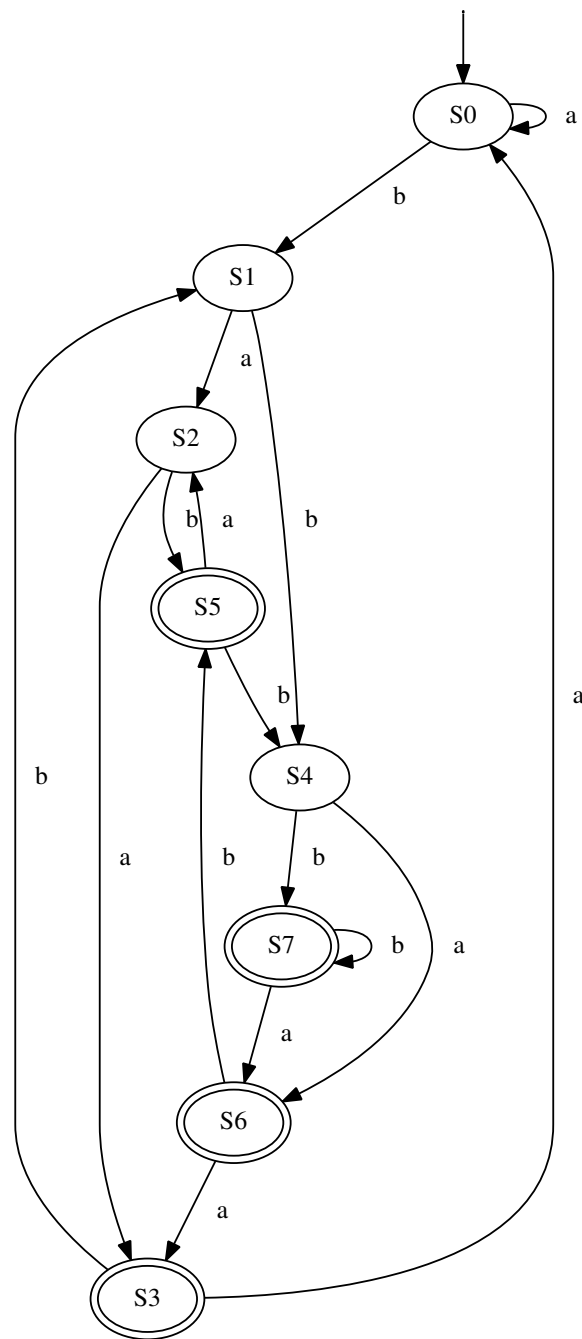
Exercise 7: Transform the non-deterministic FSM F that is shown in Figure 4.4 on page 33 into an equivalent deterministic FSM $\det(F)$. \diamond

4.3.1 Implementation

It is straightforward to implement the theory developed so far. The *Jupyter notebook*

<https://github.com/karlstroetmann/Formal-Languages/blob/master/Python/Chapter-4/NFA-2-DFA.ipynb>

contains a program that takes a non-deterministic FSM F and computes the deterministic FSM $\det(F)$.

Figure 4.5: The deterministic FSM $\det(F)$.

4.4 From Regular Expressions to Non-Deterministic Finite State Machines

In this section we show how regular expressions can be implemented as non-deterministic finite state machine. Given a regular expression r we will construct a non-deterministic FSM $A(r)$ that accepts the language that is described by the regular expression r , i.e. we will have

$$L(A(r)) = L(r).$$

The FSM $A(r)$ is defined by induction on the regular expression r . The FSM $A(r)$ will have the following properties:

1. $A(r)$ does not have a transition into its start state.
2. $A(r)$ has exactly one accepting state. We refer to this state as $\text{accept}(A(r))$. Furthermore, there are no transitions out of this state.

In the following we assume that Σ is the alphabet that has been used when constructing the regular expression r . Then we can define $A(r)$ as follows:

1. The FSM $A(\emptyset)$ is defined as

$$A(\emptyset) = \langle \{q_0, q_1\}, \Sigma, \{\}, q_0, \{q_1\} \rangle.$$

Note that this FSM has no transitions at all.

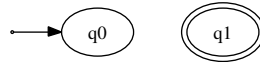


Figure 4.6: The FSM $A(\emptyset)$.

Figure 4.6 shows the FSM $A(\emptyset)$. It is obvious that we have $L(A(\emptyset)) = \{\}$.

2. The FSM $A(\varepsilon)$ is defined as

$$A(\varepsilon) = \langle \{q_0, q_1\}, \Sigma, \{\langle q_0, \varepsilon \rangle \mapsto \{q_1\}\}, q_0, \{q_1\} \rangle.$$

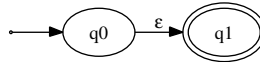


Figure 4.7: The FSM $A(\varepsilon)$.

Figure 4.7 shows the FSM $A(\varepsilon)$. We have that $L(A(\varepsilon)) = \{\lambda\}$, i.e. the FSM accepts only the empty string.

3. For a letter $c \in \Sigma$ the FSM $A(c)$ is defined as

$$A(c) = \langle \{q_0, q_1\}, \Sigma, \{\langle q_0, c \rangle \mapsto \{q_1\}\}, q_0, \{q_1\} \rangle.$$

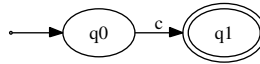


Figure 4.8: The FSM $A(c)$.

Figure 4.8 shows $A(c)$. We have that $L(A(c)) = \{c\}$, i.e. the FSM accepts only the character c .

4. In order to define the FSM $A(r_1 \cdot r_2)$ for the concatenation $r_1 \cdot r_2$ we assume that we have already constructed finite state machines $A(r_1)$ and $A(r_2)$ such that $L(A(r_1)) = L(r_1)$ and $L(A(r_2)) = L(r_2)$. Furthermore, without loss of generality we assume that the states in the FSMs $A(r_1)$ and $A(r_2)$ are different. This can always be achieved by renaming the states of $A(r_2)$. Next, we assume that $A(r_1)$ and $A(r_2)$ have the following form:

$$(a) \ A(r_1) = \langle Q_1, \Sigma, \delta_1, q_1, \{q_2\} \rangle,$$

$$(b) \ A(r_2) = \langle Q_2, \Sigma, \delta_2, q_3, \{q_4\} \rangle,$$

$$(c) \ Q_1 \cap Q_2 = \{\}.$$

Then we can build the FSM $A(r_1 \cdot r_2)$ from $A(r_1)$ and $A(r_2)$ as follows:

$$A(r_1 \cdot r_2) := \langle Q_1 \cup Q_2, \Sigma, \{ \langle q_2, \varepsilon \rangle \mapsto \{q_3\} \} \cup \delta_1 \cup \delta_2, q_1, \{q_4\} \rangle$$

Here, the notation $\{ \langle q_2, \varepsilon \rangle \mapsto \{q_3\} \} \cup \delta_1 \cup \delta_2$ specifies that $A(r_1 \cdot r_2)$ contains all transitions from both $A(r_1)$ and $A(r_2)$ and, furthermore, contains an ε -transition from q_2 to q_3 . Formally, this transition function δ can be specified as follows:

$$\delta(q, c) := \begin{cases} \{q_3\} & \text{if } q = q_2 \text{ and } c = \varepsilon, \\ \delta_1(q, c) & \text{if } q \in Q_1 \text{ and } \langle q, c \rangle \neq \langle q_2, \varepsilon \rangle, \\ \delta_2(q, c) & \text{if } q \in Q_2. \end{cases}$$

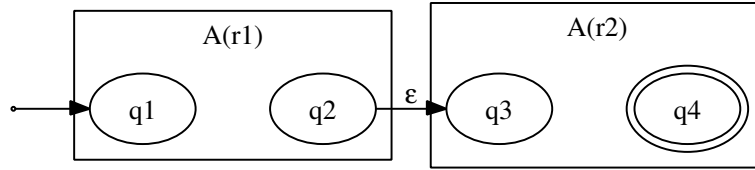


Figure 4.9: The FSM $A(r_1 \cdot r_2)$.

Figure 4.9 shows the FSM $A(r_1 \cdot r_2)$.

Instead of having an ε -transition from q_2 to q_3 we can identify the states q_2 and q_3 . The advantage is that the resulting FSM is smaller. We will do this when creating FSMs by hand.

I haven't done this identification in the definition above because both the graphical representation and the implementation get more complicated when we identify these states.

5. In order to define the FSM $A(r_1 + r_2)$ we assume that we have already constructed finite state machines $A(r_1)$ and $A(r_2)$ such that $L(A(r_1)) = L(r_1)$ and $L(A(r_2)) = L(r_2)$. Furthermore, without loss of generality we assume that the states in the FSMs $A(r_1)$ and $A(r_2)$ are different and that $A(r_1)$ and $A(r_2)$ have the following form:

- (a) $A(r_1) = \langle Q_1, \Sigma, \delta_1, q_1, \{q_3\} \rangle$,
- (b) $A(r_2) = \langle Q_2, \Sigma, \delta_2, q_2, \{q_4\} \rangle$,
- (c) $Q_1 \cap Q_2 = \{\}$.

Then the FSM $A(r_1 + r_2)$ is defined as follows:

$$\langle \{q_0, q_5\} \cup Q_1 \cup Q_2, \Sigma, \{ \langle q_0, \varepsilon \rangle \mapsto \{q_1, q_2\}, \langle q_3, \varepsilon \rangle \mapsto \{q_5\}, \langle q_4, \varepsilon \rangle \mapsto \{q_5\} \} \cup \delta_1 \cup \delta_2, q_0, \{q_5\} \rangle$$

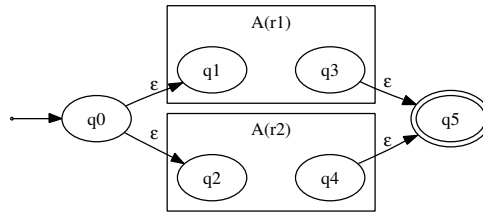


Figure 4.10: The FSM $A(r_1 + r_2)$.

Figure 4.10 shows the FSM $A(r_1 + r_2)$. In addition to the states of $A(r_1)$ and $A(r_2)$ there are two more states:

- (a) q_0 is the start state of the FSM $A(r_1 + r_2)$,
- (b) q_5 is the only accepting state of the FSM $A(r_1 + r_2)$.

In addition to the transitions of $A(r_1)$ and $A(r_2)$ the FSM $A(r_1 + r_2)$ has four more ε -transitions.

- (a) The new start state q_0 has two ε -transitions leading to the start states q_1 and q_2 of the FSMs $A(r_1)$ and $A(r_2)$.
- (b) Each of the accepting states q_3 and q_4 of the FSMs $A(r_1)$ and $A(r_2)$ has an ε -transition to the new accepting state q_5 .

In order to simplify this FSM we could identify the three states q_0 , q_1 and q_2 and the three states q_3 , q_4 and q_5 . However, the resulting FSM would be more difficult to understand and hence we are **not** doing this when creating FSMs by hand.

6. In order to define the FSM $A(r^*)$ we assume that we have already constructed a finite state machine $A(r)$ such that $L(A(r)) = L(r)$ and that $A(r)$ has the form

$$A(r) = \langle Q, \Sigma, \delta, q_1, \{q_2\} \rangle.$$

Then $A(r^*)$ is defined as follows:

$$\langle \{q_0, q_3\} \cup Q, \Sigma, \{ \langle q_0, \varepsilon \rangle \mapsto \{q_1, q_3\}, \langle q_2, \varepsilon \rangle \mapsto \{q_1, q_3\} \} \cup \delta, q_0, \{q_3\} \rangle.$$

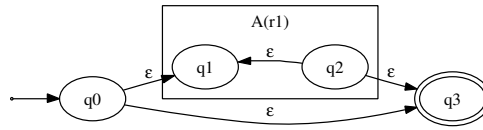


Figure 4.11: The FSM $A(r^*)$.

Figure 4.11 shows the FSM $A(r^*)$. In comparison with $A(r)$ this FSM has two additional states.

- (a) q_0 is the start state of $A(r^*)$,
- (b) q_3 is the only accepting state of $A(r^*)$.

The FSM $A(r^*)$ has four more ε -transitions than $A(r)$:

- (a) The new start state q_0 has ε -transitions to the states q_1 and q_3 .
- (b) q_2 has an ε -transition back to the state q_1 .
- (c) q_2 also has an ε -transition to the state q_3 .

Attention: If we would identify the two states q_0 and q_1 and the two states q_2 and q_3 , then the resulting FSM would no longer be correct!

Exercise 8: Construct a non-deterministic FSM that accepts the language specified by the regular expression

$$a^* \cdot b^*.$$

Consider what would happen if you would identify the two states q_0 and q_1 and the two states q_2 and q_3 in step 6 of the construction given above, while also identifying the two states q_2 and q_3 in step 4 of the construction of $A(r_1 \cdot r_2)$. ◇

Exercise 9: Construct a non-deterministic FSM for the regular expression

$$(a + b) \cdot a^* \cdot b.$$
◇

4.4.1 Implementation

The *Jupyter notebook* [Regexp-2-NFA.ipynb](#) implements the theory discussed in this section.

4.5 Translating a Deterministic Fsm into a Regular Expression

In this last section we start with a deterministic FSM F and construct a regular expression r such that we have

$$L(r) = L(F).$$

We assume that the FSM F is given as follows:

$$F = \langle \{q_0, q_1, \dots, q_n\}, \Sigma, \delta, q_0, A \rangle.$$

For every pair of states $\langle p_1, p_2 \rangle \in Q \times Q$ we define a regular expression $r(p_1, p_2)$ such that $r(p_1, p_2)$ describes those strings w that take the FSM F from the state p_1 to the state p_2 , i.e. we have

$$L(r(p_1, p_2)) = \{w \in \Sigma^* \mid \langle p_1, w \rangle \rightsquigarrow^* \langle p_2, \varepsilon \rangle\}.$$

The definition of $r(p_1, p_2)$ is done by first defining auxiliary regular expressions $r^{(k)}(p_1, p_2)$ for all $k = 0, \dots, n+1$. The regular expression $r^{(k)}(p_1, p_2)$ specifies those strings that take the FSM F from the state p_1 to the state p_2 without visiting a state from the set

$$Q_k := \{q_i \mid i \in \{k, \dots, n\}\} = \{q_k, \dots, q_n\}.$$

To this end we define the ternary relation

$$\mapsto_k \subseteq (Q \times \Sigma^* \times Q).$$

Given two states $p, q \in Q$ and a string w we have that

$$p \xrightarrow{w}_k q$$

holds iff the FSM F switches from the state p to the state q when it reads the string w but on reading w does not switch to a state from the set Q_k **in-between**. Here, “in-between” specifies that the states p and q may well be elements of the set Q_k , only the states between p and q must not be in Q_k . The formal definition of $p \xrightarrow{w}_k q$ is done by induction on w :

B.C.: $|w| \leq 1$. Then there are two cases:

- (a) $p \xrightarrow{\varepsilon}_k p$,
because when the empty string is read we can only reach the state p if we start in the state p .
- (b) $\delta(p, c) = q \Rightarrow p \xrightarrow{c}_k q$,
because when the FSM reads the character c and switches from state p directly to the state q , there are no states “inbetween”.

I.S.: $w = cv$ where $|v| \geq 1$.

$$\text{Here we have: } p \xrightarrow{c}_k q \wedge q \notin Q_k \wedge q \xrightarrow{v}_k r \Rightarrow p \xrightarrow{cv}_k r.$$

Now we are ready to define the regular expressions $r^{(k)}(p_1, p_2)$ for all $k = 0, \dots, n+1$. This definition will be done such that we have

$$L(r^{(k)}(p_1, p_2)) = \{w \in \Sigma^* \mid p_1 \xrightarrow{w}_k p_2\}.$$

The definition of the regular expressions $r^{(k)}(p_1, p_2)$ is done by induction on k .

B.C.: $k = 0$.

Then we have $Q_0 = Q$ and therefore the set Q_0 contains all states. Therefore, when the FSM switches from the state p_1 to the state p_2 it must not visit any states in-between. There are two cases.

- (a) $p_1 \neq p_2$: Then we can have $p_1 \xrightarrow{w}_0 p_2$ only if w contains but a single letter. Define

$$\{c_1, \dots, c_l\} := \{c \in \Sigma \mid \delta(p_1, c) = p_2\}$$

as the set of all letters that take the FSM from the state p_1 to the state p_2 . If this set is not empty we define

$$r^{(0)}(p_1, p_2) := c_1 + \cdots + c_l.$$

If this set is empty, then there is no direct transition from p_1 to p_2 and we define

$$r^{(0)}(p_1, p_2) := \emptyset.$$

(b) $p_1 = p_2$: Again we define

$$\{c_1, \dots, c_l\} := \{c \in \Sigma \mid \delta(p_1, c) = p_1\}.$$

If this set is not empty we have

$$r^{(0)}(p_1, p_1) := c_1 + \cdots + c_l + \varepsilon.$$

Otherwise we have

$$r^{(0)}(p_1, p_1) := \varepsilon.$$

I.S.: $k \mapsto k + 1$.

When compared to the regular expression $r^{(k)}(p_1, p_2)$, the regular expression $r^{(k+1)}(p_1, p_2)$ is allowed to use the state q_k , because q_k is the only element of the set Q_k that is not a member of the set Q_{k+1} . If the FSM reads a string w that switches the state p_1 to the state p_2 without switching into a state from the set Q_{k+1} , then there are two cases.

(a) We already have $p_1 \xrightarrow{w}_k p_2$.

(b) The string w can be written as $w = w_1 s_1 \cdots s_l w_2$ where we have:

- $p_1 \xrightarrow{w_1}_k q_k$,
- $q_k \xrightarrow{s_i}_k q_k$ for all $i = \{1, \dots, l\}$,
- $q_k \xrightarrow{w_2}_k p_2$.

Therefore we define

$$r^{(k+1)}(p_1, p_2) := r^{(k)}(p_1, p_2) + r^{(k)}(p_1, q_k) \cdot (r^{(k)}(q_k, q_k))^* \cdot r^{(k)}(q_k, p_2).$$

Now we are ready to define the regular expressions $r(p_1, p_2)$ for all states p_1 and p_2 :

$$r(p_1, p_2) := r^{(n+1)}(p_1, p_2).$$

This regular expression specifies all strings that take the FSM from the state p_1 to the state p_2 without using any state from the set Q_{n+1} in-between. Since we have

$$Q_{n+1} = \{q_i \mid i \in \{0, \dots, n\} \wedge i \geq n+1\} = \{\}$$

we know that Q_{n+1} is empty. Therefore the regular expression $r^{(n+1)}(p_1, p_2)$ does not exclude any states when switching from state p_1 to the state p_2 .

In order to construct a regular expression that specifies the language accepted by a deterministic FSM F we write the set A of accepting states of F as

$$A = \{t_1, \dots, t_m\}.$$

Then the regular expression $r(A)$ is defined as

$$r(A) := r(q_0, t_1) + \cdots + r(q_0, t_m).$$

This regular expression specifies those strings that take the FSM F from its start state q_0 into any of its accepting states. \square

Exercise 10: Take the FSM shown in Figure 4.1 and construct an equivalent regular expression.

Solution: The FSM has two states: 0 and 1. We start by computing the regular expressions $r^{(k)}(i, j)$ for all $i, j \in \{0, 1\}$ for $k = 0, 1$, and 2:

1. For $k = 0$ we have:

$$(a) \quad r^{(0)}(0, 0) = a + \varepsilon,$$

- (b) $r^{(0)}(0, 1) = \mathbf{b}$,
- (c) $r^{(0)}(1, 0) = \emptyset$,
- (d) $r^{(0)}(1, 1) = \mathbf{a} + \varepsilon$.

2. For $k = 1$ we have:

(a) For $r^{(1)}(0, 0)$ we have:

$$\begin{aligned} r^{(1)}(0, 0) &= r^{(0)}(0, 0) + r^{(0)}(0, 0) \cdot (r^{(0)}(0, 0))^* \cdot r^{(0)}(0, 0) \\ &\doteq r^{(0)}(0, 0) \cdot (r^{(0)}(0, 0))^* \end{aligned}$$

In the last step we used the fact that

$$\begin{aligned} r + r \cdot r^* \cdot r &\doteq r \cdot (\varepsilon + r^* \cdot r) \\ &\doteq r \cdot r^* \end{aligned}$$

to simplify the result. If we substitute for $r^{(0)}(0, 0)$ the expression $\mathbf{a} + \varepsilon$ we get

$$r^{(1)}(0, 0) \doteq (\mathbf{a} + \varepsilon) \cdot (\mathbf{a} + \varepsilon)^*.$$

As we have $(\mathbf{a} + \varepsilon) \cdot (\mathbf{a} + \varepsilon)^* \doteq \mathbf{a}^*$ we have

$$r^{(1)}(0, 0) \doteq \mathbf{a}^*.$$

(b) For $r^{(1)}(0, 1)$ we have:

$$\begin{aligned} r^{(1)}(0, 1) &\doteq r^{(0)}(0, 1) + r^{(0)}(0, 0) \cdot (r^{(0)}(0, 0))^* \cdot r^{(0)}(0, 1) \\ &\doteq \mathbf{b} + (\mathbf{a} + \varepsilon) \cdot (\mathbf{a} + \varepsilon)^* \cdot \mathbf{b} \\ &\doteq \mathbf{b} + \mathbf{a}^* \cdot \mathbf{b} \\ &\doteq (\varepsilon + \mathbf{a}^*) \cdot \mathbf{b} \\ &\doteq \mathbf{a}^* \cdot \mathbf{b} \end{aligned}$$

(c) For $r^{(1)}(1, 0)$ we have:

$$\begin{aligned} r^{(1)}(1, 0) &\doteq r^{(0)}(1, 0) + r^{(0)}(1, 0) \cdot (r^{(0)}(0, 0))^* \cdot r^{(0)}(0, 0) \\ &\doteq \emptyset + \emptyset \cdot (\mathbf{a} + \varepsilon)^* \cdot (\mathbf{a} + \varepsilon) \\ &\doteq \emptyset \end{aligned}$$

(d) For $r^{(1)}(1, 1)$ we have

$$\begin{aligned} r^{(1)}(1, 1) &\doteq r^{(0)}(1, 1) + r^{(0)}(1, 0) \cdot (r^{(0)}(0, 0))^* \cdot r^{(0)}(0, 1) \\ &\doteq (\mathbf{a} + \varepsilon) + \emptyset \cdot (\mathbf{a} + \varepsilon)^* \cdot \mathbf{b} \\ &\doteq (\mathbf{a} + \varepsilon) + \emptyset \\ &\doteq \mathbf{a} + \varepsilon \end{aligned}$$

3. For $k = 2$ we only have to compute the regular expression $r^{(2)}(0, 1)$:

$$\begin{aligned} r^{(2)}(0, 1) &\doteq r^{(1)}(0, 1) + r^{(1)}(0, 1) \cdot (r^{(1)}(1, 1))^* \cdot r^{(1)}(1, 1) \\ &\doteq \mathbf{a}^* \cdot \mathbf{b} + \mathbf{a}^* \cdot \mathbf{b} \cdot (\mathbf{a} + \varepsilon)^* \cdot (\mathbf{a} + \varepsilon) \\ &\doteq \mathbf{a}^* \cdot \mathbf{b} + \mathbf{a}^* \cdot \mathbf{b} \cdot \mathbf{a}^* \\ &\doteq \mathbf{a}^* \cdot \mathbf{b} \cdot (\varepsilon + \mathbf{a}^*) \\ &\doteq \mathbf{a}^* \cdot \mathbf{b} \cdot \mathbf{a}^*. \end{aligned}$$

As the state 0 is the start state and the state 1 is the only accepting state we have

$$r(F) = r^{(2)}(0, 1) \doteq \mathbf{a}^* \cdot \mathbf{b} \cdot \mathbf{a}^*.$$

□

Exercise 11: Take the FSM shown in Figure 4.12 on page 44 and construct a regular expression specifying the same language as this FSM.

Hint: In order to avoid unnecessary computations, you should only compute those regular expression $r^{(k)}(p, q)$ that are needed in order to compute $r^{(3)}(q_0, q_2)$. \diamond

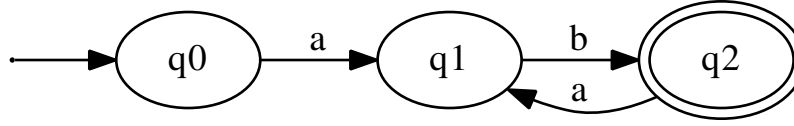


Figure 4.12: A deterministic finite state machine.

4.5.1 Implementation

The *Jupyter notebook*

<https://github.com/karlstroetmann/Formal-Languages/blob/master/Python/DFA-2-RegExp.ipynb>

contains a program that takes a deterministic FSM F and computes a regular expression r such that r specifies the language accepted by F , i.e. we have $L(r) = L(F)$.

4.6 Minimization of Finite State Machines

In this section we show how to minimize the number of states of a deterministic finite state machine

$$F = \langle Q, \Sigma, \delta, q_0, A \rangle.$$

Without loss of generality we want to assume that the FSM F is complete: Therefore, we assume that for each state $q \in Q$ and each letter $c \in \Sigma$ the expression $\delta(q, c)$ returns a state of Q . Our goal is to find a deterministic finite state machine

$$F^- = \langle Q^-, \Sigma, \delta^-, q_0, A^- \rangle,$$

which accepts the same language as F , so

$$L(F^-) = L(F)$$

and for which the number of states of the set Q^- is minimal. We start with the function

$$\delta^* : Q \times \Sigma^* \rightarrow Q$$

which extends the function δ by accepting a string instead of a single character as its first argument. The function call $\delta^*(q, s)$ calculates the state p which the FSM F enters if it reads the string s in the state q . As we have assumed the *Fsm* F to be complete, we have that

$$\delta(q, c) \notin \Omega \quad \text{for all } q \in Q \text{ and all } c \in \Sigma$$

and then $\delta^*(q, w)$ is defined by induction on w as follows:

- (a) $\delta^*(q, \varepsilon) := q$ for all $q \in Q$.
- (b) $\delta^*(q, cw) := \delta^*(\delta(q, c), w)$ for all $q \in Q$, $c \in \Sigma$, and $w \in \Sigma^*$.

Since the function δ^* is a generalization of the function δ , in the future we will not distinguish in the notation between δ and δ^* and write δ for both functions.

Obviously, in a finite state machine $F = \langle Q, \Sigma, \delta, q_0, A \rangle$ we can remove all the states $p \in Q$ which are not *reachable* from the start state. A state p is *reachable* iff a string $w \in \Sigma^*$ is given, such that

$$\delta(q_0, w) = p$$

holds. In the following we assume that all states of the FSM F can be reached from the start state.

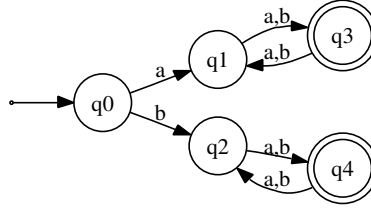


Figure 4.13: A finite state machine with equivalent states.

In general, we can minimize an FSM by identifying certain states. If we look for example at the FSM shown in figure 4.13, we can identify the states q_1 and q_2 as well as q_3 and q_4 without changing the language of the FSM. The central idea in minimizing a FSM is that we compute those states which **must not** be identified and consider all other states as equivalent. States that must not be identified are called **separable states**. This notion is defined next.

Definition 12 (Separable States) Assume $F = \langle Q, \Sigma, \delta, q_0, A \rangle$ is a deterministic finite state machine. Two states $p_1, p_2 \in Q$ are called **separable** if and only if there exists a string $s \in \Sigma^*$ such that either

1. $\delta(p_1, s) \in A$ and $\delta(p_2, s) \notin A$ or
2. $\delta(p_1, s) \notin A$ and $\delta(p_2, s) \in A$

holds. In this case, the string s **separates** p_1 and p_2 . □

If two states p_1 and p_2 are separable, then it is obvious that these states must not be identified. We define an equivalence relation \sim on the set Q of all states by setting

$$p_1 \sim p_2 \quad \text{iff} \quad \forall s \in \Sigma^* : (\delta(p_1, s) \in A \leftrightarrow \delta(p_2, s) \in A).$$

Hence, two states p_1 and p_2 are considered to be equivalent iff they are not separable. The claim is that we can identify all pairs $\langle p_1, p_2 \rangle$ of equivalent states. The **identification** of two states p_1 and p_2 is done by removing the state p_2 from the set Q and changing the transition function δ in a way that the new version of δ will return p_1 in all those cases where the old version of δ had returned p_2 .

The question remains how we can determine which states are distinguishable. One possibility is to create a set V with pairs of states. We add the pair $\langle p, q \rangle$ to the set V if we have discovered that p and q are distinguishable. We recognize p and q as distinguishable if there is a letter $c \in \Sigma$ and two states s and t that are already known to be distinguishable such that

$$\delta(p, c) = s, \quad \delta(q, c) = t, \quad \text{and} \quad \langle s, t \rangle \in V$$

holds. This idea leads to an algorithm that uses two steps:

- (a) First we initialize V with all the pairs $\langle p, q \rangle$, for which either p is an accepting state and q is not an accepting state, or q is an accepting state and p is not an accepting state, because an accepting state can be distinguished from a non-accepting state by the empty string λ :

$$V := \{ \langle p, q \rangle \in Q \times Q \mid (p \in A \wedge q \notin A) \vee (p \notin A \wedge q \in A) \}$$

This can also be written as the union of two Cartesian products. We have

$$V = A \times (Q \setminus A) \cup (Q \setminus A) \times A.$$

- (b) As long as we find a new pair $\langle p, q \rangle \in Q \times Q$ for which there is a letter c such that the states $\delta(p, c)$ and $\delta(q, c)$ are already distinguishable, we add this pair to the set V :

```

while  $\exists \langle p, q \rangle \in Q \times Q : \exists c \in \Sigma : \langle \delta(p, c), \delta(q, c) \rangle \in V \wedge \langle p, q \rangle \notin V$ 
  choose  $\langle p, q \rangle \in Q \times Q$  such that  $\langle \delta(p, c), \delta(q, c) \rangle \in V \wedge \langle p, q \rangle \notin V$ 
   $V := V \cup \{ \langle p, q \rangle, \langle q, p \rangle \}$ ;

```

If we have found all pairs $\langle p, q \rangle$ of distinguishable states, then we can identify all states p and q which are not distinguishable and therefore $\langle p, q \rangle \notin V$. The FSM constructed in this way is, in fact, minimal.

Exercise 12: We consider the FSM shown in figure 4.13 and apply the algorithm described above to this FSM. We use a table for this. The columns and rows of this table are numbered with the different states. If we have recognized in the first step that the states i and j are distinguishable, we insert a 1 in this table in the i -th row and the j -th column. Furthermore, since the states i and j can also be distinguished from the states j and i , we also insert a 1 in the j -th row and the i -th column.

1. In the first step we recognize that the two accepting states q_3 and q_4 are distinguishable from all non-accepting states. So the pairs $\langle q_0, q_3 \rangle$, $\langle q_0, q_4 \rangle$, $\langle q_1, q_3 \rangle$, $\langle q_1, q_4 \rangle$, $\langle q_2, q_3 \rangle$ and $\langle q_2, q_4 \rangle$ are distinguishable. So the table now has the following configuration:

	q_0	q_1	q_2	q_3	q_4
q_0				1	1
q_1				1	1
q_2				1	1
q_3	1	1	1		
q_4	1	1	1		

2. Next we see that the states q_0 and q_1 are distinguishable because

$$\delta(q_0, a) = q_1, \quad \delta(q_1, a) = q_3 \quad \text{and} \quad q_1 \not\sim q_3.$$

Also we see that the states q_0 and q_2 are distinguishable, because

$$\delta(q_0, b) = q_2, \quad \delta(q_2, b) = q_4 \quad \text{and} \quad q_2 \not\sim q_4.$$

Since we have found in the second step that $q_0 \not\sim q_1$ and $q_0 \not\sim q_2$ holds true, we will insert a 2 at the corresponding positions in the table. Therefore the table now has the following form:

	q_0	q_1	q_2	q_3	q_4
q_0		2	2	1	1
q_1	2			1	1
q_2	2			1	1
q_3	1	1	1		
q_4	1	1	1		

3. Now we do not find any more pairs of distinguishable states, because if we take a look at the pair $\langle q_1, q_2 \rangle$ we can see that

$$\delta(q_1, a) = q_3 \quad \text{and} \quad \delta(q_2, a) = q_4,$$

but because the states q_3 and q_4 are not distinguishable so far, this does not yield a new distinguishable pair. Neither does

$$\delta(q_1, b) = q_3 \quad \text{and} \quad \delta(q_2, b) = q_4$$

yield a new distinguishable pair. At this point, the two states q_3 and q_4 remain. We find

$$\delta(q_3, c) = q_1 \quad \text{and} \quad \delta(q_4, c) = q_2 \quad \text{for all } c \in \{a, b\}$$

and because the states q_1 and q_2 are not yet known to be distinguishable, we have not found any new distinguishable states. So we can read the equivalent states from the table, it holds that:

- (a) $q_1 \sim q_2$
- (b) $q_3 \sim q_4$

Figure 4.14 shows the corresponding reduced finite FSM.

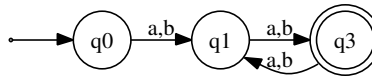


Figure 4.14: The reduced FSM.

Exercise 13: Construct the minimal deterministic FSM that recognizes the language $L(a \cdot (b \cdot a)^*)$. Perform the following steps:

- (a) Construct a non-deterministic FSM which recognizes this language.
- (b) Transform this non-deterministic FSM into a deterministic FSM.
- (c) Minimize the number of states of this FSM with the algorithm given above.

4.6.1 Implementation

The *Jupyter notebook*

<https://github.com/karlstroetmann/Formal-Languages/blob/master/Python/Minimize.ipynb>

contains a program that takes a deterministic FSM F and returns an equivalent FSM that has the minimum number of states.

4.7 Conclusion

In this chapter we have shown that the concept of a [deterministic finite state machine](#) and a [regular expression](#) are equivalent.

- (a) Every deterministic finite state machine can be translated into an equivalent regular expression.
- (b) Every regular expression can be translated into an equivalent non-deterministic FSM.
- (c) Every non-deterministic FSM can be transformed into an equivalent deterministic FSM.

Furthermore, we have shown that every deterministic finite state machine can be minimized.

Historical Remark [Stephen C. Kleene](#) (1909 – 1994) has shown in 1956 that the concepts of [finite state machines](#) and [regular expression](#) have the same strength [[Kle56](#)].

4.8 Check your Understanding

- (a) How are deterministic and non-deterministic finite state machines defined?
- (b) Given a non-deterministic FSM F , can you convert it into a deterministic FSM $\text{det}(F)$ that accepts the same language as the FSM F ?
- (c) Given a regular expression r , can you describe the steps necessary to construct a non-deterministic FSM F that accepts the language specified by r ?
- (d) Given a deterministic FSM F can you describe how to construct a regular expression r that specifies the language accepted by F ?
- (e) How do we minimize a finite state machine?

Chapter 5

The Theory of Regular Languages

A formal language $L \subseteq \Sigma^*$ is called a **regular language** if there is a regular expression r such that the language L is specified by r , i.e. if

$$L = L(r)$$

holds. In Chapter 4 we have shown that the regular languages are those languages that are recognized by a finite state machine. In this chapter, we show that the class of regular languages has certain **closure properties**:

- (a) The **union** $L_1 \cup L_2$ of two regular languages L_1 and L_2 is a regular language, too.
- (b) The **intersection** $L_1 \cap L_2$ of two regular languages L_1 and L_2 is again a regular language.
- (c) The **complement** $\Sigma^* \setminus L$ of a regular language L is also a regular language.

As an application of these closure properties we will then show how it is possible to decide whether two regular expressions are **equivalent**, i.e. we present an algorithm that takes two regular expressions r_1 and r_2 as input and checks, whether

$$r_1 \doteq r_2$$

holds. After that, we discuss the **limits** of regular languages. To this end, we prove the **pumping lemma**. Using the pumping lemma we will be able to show that, for example, the language

$$\{a^n b^n \mid n \in \mathbb{N}\}$$

is not regular. To summarize, this chapter

- discusses closure properties of regular languages
- presents an algorithm for checking the equivalence of regular expressions, and
- shows that certain languages are not regular.

5.1 Closure Properties of Regular Languages

In this section we show that regular languages are closed under the Boolean operations of **union**, **intersection** and **complement**. We start with the union.

Proposition 13 If L_1 and L_2 are regular languages, then the union of the sets L_1 and L_2 , which is the set $L_1 \cup L_2$, is a regular language, too.

Proof: As L_1 and L_2 are regular languages, there exist regular expressions r_1 and r_2 such that

$$L_1 = L(r_1) \quad \text{and} \quad L_2 = L(r_2)$$

holds. We define $r := r_1 + r_2$. Then we have

$$L(r) = L(r_1 + r_2) = L(r_1) \cup L(r_2) = L_1 \cup L_2.$$

Since $L_1 \cup L_2$ is the language that is generated by the regular expression $r_1 + r_2$, it is a regular language. \square

Proposition 14 If L_1 and L_2 are regular languages, then the intersection of the sets L_1 and L_2 , which is the set $L_1 \cap L_2$, is a regular language, too.

Proof: While the proof of Proposition 13 follows directly from the definition of regular expressions, we have to do a little more work here. In the previous chapter we saw that for every regular expression r there is an equivalent deterministic finite state machine F , which accepts the language specified by r , and we can also assume that this DFA is complete. Let r_1 and r_2 be the regular expressions that define the languages L_1 and L_2 , i.e. we have

$$L_1 = L(r_1) \quad \text{und} \quad L_2 = L(r_2).$$

First, we construct two complete deterministic FSMs F_1 and F_2 that accept these languages, i.e. such that we have

$$L(F_1) = L_1(r_1) \quad \text{and} \quad L(F_2) = L_2(r_2).$$

Our goal is to construct a finite state machine F such that F accepts the language $L_1 \cap L_2$ and nothing else. As every finite state machine can be converted into an equivalent regular expression we will then have shown that the language $L_1 \cap L_2$ is regular. We will use the FSMs F_1 and F_2 to construct F . Assume that

$$F_1 = \langle Q_1, \Sigma, \delta_1, q_1, A_1 \rangle \quad \text{and} \quad F_2 = \langle Q_2, \Sigma, \delta_2, q_2, A_2 \rangle$$

holds. We define F as follows:

$$F := \langle Q_1 \times Q_2, \Sigma, \delta, \langle q_1, q_2 \rangle, A_1 \times A_2 \rangle,$$

where the state transition function

$$\delta : (Q_1 \times Q_2) \times \Sigma \rightarrow Q_1 \times Q_2$$

is defined as

$$\delta(\langle p_1, p_2 \rangle, c) := \langle \delta_1(p_1, c), \delta_2(p_2, c) \rangle.$$

Effectively, the FSM F runs the FSM F_1 and F_2 in parallel. In order to do so, the states of F are pairs of the form $\langle p_1, p_2 \rangle$ where p_1 is a state from F_1 and p_2 is a state from F_2 and the transition function δ computes the state $\langle \delta_1(p_1, c), \delta_2(p_2, c) \rangle$ by simultaneously computing the next states of p_1 and p_2 in the FSMs F_1 and F_2 when these FSMs read the character c . A string s is accepted by F if and only if both F_1 and F_2 accept s . Therefore, the set A of accepting states of the FSM F is defined as:

$$A := \{ \langle p_1, p_2 \rangle \in Q_1 \times Q_2 \mid p_1 \in A_1 \wedge p_2 \in A_2 \} = A_1 \times A_2.$$

Then for all $s \in \Sigma^*$ we have:

$$\begin{aligned} s &\in L(F) \\ \Leftrightarrow \delta(\langle q_1, q_2 \rangle, s) &\in A \\ \Leftrightarrow \langle \delta_1(q_1, s), \delta_2(q_2, s) \rangle &\in A_1 \times A_2 \\ \Leftrightarrow \delta_1(q_1, s) \in A_1 \wedge \delta_2(q_2, s) &\in A_2 \\ \Leftrightarrow s \in L(F_1) \wedge s \in L(F_2) \\ \Leftrightarrow s \in L(F_1) \cap L(F_2) \\ \Leftrightarrow s \in L_1 \cap L_2 \end{aligned}$$

Therefore we have shown that

$$L(F) = L_1 \cap L_2$$

and this completes the proof. \square

Remark: In principle it would be possible to define a function

$$\wedge : \text{RegExp} \times \text{RegExp} \rightarrow \text{RegExp}$$

that takes two regular expressions r_1 and r_2 and returns a regular expression $r_1 \wedge r_2$ such that we have

$$L(r_1 \wedge r_2) = L(r_1) \cap L(r_2).$$

To compute $r_1 \wedge r_2$ we would first compute non-deterministic FSMs F_1 and F_2 such that

$$L(F_1) = L(r_1) \quad \text{and} \quad L(F_2) = L(r_2)$$

holds. Then we would transform the FSMs F_1 and F_2 into equivalent deterministic FSMs $\det(F_1)$ and $\det(F_2)$. After that, we would build the extended Cartesian product of $\det(F_1)$ and $\det(F_2)$ as shown above. Finally, we would convert the FSM $\det(F_1) \times \det(F_2)$ into an equivalent regular expression. However, most of the time the resulting regular expression would be absurdly large. Therefore, it is not practical to implement the function \wedge . \diamond

Proposition 15 If L is a regular language with the alphabet Σ , then the **complement** of L , which is defined as the language $\Sigma^* \setminus L$, is regular.

Proof: We assume that r is a regular expression describing the language L , i.e. $L = L(r)$. We construct a non-deterministic FSM F such that $L(F) = L(r) = L$. We transform this non-deterministic FSM into the deterministic FSM $\det(F)$ as discussed in the previous chapter and note that this DFA is complete. Assume that we then have

$$\det(F) = \langle Q, \Sigma, \delta, q_0, A \rangle.$$

We define the deterministic FSM $\overline{\det(F)}$ as follows:

$$\overline{\det(F)} = \langle Q, \Sigma, \delta, q_0, Q \setminus A \rangle.$$

i.e. the set of accepting states of $\overline{\det(F)}$ is the complement of the set of accepting states of F . Then we have

$$\begin{aligned} w &\in L(\overline{\det(F)}) \\ \Leftrightarrow \delta(q_0, w) &\in Q \setminus A \\ \Leftrightarrow \delta(q_0, w) &\notin A \\ \Leftrightarrow w &\notin L(\det(F)) \\ \Leftrightarrow w &\notin L(F) \\ \Leftrightarrow w &\notin L(r) \\ \Leftrightarrow w &\notin L \\ \Leftrightarrow w &\in \Sigma^* \setminus L \end{aligned}$$

This shows that the language $\Sigma^* \setminus L$ is accepted by the FSM $\overline{\det(F)}$ and hence it is a regular language. \square

Corollary 16 If L_1 and L_2 are regular languages over the common alphabet Σ , then the **set difference** $L_1 \setminus L_2$ is a regular language.

Proof: A string w is a member of $L_1 \setminus L_2$ iff w is a member of L_1 and w is also a member of the complement of L_2 . Therefore we have

$$L_1 \setminus L_2 = L_1 \cap (\Sigma^* \setminus L_2),$$

The previous proposition shows that for a regular language L_2 the complement $\Sigma^* \setminus L_2$ is also a regular language. Since the intersection of regular languages is regular, too, we see that $L_1 \setminus L_2$ is also regular. \square

In summary, we have demonstrated that regular languages are closed under the operations of Boolean algebra on sets.

Exercise 14: Let Σ be a given alphabet. For a string $s = c_1c_2 \dots c_{n-1}c_n \in \Sigma^*$, the **reversal** of s is denoted by s^R and is defined as follows:

$$s^R := c_nc_{n-1} \dots c_2c_1$$

For instance, if $s = abc$, then $s^R = cba$. The reversal L^R of a language $L \subseteq \Sigma^*$ is formally defined as:

$$L^R := \{s^R \mid s \in L\}$$

Subsequently, let us consider a language $L \subseteq \Sigma^*$ that is regular. Your task is to prove that L^R is also a regular language. \diamond

5.2 Recognizing Empty Languages

In this section we develop an algorithm that takes a deterministic FSM

$$F = \langle Q, \Sigma, \delta, q_0, A \rangle$$

as its input and then checks, whether the language accepted by F is empty, i.e. it checks whether $L(F) = \{\}$. To this end we interpret the FSM F as a directed graph. The nodes of this graph are the states of the set Q and for two states q_1 and q_2 there is an edge q_1 from to q_2 iff there exists a character $c \in \Sigma$, such that $\delta(q_1, c) = q_2$. The language $L(F)$ is empty if and only if this graph has no path that starts in the state q_0 and leads to an accepting state.

Therefore, in order to answer the question whether $L(F) = \{\}$ holds, we have to compute the set R of all those states that are **reachable** from the start state q_0 . The computation of the set R of all reachable states can be done iteratively as follows:

1. $q_0 \in R$, because the start state is obviously reachable from the start state.
2. $p_1 \in R \wedge \delta(p_1, c) = p_2 \rightarrow p_2 \in R$,
because if p_1 is reachable from q_0 and there is a transition from p_1 to p_2 while reading the character c , then p_2 is reachable from q_0 .

This last step is repeated until there are no more states that can be added to the set R .

Then we have $L(F) = \{\}$ if and only if none of the accepting states is reachable, i.e. we have

$$L(F) = \{\} \Leftrightarrow R \cap A = \{\}.$$

Hence we now have an algorithm for checking whether $L(F) = \{\}$ holds: We compute the states that are reachable from the start state q_0 and then we check whether this set contains any accepting states.

Remark: If the regular language L is not specified via an FSM F , but rather is defined via a regular expression r , then there is a simple recursive algorithm for checking whether $L(r)$ is empty:

1. $L(\emptyset) = \{\}$.
2. $L(\varepsilon) \neq \{\}$.
3. $L(c) \neq \{\}$ for all $c \in \Sigma$.
4. $L(r_1 \cdot r_2) = \{\} \Leftrightarrow L(r_1) = \{\} \vee L(r_2) = \{\}$.
5. $L(r_1 + r_2) = \{\} \Leftrightarrow L(r_1) = \{\} \wedge L(r_2) = \{\}$.
6. $L(r^*) \neq \{\}$.

\diamond

5.3 Equivalence of Regular Expressions

In Chapter 2 we had defined two regular expressions r_1 and r_2 to be **equivalent** (written $r_1 \doteq r_2$), if the languages specified by r_1 and r_2 are identical:

$$r_1 \doteq r_2 \stackrel{\text{def}}{\iff} L(r_1) = L(r_2).$$

In this section, we present an algorithm that takes two regular expressions r_1 and r_2 as input and then checks whether $r_1 \doteq r_2$ holds.

Theorem 17 If r_1 and r_2 are regular expressions, then the question whether $r_1 \doteq r_2$ holds is decidable.

Proof: We present an algorithm that decides whether $L(r_1) = L(r_2)$ holds. First, we observe that the sets $L(r_1)$ and $L(r_2)$ are identical iff the set differences $L(r_2) \setminus L(r_1)$ and $L(r_1) \setminus L(r_2)$ are both empty:

$$\begin{aligned} L(r_1) = L(r_2) &\iff L(r_1) \subseteq L(r_2) \wedge L(r_2) \subseteq L(r_1) \\ &\iff L(r_1) \setminus L(r_2) = \{\} \wedge L(r_2) \setminus L(r_1) = \{\} \end{aligned}$$

Next, assume that F_1 and F_2 are deterministic FSMS such that

$$L(F_1) = L(r_1) \quad \text{and} \quad L(F_2) = L(r_2)$$

holds. We have seen in Chapter 4 how F_1 and F_2 can be constructed from r_1 and r_2 . According to the corollary 16 the languages $L(r_1) \setminus L(r_2)$ and $L(r_2) \setminus L(r_1)$ are regular and we have seen how to construct FSMS $F_{1,2}$ and $F_{2,1}$ such that

$$L(r_1) \setminus L(r_2) = L(F_{1,2}) \quad \text{and} \quad L(r_2) \setminus L(r_1) = L(F_{2,1})$$

holds. Hence we have

$$r_1 \doteq r_2 \iff L(F_{1,2}) = \{\} \wedge L(F_{2,1}) = \{\}$$

and according to Section 5.2 this question is decidable by checking whether any of the accepting states of $F_{1,2}$ or $F_{2,1}$ are reachable from the start state. \square

Remark: The Jupyter notebook `Equivalence.ipynb`, which is available at

<https://github.com/karlstroetmann/Formal-Languages/blob/master/Python/Equivalence.ipynb>

implements this algorithm.

5.4 Limits of Regular Languages

In this section we present a theorem that can be used to show that certain languages are not regular. This theorem is known as the **pumping lemma for regular languages**.

Theorem 18 (Pumping Lemma for Regular Languages, Michael Rabin and Dana Scott 1959, [RS59])

Assume L is a regular language. Then there exists a positive natural number $n \in \mathbb{N}$ such that every string $s \in L$ that has a length of at least n can be split into three substrings u , v , and w such that the following holds:

1. $s = uvw$,
2. $v \neq \varepsilon$,
3. $|uv| \leq n$,
4. $\forall h \in \mathbb{N} : uv^h w \in L$.

This theorem can be written as a single formula: If L is a regular language, then

$$\exists n \in \mathbb{N} : \left(n > 0 \wedge \forall s \in L : (|s| \geq n \rightarrow \exists u, v, w \in \Sigma^* : s = uvw \wedge v \neq \varepsilon \wedge |uv| \leq n \wedge \forall h \in \mathbb{N} : uv^h w \in L) \right).$$

Proof: As L is a regular language, there exists a deterministic FSM

$$F = \langle Q, \Sigma, \delta, q_0, A \rangle,$$

such that $L = L(F)$. The number n whose existence is claimed in the Pumping Lemma is defined as the number of states of F , i.e. as the cardinality of the set Q :

$$n := \text{card}(Q).$$

Next, assume a string $s \in L$ is given such that $|s| \geq n$. Define $m := |s|$. Then there are m characters c_i such that

$$s = c_1 c_2 \cdots c_m.$$

Since $|s| \geq n$, we have $m \geq n$. On reading the characters c_i the FSM changes its states as follows:

$$q_0 \xrightarrow{c_1} q_1 \xrightarrow{c_2} q_2 \xrightarrow{c_3} \cdots \xrightarrow{c_m} q_m,$$

i.e. on reading the character c_{i+1} in the state q_i the FSM switches into the state q_{i+1} for all $i = 0, \dots, m-1$. Since we have $s \in L$ we conclude that q_m must be an accepting state, i.e. $q_m \in A$. As $m \geq n$ and n is the total number of states of F , not all of the states

$$q_0, q_1, q_2, \dots, q_m$$

can be different. Because of

$$\text{card}(\{0, 1, \dots, n\}) = n + 1$$

we know, that even in the list

$$[q_0, q_1, q_2, \dots, q_n]$$

at least one state has to occur at least twice. Hence there are natural numbers $k, l \in \{0, \dots, n\}$ such that

$$q_k = q_l \wedge k < l.$$

Next, we define the strings u , v , and w as follows:

$$u := c_1 \cdots c_k, \quad v := c_{k+1} \cdots c_l, \quad \text{and} \quad w := c_{l+1} \cdots c_m.$$

As $k < l$ we have that $v \neq \varepsilon$ and $l \leq n$ implies $|uv| \leq n$. Furthermore, we have the following:

1. Reading the string u changes the state of the FSM F from the start state q_0 to the state q_k , we have

$$q_0 \xrightarrow{u} q_k. \tag{5.1}$$

2. Reading the string v changes the state of the FSM F from the state q_k to the state q_l . As we have $q_l = q_k$, this implies

$$q_k \xrightarrow{v} q_k. \tag{5.2}$$

3. Reading the string w changes the state of the FSM F from the state $q_l = q_k$ to the accepting state q_m :

$$q_k \xrightarrow{w} q_m. \tag{5.3}$$

From $q_k \xrightarrow{v} q_k$ we conclude

$$q_k \xrightarrow{v} q_k \xrightarrow{v} q_k, \quad \text{hence} \quad q_k \xrightarrow{v^2} q_k.$$

As we can repeat reading v in state q_k any number of times, we have

$$q_k \xrightarrow{v^h} q_k \quad \text{for all } h \in \mathbb{N}. \tag{5.4}$$

Combining the equations (5.1), (5.3), and (5.4) we have

$$q_0 \xrightarrow{u} q_k \xrightarrow{v^h} q_k \xrightarrow{w} q_m.$$

This can be condensed to

$$q_0 \xrightarrow{uv^hw} q_m$$

and since the state q_m is an accepting state we conclude that $uv^hw \in L$ holds for any $h \in \mathbb{N}$. \square

Proposition 19 The alphabet Σ is defined as $\Sigma = \{ \text{"a"}, \text{"b"} \}$. Define the language L as the set of all strings of the form $a^k b^k$ where k is some natural number:

$$L = \{ a^k b^k \mid k \in \mathbb{N} \}.$$

Then the language L is not regular.

Proof: This is a proof by contradiction. We assume that L is a regular language. According to the Pumping Lemma there exists a fixed natural number $n > 0$ such that every $s \in L$ that satisfies $|s| \geq n$ can be written as

$$s = uvw$$

where, furthermore, we have that

$$|uv| \leq n, \quad v \neq \varepsilon, \quad \text{and} \quad \forall h \in \mathbb{N} : uv^h w \in L$$

holds. Let us define the string s as

$$s := a^n b^n.$$

Obviously we have $|s| = 2 \cdot n \geq n$. Hence there are strings u , v , and w such that

$$a^n b^n = uvw, \quad |uv| \leq n, \quad v \neq \varepsilon, \quad \text{and} \quad \forall h \in \mathbb{N} : uv^h w \in L.$$

As $|uv| \leq n$, the string uv is a prefix not only of s but even of a^n . Therefore, and since $v \neq \varepsilon$ we know that the string v must have the form

$$v = a^k \quad \text{for some } k \in \mathbb{N} \text{ such that } k > 0.$$

If we take the formula $\forall h \in \mathbb{N} : uv^h w \in L$ and set $h := 0$, we conclude that

$$uw \in L. \tag{5.5}$$

In order to facilitate our argument, we define the function

$$\text{count} : \Sigma^* \times \Sigma \rightarrow \mathbb{N}.$$

Given a string t and a character c the function $\text{count}(t, c)$ counts how often the character c occurs in the string t . For the language L we have

$$t \in L \Rightarrow \text{count}(t, 'a') = \text{count}(t, 'b').$$

On one hand we have:

$$\begin{aligned} \text{count}(uw, 'a') &= \text{count}(uvw, 'a') - \text{count}(v, 'a') \\ &= \text{count}(s, 'a') - \text{count}(v, 'a') \\ &= \text{count}(a^n b^n, 'a') - \text{count}(a^k, 'a') \\ &= n - k \\ &< n \end{aligned}$$

But on the other hand we have

$$\begin{aligned} \text{count}(uw, 'b') &= \text{count}(uvw, 'b') - \text{count}(v, 'b') \\ &= \text{count}(s, 'b') - \text{count}(v, 'b') \\ &= \text{count}(a^n b^n, 'b') - \text{count}(a^k, 'b') \\ &= n - 0 \\ &= n \end{aligned}$$

Therefore, we have

$$\text{count}(uw, 'a') < \text{count}(uw, 'b')$$

and this shows that the string uw is not a member of the language L because for all strings in L the number of occurrences of the character “a” is the same as the number of occurrences of the character “b”. This contradiction shows that the language L cannot be regular. \square

Exercise 15: Let us define the language L as the set of all those strings that contain an equal number of occurrences of the characters “a” and “b”:

$$L := \{w \in \{ 'a', 'b' \}^* \mid \text{count}(w, 'a') = \text{count}(w, 'b')\}$$

Prove that the language L is not regular. \diamond

Remark: As the language L defined in the previous exercise is not regular we can conclude that regular expressions are unable to check even such simple questions as to whether the parentheses in an expression are balanced. Therefore, the concept of regular expressions is not strong enough to describe the syntax of a programming language. The next chapter introduces the notion of [context-free languages](#). These languages are powerful enough to describe modern programming languages.

Exercise 16: The language L_{square} is the set of all strings of the form a^n where n is a square, we have

$$L_{\text{square}} = \{a^m \mid \exists k \in \mathbb{N} : m = k^2\}$$

Prove that the language L_{square} is not a regular language. \diamond

Solution: The proof employs the method of contradiction. We begin by assuming that L_{square} is a regular language. According to the Pumping Lemma, there exists a natural number $n > 0$ such that for any string $s \in L_{\text{square}}$ with $|s| \geq n$, the string can be decomposed into three parts u , v , and w satisfying the following conditions:

1. $s = uvw$,
2. $|uv| \leq n$,
3. $v \neq \varepsilon$,
4. $\forall h \in \mathbb{N} : uv^h w \in L_{\text{square}}$.

Let us define $s := a^{n^2}$. We have $s \in L_{\text{square}}$ and we see that

$$|s| = |a^{n^2}| = n^2 \geq n.$$

Hence there have to be strings u , v and w such that $s = uvw$ and u , v , and w have the properties specified above. As ‘a’ is the only character that occurs in s , the strings u , v , and w also contain only this character. Hence there must be natural numbers x , y , and z such that

$$u = a^x, v = a^y \text{ und } w = a^z$$

holds. Then we have the following.

(a) The partition $s = uvw$ has the form $a^{n^2} = a^x a^y a^z$ and hence we have

$$n^2 = x + y + z. \tag{5.6}$$

(b) The inequality $|uv| \leq n$ implies $x + y \leq n$, which implies

$$y \leq n. \tag{5.7}$$

(c) From the condition $v \neq \varepsilon$ we get

$$y > 0. \tag{5.8}$$

(d) The formula $\forall h \in \mathbb{N} : uv^h w \in L_{\text{square}}$ implies

$$\forall h \in \mathbb{N} : a^x a^{y \cdot h} a^z \in L_{\text{square}}. \quad (5.9)$$

In particular, this must hold for $h = 2$:

$$a^x a^{y \cdot 2} a^z \in L_{\text{square}}.$$

According to the definition of L_{square} there is a natural number k such that

$$x + 2 \cdot y + z = k^2. \quad (5.10)$$

If we add y on both sides of equation (5.6) we get

$$n^2 + y = x + 2 \cdot y + z = k^2.$$

Because of $y > 0$ this implies

$$n < k. \quad (5.11)$$

On the other hand we have

$$\begin{aligned} k^2 &= x + 2 \cdot y + z && \text{according to (5.10)} \\ &= x + y + z + y \\ &\leq x + y + z + n && \text{according to (5.7)} \\ &= n^2 + n && \text{according to (5.6)} \\ &< n^2 + 2 \cdot n + 1 && \text{since } n + 1 > 0 \\ &= (n + 1)^2 \end{aligned}$$

This shows that $k^2 < (n + 1)^2$ holds and therefore we have

$$k < n + 1. \quad (5.12)$$

The inequalities (5.11) and (5.12) imply

$$n < k < n + 1.$$

Since k is a natural number and n is also a natural number this is a contradiction because there is no natural number between n and $n + 1$. \square

Exercise 17: Define $\Sigma := \{a\}$. Prove that the language

$$L := \{a^p \mid p \text{ is a prime number}\}$$

is not regular. \diamond

Proof: The proof is done by contradiction. Assume that L is regular. According to the Pumping Lemma there is a positive natural number n such that all strings $s \in L$ that have a length of at least n can be written as

$$s = uvw$$

where, furthermore, the substrings u , v , and w have the following properties:

1. $v \neq \varepsilon$,
2. $|uv| \leq n$, and
3. $\forall h \in \mathbb{N} : uv^h w \in L$.

Let us choose a prime number p such that $p \geq n + 2$. Since there are infinitely many prime numbers, this is always possible. Next, define $s := a^p$. Then we have $|s| = p \geq n$ and we can use the Pumping Lemma to conclude that there must be substrings u , v , and w such that

$$a^p = uvw$$

holds. Hence the substrings u , v , and w can only contain the letter “a”. Therefore there exist natural numbers x , y , and z such that

$$u = a^x, \quad v = a^y, \quad \text{and} \quad w = a^z.$$

We can conclude that x , y , and z have the following properties:

1. $x + y + z = p$,
2. $y \neq 0$,
3. $x + y \leq n$,
4. $\forall h \in \mathbb{N} : x + h \cdot y + z \in \mathbb{P}$.

Here, \mathbb{P} denotes the set of prime numbers.

If we substitute $h := x + z$ in the last formula, we get

$$x + (x + z) \cdot y + z \in \mathbb{P}.$$

Because of $x + (x + z) \cdot y + z = (x + z) \cdot (1 + y)$ this leads to

$$(x + z) \cdot (1 + y) \in \mathbb{P}.$$

This is impossible. As $y > 0$ the factor $1 + y$ is different from 1 and because of $x + y \leq n$, $x + y + z = p$, and $p \geq n + 2$ it follows that $z \geq 2$. Hence the factor $x + z$ is also greater than 1. But then the product $(x + z) \cdot (1 + y)$ is not a prime number and this contradiction shows that L can not be regular. \square

Exercise 18: The language L_{power} is the set of all strings of the form a^n where n is a power of 2, i.e. we have

$$L_{\text{power}} = \{a^{2^k} \mid k \in \mathbb{N}\}$$

Prove that the language L_{power} is not regular. \diamond

5.5 Check your Understanding

- (a) What are the closure properties of regular languages?
- (b) How can we check whether two regular expressions are equivalent?
- (c) Specify the exact wording of the the [Pumping Lemma](#).
- (d) Can you prove that the language

$$L := \{w \in \{ 'a', 'b' \}^* \mid \text{count}(w, 'a') = \text{count}(w, 'b')\}$$

is not regular?

Chapter 6

Context-Free Languages

In this chapter we present the notion of a *context-free language*. This concept is much more powerful than the notion of a regular language. The syntax of most modern programming languages can be described by a context-free language. Furthermore, checking whether a string is a member of a context-free language structures the string into a recursive structure known as a *parse tree*. These parse trees are the basis for understanding the meaning of a string that is to be interpreted as a program fragment. A program that checks whether a given string is an element of a context-free language is called a *parser*. The task of a parser is to build a *parse tree* from a given string. Parsing is therefore the first step in an interpreter or a compiler. In this chapter, we first define the notion of context-free languages. Next, we discuss parse trees. We conclude this chapter by introducing *top down parsing*, which is one of the less complex algorithms that are available for parsing a string.

6.1 Context-Free Grammars

Context-free languages are used to describe programming languages. Context-free languages are formal languages like regular languages, but they are much more expressive than regular languages. When we process a program, we not only want to decide whether the program is syntactically correct, but we also want to understand the *structure* of the program. The process of *structuring* is also referred to as *parsing* and the program that does this structuring is called a *parser*. As input a parser usually does not receive the text of a program, but instead a sequence of so-called *terminals*, which are also called *tokens*. These tokens are generated by a scanner, which uses regular expressions to split the program text into single words, which we call tokens in this context.

The parser receives a sequence of tokens from the scanner and has the task of constructing a so-called *syntax tree*. For this purpose the parser uses a grammar which specifies how the input is to be structured. As an example, consider parsing arithmetic expressions. We define the set *ArithExpr* of arithmetic expressions inductively. In order to correctly represent the structure of arithmetic expressions, we define simultaneously the sets *Product* and *Factor*. The set *Product* encompasses arithmetic expressions representing products and quotients, while the set *Factor* includes numbers and parenthesized expressions. Defining these additional sets is crucial for ensuring the correct precedence of operators. The basic building blocks of arithmetic expressions are variables, numbers, the operator symbols “+”, “-”, “*”, “/”, and the bracket symbols “(” and “)”. Based on these symbols the inductive definition of the sets *Factor*, *Product* and *ArithExpr* proceeds as follows:

1. Each number is a factor:

$$C \in \text{number} \Rightarrow C \in \text{Factor}.$$

2. Each variable is a factor:

$$V \in \text{variable} \Rightarrow V \in \text{Factor}.$$

3. If *A* is an arithmetic expression and we enclose this expression in parentheses we get an expression that we can use as a factor:

$$A \in \text{ArithExpr} \Rightarrow “(” A “)” \in \text{Factor}.$$

A note on notation: In the preceding formula, A serves as a meta-variable representing an arbitrary arithmetic expression. The strings “(” and “)” should be taken literally and are thus enclosed in quotation marks. These quotation marks are not part of the arithmetic expression; they are used solely for notational clarity.

4. If F is a factor, then F is also a product:

$$F \in \text{Factor} \Rightarrow F \in \text{Product}.$$

5. If P is a product and if F is a factor, then the strings $P \text{ “*” } F$ and $P \text{ “/” } F$ are also products:

$$P \in \text{Product} \wedge F \in \text{Factor} \Rightarrow P \text{ “*” } F \in \text{Product} \wedge P \text{ “/” } F \in \text{Product}.$$

6. Each product is also an arithmetic expression

$$P \in \text{Product} \Rightarrow P \in \text{ArithExpr}.$$

7. If A is an arithmetic expression and P is a product, then the strings $A \text{ “+” } P$ and $A \text{ “-” } P$ are arithmetic expressions:

$$A \in \text{ArithExpr} \wedge P \in \text{Product} \Rightarrow A \text{ “+” } P \in \text{ArithExpr} \wedge A \text{ “-” } P \in \text{ArithExpr}.$$

The sets *Factor*, *Product*, and *ArithExpr* are defined above through mutual recursion. This definition can be succinctly represented using what are known as [grammar rules](#).

```

arithExpr → arithExpr “+” product
arithExpr → arithExpr “-” product
arithExpr → product

product → product “*” factor
product → product “/” factor
product → factor

factor → “(” arithExpr “)”
factor → VARIABLE
factor → NUMBER

```

Expressions on the left hand side of a grammar rule are known as [syntactic variables](#) or [non-terminals](#), while all other expressions are termed [terminals](#). We adopt the convention of writing syntactic variables in lowercase to align with the syntax used by parser generators such as [ANTLR](#) and [PLY](#), which we will explore later. However, it is worth noting that in much of the existing literature, the convention is reversed: syntactic variables are typically capitalized, and terminals are presented in lowercase. Additionally, syntactic variables may occasionally be referred to as [syntactic categories](#).

In the example, *arithExpr*, *product*, and *factor* function as the [syntactic variables](#). The remaining elements, namely `NUMBER`, `VARIABLE`, and the symbols “+”, “-”, “*”, “/”, “(”, and “)”, are the [terminals](#) or [tokens](#). These terminals are precisely the characters that are not found on the left side of any grammar rule. Terminals can be classified into two types:

1. Operator symbols and separators, like “/” and “(”, are used in their literal sense.
2. Tokens such as `NUMBER` or `VARIABLE` carry associated values. For `NUMBER`, the value is numeric; for `VARIABLE`, it is a string representing the variable’s name. To distinguish them from syntactic variables, these token types are always written in uppercase letters.

Grammar rules are often expressed in a notation more compact than that introduced previously. For the given example, the compact notation is as follows:

```

arithExpr → arithExpr “+” product | arithExpr “-” product | product
product → product “*” factor | product “/” factor | factor
factor → “(” arithExpr “)” | NUMBER | VARIABLE

```

In this format, individual alternatives within a rule are demarcated by the metacharacter "|". Building on the preceding example, we now introduce the formal definition of a [context-free grammar](#).

Definition 20 (Context-Free Grammar) A [context-free grammar](#) G is a quadruple

$$G = \langle V, T, R, S \rangle,$$

where

1. V is a set called [syntactic variables](#) or [non-terminals](#). In the preceding example, we have

$$V = \{arithExpr, product, factor\}.$$

2. T is a set referred to as [terminals](#) or [tokens](#). The sets T and V are disjoint, hence

$$T \cap V = \emptyset.$$

For the aforementioned example, we have

$$T = \{\text{NUMBER}, \text{VARIABLE}, "+", "-", "*", "/", "(", ")"\}.$$

3. R is a set of [grammar rules](#). A grammar rule is formally a pair $\langle A, \alpha \rangle$ where:

- (a) The first component, A , is a syntactic variable:

$$A \in V.$$

- (b) The second component, α , is a string composed of syntactic variables and terminals:

$$\alpha \in (V \cup T)^*.$$

Consequently, we have

$$R \subseteq V \times (V \cup T)^*.$$

A rule $\langle x, \alpha \rangle$ is typically written as

$$x \rightarrow \alpha.$$

In the example, the first rule is

$$arithExpr \rightarrow arithExpr "+" product,$$

which formally corresponds to the pair

$$\langle arithExpr, [arithExpr, "+", product] \rangle.$$

4. S is a member of V designated as the [start symbol](#). In the given example, *arithExpr* is the start symbol. \diamond

When presenting a grammar, we typically enumerate the grammar rules and apply the following conventions to identify the start symbol and to differentiate between syntactic variables and terminals:

1. The start symbol is the syntactic variable appearing on the left side of the first rule.
2. Symbols enclosed within quotes are terminals.
3. Names beginning with an uppercase letter are terminals.
4. Names beginning with a lowercase letter are syntactic variables.

Example: The set of grammar rules below defines arithmetic expressions constructed from numbers, variables, and the binary operators “+”, “-”, “.”, and “/”.

$$\begin{aligned}
 \text{arithExpr} &\rightarrow \text{arithExpr} \text{ “+” } \text{product} \\
 &\quad | \text{arithExpr} \text{ “-” } \text{product} \\
 &\quad | \text{product} \\
 \text{product} &\rightarrow \text{product} \text{ “.” } \text{factor} \\
 &\quad | \text{product} \text{ “/” } \text{factor} \\
 &\quad | \text{factor} \\
 \text{factor} &\rightarrow \text{ “(” arithExpr “)” } \\
 &\quad | \text{NUMBER} \\
 &\quad | \text{VARIABLE}
 \end{aligned}$$

In this case, the following observations can be made:

1. The start symbol is *arithExpr*, as it is the syntactic variable on the left in the first grammar rule.
2. *arithExpr*, *product*, and *factor* are recognized as syntactic variables because they start with a lowercase letter.
3. The symbols “+”, “-”, “.”, and “/” are identified as terminals since they are enclosed in quotes.
4. NUMBER and VARIABLE are terminals, indicated by their initial uppercase letters.

6.1.1 Derivations

Next, we aim to identify the [language](#) defined by a given grammar G . For this purpose, we first define the concept of a [derivation step](#). Consider the following:

1. $G = \langle V, T, R, S \rangle$ is a grammar,
2. a is a syntactic variable in V ,
3. $\alpha a \beta$ is a string composed of terminals and syntactic variables from $(V \cup T)^*$, which includes the variable a , and
4. $(a \rightarrow \gamma)$ is a rule in R .

Under these conditions, the string $\alpha a \beta$ can undergo a derivation step to transform into the string $\alpha \gamma \beta$. This process involves substituting one occurrence of the syntactic variable a with the right-hand side γ of the rule $a \rightarrow \gamma$. We denote this derivation step as

$$\alpha a \beta \Rightarrow_G \alpha \gamma \beta.$$

When the grammar G is clear from the context, we may drop the subscript G and simply use \Rightarrow instead of \Rightarrow_G . The transitive and reflexive closure of the relation \Rightarrow_G is indicated by \Rightarrow_G^* . To specify that the derivation of string w from the non-terminal a encompasses n derivation steps, we write:

We illustrate with an example:

$$\begin{aligned}
 \text{arithExpr} &\Rightarrow \text{arithExpr "+" product} \\
 &\Rightarrow \text{product "+" product} \\
 &\Rightarrow \text{product "." factor "+" product} \\
 &\Rightarrow \text{factor "." factor "+" product} \\
 &\Rightarrow \text{NUMBER "." factor "+" product} \\
 &\Rightarrow \text{NUMBER "." NUMBER "+" product} \\
 &\Rightarrow \text{NUMBER "." NUMBER "+" factor} \\
 &\Rightarrow \text{NUMBER "." NUMBER "+" NUMBER}
 \end{aligned}$$

Hence, we have demonstrated that

$$\text{arithExpr} \Rightarrow^* \text{NUMBER "." NUMBER "+" NUMBER}$$

is valid. To be more precise, we could express this as

$$\text{arithExpr} \Rightarrow^8 \text{NUMBER "." NUMBER "+" NUMBER}$$

since the derivation comprises eight steps. By substituting the terminal NUMBER with distinct numerals, we demonstrate that the string

$$2 \cdot 3 + 4$$

is an arithmetic expression. Generally, the language $L(G)$ defined by a grammar G is the set of all strings that are exclusively composed of terminals and can be derived from the start symbol S of the grammar; that is, we define

$$L(G) := \{w \in T^* \mid S \Rightarrow^* w\}.$$

Example: The language

$$L = \{(^n)^n \mid n \in \mathbb{N}\}$$

is generated by the grammar

$$G = \langle \{s\}, \{ "(" , ")" \}, R, s \rangle,$$

where the set of rules R is defined as:

$$s \rightarrow "(" s ")" \mid \lambda.$$

Proof: We first establish that each string w in L can be derived from the start symbol s :

$$\text{If } w \in L, \text{ then } s \Rightarrow^* w.$$

Let $w_n = ({}^n)^n$. We use induction on $n \in \mathbb{N}$ to show that $w_n \in L(G)$.

B.C.: $n = 0$.

We have $w_0 = \lambda$. Since the grammar includes the rule $s \rightarrow \lambda$, it follows that

$$s \Rightarrow \lambda,$$

and thus $w_0 \in L(G)$.

I.S.: $n \mapsto n + 1$.

The string w_{n+1} has the form $w_{n+1} = "("w_n")"$, where w_n is also in L . By the inductive hypothesis, there exists a derivation for w_n :

$$s \Rightarrow^* w_n.$$

Thus, we obtain the derivation

$$s \Rightarrow "("s")" \Rightarrow^* "("w_n")" = w_{n+1},$$

which confirms $w_{n+1} \in L(G)$.

Next, we prove that any string w derived from s belongs to L , using induction on the number n of derivation steps:

B.C.: $n = 1$.

The sole derivation from terminals that is one step long is

$$s \Rightarrow \lambda.$$

Hence, $w = \lambda$, and since $\lambda = ({}^0)^0$ belongs to L , we deduce $w \in L$.

I.S.: $n \mapsto n + 1$.

For a derivation exceeding one step, it must begin as

$$s \Rightarrow "("s")" \Rightarrow^n w.$$

This implies

$$w = "("v")" \quad \text{and} \quad s \Rightarrow^n v.$$

By the inductive hypothesis, $v \in L$. Therefore, there exists a $k \in \mathbb{N}$ such that $v = ({}^k)^k$. Consequently,

$$w = "("v")" = (({}^k)^k) = ({}^{k+1})^{k+1} \in L.$$

□

Exercise 19: We define for $w \in \Sigma^*$ and $c \in \Sigma$ the function

$$\text{count}(w, c).$$

The function counts how many times the letter c occurs in the word w . The definition is done by induction on the string w .

B.C.: $w = \lambda$.

We set

$$\text{count}(\lambda, c) := 0.$$

I.S.: $w = dv$ with $d \in \Sigma$ and $v \in \Sigma^*$.

Then $\text{count}(dv, c)$ is defined by case distinction:

$$\text{count}(dv, c) := \begin{cases} \text{count}(v, c) + 1 & \text{if } c = d; \\ \text{count}(v, c) & \text{falls } c \neq d. \end{cases}$$

◇

We define $\Sigma = \{“A”, “B”\}$ and define the language L as the set of words $w \in \Sigma^*$ in which the letters “A” and “B” occur with the same frequency:

$$L := \{w \in \Sigma^* \mid \text{count}(w, “A”) = \text{count}(w, “B”) \}$$

Give a grammar G such that $L = L(G)$ and prove that this grammar indeed generates L . \diamond

Exercise 20: Define $\Sigma := \{“A”, “B”\}$. In the previous chapter, we have already defined the reversal of a string $w = c_1c_2 \cdots c_{n-1}c_n \in \Sigma^*$ as the string

$$w^R := c_nc_{n-1} \cdots c_2c_1.$$

A string $w \in \Sigma^*$ is called a **palindrome** if the string is identical to its reversal, i.e. if

$$w = w^R$$

holds true. For example, the strings

$$w_1 = ABABA \quad \text{and} \quad w_2 = ABBA$$

are both palindromes, while the string ABB is not a palindrome. The **language of palindromes** $L_{\text{palindrome}}$ is the set of all strings in Σ^* that are palindromes, i.e. we have

$$L_{\text{palindrome}} := \{w \in \Sigma^* \mid w = w^R\}.$$

(a) Prove that the language $L_{\text{palindrome}}$ is a context-free language.

(b) Prove that the language $L_{\text{palindrome}}$ is not regular. \diamond

6.1.2 Parse Trees

Using a grammar G , we can not only tell whether a given string s is an element of the language $L(G)$ generated by the grammar, we can also **structure** the string by building a **parse tree**. If a grammar is

$$G = \langle V, T, R, S \rangle$$

given, a **parse tree** for this grammar is a tree satisfying the following conditions:

1. The tree consists of two types of nodes:
 - (a) The **leaf nodes** are those nodes that have no outgoing edges.
 - (b) The **inner nodes** are all those nodes that have outgoing edges.
2. Each **inner node** is labeled with a variable.
3. Each **leaf node** is labeled with a terminal or with a variable.
4. If a leaf node is labeled with a variable a , then the grammar contains a rule of the form

$$a \rightarrow \lambda.$$
5. If an inner node is labeled with a variable a and the children of this node are labeled with the symbols X_1, X_2, \dots, X_n , then the grammar G contains a rule of the form

$$a \rightarrow X_1X_2 \cdots X_n.$$

If we read the leaf nodes of a parse tree from left to right, they yield a word that is derived from the grammar G . Figure 6.1 shows a parse tree for the word “2*3+4”. It is derived using the grammar given above for arithmetic expressions. The only caveat here is that instead of the symbol “.” the parse tree uses the symbol “*”.

Since trees of the type shown in Figure 6.1 become too large very quickly we simplify these trees using the following rules:

1. Is n an interior node labeled with the variable A and among the children of this node there is exactly one child labeled with a terminal o then we remove this child and label the node n instead with the terminal o .

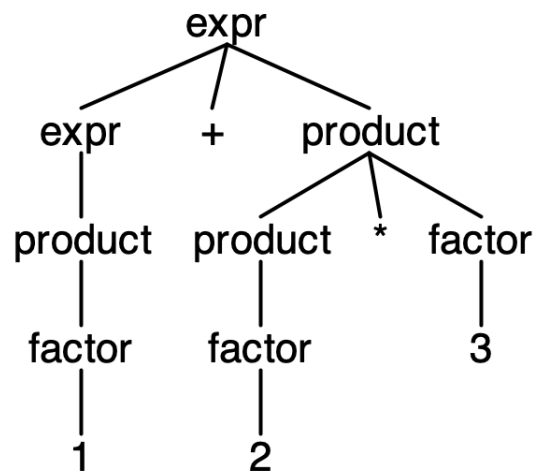


Figure 6.1: A parse tree for the string “2*3+4”.

2. If an inner node has only one child, we replace that node with its child.

We call the tree obtained in this way the [abstract syntax tree](#). Figure 6.2 shows the abstract syntax tree that results from the tree in Figure 6.1. The structure stored in this tree is exactly what we need to evaluate the arithmetic expression “2*3+4”, because the tree shows us the order for evaluating the operators.

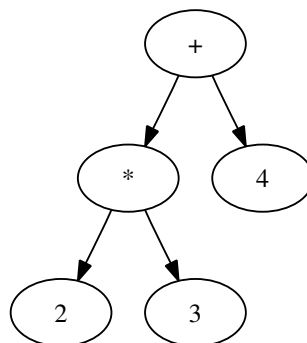


Figure 6.2: The abstract syntax tree for the string “2*3+4”.

6.1.3 Ambiguous Grammars

The grammar given at the beginning of section 6.1 to describe arithmetic expressions seems very complicated because it uses three different syntactic categories: *arithExpr*, *product*, and *factor*. We introduce a simpler grammar G which describes the same language:

$$G = \langle \{expr\}, \{\text{NUMBER, VARIABLE, “+”, “-”, “.”, “/”, “(”, “)”}\}, R, expr \rangle,$$

The rules R are given as follows:

$$\begin{aligned} \text{expr} &\rightarrow \text{expr "+" expr} \\ &| \text{expr "-" expr} \\ &| \text{expr "." expr} \\ &| \text{expr "/" expr} \\ &| "(" \text{expr} ")" \\ &| \text{NUMBER} \\ &| \text{VARIABLE} \end{aligned}$$

In order to show that the string "2·3+4" is in the language generated by this grammar, we give the following derivation:

$$\begin{aligned} \text{expr} &\Rightarrow \text{expr "+" expr} \\ &\Rightarrow \text{expr "." expr "+" expr} \\ &\Rightarrow 2 \text{ "." expr "+" expr} \\ &\Rightarrow 2 \text{ "." } 3 \text{ "+" expr} \\ &\Rightarrow 2 \text{ "." } 3 \text{ "+" } 4 \end{aligned}$$

This derivation corresponds to the abstract syntax tree shown in Fig. 6.2 is shown. However, there is another derivation of the string "2·3+4" with this grammar:

$$\begin{aligned} \text{expr} &\Rightarrow \text{expr "." expr} \\ &\Rightarrow \text{expr "." expr "+" expr} \\ &\Rightarrow 2 \text{ "." expr "+" expr} \\ &\Rightarrow 2 \text{ "." } 3 \text{ "+" expr} \\ &\Rightarrow 2 \text{ "." } 3 \text{ "+" } 4 \end{aligned}$$

This derivation corresponds to the abstract syntax tree shown in Fig. 6.3. In this derivation, the string "2·3+4" is apparently taken to be a product, which contradicts the convention that the operator "." binds stronger than the operator "+". If we were to evaluate the string using the last syntax tree, we would obviously get the wrong result! The

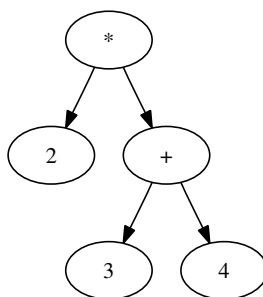


Figure 6.3: Another abstract syntax tree for the string "2·3+4". Here "." is denoted as "*".

reason for this problem is the fact that the last specified grammar is [ambiguous](#). An ambiguous grammar is unsuitable for parsing. Unfortunately, the question of whether a given grammar is ambiguous is, in general, not [decidable](#): It can be shown that this question is equivalent to the [Post correspondence problem](#). Since Post's correspondence problem has been shown to be undecidable, the question whether a grammar is ambiguous is also unsolvable. Proofs of these claims can be found, for example, in the book by Hopcroft, Motwani, and Ullman [HMU06].

6.2 Top-Down Parser

In this section, we present a method that can be used to conveniently parse a whole range of grammars. The basic idea is simple: In order to parse a string w using a grammar rule of the form

$$a \rightarrow X_1 X_2 \cdots X_n$$

we try to parse an X_1 first. In doing so, we decompose the string w into the form $w = w_1 r_1$ such that $w_1 \in L(X_1)$ holds. Then we try to find an X_2 in the residual string r_1 , thus decomposing r_1 as $r_1 = w_2 r_2$ where $w_2 \in L(X_2)$ holds. Continuing this process, we end up with the string w being split as

$$w = w_1 w_2 \cdots w_n \quad \text{with } w_i \in L(X_i) \text{ for all } i = 1, \dots, n.$$

Unfortunately, this procedure does not work when the grammar is [left-recursive](#), that is, a rule has the form

$$a \rightarrow a \beta$$

because then to parse an a we would immediately try again to parse an a and thus we would be stuck in an infinite loop. There are two ways to deal with this kind of problem:

- (a) We can rewrite the grammar so that it no longer is left-recursive.
- (b) A simpler method is to extend the notion of a context-free grammar. We will use the notion of a so called [extended Backus Naur form](#) grammar (abbreviated as EBNF-grammar). Theoretically, the expressive power of EBNF grammars is the same as the expressive power of context-free grammars. In practice, however, it turns out that the construction of top-down parsers for EBNF grammars is easier, because in an EBNF grammar the left recursion can be replaced by iteration.

In the rest of this chapter we will discuss these two procedures in more detail using the grammar for arithmetic expressions as an example.

6.2.1 Rewriting a Grammar to Eliminate Left Recursion

In the following, assume that a grammar $G = \langle V, T, R, S \rangle$ is given, $a \in V$ is a syntactic variable and the Greek letters β and γ stand for any strings consisting of syntactic variables and tokens, i.e. we have $\beta, \gamma \in (V \cup T)^*$. If a is defined by the two rules

$$\begin{array}{lcl} a & \rightarrow & a\beta \\ & | & \gamma \end{array}$$

then a derivation of a , where we always replace the syntactic variable a first, has the form

$$a \Rightarrow a\beta \Rightarrow a\beta\beta \Rightarrow a\beta\beta\beta \Rightarrow \cdots \Rightarrow a\beta^n \Rightarrow \gamma\beta^n.$$

Thus we see that the language $L(a)$ described by the syntactic variable a consists of all the strings that can be derived from the expression $\gamma\beta^n$:

$$L(a) = \{w \in \Sigma^* \mid \exists n \in \mathbb{N} : \gamma\beta^n \Rightarrow^* w\}.$$

This language can also be described by the following rules for a :

$$\begin{array}{lcl} a & \rightarrow & \gamma b \\ b & \rightarrow & \beta b \\ & | & \lambda \end{array}$$

Here we have introduced the auxiliary variable b . The derivations resulting from the syntactic variable b have the form

$$b \Rightarrow \beta b \Rightarrow \beta\beta b \Rightarrow \cdots \Rightarrow \beta^n b \Rightarrow \beta^n.$$

Hence the variable b describes the language

$$L(b) = \{w \in \Sigma \mid \exists n \in \mathbb{N} : \beta^n \Rightarrow^* w\}.$$

Thus it is clear that with the grammar rules given above we have

$$L(a) = \{w \in \Sigma^* \mid \exists n \in \mathbb{N} : \gamma\beta^n \Rightarrow^* w\}.$$

To remove the left recursion from the grammar shown in Figure 6.4 on page 69, we need to generalize the example given above. We now consider the general case and assume that a non-terminal a is defined by rules of the following

form:

$$\begin{array}{lcl}
 a & \rightarrow & a\beta_1 \\
 & | & a\beta_2 \\
 & \vdots & \vdots \\
 & | & a\beta_k \\
 & | & \gamma_1 \\
 & \vdots & \vdots \\
 & | & \gamma_l
 \end{array}$$

We can reduce this case to the first case by introducing two auxiliary variables b and c :

$$\begin{array}{lcl}
 a & \rightarrow & ab \mid c \\
 b & \rightarrow & \beta_1 \mid \cdots \mid \beta_k \\
 c & \rightarrow & \gamma_1 \mid \cdots \mid \gamma_l
 \end{array}$$

Then we can rewrite this grammar by introducing a new auxiliary variable, let's call it l for list, and get

$$\begin{array}{lcl}
 a & \rightarrow & cl \\
 l & \rightarrow & bl \mid \lambda.
 \end{array}$$

At this point, the auxiliary variables b and c can now be eliminated. This yields the following grammar rules:

$$\begin{array}{lcl}
 a & \rightarrow & \gamma_1 l \mid \gamma_2 l \mid \cdots \mid \gamma_l l \\
 l & \rightarrow & \beta_1 l \mid \beta_2 l \mid \cdots \mid \beta_k l \mid \lambda
 \end{array}$$

$ \begin{array}{lcl} \text{expr} & \rightarrow & \text{expr "+" product} \\ & & \text{expr "-" product} \\ & & \text{product} \\ \text{product} & \rightarrow & \text{product "." factor} \\ & & \text{product "/" factor} \\ & & \text{factor} \\ \text{factor} & \rightarrow & "(" \text{ expr } ")" \\ & & \text{NUMBER} \end{array} $
--

Figure 6.4: Left-recursive grammar for arithmetic expressions.

If we apply this procedure to the grammar for arithmetic expressions shown in Figure 6.4, we obtain the grammar shown in Figure 6.5. The variables *exprRest* and *productRest* can be interpreted as follows:

1. *exprRest* describes a list of the form.

$$op \text{ product } \cdots op \text{ product},$$

where $op \in \{ "+", "-" \}$.

2. *productRest* describes a list of the form

$$op \text{ factor } \cdots op \text{ factor},$$

where $op \in \{ ".", "/" \}$ holds.

<i>expr</i>	→	<i>product exprRest</i>
<i>exprRest</i>	→	"+" <i>product exprRest</i>
		"-" <i>product exprRest</i>
		λ
<i>product</i>	→	<i>factor productRest</i>
<i>productRest</i>	→	"." <i>factor productRest</i>
		"/" <i>factor productRest</i>
		λ
<i>factor</i>	→	"(" <i>expr</i> ")"
		NUMBER

Figure 6.5: Grammar for arithmetic expressions without left-recursion.

Exercise 21:

(a) The following grammar describes regular expressions:

<i>regExp</i>	→	<i>regExp</i> "+" <i>regExp</i>
		<i>regExp regExp</i>
		<i>regExp</i> "*" <i>regExp</i>
		"(" <i>regExp</i> ")"
		LETTER

This grammar uses only the syntactic variable $\{regExp\}$ and the following Terminals

$\{ "+", "*", "(", ")", \text{LETTER} \}$.

Since the grammar is ambiguous, this grammar is unsuitable for parsing. Transform this grammar into an unambiguous grammar where the postfix operator "*" binds more strongly than the concatenation of two regular expressions, while the "+" operator binds weaker than concatenation. Use the grammar for arithmetic expressions as a guide and introduce suitable new syntactic variables.

(b) Remove the left recursion from the grammar created in part (a) of this task. ◇

6.2.2 Implementing a Top Down Parser in Python

Now we are ready to implement a parser for recognizing arithmetic expressions. We will use the grammar that is shown in Figure 6.5 on page 70. Before we can implement the parser, we need a scanner. We will use a hand-coded scanner that is shown in Figure 6.6 on page 71. The function `tokenize` implemented in this scanner receives a string `s` as argument and returns a list of tokens. The string `s` is supposed to represent an arithmetical expression. In order to understand the implementation, you need to know the following:

(a) We need to set the flag `re.VERBOSE` in our call of the function `findall` below because otherwise we are not able to format the regular expression `lexSpec` the way we have done it.

```

1  import re
2
3  def tokenize(s):
4      lexSpec = r'''([\t]+)      | # blanks and tabs
5                  ([1-9][0-9]*|0) | # number
6                  ([()])        | # parentheses
7                  ([-+*/])      | # arithmetical operators
8                  (.)           | # unrecognized character
9                  '''
10     tokenList = re.findall(lexSpec, s, re.VERBOSE)
11     result = []
12     for ws, number, parenthesis, operator, error in tokenList:
13         if ws:          # skip blanks and tabs
14             pass
15         if number:
16             result += [ number ]
17         if parenthesis:
18             result += [ parenthesis ]
19         if operator:
20             result += [ operator ]
21         if error:
22             result += [ f'ERROR({error})' ]
23     return result

```

Figure 6.6: A scanner for arithmetic expressions.

- (b) The regular expression `lexSpec` contains 5 parenthesized groups. Therefore, `findall` returns a list of 5-tuples where the 5 components correspond to the 5 groups of the regular expression. As the 5 groups are non-overlapping, exactly one of the 5 components will be a non-empty string.

Figure 6.7 on page 72 shows an implementation of a recursive descent parser in PYTHON.

- (a) The main function is the function `parse`. This function takes a string `s` representing an arithmetic expression. This string is tokenized using the function `tokenize`. The function `tokenize` turns a string into a list of tokens. For example, the expression

```
tokenize('(1 + 2) * 3')
```

returns the result

```
['(', 1, '+', 2, ')', '*', 3].
```

This list of tokens is then parsed by the function `parseExpr`. That function returns a pair:

- (a) The first component is the value of the arithmetic expression.
 (b) The second component is the list of those tokens that have not been consumed when parsing the expression. Of course, on a successful parse this list should be empty.
- (b) The function `parseExpr` implements the grammar rule

$$\text{expr} \rightarrow \text{product exprRest}.$$

It takes a token list `TL` as input. It will return a pair of the form

```
(v, Rest),
```



```

1  def parse(s):
2      TL          = tokenize(s)
3      result, Rest = parseExpr(TL)
4      assert Rest == [], f'Parse Error: could not parse {TL}'
5      return result
6
7  def parseExpr(TL):
8      product, Rest = parseProduct(TL)
9      return parseExprRest(product, Rest)
10
11 def parseExprRest(Sum, TL):
12     if TL == []:
13         return Sum, []
14     elif TL[0] == '+':
15         product, Rest = parseProduct(TL[1:])
16         return parseExprRest(Sum + product, Rest)
17     elif TL[0] == '-':
18         product, Rest = parseProduct(TL[1:])
19         return parseExprRest(Sum - product, Rest)
20     else:
21         return Sum, TL
22
23 def parseProduct(TL):
24     factor, Rest = parseFactor(TL)
25     return parseProductRest(factor, Rest)
26
27 def parseProductRest(product, TL):
28     if TL == []:
29         return product, []
30     elif TL[0] == '*':
31         factor, Rest = parseFactor(TL[1:])
32         return parseProductRest(product * factor, Rest)
33     elif TL[0] == '/':
34         factor, Rest = parseFactor(TL[1:])
35         return parseProductRest(product / factor, Rest)
36     else:
37         return product, TL
38
39 def parseFactor(TL):
40     if TL[0] == '(':
41         expr, Rest = parseExpr(TL[1:])
42         assert Rest[0] == ')', 'Parse Error: expected ")"'
43         return expr, Rest[1:]
44     else:
45         return int(TL[0]), TL[1:]

```

Figure 6.7: A top down parser for arithmetic expressions.

where v is the value of the arithmetic expression that has been parsed, while `Rest` is the list of the remaining tokens. For example, the expression

```
parseExpr(['(', 1, '+', 2, ')', '*', 3, ')', '*', 2])
```

returns the result

```
[9, [')', '*', 2]].
```

Here, the part ['(', 1, '+', 2, ')', '*', 3] has been parsed and evaluated as the number 9 and [')', '*', 2] is the list of tokens that have not yet been processed.

In order to parse an arithmetic expression, the function first parses a *product* and then it tries to parse the remaining tokens as an *exprRest*. The function `parseExprRest` that is used to parse an *exprRest* needs two arguments:

- (a) The first argument is the value of the product that has been parsed by the function `parseProduct`.
- (b) The second argument is the list of tokens that can be used.

To understand the mechanics of `parseExpr`, consider the evaluation of

```
[1, '*', 2, '+', 3].
```

Here, the function `parseProduct` will return the result

```
(2, ['+', 3]),
```

where 2 is the result of parsing and evaluating the token list [1, '*', 2], while ['+', 3] is the part of the input token list that is not used by `parseProduct`. Next, the list ['+', 3] needs to be parsed as the rest of an expression and then 3 needs to be added to 2.

- (c) The function `parseExprRest` takes a number and a list of tokens. It implements the following grammar rules:

$$\begin{array}{lcl} \text{exprRest} & \rightarrow & "+" \text{ product exprRest} \\ & | & "-" \text{ product exprRest} \\ & | & \lambda \end{array}$$

Therefore, it checks whether the first token is either "+" or "-". If the token is "+", it parses a *product*, adds the result of this product to the sum of values parsed already and proceeds to parse the rest of the tokens.

The case that the first token is "-" is similar to the previous case. If the next token is neither "+" nor "-", then it could be either the token ")" or else it might be the case that the list of tokens is already exhausted. In either case, the rule

$$\text{exprRest} \rightarrow \lambda$$

is used. Therefore, in that case we have not consumed any tokens and hence the input argument is already the result.

- (d) The function `parseProduct` implements the rule

$$\text{product} \rightarrow \text{factor exprRest}.$$

The implementation is similar to the implementation of `parseExpr`.

- (e) The function `parseProductRest` implements the rules

$$\begin{array}{lcl} \text{productRest} & \rightarrow & "." \text{ factor productRest} \\ & | & "/" \text{ factor productRest} \\ & | & \lambda \end{array}$$

The implementation is similar to the implementation of `parseExprRest`.

(f) The function `parseFactor` implements the rules

$$\begin{array}{lcl} \text{factor} & \rightarrow & "(" \text{ expr } ")" \\ & | & \text{NUMBER} \end{array}$$

Therefore, we first check whether the next token is "(" because in that case, we have to use the first grammar rule, otherwise we use the second.

The parser shown in Figure 6.7 does not contain any error handling. Appropriate error handling will be discussed once we have covered the theory of top-down parsing.

Exercise 22: In Exercise 21 on page 70 you have developed a grammar for regular expressions that does not contain left recursion. Implement a top down parser for this grammar. The resulting grammar should return a nested tuple that represents a regular expression. ◇

6.2.3 Implementing a Recursive Descent Parser that Uses an EBNF Grammar

The previous solution to parse an arithmetical expression was not completely satisfying: The reason is that we did not really fix the problem of left recursion but rather cured the symptoms. The underlying reason for left recursion is that context free grammars are not that convenient to describe the structure of programming languages since a description of this structure needs both recursion and iteration, but context-free grammars provide no direct way to describe iteration. Rather, they simulate iteration via recursion. Let us therefore improve the convenience of context-free languages by admitting the regular expression operators "*", "|", and "?" on the right hand side of grammar rules. The meaning of these operators is the same as when these operators are used in the regular expressions of the programming language *Python*. Furthermore, the right hand side of a grammar rule can be structured using parentheses. These new type of grammars are known as *extended Backus Naur form* grammars, which is abbreviated as EBNF grammars.

$$\begin{array}{lcl} \text{expr} & \rightarrow & \text{product } (('+' | '-') \text{ product})^* \\ \text{product} & \rightarrow & \text{factor } (('*' | '/') \text{ factor})^* \\ \text{factor} & \rightarrow & '(' \text{ expr } ') ' \\ & | & \text{NUMBER} \end{array}$$

Figure 6.8: EBNF grammar for arithmetical expressions.

It can be shown that the languages described by EBNF grammars are still context-free languages. Therefore, these operators do not change the expressive power of context-free grammars. However, it is often much more convenient to describe a language using an EBNF grammar rather than using a context-free grammar. Figure 6.8 displays an EBNF grammar for arithmetical expressions.

Obviously, the grammar in Figure 6.8 is more concise than the context-free grammar shown in Figure 6.5 on page 70. For example, the first rule clearly expresses that an arithmetical expression is a list of products that are separated by the operators "+" and "-".

Figure 6.9 shows a recursive descent parser that implements this grammar.

1. The function `parseExpr` recognizes a product in line 2. The value of this product is stored in the variable `result` together with the list `Rest` of those tokens that have not been consumed yet. If the list `Rest` is not empty and the first token in this list is either the operator "+" or the operator "-", then the function `parseExpr` tries to recognize more products. These are added to or subtracted from the `result` computed so far in line 7 or 9. If there are no more products to be parsed, the `while` loop terminates and the function returns the `result` together with the list of the remaining tokens `Rest`.

2. The function `parseProduct` recognizes a factor in line 13. The value of this factor is stored in the variable `result` together with the list `Rest` of those tokens that have not been consumed yet. If the list `Rest` is not empty and the first token in this list is either the operator `"*"` or the operator `"/"`, then the function `parseProduct` tries to recognize more factors. The result computed so far is multiplied with or divided by these factors in line 18 or 20. If there are no more products to be parsed, the `while` loop terminates and the function returns the `result` together with the list `Rest` of tokens that have not been consumed.
3. The function `parseFactor` recognizes a factor. This is either an expression in parentheses or a number.
 - If the first token is a an opening parenthesis, the function tries to parse an expression next. This expression has to be followed by a closing parenthesis. The tokens following this closing parenthesis are not consumed but rather are returned together with the result of evaluating the expression.
 - If the first token is a number, this number is returned together with the list of all those tokens that have not been consumed.

```

1  def parseExpr(TL):
2      result, Rest = parseProduct(TL)
3      while len(Rest) > 1 and Rest[0] in {'+', '-'}:
4          operator = Rest[0]
5          arg, Rest = parseProduct(Rest[1:])
6          if operator == '+':
7              result += arg
8          else:
9              # operator == '-'
10             result -= arg
11     return result, Rest
12
13 def parseProduct(TL):
14     result, Rest = parseFactor(TL)
15     while len(Rest) > 1 and Rest[0] in {'*', '/'}:
16         operator = Rest[0]
17         arg, Rest = parseFactor(Rest[1:])
18         if operator == '*':
19             result *= arg
20         else:
21             # operator == '/'
22             result /= arg
23     return result, Rest
24
25 def parseFactor(TL):
26     if TL[0] == '(':
27         expr, Rest = parseExpr(TL[1:])
28         assert Rest[0] == ')', "ERROR: ')' expected, got {Rest[0]}"
29         return expr, Rest[1:]
30     else:
31         assert isinstance(TL[0], int), "ERROR: Number expected, got {TL[0]}"
32         return TL[0], TL[1:]

```

Figure 6.9: A recursive descent parser for the grammar in Figure 6.8.

Exercise 23: In Exercise 21 on page 70 you have developed an EBNF grammar for regular expressions. Implement a top down parser for this grammar. The resulting grammar should return a nested tuple that represents a regular expression. ◇

Historical Notes The language ALGOL [Bac59, NBB⁺60] was the first programming language with a syntax that was based on an EBNF grammar.

6.3 Check your Understanding

- (a) Define the concept of a [context-free grammar](#).
- (b) Assume that G is a context-free grammar. How is the language $L(G)$ defined?
- (c) What is the definition of a [parse tree](#)?
- (d) How do we transform a parse tree into an [abstract syntax tree](#)?
- (e) What are ambiguous grammars?
- (f) How can a context free grammar that contains left recursion be transformed into an equivalent grammar that does not contain left recursion.
- (g) How does a top-down parser work?
- (h) Why do we have to eliminate left recursion from a grammar in order to build a top-down parser?
- (i) Define the notion of an EBNF grammar.

6.4 Implementing a Simple Interpreter

ANTLR supports EBNF grammars. For example, the postfix operators “*”, “+” and “?” known from regular expressions are supported. We demonstrate the use of these operators by developing an interpreter for a simple programming language using the parser generator ANTLR. Figure 6.10 shows the ANTLR grammar of the programming language for which we are developing an interpreter in this section. The syntax of this language is modeled after the syntax of the language C, but instead of the simple equality sign “=”, we use the string “:=” as the assignment operator.

```

1  grammar Pure;
2
3  program : statement+
4          ;
5  statement: VAR ':=' expr ';'
6          | 'print' '(' expr ')' ';'
7          | 'if' '(' boolExpr ')' '{' statement* '}'
8          | 'while' '(' boolExpr ')' '{' statement* '}'
9          ;
10 boolExpr : expr '==' expr
11          | expr '<' expr
12          ;
13 expr      : expr '+' product
14          | expr '-' product
15          | product
16          ;
17 product   : product '*' factor
18          | product '/' factor
19          | factor
20          ;
21 factor    : 'read' '(' ')'
22          | '(' expr ')'
23          | VAR
24          | NUMBER
25          ;
26 VAR       : [a-zA-Z][a-zA-Z_0-9]*;
27 NUMBER    : '0'|[1-9][0-9]*;
28
29 MULTI_COMMENT : '/*' .*? '*/' -> skip;
30 LINE_COMMENT  : '//' ~( '\n' )* -> skip;
31 WS           : [ \t\n\r] -> skip;

```

Figure 6.10: ANTLR-Grammar for a simple programming language.

The commands of the language to be implemented include assignments, print commands, if statements, and while loops. Figure 6.11 shows an example program that conforms to this grammar. This program first reads a number which is stored in the variable *n*. Subsequently, the sum

$$\sum_{i=1}^{n^2} i$$

is computed and printed using a while loop.

```

1  n := read();
2  s := 0;
3  i := 0;
4  while (i < n * n) {
5      i := i + 1;
6      s := s + i;
7  }
8  print(s);

```

Figure 6.11: A program to compute the sum $\sum_{i=0}^{n^2} i$.

Similar to our program for symbolic differentiation, we will represent the individual commands as nested tuples. The program shown in Figure 6.11 is represented by the nested tuple displayed in Figure 6.12. This nested tuple is nothing other than the abstract syntax tree corresponding to the original program.

```

1  ('program',
2    ('read', 'n'),
3    (':=', 's', 0),
4    (':=', 'i', 0),
5    ('while', ('<', 'i', ('*', 'n', 'n')),
6      (':=', 'i', ('+', 'i', 1)),
7      (':=', 's', ('+', 's', 'i'))
8    ),
9    ('print', 's')
10 )

```

Figure 6.12: A nested tuple that represents the program from Figure 6.11.

Figure 6.13 shows the implementation of the parser using the ANTLR tool.

1. The start symbol of the grammar is the variable `program`. When parsing this variable, the parser returns a tuple of commands in the variable `stmt_list`. To this end, we initialize the variable `SL` as a list containing only the string `'program'`. Subsequently, each recognized command is appended to this list. Finally, this list is converted into a tuple and stored in the attribute variable `stmt_list`.
2. The syntactic variable `statement` describes the various commands that are supported in our simple language.
 - (a) The simplest commands are the assignments. These have the form:

$$v := e;$$

Here, v is a variable and e is an arithmetic expression. Such an assignment is represented by the nested tuple

$$(':=', v, e).$$

- (b) The command for printing a value has the form:

$$\text{print}(e);$$

The purpose of this command is to compute the value of the expression e and print it. This command is represented by the nested tuple

$$('print', v).$$

```

1  grammar Simple;
2
3  program returns [stmtnt_list]
4      : {SL = ['program']} (s = statement {SL.append($s.stmnt)})+ {$stmtnt_list = tuple(SL)}
5      ;
6
7  statement returns [stmtnt]
8      : v = VAR ':' e = expr ';' {$stmtnt = (':=', $v.text, $e.result)}
9      | 'print' '(' r = expr ')' ';' {$stmtnt = ('print', $r.result)}
10     | 'if' '(' b = boolExpr ')' {SL = []} '{' (l = statement {SL.append($l.stmnt)}) * '}'
11     {$stmtnt = ('if', $b.result) + tuple(SL)}
12     | 'while' '(' b = boolExpr ')' {SL = []} '{' (l = statement {SL.append($l.stmnt)}) * '}'
13     {$stmtnt = ('while', $b.result) + tuple(SL)}
14     ;
15
16  boolExpr returns [result]
17      : l = expr '==' r = expr {$result = ('==', $l.result, $r.result)}
18      | l = expr '<' r = expr {$result = ('<', $l.result, $r.result)}
19      ;
20
21  expr returns [result]
22      : e = expr '+' p = product {$result = ('+', $e.result, $p.result)}
23      | e = expr '-' p = product {$result = ('-', $e.result, $p.result)}
24      | p = product {$result = $p.result}
25      ;
26
27  product returns [result]
28      : p = product '*' f = factor {$result = ('*', $p.result, $f.result)}
29      | p = product '/' f = factor {$result = ('/', $p.result, $f.result)}
30      | f = factor {$result = $f.result}
31      ;
32
33  factor returns [result]
34      : 'read' '(' ')' {$result = ('read',)}
35      | '(' expr ')' {$result = $expr.result}
36      | v = VAR {$result = $v.text}
37      | n = NUMBER {$result = int($n.text)}
38      ;
39
40  VAR      : [a-zA-Z][a-zA-Z_0-9]*;
41  NUMBER   : '0'|[1-9][0-9]*;
42
43  MULTI_COMMENT : '/*' .*? '*/' -> skip;
44  LINE_COMMENT  : '//' ~( '\n' ) * -> skip;
45  WS            : [ \t\n\r] -> skip;

```

Figure 6.13: ANTLR-Spezifikation der Grammatik.

(c) A conditional test has the syntax:

$$\text{if } (b) \{ \text{statements} \}$$

Here, b is an expression that evaluates to True or False and statements is a list of commands that are

executed if b evaluates to True. This command is represented by the nested tuple

`('if', b , $statements$).`

Here, b is a nested tuple representing the Boolean expression.

(d) A loop has the syntax:

`while (b) { $statements$ }.`

Here, b is an expression that evaluates to True or False and $statements$ is a list of commands that are executed as long as b evaluates to True. This command is represented by the nested tuple

`('while', b , $statements$).`

3. The syntactic variable `boolExpr` describes an expression that takes a Boolean value. There are two ways to generate such a value.

(a) An expression of the form

`$l == r$`

tests whether the evaluations of l and r yield the same values. This expression is represented by the nested tuple

`('==', l , r).`

(b) An expression of the form

`$l < r$`

tests whether the evaluation of l yields a value that is smaller than the value obtained from the evaluation of r . This expression is represented by the nested tuple

`('<', l , r)`

4. In a similar manner, the syntactic variables `expr`, `product`, and `factor` describe arithmetic expressions. Since we have already discussed this in sufficient detail earlier, we will not delve further into the grammar rules that define these syntactic variables at this point.

The specification of the tokens is shown in lines 40–45 of Figure 6.13. The scanner primarily differentiates between variables and numbers. Variables start with an uppercase or lowercase letter, followed by additional digits and the underscore. Sequences of digits are interpreted as numbers. If such a sequence contains more than one character, the first digit must not be 0. Additionally, the scanner removes whitespace and comments. We have also used the so-called *non-greedy* version of the operator “*” in the specification of multiline comments. The non-greedy version of the operator “*” is written as “*?” and matches as little as possible. Therefore, the regular expression

`'/*' .*? '*/'`

represents a string that starts with the character sequence “/*”, ends with the character sequence “*/”, and is as short as possible. This ensures that in a line like

`/* Hugo */ i := i + 1; /* Anton */`

two separate comments are recognized.

Figure 6.14 shows the function `main`, which receives the name of a file containing a program in our simple programming language as input. This program is parsed and thus transformed into the nested tuple `Statements`. The function `execute_tuple` executes the individual commands in the tuple `Statements`. It uses the function `execute` to execute a single command. Figure 6.15 shows the implementation of the function `execute`. This implementation consists mainly of a large case distinction based on the type of command to be executed.

1. First, we check if `statement` is the string `'program'`, which marks the beginning of the program.¹ Since this is only a marker and not a real command, nothing further needs to be done.

¹This marking is only needed for the representation of the program as an abstract syntax tree.

```

1  def main(file):
2      with open(file, 'r') as handle:
3          program_text = handle.read()
4          input_stream = antlr4.InputStream(program_text)
5          lexer        = SimpleLexer(input_stream)
6          token_stream = antlr4.CommonTokenStream(lexer)
7          parser        = SimpleParser(token_stream)
8          result        = parser.program()
9          Statements    = result.stmnt_list
10         execute_tuple(Statements)

```

Figure 6.14: Die Funktion main.

```

1  def execute_tuple(Statement_List, Values={}):
2      for stmnt in Statement_List:
3          execute(stmnt, Values)
4
5  def execute(stmnt: NestedTuple, Values: dict[str, Number]) -> None:
6      match stmnt:
7          case 'program':
8              pass
9          case (':=' , var, value):
10             Values[var] = evaluate(value, Values)
11          case ('print', expr):
12             print(evaluate(expr, Values))
13          case ('if', test, *SL):
14             if evaluate(test, Values):
15                 execute_tuple(tuple(SL), Values)
16          case ('while', test, *SL):
17             while evaluate(test, Values):
18                 execute_tuple(tuple(SL), Values)
19          case _:
20             assert False, f'{stmnt} unexpected'

```

Figure 6.15: The function execute execute.

2. If the command is an assignment of the form

(':', '=', var, value)

then the value of the expression value is computed using the function evaluate. This value is then stored in the dictionary Values under the key var.

3. If the command is an operation of the form

('print', expr)

then the expression expr is first evaluated using the function evaluate. The resulting value is then printed.

4. If the command is of the form

('if', test, s_1 , \dots , s_n)

then test is a nested tuple representing a Boolean expression and (s_1, \dots, s_n) is a tuple of commands

stored in the variable `SL`. In this case, the expression `test` is first evaluated using the function `evaluate`. If this evaluation yields the value `True`, then the commands in the tuple `SL` are executed in sequence.

5. If the command is of the form

`('while', test, s_1, \dots, s_n)`

then `test` is a Boolean expression and (s_1, \dots, s_n) is a tuple of commands stored in the variable `SL`. In this case, the expression `test` is first evaluated using the function `evaluate`. If this evaluation yields the value `True`, then the commands in the tuple `SL` are executed in sequence. Then the expression `test` is evaluated again. If the result is `False`, then the execution of the command is finished. Otherwise, the commands in the list `SL` are executed as long as the evaluation of `test` yields `False`.

```

1 def evaluate(expr: NestedTuple, Values: dict[str, Number]) -> Number:
2     match expr:
3         case int():
4             return expr
5         case str():
6             return Values[expr]
7         case ('read',):
8             return int(input('Please enter a natural number: '))
9         case ('==', lhs, rhs):
10            return evaluate(lhs, Values) == evaluate(rhs, Values)
11        case ('<', lhs, rhs):
12            return evaluate(lhs, Values) < evaluate(rhs, Values)
13        case ('+', lhs, rhs):
14            return evaluate(lhs, Values) + evaluate(rhs, Values)
15        case ('-', lhs, rhs):
16            return evaluate(lhs, Values) - evaluate(rhs, Values)
17        case ('*', lhs, rhs):
18            return evaluate(lhs, Values) * evaluate(rhs, Values)
19        case ('/', lhs, rhs):
20            return evaluate(lhs, Values) / evaluate(rhs, Values)
21        case _:
22            assert False, f'{expr} unexpected'

```

Figure 6.16: Die Funktion `evaluate`.

Figure 6.16 shows the implementation of the function `evaluate`. This function receives an arithmetic expression and a dictionary storing the values of the variables as input.

- (a) If the expression to be evaluated is a number, we return this number as the result.
- (b) If the expression to be evaluated is a variable, we look up the value of this variable in the dictionary `Values` and return this value as the result.
- (c) If the expression to be evaluated is a call to the function `read`, we prompt the user to enter a natural number. We then convert the string entered by the user into a number.
- (d) If the expression to be evaluated is a Boolean expression of the form

`('==', lhs, rhs)`

we recursively evaluate the expressions `lhs` and `rhs` and return `True` if and only if both expressions yield the same value.

- (e) If the expression to be evaluated is a Boolean expression of the form

$('<', lhs, rhs)$

we recursively evaluate the expressions `lhs` and `rhs` and return `True` if and only if the value obtained from the evaluation of `lhs` is less than the value obtained from the evaluation of `rhs`.

- (f) If the expression to be evaluated is a sum of the form

$('+', lhs, rhs)$

we recursively evaluate the expressions `lhs` and `rhs` and return the sum of these values.

- (g) The evaluation of the arithmetic operators `'-'`, `'*'`, and `'/'` is analogous to the evaluation of the operator `'+'`.

Exercise 24:

- (a) Extend the interpreter to support the operator `"<="`.
- (b) Add `for` loops to the interpreter.
- (c) Expand the interpreter to include logical operators `"&&"` for logical *AND*, `"||"` for logical *OR*, and `"!"` for negation. The operator `"!"` should bind the strongest and the operator `"||"` the weakest.
- (d) Extend the syntax of arithmetic expressions to include predefined mathematical functions like `exp()` or `ln()`.
- (e) Enhance the interpreter to allow for user-defined functions.

- (a) A function should only have access to its parameters and those variables that are defined locally inside the function.
- (b) A function should always return a value using a `return` statement. In order to facilitate that a function can contain multiple `return` statements and that a `return` statement can occur anywhere in the function, use an [exception](#) to communicate the return value and to transfer the control flow out of the function.

- (f) Support the data type of [sets](#) of integers. Sets can be created in the following ways:

1. The elements of a set can be enumerated as in the following expression:

$\{ 1, 2, 3, 4 \}.$

2. A set can be specified as a range as shown below:

$\{ 1, \dots, 10 \}.$

3. A set can be created as subset of a given set as shown in the following example:

$\{ x \text{ in } S \mid \text{mod}(x, 2) == 0 \}.$

4. A set can be created as the image of several sets, for example:

$\{ f(x,y) : x \text{ in } S, y \text{ in } T \}$

5. Sets can be combined using binary operators:

- $S + T$ is the union of S and T .
- $S * T$ is the intersection of S and T .
- $S - T$ is the set difference of S and T .

◇

6.5 Check your Understanding

- (a) What is the difference between an EBNF-grammar and a context-free grammar?
- (b) Are you able to write an ANTLR parser for a simple programming language?
- (c) Are you able to implement an ANTLR parser that constructs an abstract syntax tree?

Chapter 7

Earley-Parser

In diesem Kapitel stellen wir ein effizientes Verfahren vor, mit dem es möglich ist, für eine beliebige vorgegebene kontextfreie Grammatik

$$G = \langle V, \Sigma, R, S \rangle \quad \text{und einen vorgegebenen String } s \in \Sigma^*$$

zu entscheiden, ob s ein Element der Sprache $L(G)$ ist, ob also $s \in L(G)$ gilt. Der Algorithmus, den wir gleich diskutieren werden, wurde 1970 von Jay Earley publiziert [Ear70]. Neben dem Algorithmus von Earley gibt es noch den [Cocke-Younger-Kasami-Algorithmus](#), in der Literatur auch als [CYK-Algorithmus](#) bekannt, der unabhängig von John Cocke [CS70], Daniel H. Younger [You67] und Tadao Kasami [Kas65] entdeckt wurde. Der CYK-Algorithmus ist nur anwendbar, wenn die Grammatik in [Chomsky-Normalform](#) vorliegt. Da es sehr aufwendig ist, eine Grammatik in Chomsky-Normalform zu transformieren, wird der CYK-Algorithmus in der Praxis nicht eingesetzt. Demgegenüber kann der von Earley angegebene Algorithmus auf beliebige kontextfreie Grammatiken angewendet werden. Im allgemeinen Fall hat dieser Algorithmus die Komplexität $\mathcal{O}(n^3)$, aber falls die vorgegebene Grammatik eindeutig ist, dann ist die Komplexität lediglich $\mathcal{O}(n^2)$. Geschickte Implementierungen von Earley's Algorithmus erreichen für viele praktisch relevante Grammatiken sogar eine lineare Laufzeit. Im Gegensatz hat der CYK-Algorithmus **immer** die Komplexität $\mathcal{O}(n^3)$. Earley's Algorithmus hat sowohl für $LL(k)$ -Grammatiken als auch für $LR(1)$ -Grammatiken, die wir in einem späteren Kapitel analysieren werden, nur eine lineare Laufzeit.

Dieses Kapitel gliedert sich in die folgenden Abschnitte:

- (a) Zunächst skizzieren wir die Theorie, die Earley's Algorithmus zu Grunde liegt.
- (b) Danach geben wir eine einfache Implementierung des Algorithmus in *Python* an.

7.1 Der Algorithmus von Earley

Der zentrale Begriff des von Earley angegebenen Algorithmus ist der Begriff des [Earley-Objekts](#), das wie folgt definiert ist:

Definition 21 (Earley-Objekt) Gegeben sei eine kontextfreie Grammatik $G = \langle V, \Sigma, R, S \rangle$ und ein String $s = x_1 x_2 \cdots x_n \in \Sigma^*$ der Länge n . Wir bezeichnen ein Paar der Form

$$\langle A \rightarrow \alpha \bullet \beta, k \rangle$$

genau dann als ein [Earley-Objekt](#), falls folgendes gilt:

- (a) $(A \rightarrow \alpha \beta) \in R$ und
- (b) $k \in \{0, 1, \dots, n\}$. □

Erklärung: Ein Earley-Objekt beschreibt einen Zustand, in dem ein Parser sich befinden kann. Ein Earley-Parser, der einen String $x_1 \cdots x_n$ parsen soll, verwaltet $n + 1$ Mengen von Earley-Objekten. Diese Mengen bezeichnen wir mit

$$Q_0, Q_1, \dots, Q_n.$$

Die Interpretation von

$$\langle A \rightarrow \alpha \bullet \beta, k \rangle \in Q_j \quad \text{mit } j \geq k$$

ist dann wie folgt:

1. Der Parser versucht, die Regel $A \rightarrow \alpha\beta$ auf den Teilstring $x_{k+1} \cdots x_n$ anzuwenden und am Anfang dieses Teilstrings ein A mit Hilfe der Regel $A \rightarrow \alpha\beta$ zu erkennen.
2. In dem Teilstrings $x_{k+1} \cdots x_j$ hat der Parser bereits α erkannt, es gilt also

$$\alpha \Rightarrow^* x_{k+1} \cdots x_j.$$
3. Folglich versucht der Parser, am Anfang des Teilstrings $x_{j+1} \cdots x_n$ ein β zu erkennen.

Der Algorithmus von Earley verwaltet für $j = 0, 1, \dots, n$ Mengen Q_j von Earley-Objekten, die den Zustand beschreiben, in dem der Parser ist, wenn der Teilstring $x_1 \cdots x_j$ verarbeitet ist. Zu Beginn des Algorithmus wird der Grammatik ein neues Start-Symbol \hat{S} sowie die Regel $\hat{S} \rightarrow S$ hinzugefügt. Die Menge Q_0 wird definiert als

$$Q_0 := \{ \langle \hat{S} \rightarrow \bullet S, 0 \rangle \},$$

denn der Parser soll ja das Start-Symbol S am Anfang des Strings $x_1 \cdots x_n$ erkennen. Die restlichen Mengen Q_j sind für $j = 1, \dots, n$ zunächst leer. Die Mengen Q_j werden nun durch die folgenden drei Operationen so lange wie möglich erweitert:

1. Lese-Operation

Falls der Zustand Q_j ein Earley-Objekt der Form $\langle A \rightarrow \beta \bullet a\gamma, k \rangle$ enthält, wobei a ein Terminal ist, so versucht der Parser, die rechte Seite der Regel $A \rightarrow \beta a\gamma$ zu erkennen und hat bis zur Position j bereits den Teil β erkannt. Folgt auf dieses β nun, wie in der Regel $A \rightarrow \beta a\gamma$ vorgesehen, an der Position $j+1$ das Terminal a , so muss der Parser nach der Position $j+1$ nur noch γ erkennen. Daher wird in diesem Fall das Earley-Objekt

$$\langle A \rightarrow \beta a \bullet \gamma, k \rangle$$

dem Zustand Q_{j+1} hinzugefügt:

$$\langle A \rightarrow \beta \bullet a\gamma, k \rangle \in Q_j \wedge x_{j+1} = a \Rightarrow Q_{j+1} := Q_{j+1} \cup \{ \langle A \rightarrow \beta a \bullet \gamma, k \rangle \}.$$

2. Vorhersage-Operation

Falls der Zustand Q_j ein Earley-Objekt der Form $\langle A \rightarrow \beta \bullet C\delta, k \rangle$ enthält, wobei C eine syntaktische Variable ist, so versucht der Parser im Zustand Q_j den Teilstring $C\delta$ zu erkennen. Dazu muss der Parser an diesem Punkt ein C erkennen. Wir fügen daher für jede Regel $C \rightarrow \gamma$ der Grammatik das Earley-Objekt $\langle C \rightarrow \bullet \gamma, j \rangle$ zu der Menge Q_j hinzu:

$$\langle A \rightarrow \beta \bullet C\delta, k \rangle \in Q_j \wedge (C \rightarrow \gamma) \in R \Rightarrow Q_j := Q_j \cup \{ \langle C \rightarrow \bullet \gamma, j \rangle \}.$$

3. Vervollständigungs-Operation

Falls der Zustand Q_i ein Earley-Objekt der Form $\langle C \rightarrow \gamma \bullet, j \rangle$ enthält und weiter der Zustand Q_j ein Earley-Objekt der Form $\langle A \rightarrow \beta \bullet C\delta, k \rangle$ enthält, dann hat der Parser im Zustand Q_j versucht, ein C zu parsen und das C ist im Zustand Q_i erkannt worden. Daher fügen wir dem Zustand Q_i nun das Earley-Objekt $\langle A \rightarrow \beta C \bullet \delta, k \rangle$ hinzu:

$$\langle C \rightarrow \gamma \bullet, j \rangle \in Q_i \wedge \langle A \rightarrow \beta \bullet C\delta, k \rangle \in Q_j \Rightarrow Q_i := Q_i \cup \{ \langle A \rightarrow \beta C \bullet \delta, k \rangle \}.$$

Der Algorithmus von Earley, der einen String der Form $s = x_1 \cdots x_n$ parsen will, funktioniert nun so:

1. Wir initialisieren die Zustände Q_i wie folgt:

$$Q_0 := \{ \langle \hat{S} \rightarrow \bullet S, 0 \rangle \},$$

$$Q_i := \{ \} \quad \text{für } i = 1, \dots, n.$$

2. Anschließend lassen wir in einer Schleife i von 0 bis n laufen und führen die folgenden Schritte durch:

- (a) Wir vergrößern Q_i mit der Vervollständigungs-Operation so lange, bis mit dieser Operation keine neuen Earley-Objekte mehr gefunden werden können.
- (b) Anschließend vergrößern wir Q_i mit Hilfe der Vorhersage-Operation. Diese Operation wird ebenfalls so lange durchgeführt, wie neue Earley-Objekte gefunden werden.
- (c) Falls $i < n$ ist, wenden wir die Lese-Operation auf Q_i an und initialisierend damit Q_{i+1} .

Falls die betrachtete Grammatik G auch ε -Regeln enthält, also Regeln der Form

$$C \rightarrow \varepsilon,$$

dann kann es passieren, dass durch die Anwendung einer Vorhersage-Operation eine neue Anwendung der Vervollständigungs-Operation möglich wird. In diesem Fall müssen Vorhersage-Operation und Vervollständigungs-Operation so lange iteriert werden, bis durch Anwendung dieser beiden Operationen keine neuen Earley-Objekte mehr erzeugt werden können.

- 3. Falls nach Beendigung des Algorithmus die Menge Q_n das Earley-Objekt $\langle \hat{S} \rightarrow S\bullet, 0 \rangle$ enthält, dann war das Parsen erfolgreich und der String $x_1 \cdots x_n$ liegt in der von der Grammatik erzeugten Sprache.

Beispiel: Abbildung 7.1 zeigt eine vereinfachte Grammatik für arithmetische Ausdrücke, die nur aus den Zahlen "1", "2" und "3" und den beiden Operator-Symbolen "+" und "*" aufgebaut ist. Die Menge T der Terminale dieser Grammatik ist also durch

$$T = \{ "1" , "2" , "3" , "+" , "*" \}$$

gegeben. Wir zeigen, wie sich der String "1+2*3" mit dieser Grammatik und dem Algorithmus von Earley parsen lässt. In der folgenden Darstellung werden wir die syntaktische Variable `expr` mit dem Buchstaben E abkürzen, für `prod` schreiben wir P und für `fact` verwenden wir die Abkürzung F .

```

1  expr : expr '+' prod
2      | prod
3      ;
4
5  prod : prod '*' fact
6      | fact
7      ;
8
9  fact : '1'
10     | '2'
11     | '3'
12     ;

```

Figure 7.1: Eine vereinfachte Grammatik für arithmetische Ausdrücke.

- 1. Wir initialisieren Q_0 als

$$Q_0 = \{ \langle \hat{S} \rightarrow \bullet E, 0 \rangle \}.$$

Die Mengen Q_1 , Q_2 , Q_3 , Q_4 und Q_5 sind zunächst alle leer. Wenden wir die Vervollständigungs-Operation auf Q_0 an, so finden wir keine neuen Earley-Objekte.

Anschließend wenden wir die Vorhersage-Operation auf das Earley-Objekt $\langle \hat{S} \rightarrow \bullet E, 0 \rangle$ an. Dadurch werden der Menge Q_0 zunächst die beiden Earley-Objekte

$$\langle E \rightarrow \bullet E "+" P, 0 \rangle \quad \text{und} \quad \langle E \rightarrow \bullet P, 0 \rangle$$

hinzugefügt. Auf das Earley-Objekt $\langle E \rightarrow \bullet P, 0 \rangle$ können wir die Vorhersage-Operation ein weiteres Mal anwenden und erhalten dann die beiden neuen Earley-Objekte

$$\langle P \rightarrow \bullet P "*" F, 0 \rangle \quad \text{und} \quad \langle P \rightarrow \bullet F, 0 \rangle.$$

Wenden wir auf das Earley-Objekt $\langle P \rightarrow \bullet F, 0 \rangle$ die Vorhersage-Operation an, so erhalten wir schließlich noch die folgenden Earley-Objekte in Q_0 :

$$\langle F \rightarrow \bullet "1", 0 \rangle, \quad \langle F \rightarrow \bullet "2", 0 \rangle, \quad \text{und} \quad \langle F \rightarrow \bullet "3", 0 \rangle.$$

Insgesamt enthält Q_0 nun die folgenden Earley-Objekte:

- (a) $\langle \hat{S} \rightarrow \bullet E, 0 \rangle,$
- (b) $\langle E \rightarrow \bullet E "+" P, 0 \rangle$
- (c) $\langle E \rightarrow \bullet P, 0 \rangle,$
- (d) $\langle P \rightarrow \bullet P "*" F, 0 \rangle,$
- (e) $\langle P \rightarrow \bullet F, 0 \rangle,$
- (f) $\langle F \rightarrow \bullet "1", 0 \rangle,$
- (g) $\langle F \rightarrow \bullet "2", 0 \rangle,$
- (h) $\langle F \rightarrow \bullet "3", 0 \rangle.$

Jetzt wenden wir die Lese-Operation auf Q_0 an. Da das erste Zeichen des zu parsenden Strings eine "1" ist, hat die Menge Q_1 danach die folgende Form:

$$Q_1 = \{ \langle F \rightarrow "1" \bullet, 0 \rangle \}.$$

2. Nun setzen wir $i = 1$ und wenden zunächst auf Q_1 die Vervollständigungs-Operation an. Aufgrund des Earley-Objekts $\langle F \rightarrow "1" \bullet, 0 \rangle$ in Q_1 suchen wir in Q_0 ein Earley-Objekt, bei dem die Markierung " \bullet " vor der Variablen F steht. Wir finden das Earley-Objekt $\langle P \rightarrow \bullet F, 0 \rangle$. Daher fügen wir nun Q_1 das Earley-Objekt

$$\langle P \rightarrow F \bullet, 0 \rangle$$

hinzu. Hierauf können wir wieder die Vervollständigungs-Operation anwenden und finden (nach mehrmaliger Anwendung) für Q_1 insgesamt die folgenden Earley-Objekte:

- (a) $\langle P \rightarrow F \bullet, 0 \rangle,$
- (b) $\langle P \rightarrow P \bullet "*" F, 0 \rangle,$
- (c) $\langle E \rightarrow P \bullet, 0 \rangle,$
- (d) $\langle E \rightarrow E \bullet "+" P, 0 \rangle,$
- (e) $\langle \hat{S} \rightarrow E \bullet, 0 \rangle.$

Als nächstes wenden wir auf diese Earley-Objekte die Vorhersage-Operation an. Da das Markierungs-Zeichen " \bullet " aber in keinem der in Q_i auftretenden Earley-Objekte vor einer Variablen steht, ergeben sich hierbei keine neuen Earley-Objekte.

Als letztes wenden wir die Lese-Operation auf Q_1 an. Da in dem String "1+2*3" das Zeichen "+" an der Position 2 liegt ist und Q_1 das Earley-Objekt

$$\langle E \rightarrow E \bullet "+" P, 0 \rangle$$

enthält, fügen wir in Q_2 das Earley-Objekt

$$\langle E \rightarrow E "+" \bullet P, 0 \rangle$$

ein.

3. Nun setzen wir $i = 2$ und wenden zunächst auf Q_2 die Vervollständigungs-Operation an. Zu diesem Zeitpunkt gilt

$$Q_2 = \{ \langle E \rightarrow E "+" \bullet P, 0 \rangle \}.$$

Da in dem einzigen Earley-Objekt, das hier auftritt, das Markierungs-Zeichen " \bullet " nicht am Ende der Grammatik-Regel steht, finden wir durch die Vervollständigungs-Operation in diesem Schritt keine neuen Earley-Objekte.

Als nächstes wenden wir auf Q_2 die Vorhersage-Operation an. Da das Markierungs-Zeichen vor der Variablen P steht, finden wir zunächst die beiden Earley-Objekte

$$\langle P \rightarrow \bullet F, 2 \rangle \quad \text{und} \quad \langle P \rightarrow \bullet P "*" F, 2 \rangle.$$

Da in dem ersten Earley-Objekt das Markierungs-Zeichen vor der Variablen F steht, kann die Vorhersage-Operation ein weiteres Mal angewendet werden und wir finden noch die folgenden Earley-Objekte:

- (a) $\langle F \rightarrow \bullet "1", 2 \rangle$,
- (b) $\langle F \rightarrow \bullet "2", 2 \rangle$,
- (c) $\langle F \rightarrow \bullet "3", 2 \rangle$.

Als letztes wenden wir die Lese-Operation auf Q_2 an. Da das dritte Zeichen in dem zu lesenden String "1+2*3" die Ziffer "2" ist, hat Q_3 nun die Form

$$Q_3 = \{ \langle F \rightarrow "2" \bullet, 2 \rangle \}.$$

4. Wir setzen $i = 3$ und wenden auf Q_3 die Vervollständigungs-Operation an. Dadurch fügen wir

$$\langle P \rightarrow F \bullet, 2 \rangle$$

in Q_3 ein. Hier können wir ein weiteres Mal die Vervollständigungs-Operation anwenden. Durch iterierte Anwendung der Vervollständigungs-Operation erhalten wir zusätzlich die folgenden Earley-Objekte:

- (a) $\langle P \rightarrow P \bullet "*" F, 2 \rangle$,
- (b) $\langle E \rightarrow E "+" P \bullet, 0 \rangle$,
- (c) $\langle E \rightarrow E \bullet "+" P, 0 \rangle$
- (d) $\langle \hat{S} \rightarrow E \bullet, 0 \rangle$.

Als letztes wenden wir die Lese-Operation an. Da der nächste zu lesende Buchstabe das Zeichen "*" ist, erhalten wir

$$Q_4 = \{ \langle P \rightarrow P "*" \bullet F, 2 \rangle \}.$$

5. Wir setzen $i = 4$. Die Vervollständigungs-Operation liefert keine neuen Earley-Objekte. Die Vorhersage-Operation liefert folgende Earley-Objekte:

- (a) $\langle F \rightarrow \bullet "1", 4 \rangle$,
- (b) $\langle F \rightarrow \bullet "2", 4 \rangle$,
- (c) $\langle F \rightarrow \bullet "3", 4 \rangle$.

Da das nächste Zeichen die Ziffer "3" ist, liefert die Lese-Operation für Q_5 :

$$Q_5 = \langle F \rightarrow "3" \bullet, 4 \rangle.$$

6. Wir setzen $i = 5$. Die Vervollständigungs-Operation liefert nacheinander die folgenden Earley-Objekte:

- (a) $\langle P \rightarrow P "*" F \bullet, 2 \rangle$,
- (b) $\langle E \rightarrow E "+" P \bullet, 0 \rangle$,
- (c) $\langle P \rightarrow P \bullet "*" F, 2 \rangle$,
- (d) $\langle E \rightarrow E \bullet "+" P, 0 \rangle$
- (e) $\langle \hat{S} \rightarrow E \bullet, 0 \rangle$.

Da die Menge Q_5 das Earley-Objekt $\langle \hat{S} \rightarrow E \bullet, 0 \rangle$ enthält, können wir schließen, dass der String "1+2*3" tatsächlich in der von der Grammatik erzeugten Sprache liegt.

Aufgabe 25: Zeigen Sie, dass der String "1*2+3" in der Sprache der Grammatik liegt, die in Abbildung 7.1 gezeigt wird. Benutzen Sie dazu den von Earley angegebenen Algorithmus.

7.2 Implementing Earley's Algorithm in Python

The *Jupyter* notebook

<https://github.com/karlstroetmann/Formal-Languages/blob/master/ANTLR4-Python/Earley-Parser/Earley-Parser.ipynb>
contains an implementation of Earley's algorithm.

7.3 Prüfen Sie Ihr Verständnis

- (a) Was ist ein Earley-Objekt und wie werden die Komponenten eines Earley-Objekts interpretiert?
- (b) Wie funktioniert die Lese-Operation beim Algorithmus von Earley?
- (c) Wie funktioniert die Vorhersage-Operation beim Algorithmus von Earley?
- (d) Wie funktioniert die Vervollständigungs-Operation beim Algorithmus von Earley?
- (e) Skizzieren Sie den Algorithmus von Earley für den Fall, dass die Grammatik keine ε -Regeln enthält.
- (f) Welche Komplexität hat der Algorithmus von Earley im allgemeinen Fall?
- (g) Welche Komplexität hat der Algorithmus von Earley in dem Fall, dass die Grammatik eindeutig ist?

Chapter 8

Bottom-up Parsers

In constructing a parser, there are generally two approaches: [top-down](#) and [bottom-up](#). We have already discussed the top-down approach. In this chapter, we will explain the bottom-up approach.

1. In the next section, we present the general concept underlying a [bottom-up parser](#).
2. In the following section, we show how bottom-up parsers can be implemented and introduce [shift-reduce parsers](#) as one possible implementation.
A *shift-reduce parser* operates with a table that specifies how the parser should process inputs in a given state.
3. We then develop the theory of how such a table can be meaningfully filled with information in the following section: First, we discuss the [SLR parsers](#) (simple LR parsers), which are the simplest class of shift-reduce parsers.
4. Unfortunately, the concept of SLR parsers is not powerful enough for practical use. Therefore, we refine this concept to obtain the class of [canonical LR parsers](#).
5. As the tables for LR parsers often become large in practice, we simplify these tables slightly and then obtain the concept of LALR parsers, which in terms of power lies between the concept of SLR parsers and that of LR parsers.

In the following chapter, we will discuss the parser generator `PLY`, which is an LALR parser.

8.1 Bottom-Up-Parser

The parsers created with `ANTLR` are known as [top-down parsers](#): starting from the start symbol of the grammar, they attempt to parse a given input by applying various grammar rules. In contrast, the parsers we are now going to develop are [bottom-up parsers](#). In such parsers, the idea is to start with the string to be parsed and group terminals based on the right-hand sides of the grammar rules.

As an example, consider parsing the string “1 + 2 * 3” using the grammar defined by the following rules:

$$\begin{aligned} e &\rightarrow e \text{ “+” } p \mid p \\ p &\rightarrow p \text{ “*” } f \mid f \\ f &\rightarrow \text{ “1” } \mid \text{ “2” } \mid \text{ “3” } \end{aligned}$$

To parse this string, we look for substrings that correspond to the right-hand sides of grammar rules, scanning the string from left to right. This way, we try to construct a parse tree in reverse from bottom to top:

$$\begin{aligned}
1 + 2 * 3 &\Leftarrow f + 2 * 3 && (\text{rule: } f \rightarrow "1") \\
&\Leftarrow p + 2 * 3 && (\text{rule: } p \rightarrow f) \\
&\Leftarrow e + 2 * 3 && (\text{rule: } e \rightarrow p) \\
&\Leftarrow e + f * 3 && (\text{rule: } f \rightarrow "2") \\
&\Leftarrow e + p * 3 && (\text{rule: } p \rightarrow f) \\
&\Leftarrow e + p * f && (\text{rule: } f \rightarrow "3") \\
&\Leftarrow e + p && (\text{rule: } p \rightarrow p "*" f) \\
&\Leftarrow e && (\text{rule: } e \rightarrow e "+" P)
\end{aligned}$$

In the first step, we used the grammar rule $f \rightarrow "1"$ to replace the string "1" with f , thus obtaining the string " $f + 2 * 3$ ". In the second step, we used the rule $p \rightarrow f$ to replace f with p . In this manner, we ultimately traced the original string " $1 + 2 * 3$ " back to e . At this point, we can make two observations:

- (a) In our approach, we always replace the leftmost substring that can be replaced when we want to trace the initially given string back to the start symbol of the grammar.
- (b) If we write down the derivation that we have constructed backwards, in the correct order, we obtain:

$$\begin{aligned}
e &\Rightarrow e + p \\
&\Rightarrow e + p * f \\
&\Rightarrow e + p * 3 \\
&\Rightarrow e + f * 3 \\
&\Rightarrow e + 2 * 3 \\
&\Rightarrow p + 2 * 3 \\
&\Rightarrow f + 2 * 3 \\
&\Rightarrow 1 + 2 * 3
\end{aligned}$$

We see here that in this derivation, the rightmost syntactic variable is always the one being replaced. Such a derivation is referred to as a [rightmost derivation](#).

In contrast, the derivations produced by a [top-down parser](#) are exactly the opposite: there, the leftmost syntactic variable is always replaced. Therefore, the derivations created with such a parser are called [leftmost derivations](#).

The above two observations are the reason why the parsers discussed in this chapter are referred to as [LR Parsers](#). The [L](#) stands for [left to right](#) and describes the approach of always scanning the string from left to right, while the [R](#) stands for [reverse rightmost derivation](#) and indicates that such parsers construct a rightmost derivation in reverse.

In the implementation of an LR parser, two questions arise:

1. Which substrings do we replace?
2. Which rules do we use in the process?

Answering these questions is generally non-trivial. Although we always scan the strings from left to right, it is not immediately clear which substring we should replace, as the potentially replaceable substrings can overlap. Consider, for example, the intermediate result

$$e + p * 3,$$

that we obtained in the fifth step above. Here, we could replace the substring "p" using the rule

$$e \rightarrow p$$

with "e". Then, we would obtain the string

$$e + e * 3.$$

The only reductions we can now perform, via the intermediate results $e + e * f$ and $e + e * p$, lead to the string

$$e + e * e,$$

which then cannot be reduced any further with the grammar given above. The answers to the above questions, which substring we replace and which rule we use, require a fair amount of theory, which we will develop in the following sections.

8.2 Shift-Reduce-Parser

Shift-reduce parsing is one way to implement bottom up parsing. Assume a grammar $G = \langle V, T, R, S \rangle$ is given. A **shift-reduce parser** is defined as a 4-Tuple

$$P = \langle Q, q_0, action, goto \rangle$$

where

1. Q is the set of **states** of the shift-reduce parser.
At first, states are purely abstract.
2. $q_0 \in Q$ is the start state.
3. *action* is a function taking two arguments. The first argument is a state $q \in Q$ and the second argument is a terminal $t \in T$. The result of this function is an element from the set

$$Action := \{ \langle \text{shift}, q \rangle \mid q \in Q \} \cup \{ \langle \text{reduce}, r \rangle \mid r \in R \} \cup \{ \text{accept} \} \cup \{ \text{error} \}.$$

Here **shift**, **reduce**, **accept**, and **error** are strings that serve to distinguish the different kinds of result of the function *action*. Therefore the signature of the function *action* is given as follows:

$$action : Q \times T \rightarrow Action.$$

4. *goto* is a function that takes a state $q \in Q$ and a syntactical variable $v \in V$ and computes a new state. Therefore the signature of *goto* is as follows:

$$goto : Q \times V \rightarrow Q.$$

A shift-reduce parser uses two stacks:

- (a) *States* is a stack of states from the set Q :

$$States \in Stack[Q].$$

- (b) *Symbols* is a stack of grammar symbols, i.e. this stack contains both terminals and syntactical variables:

$$Symbols \in Stack[T \cup V].$$

In order to simplify the exposition of shift-reduce parsing we assume that the set T of terminals contains the special symbol “EOF” (short for **end of file**). This symbol is assumed to occur at the end of the input string but does not occur elsewhere.

In order to understand how a shift-reduce parser works we introduce the notion of a **parser configuration**. A parser configuration is a triple of the form

$$(States, Symbols, Tokens)$$

where *States* and *Symbols* are the aforementioned stacks of states and grammar symbols, while *Tokens* is the rest of the tokens from the input string that have not been processed. The stack *States* always starts with the start state q_0 and has a length that is one more than the length of the stack *Symbols*. If the input string that is to be parsed has the form

$$[t_1, \dots, t_n]$$

and we have already reduced the first part $[t_1, \dots, t_k]$ of the input string to produce the symbols

$$[X_1, \dots, X_m],$$

while $[t_{k+1}, \dots, t_n]$ is the part of the input string that still needs to be processed, then we have

$$States = [q_0, q_1, \dots, q_m], \quad Symbols = [X_1, \dots, X_m], \quad \text{and} \quad Tokens = [t_{k+1}, \dots, t_n, \text{EOF}],$$

and the parser configuration $\langle States, Symbols, Tokens \rangle$ is written as

$$q_0, q_1, \dots, q_m \mid X_1, \dots, X_m \mid t_{k+1}, \dots, t_n, \text{EOF}.$$

Shift-reduce parsing starts out with the configuration

$$q_0 \mid t_1, \dots, t_n, \text{EOF}.$$

Then, parsing proceeds iteratively. If the current configuration is

$$q_0, q_1, \dots, q_m \mid X_1, \dots, X_m \mid t_{k+1}, \dots, t_n, \text{EOF},$$

then there is a case distinction according to the value of $\text{action}(q_m, t_{k+1})$.

- (a) If $\text{action}(q_m, t_{k+1}) = \text{error}$, then we know that the given string $t_1 \dots t_n$ is not generated by the given grammar and parsing is aborted with an error message.
- (b) If $\text{action}(q_m, t_{k+1}) = \text{accept}$, then we must have $k = n$ and hence $t_{k+1} = t_{n+1} = \text{EOF}$ and we also must have $X_1, \dots, X_m = S$. In this case, we have reduced the string $t_1 \dots t_n$ to the start symbol S of the given grammar and parsing finishes with success.
- (c) If $\text{action}(q_m, t_{k+1}) = \langle \text{shift}, q \rangle$, then the current configuration is changed into the new configuration

$$q_0, q_1, \dots, q_m, q \mid X_1, \dots, X_m, t_{k+1} \mid t_{k+2}, \dots, t_n, \text{EOF},$$

i.e. the next token t_{k+1} is moved from the unread input to the top of the symbol stack and the new state q is pushed onto the stack *States*.

- (d) If $\text{action}(q_m, t_{k+1}) = \langle \text{reduce}, r \rangle$, where r is a grammar rule, then the grammar rule r must have the form

$$a \rightarrow X_{m-l} \dots X_m,$$

i.e. the right hand side of the grammar rule matches the end of the stack *Symbols*. In this case, the symbols stack is reduced with this grammar rule, i.e. the symbols X_{m-l}, \dots, X_m are replaced by a . Furthermore, in the stack *States* the states $q_{m-l} \dots q_m$ are replaced by the state $\text{goto}(q_{m-l-1}, a)$. Therefore, the configuration changes as follows:

$$q_0, q_1, \dots, q_{m-l-1}, \text{goto}(q_{m-l-1}, a) \mid X_1, \dots, X_{m-l-1}, a \mid t_{k+1}, \dots, t_n, \text{EOF}.$$

```

1  class ShiftReduceParser():
2      def __init__(self, actionTable, gotoTable):
3          self.mActionTable = actionTable
4          self.mGotoTable   = gotoTable

```

Figure 8.1: Implementation of a shift-reduce parser in *Python*

The class `Shift-Reduce-Parser-Pure` that is shown in Figure 8.1 on page 93 displays the class `ShiftReduceParser`, which maintains two dictionaries.

- (a) `mActionTable` stores the function $\text{action} : Q \times T \rightarrow \text{Action}$.
- (b) `mGotoTable` stores the function $\text{goto} : Q \times V \rightarrow Q$.

Figure 8.2 on page 94 shows the implementation of the method `parse` that implements [shift-reduce parsing](#). This method assumes that the function `action` is coded as a dictionary that is stored in the member variable `mActionTable`. The function `goto` is also represented as a dictionary. It is stored in the member variable `mGotoTable`. The method `parse` is called with one argument `TL`. This is the list of tokens that have to be parsed. We append the special token “EOF” at the end of this list. The invocation `parse(TL)` returns `True` if the token list `TL` can be parsed successfully and `false` otherwise. The implementation of `parse` works as follows:

1. The variable `index` points to the next token in the token list that is to be read. Therefore, this variable is initialized to 0.

```

def parse(self, TL):
2   index    = 0          # points to next token
3   Symbols  = []         # stack of symbols
4   States   = ['s0']     # stack of states, s0 is start state
5   TL      += ['EOF']
6   while True:
7       q = States[-1]
8       t = TL[index]
9       # Below, an undefined table entry is interpreted as an error entry.
10      match self.mActionTable.get((q, t), 'error'):
11          case 'error':
12              return False
13          case 'accept':
14              return True
15          case 'shift', s:
16              Symbols += [t]
17              States  += [s]
18              index   += 1
19          case 'reduce', rule:
20              head, body = rule
21              n          = len(body)
22              Symbols = Symbols[:-n]
23              States  = States[:-n]
24              Symbols = Symbols + [head]
25              state   = States[-1]
26              States += [ self.mGotoTable[state, head] ]

```

Figure 8.2: Implementation of a shift-reduce parser in *Python*

2. The variable *Symbols* stores the stack of symbols. The top of this stack is at the end of this list. Initially, the stack of symbols is empty.
3. The variable *States* is the stack of states. The start state is assumed to be the state “s0”. Therefore this stack is initialized to contain only this state.
4. The main loop of the parser
 - sets the variable q to the current state,
 - initializes t to the next token, and then
 - looks up the appropriate action in the action table.

What happens next depends on this value of $action(q, t)$.

(a) $action(q, t) = \text{error}$.

In this case the parser has found a syntax error and returns False.

(b) $action(q, t) = \text{accept}$.

In this case parsing is successful and therefore the function returns True.

(c) $action(q, t) = \langle \text{shift}, s \rangle$.

In this case, the token t is pushed onto the symbol stack in line 16, while the state s is pushed onto the stack of states. Furthermore, the variable *index* is incremented to point to the next unread token.

(d) $action(q, t) = \langle \text{reduce}, A \rightarrow X_1 \cdots X_n \rangle$.

In this case, we use the grammar rule

$$r = (A \rightarrow X_1 \cdots X_n)$$

to reduce the symbol stack. The variable *head* represents the left hand side A of this rule, while the list $[X_1, \dots, X_n]$ is represented by the variable *body*.

In this case, it can be shown that the symbols X_1, \dots, X_n are on top of the symbol stack. As we are going to reduce the symbol stack with the rule r , we remove these n symbols from the symbol stack and replace them with the variable A .

Furthermore, we have to remove n states from the stack of states. After that, we set the variable *state* to the state that is on top of the stack of states after we have removed n states. Next, the new state $goto(state, A)$ is put on top of the stack of states in line 26.

In order to make the function *parse* work we have to provide an implementation of the functions *action* and *goto*. The tables 8.1 and 8.2 show these functions for the grammar given in Figure 8.3. For this grammar, there are 16 different states, which have been baptized as s_0, s_1, \dots, s_{15} . The tables use two different abbreviations:

1. $\langle shft, s_i \rangle$ is short for $\langle \text{shift}, s_i \rangle$.
2. $\langle rdc, r_i \rangle$ is short for $\langle \text{reduce}, r_i \rangle$, where r_i denotes the grammar rule number i . Here, we have numbered the rules as follows:
 - (a) $r_1 = (expr \rightarrow expr \text{ "+" } product)$
 - (b) $r_2 = (expr \rightarrow expr \text{ "-" } product)$
 - (c) $r_3 = (expr \rightarrow product)$
 - (d) $r_4 = (product \rightarrow product \text{ "*" } factor)$
 - (e) $r_5 = (product \rightarrow product \text{ "/" } factor)$
 - (f) $r_6 = (product \rightarrow factor)$
 - (g) $r_7 = (factor \rightarrow "(" expr ")")$
 - (h) $r_8 = (factor \rightarrow \text{NUMBER})$

The corresponding grammar is shown in Figure 8.3. The definition of the grammar rules and the coding of the functions *action* and *goto* is shown in the Figures 8.4, 8.6, and 8.5 on the following pages. Of course, at present we do not have any idea how the functions *action* and *goto* are computed. This requires some theory that will be presented in the next section.

<i>expr</i>	\rightarrow	<i>expr</i> "+" <i>product</i>
		<i>expr</i> "-" <i>product</i>
		<i>product</i>
<i>product</i>	\rightarrow	<i>product</i> "*" <i>factor</i>
		<i>product</i> "/" <i>factor</i>
		<i>factor</i>
<i>factor</i>	\rightarrow	"(" <i>expr</i> ")"
		NUMBER

Figure 8.3: A grammar for arithmetical expressions.

State	EOF	+	-	*	/	()	NUMBER
s_0						$\langle shft, s_5 \rangle$		$\langle shft, s_2 \rangle$
s_1	$\langle rdc, r_6 \rangle$	$\langle rdc, r_6 \rangle$	$\langle rdc, r_6 \rangle$	$\langle rdc, r_6 \rangle$	$\langle rdc, r_6 \rangle$		$\langle rdc, r_6 \rangle$	
s_2	$\langle rdc, r_8 \rangle$	$\langle rdc, r_8 \rangle$	$\langle rdc, r_8 \rangle$	$\langle rdc, r_8 \rangle$	$\langle rdc, r_8 \rangle$		$\langle rdc, r_8 \rangle$	
s_3	$\langle rdc, r_3 \rangle$	$\langle rdc, r_3 \rangle$	$\langle rdc, r_3 \rangle$	$\langle shft, s_{12} \rangle$	$\langle shft, s_{11} \rangle$		$\langle rdc, r_3 \rangle$	
s_4	<i>accept</i>	$\langle shft, s_8 \rangle$	$\langle shft, s_9 \rangle$					
s_5						$\langle shft, s_5 \rangle$		$\langle shft, s_2 \rangle$
s_6		$\langle shft, s_8 \rangle$	$\langle shft, s_9 \rangle$				$\langle shft, s_7 \rangle$	
s_7	$\langle rdc, r_7 \rangle$	$\langle rdc, r_7 \rangle$	$\langle rdc, r_7 \rangle$	$\langle rdc, r_7 \rangle$	$\langle rdc, r_7 \rangle$		$\langle rdc, r_7 \rangle$	
s_8						$\langle shft, s_5 \rangle$		$\langle shft, s_2 \rangle$
s_9						$\langle shft, s_5 \rangle$		$\langle shft, s_2 \rangle$
s_{10}	$\langle rdc, r_2 \rangle$	$\langle rdc, r_2 \rangle$	$\langle rdc, r_2 \rangle$	$\langle shft, s_{12} \rangle$	$\langle shft, s_{11} \rangle$		$\langle rdc, r_2 \rangle$	
s_{11}						$\langle shft, s_5 \rangle$		$\langle shft, s_2 \rangle$
s_{12}						$\langle shft, s_5 \rangle$		$\langle shft, s_2 \rangle$
s_{13}	$\langle rdc, r_4 \rangle$	$\langle rdc, r_4 \rangle$	$\langle rdc, r_4 \rangle$	$\langle rdc, r_4 \rangle$	$\langle rdc, r_4 \rangle$		$\langle rdc, r_4 \rangle$	
s_{14}	$\langle rdc, r_5 \rangle$	$\langle rdc, r_5 \rangle$	$\langle rdc, r_5 \rangle$	$\langle rdc, r_5 \rangle$	$\langle rdc, r_5 \rangle$		$\langle rdc, r_5 \rangle$	
s_{15}	$\langle rdc, r_1 \rangle$	$\langle rdc, r_1 \rangle$	$\langle rdc, r_1 \rangle$	$\langle shft, s_{12} \rangle$	$\langle shft, s_{11} \rangle$		$\langle rdc, r_1 \rangle$	

Table 8.1: The function *action()*.

State	expr	product	factor
s_0	s_4	s_3	s_1
s_1			
s_2			
s_3			
s_4			
s_5	s_6	s_3	s_1
s_6			
s_7			
s_8		s_{15}	s_1
s_9		s_{10}	s_1
s_{10}			
s_{11}			s_{14}
s_{12}			s_{13}
s_{13}			
s_{14}			
s_{15}			

Table 8.2: The function *goto()*.

```

1  r1 = ('e', ('e', '+', 'p'))
2  r2 = ('e', ('e', '-', 'p'))
3  r3 = ('e', ('p'))
4
5  r4 = ('p', ('p', '*', 'f'))
6  r5 = ('p', ('p', '/', 'f'))
7  r6 = ('p', ('f'))
8
9  r7 = ('f', (('(', 'e', ')'))
10 r8 = ('f', ('NUMBER',))

```

Figure 8.4: Grammar rules coded in *Python*.

```

1  gotoTable := {};
2
3  gotoTable["s0", "e"] := "s4";
4  gotoTable["s0", "p"] := "s3";
5  gotoTable["s0", "f"] := "s1";
6
7  gotoTable["s5", "e"] := "s6";
8  gotoTable["s5", "p"] := "s3";
9  gotoTable["s5", "f"] := "s1";
10
11 gotoTable["s8", "p"] := "s15";
12 gotoTable["s8", "f"] := "s1";
13
14 gotoTable["s9", "p"] := "s10";
15 gotoTable["s9", "f"] := "s1";
16
17 gotoTable["s11", "f"] := "s14";
18 gotoTable["s12", "f"] := "s13";

```

Figure 8.5: Goto table coded in PYTHON.

```

actionTable = {}

actionTable['s0', '('] = ('shift', 's5'); actionTable['s8', '('] = ('shift', 's5')
actionTable['s0', 'NUMBER'] = ('shift', 's2'); actionTable['s8', 'NUMBER'] = ('shift', 's2')

actionTable['s1', 'EOF'] = ('reduce', r6); actionTable['s9', '('] = ('shift', 's5')
actionTable['s1', '+' ] = ('reduce', r6); actionTable['s9', 'NUMBER'] = ('shift', 's2')
actionTable['s1', '-' ] = ('reduce', r6);
actionTable['s1', '*' ] = ('reduce', r6);
actionTable['s1', '/' ] = ('reduce', r6);
actionTable['s1', ')' ] = ('reduce', r6);

actionTable['s2', 'EOF'] = ('reduce', r8);
actionTable['s2', '+' ] = ('reduce', r8);
actionTable['s2', '-' ] = ('reduce', r8);
actionTable['s2', '*' ] = ('reduce', r8);
actionTable['s2', '/' ] = ('reduce', r8);
actionTable['s2', ')' ] = ('reduce', r8);

actionTable['s3', 'EOF'] = ('reduce', r3);
actionTable['s3', '+' ] = ('reduce', r3);
actionTable['s3', '-' ] = ('reduce', r3);
actionTable['s3', '*' ] = ('shift', 's12');
actionTable['s3', '/' ] = ('shift', 's11');
actionTable['s3', ')' ] = ('reduce', r3);

actionTable['s4', 'EOF'] = 'accept';
actionTable['s4', '+' ] = ('shift', 's8');
actionTable['s4', '-' ] = ('shift', 's9');

actionTable['s5', '('] = ('shift', 's5');
actionTable['s5', 'NUMBER'] = ('shift', 's2');

actionTable['s6', '+' ] = ('shift', 's8');
actionTable['s6', '-' ] = ('shift', 's9');
actionTable['s6', ')' ] = ('shift', 's7');

actionTable['s7', 'EOF'] = ('reduce', r7);
actionTable['s7', '+' ] = ('reduce', r7);
actionTable['s7', '-' ] = ('reduce', r7);
actionTable['s7', '*' ] = ('reduce', r7);
actionTable['s7', '/' ] = ('reduce', r7);
actionTable['s7', ')' ] = ('reduce', r7);

actionTable['s10', 'EOF'] = ('reduce', r2)
actionTable['s10', '+' ] = ('reduce', r2)
actionTable['s10', '-' ] = ('reduce', r2)
actionTable['s10', '*' ] = ('shift', 's12')
actionTable['s10', '/' ] = ('shift', 's11')
actionTable['s10', ')' ] = ('reduce', r2)

actionTable['s11', '('] = ('shift', 's5')
actionTable['s11', 'NUMBER'] = ('shift', 's2')

actionTable['s12', '('] = ('shift', 's5')
actionTable['s12', 'NUMBER'] = ('shift', 's2')

actionTable['s13', 'EOF'] = ('reduce', r4)
actionTable['s13', '+' ] = ('reduce', r4)
actionTable['s13', '-' ] = ('reduce', r4)
actionTable['s13', '*' ] = ('reduce', r4)
actionTable['s13', '/' ] = ('reduce', r4)
actionTable['s13', ')' ] = ('reduce', r4)

actionTable['s14', 'EOF'] = ('reduce', r5)
actionTable['s14', '+' ] = ('reduce', r5)
actionTable['s14', '-' ] = ('reduce', r5)
actionTable['s14', '*' ] = ('reduce', r5)
actionTable['s14', '/' ] = ('reduce', r5)
actionTable['s14', ')' ] = ('reduce', r5)

actionTable['s15', 'EOF'] = ('reduce', r1)
actionTable['s15', '+' ] = ('reduce', r1)
actionTable['s15', '-' ] = ('reduce', r1)
actionTable['s15', '*' ] = ('shift', 's12')
actionTable['s15', '/' ] = ('shift', 's11')
actionTable['s15', ')' ] = ('reduce', r1)

```

Figure 8.6: Action table coded in PYTHON.

8.3 SLR-Parser

In this section, we demonstrate how to compute the functions

$$action : Q \times T \rightarrow Action \quad \text{and} \quad goto : Q \times V \rightarrow Q$$

that have been used in the last section to build a shift-reduce parser for a given context-free grammar G . For this purpose, we first clarify what information the states contained in the set Q should hold. We will define these states in such a way that they contain information about which rule the shift-reduce parser is attempting to apply, which part of the right hand side of a grammar rule has already been recognized, and what are the symbols that are still expected. To this end, we define the concept of a **marked rule**. Unfortunately, the original English literature [Knu65] uses the rather meaningless term “**item**” instead of the notion of a *marked rule*.

Definition 22 (Marked Rule) A **marked rule** of a context-free grammar $G = \langle V, T, R, s \rangle$ is a triple

$$(a, \beta, \gamma),$$

such that

$$(a \rightarrow \beta\gamma) \in R.$$

A marked rule of the form $\langle a, \beta, \gamma \rangle$ is written as

$$a \rightarrow \beta \bullet \gamma.$$

◇

The marked rule $a \rightarrow \beta \bullet \gamma$ indicates that the parser is trying to parse an a with the rule $a \rightarrow \beta\gamma$, having already seen β and is next attempting to recognize γ . The symbol \bullet thus marks the part of the right side of the rule that we have already read. Now the idea is to represent a state of an SLR parser as a **set of marked rules**. To illustrate this idea, let's consider a specific example: we start with the grammar for arithmetic expressions shown in Figure 8.3 on page 95. We extend this grammar by a new start symbol \hat{s} and the rule

$$\hat{s} \rightarrow expr \$$$

where the symbol “\$” is used to denote EOF, i.e. the end of the input. The start state clearly contains the marked rule

$$\hat{s} \rightarrow \lambda \bullet expr \$,$$

since at the beginning we are trying to derive the start symbol \hat{s} . The component λ indicates that we have not yet processed anything. In addition to this marked rule, the start state must also contain the marked rules

1. $expr \rightarrow \lambda \bullet expr '+' product$,
2. $expr \rightarrow \lambda \bullet expr '-' product$ and
3. $expr \rightarrow \lambda \bullet product$

because, for example, it may be necessary to use the rule

$$expr \rightarrow expr '+' product$$

to derive the sought $expr$. Similarly, it could be that instead we need to use the rule

$$expr \rightarrow product$$

This explains why we must include the marked rule

$$expr \rightarrow \lambda \bullet product$$

in the start state, because at the beginning we cannot yet know which rule we will need, so the start state must contain all these rules. Once we have added the marked rule

$$expr \rightarrow \lambda \bullet product$$

to the start state, we see that we might need to read a *product* next. Therefore, the start state also contains the following marked rules¹:

4. $product \rightarrow \bullet product '*' factor,$
5. $product \rightarrow \bullet product '/' factor,$
6. $product \rightarrow \bullet factor,$

Now, the sixth rule indicates that we might first read a *factor*. Therefore, we add the following two marked rules to the start state:

7. $factor \rightarrow \bullet '(' expr ')',$
8. $factor \rightarrow \bullet NUMBER.$

Overall, we see that the start state consists of a set of 8 marked rules. The system shown above, for deriving further rules from a given rule, is formalized in the concept of the **closure** of a set of marked rules.

Definition 23 ($\text{closure}(\mathcal{M})$) Let \mathcal{M} be a set of marked rules. Then we define the **closure** of this set as the smallest set \mathcal{K} of marked rules such that:

1. $\mathcal{M} \subseteq \mathcal{K}$,
i.e. the closure includes the original set of rules.
2. If on the one hand

$$a \rightarrow \beta \bullet c \delta$$

is a marked rule from the set \mathcal{K} , where c is a syntactic variable, and on the other hand

$$c \rightarrow \gamma$$

is a grammar rule of the underlying grammar G , then the marked rule

$$c \rightarrow \bullet \gamma$$

is also an element of the set \mathcal{K} . Formally, this is written as follows:

$$(a \rightarrow \beta \bullet c \delta) \in \mathcal{K} \wedge (c \rightarrow \gamma) \in R \Rightarrow (c \rightarrow \bullet \gamma) \in \mathcal{K}$$

The set \mathcal{K} defined in this way is uniquely determined and is referred to as $\text{closure}(\mathcal{M})$ in the following. ◇

Remark: If you recall the Earley algorithm, you will notice that the calculation of the closure is the same as the *prediction operation* in the Earley algorithm. ◇

For a given set \mathcal{M} of marked rules, the computation of $\mathcal{K} := \text{closure}(\mathcal{M})$ can be done iteratively:

1. Initially, set $\mathcal{K} := \mathcal{M}$.
2. Then, find all rules of the form

$$a \rightarrow \beta \bullet c \delta$$

within the set \mathcal{K} , where c is a syntactic variable. Subsequently, for all rules of the form $c \rightarrow \gamma$, add the new marked rule

$$c \rightarrow \bullet \gamma$$

into the set \mathcal{K} . Repeat this step until no new rules are found.

¹In order to be more concise, we drop the empty string λ in the following rules.

Example: Starting from the grammar for arithmetic expressions shown in Figure 8.3 on page 95, let's consider the set

$$\mathcal{M} := \{product \rightarrow product '*' \bullet factor\}$$

For the set $closure(\mathcal{M})$, we then find

$$closure(\mathcal{M}) = \left\{ \begin{array}{l} product \rightarrow product '*' \bullet factor, \\ factor \rightarrow \bullet '(' expr ')', \\ factor \rightarrow \bullet NUMBER \end{array} \right\}.$$

◇

Our goal is to define a Shift-Reduce-Parser

$$P = \langle Q, q_0, action, goto \rangle$$

for a given context-free grammar $G = \langle V, T, R, s \rangle$. To achieve this goal, we first need to determine how to define the states in the set Q , as this will almost automatically define the remaining components. As mentioned earlier, we define the states as sets of marked rules. Let's first define

$$\Gamma := \{a \rightarrow \beta \bullet \gamma \mid (a \rightarrow \beta \gamma) \in R\}$$

as the set of all marked rules of the grammar. However, it is not sensible to allow arbitrary subsets of Γ as states: A subset $\mathcal{M} \subseteq \Gamma$ is only considered as a state if the set \mathcal{M} is [closed](#) under the function $closure()$, meaning $closure(\mathcal{M}) = \mathcal{M}$. Therefore, we define

$$Q := \{\mathcal{M} \in 2^\Gamma \mid closure(\mathcal{M}) = \mathcal{M}\}.$$

The interpretation of the sets $\mathcal{M} \in Q$ is that a state \mathcal{M} contains exactly those marked rules that can be applied in the situation described by the state.

To simplify the following constructions, we extend the grammar $G = \langle V, T, R, s \rangle$ by introducing a new start symbol \hat{s} and a new token $\$$ to the grammar

$$\hat{G} = \langle V \cup \{\hat{s}\}, T \cup \{\$\}, R \cup \{\hat{s} \rightarrow s \$\}, \hat{s} \rangle.$$

We refer to the grammar \hat{G} as the [augmented grammar](#). The use of the augmented grammar enables the following definition of the start state. We set:

$$q_0 := closure(\{\hat{s} \rightarrow \bullet s \$\}).$$

Next, we construct the function $goto()$. The definition is as follows:

$$goto(\mathcal{M}, c) := closure(\{a \rightarrow \beta c \bullet \delta \mid (a \rightarrow \beta \bullet c \delta) \in \mathcal{M}\}).$$

To understand this definition, assume that the parser is in a state where it tries to parse an a using the rule $a \rightarrow \beta c \delta$ and, furthermore, assume that the substring β has already been recognized. This state is described by the marked rule

$$a \rightarrow \beta \bullet c \delta$$

If a c is now recognized, the parser can transition from the state containing the rule $a \rightarrow \beta \bullet c \delta$ to a state containing the rule $a \rightarrow \beta c \bullet \delta$. Therefore, we get the above definition of the function $goto(\mathcal{M}, c)$. For the subsequent definition of the function $action(\mathcal{M}, t)$, it is useful to extend the definition of the function $goto$ to terminals. For terminals $t \in T$ we set:

$$goto(\mathcal{M}, t) := closure(\{a \rightarrow \beta t \bullet \delta \mid (a \rightarrow \beta \bullet t \delta) \in \mathcal{M}\}).$$

Before we can compute the function $action$, we need to define two more functions, the functions *First* and *Follow*.

8.3.1 The Functions First and Follow

In this section, we define the two functions *First* and *Follow*. These functions are needed to compute the function *action*. We consider a given context-free grammar $G = \langle V, T, R, s \rangle$.

- (a) For a syntactic variable a , $First(a)$ calculates the set of all tokens with which a string w can begin, which is derived from the variable a , for which $a \Rightarrow_G^* w$ holds.
- (b) For a syntactic variable a , $Follow(a)$ calculates the set of tokens that can follow an a in a string derived from s , i.e., $t \in Follow(a)$ if there is a derivation of the following form:

$$s \Rightarrow_G^* \beta a t \gamma.$$

Here, β and γ are strings consisting of tokens and variables, while t denotes a single token.

Definition 24 (λ -generating) Let $G = \langle V, T, R, S \rangle$ be a context-free grammar and a be a syntactic variable, i.e., $a \in V$. The variable a is called **λ -generating** precisely when

$$a \Rightarrow^* \lambda$$

holds, which means that the empty word can be derived from the variable a . We write $nullab(a)$ if the variable a is λ -generating. \diamond

Examples:

- (a) In the grammar shown in Figure 6.5 on page 70, the variables *exprRest* and *productRest* are λ -generating.
- (b) Now let's consider a less obvious example. Suppose the grammar G contains the following rules:

$$\begin{aligned} S &\rightarrow a b c \\ a &\rightarrow 'X' b \mid a 'Y' \mid b c \\ b &\rightarrow 'X' b \mid a 'Y' \mid c c \\ c &\rightarrow a b c \mid \lambda \end{aligned}$$

Initially, it's clear that the variable c is λ -generating. Then, due to the rule $b \rightarrow c c$, b is also λ -generating, and from the rule $a \rightarrow b c$, a becomes λ -generating as well. Finally, S is recognized as λ -generating, since the first rule states

$$S \rightarrow a b c$$

and here, all variables on the right side of the rule have already been proven to be λ -generating variables.

Definition 25 (First()) Let $G = \langle V, T, R, s \rangle$ be a context-free grammar and $a \in V$. We define $First(a)$ as the set of all tokens t with which a word derived from a can begin:

$$First(a) := \{t \in T \mid \exists \gamma \in (V \cup T)^* : a \Rightarrow^* t \gamma\}.$$

The definition of the function $First()$ can be extended to strings in $(V \cup T)^*$ as follows:

1. $First(\lambda) = \{\}$.
2. $First(t\beta) = \{t\}$ if $t \in T$.
3. $First(a\beta) = \begin{cases} First(a) \cup First(\beta) & \text{if } a \Rightarrow^* \lambda; \\ First(a) & \text{otherwise.} \end{cases}$

If a is a variable of G and the rules defining a are given as

$$a \rightarrow \alpha_1 \mid \dots \mid \alpha_n,$$

then we have

$$First(a) = \bigcup_{i=1}^n First(\alpha_i).$$

\diamond

Remark: Note that the definitions of the function $First(a)$ for variables $a \in V$ and the function $First(\alpha)$ for strings $\alpha \in (V \cup T)^*$ are mutually recursive. The computation of $First(a)$ is best done via a fixpoint computation: Start by setting $First(a) := \{\}$ for all variables $a \in V$ and then continue to iterate the equations defining $First(a)$ until none of the sets $First(a)$ changes any more. The next example clarifies this idea.

Example: We can iteratively compute the sets $First(a)$ for the variables a of the grammar shown in Figure 6.5. It's best to compute the function $First(a)$ for the individual variables a starting with those at the bottom of the hierarchy.

1. First, the rules

$$factor \rightarrow '(' \text{ expr } ')' \mid \text{NUMBER}$$

imply that every string derived from $factor$ either starts with an opening parenthesis or a number:

$$First(factor) = \{ '(', \text{NUMBER} \}.$$

2. Similarly, from the rules

$$productRest \rightarrow '*' factor productRest \mid '/' factor productRest \mid \lambda$$

we find that a $productRest$ either starts with the character "*" or "/":

$$First(productRest) = \{ '*', '/' \}.$$

3. The rule for the variable $product$ is

$$product \rightarrow factor productRest.$$

Since the variable $factor$ is not λ -generating, we see that the set $First(product)$ matches the set $First(factor)$:

$$First(product) = \{ '(', \text{NUMBER} \}.$$

4. From the rules

$$exprRest \rightarrow '+' product exprRest \mid '-' product exprRest \mid \lambda,$$

we can compute $First(exprRest)$ as follows:

$$First(exprRest) = \{ '+', '-' \}.$$

5. Further, from the rule

$$expr \rightarrow product exprRest,$$

and the fact that the variable $product$ is not λ -generating, the set $First(expr)$ matches the set $First(product)$:

$$First(expr) = \{ '(', \text{NUMBER} \}.$$

Since we have computed the sets $First(a)$ in a clever order, we did not have to perform a proper fixpoint iteration in this example

Definition 26 (Follow()) Let $G = \langle V, T, R, s \rangle$ be a context-free grammar and $a \in V$. In computing $Follow(a)$, the grammar is first **augmented** by adding the symbol $\$$ as a new symbol to the set T of terminals and adding the new symbol \hat{s} to the variables. This symbol is also the new start symbol of the grammar. To the set R of rules, we add the following new rule:

$$\hat{s} \rightarrow s \$.$$

Furthermore, we define

$$\hat{T} := T \cup \{\$ \}.$$

We refer to the thus modified grammar $\hat{G} = \langle V, \hat{T}, R, \hat{s} \rangle$ as the *augmented* grammar. Then we define $Follow(a)$ as the set of all tokens t that can follow the variable a in a derivation:

$$Follow(a) := \{ t \in \hat{T} \mid \exists \beta, \gamma \in (V \cup \hat{T})^* : \hat{s} \Rightarrow^* \beta a t \gamma \}.$$

If the start symbol \hat{s} can somehow derive a string $\beta a t \gamma$ in which the token t follows the variable a , then t is an element of the set $\text{Follow}(a)$. \diamond

Example: Let's reexamine the grammar for arithmetic expressions shown in Figure 6.5.

1. Due to the newly added rule

$$\hat{S} \rightarrow \text{expr } \$$$

the set $\text{Follow}(\text{expr})$ must contain the symbol $\$$. Furthermore, due to the rule

$$\text{factor} \rightarrow '(\text{ expr })'$$

the set $\text{Follow}(\text{expr})$ must also contain the symbol $)'$. Thus, we have in total

$$\text{Follow}(\text{expr}) = \{ \$, ')'\}.$$

2. Due to the rule

$$\text{expr} \rightarrow \text{product exprRest}$$

we know that all terminals that can follow an expr can also follow an exprRest . Thus, we already know that $\text{Follow}(\text{exprRest})$ contains the tokens $\$$ and $)'$. Since exprRest otherwise only appears at the end of its defining rules, these are all the tokens that can follow exprRest , and we have

$$\text{Follow}(\text{exprRest}) = \{ \$, ')'\}.$$

3. The rules

$$\text{exprRest} \rightarrow '+' \text{ product exprRest} \mid '-' \text{ product exprRest}$$

show that all elements from $\text{First}(\text{exprRest})$ can follow a product . However, that's not all: since the variable exprRest is λ -generating, additionally, all tokens that can follow exprRest can also follow product . Thus, we have in total

$$\text{Follow}(\text{product}) = \{ '+', '-', \$, ')'\}.$$

4. The rule

$$\text{product} \rightarrow \text{factor productRest}$$

shows that all terminals that can follow a product can also follow a productRest . Since productRest otherwise only appears at the end of its defining rules, these are all the tokens that can follow productRest , and we have in total

$$\text{Follow}(\text{productRest}) = \{ '+', '-', \$, ')'\}.$$

5. The rules

$$\text{productRest} \rightarrow '*' \text{ factor productRest} \mid '/' \text{ factor productRest}$$

show that all elements from $\text{First}(\text{productRest})$ can follow a factor . However, that's not all: since the variable productRest is λ -generating, additionally, all tokens that can follow productRest can also follow factor . Thus, we have in total

$$\text{Follow}(\text{factor}) = \{ '*', '/', '+', '-', \$, ')'\}. \quad \diamond$$

The last example demonstrates that the computation of the $\text{nullable}()$ predicate and the computation of the sets $\text{First}(a)$ and $\text{Follow}(a)$ for a syntactic variable a are closely interconnected. Suppose we have a grammar rule:

$$a \rightarrow Y_1 Y_2 \cdots Y_k$$

Then the following relationships exist between the $\text{nullable}()$ predicate and the $\text{First}()$ and $\text{Follow}()$ functions:

1. $\forall t \in T : \neg \text{nullable}(t)$.
2. $k = 0 \Rightarrow \text{nullable}(a)$.

3. $(\forall i \in \{1, \dots, k\} : \text{nullable}(Y_i)) \Rightarrow \text{nullable}(a)$.

Setting $k = 0$ here, we see that 2. is a special case of 3.

4. $\text{First}(Y_1) \subseteq \text{First}(a)$.

5. $(\forall j \in \{1, \dots, i-1\} : \text{nullable}(Y_j)) \Rightarrow \text{First}(Y_i) \subseteq \text{First}(a)$.

Setting $i = 1$ above, we see that 4. is a special case of 5.

6. $\text{Follow}(a) \subseteq \text{Follow}(Y_k)$.

7. $(\forall j \in \{i+1, \dots, k\} : \text{nullable}(Y_j)) \Rightarrow \text{Follow}(a) \subseteq \text{Follow}(Y_i)$.

Setting $i = k$ here, we see that 6. is a special case of 7.

8. $\forall i \in \{1, \dots, k-1\} : \text{First}(Y_{i+1}) \subseteq \text{Follow}(Y_i)$.

9. $(\forall j \in \{i+1, \dots, l-1\} : \text{nullable}(Y_j)) \Rightarrow \text{First}(Y_l) \subseteq \text{Follow}(Y_i)$.

Setting $l = i+1$ here, we see that 8. is a special case of 9.

Using these relationships, $\text{nullable}()$, $\text{First}()$, and $\text{Follow}()$ can be computed iteratively through a fixpoint iteration:

- Initially, the functions $\text{First}(a)$ and $\text{Follow}(a)$ for each syntactic variable a are initialized with the empty set. The predicate $\text{nullable}(a)$ is set to false for each syntactic variable.
- Subsequently, the rules outlined above are applied repeatedly until no further changes occur from their application.

8.3.2 Computing the Function action

Finally, we specify how the function $\text{action}(\mathcal{M}, t)$ is computed for a set of marked rules \mathcal{M} and a token t . When defining $\text{action}(\mathcal{M}, t)$, we distinguish four cases.

- If \mathcal{M} contains a marked rule of the form $a \rightarrow \beta \bullet t \delta$ and $t \neq \$$, then we set

$$\text{action}(\mathcal{M}, t) := \langle \text{shift}, \text{goto}(\mathcal{M}, t) \rangle,$$

because in this case the parser is trying to recognize an a using the rule $a \rightarrow \beta t \delta$ and has already recognized β . If the next token in the input string is indeed the token t , the parser can read this t and transitions from the state $a \rightarrow \beta \bullet t \delta$ to the state $a \rightarrow \beta t \bullet \delta$, which is computed by the function $\text{goto}(\mathcal{M}, t)$. So, we have

$$\text{action}(\mathcal{M}, t) := \langle \text{shift}, \text{goto}(\mathcal{M}, t) \rangle \quad \text{if } (a \rightarrow \beta \bullet t \delta) \in \mathcal{M} \text{ and } t \neq \$.$$

- If \mathcal{M} contains a marked rule of the form $a \rightarrow \beta \bullet$ and additionally $t \in \text{Follow}(a)$, then we set

$$\text{action}(\mathcal{M}, t) := \langle \text{reduce}, a \rightarrow \beta \rangle,$$

because in this case the parser is trying to recognize an a using the rule $a \rightarrow \beta$ and has already recognized β . If the next token in the input string is the token t and t is a token that can follow a , i.e., $t \in \text{Follow}(a)$, then the parser can apply the rule $a \rightarrow \beta$ and reduce the symbol stack with this rule. Therefore,

$$\text{action}(\mathcal{M}, t) := \langle \text{reduce}, a \rightarrow \beta \rangle \quad \text{if } (a \rightarrow \beta \bullet) \in \mathcal{M}, a \neq \hat{s} \text{ and } t \in \text{Follow}(a).$$

- If \mathcal{M} contains the marked rule $\hat{s} \rightarrow s \bullet \$$ and we have completely read the string to be parsed, we set

$$\text{action}(\mathcal{M}, \$) := \text{accept},$$

because in this case the parser is trying to recognize \hat{s} using the rule $\hat{s} \rightarrow s \$$ and has already recognized s . If the next token in the input string is the end-of-file character $\$$, then the string being parsed belongs to the language specified by the grammar G , $L(G)$. Therefore, we have

$$\text{action}(\mathcal{M}, \$) := \text{accept}, \quad \text{if } (\hat{s} \rightarrow s \bullet \$) \in \mathcal{M}.$$

4. In all other cases, we set

$$\text{action}(\mathcal{M}, t) := \text{error}.$$

Conflicts can arise between the first two rules. We distinguish between two types of conflicts.

1. A **shift-reduce conflict** occurs when both the first and second cases apply. In this situation, the set \mathcal{M} contains, on one hand, a marked rule of the form

$$a \rightarrow \beta \bullet t \gamma,$$

and on the other hand, \mathcal{M} contains a rule of the form

$$c \rightarrow \delta \bullet \quad \text{with } t \in \text{Follow}(c).$$

If the next token is t , it is unclear whether this token should be pushed onto the symbol stack and the parser transition to a state with the marked rule $a \rightarrow \beta t \bullet \gamma$, or whether the symbol stack should be reduced using the rule $c \rightarrow \delta$ instead.

2. A **reduce-reduce conflict** exists if the set \mathcal{M} contains two different marked rules of the form

$$c_1 \rightarrow \gamma_1 \bullet \quad \text{and} \quad c_2 \rightarrow \gamma_2 \bullet$$

and if at the same time $t \in \text{Follow}(c_1) \cap \text{Follow}(c_2)$, because then it is not clear which of the two rules the parser should apply when the next token to be read is t .

If either of these conflicts occurs, we say that the grammar is not an SLR grammar. Such a grammar cannot be parsed using an SLR parser. We will later provide examples of both types of conflicts, but first, we want to examine a grammar where no conflicts occur and actually compute the functions $\text{goto}()$ and $\text{action}()$ for it. We use the grammar for arithmetic expressions that is shown in Figure 8.3 as a basis.

Since the syntactic variable expr appears on the right side of grammar rules, we define start as a new start symbol and add the rule

$$\text{start} \rightarrow \text{expr} \$$$

to the grammar. This step corresponds to the previously discussed process of **augmenting** the grammar.

First, we compute the set of states Q . We had previously provided the following formula for this:

$$Q := \{\mathcal{M} \in 2^\Gamma \mid \text{closure}(\mathcal{M}) = \mathcal{M}\}.$$

However, this set also contains states that cannot be reached from the start state via the function $\text{goto}()$. Therefore, we only compute the states that can actually be reached from the start state using the function $\text{goto}()$. To avoid making the calculation too wordy, we introduce the following abbreviations:

$$s := \text{start}, \quad e := \text{expr}, \quad p := \text{product}, \quad f := \text{factor}, \quad \text{and } N := \text{NUMBER}.$$

We begin with the start state:

1. $s_0 := \text{closure}(\{ s \rightarrow \bullet e \$ \})$
 $= \{ s \rightarrow \bullet e \$, \\ e \rightarrow \bullet e '+' p, e \rightarrow \bullet e '-' p, e \rightarrow \bullet p, \\ p \rightarrow \bullet p '*' f, p \rightarrow \bullet p '/' f, p \rightarrow \bullet f, \\ f \rightarrow \bullet '(' e ')', f \rightarrow \bullet N \}.$
2. $s_1 := \text{goto}(s_0, f)$
 $= \text{closure}(\{ p \rightarrow f \bullet \})$
 $= \{ p \rightarrow f \bullet \}.$
3. $s_2 := \text{goto}(s_0, N)$
 $= \text{closure}(\{ f \rightarrow N \bullet \})$
 $= \{ f \rightarrow N \bullet \}.$

4. $s_3 := \text{goto}(s_0, p)$
 $= \text{closure}(\{p \rightarrow p \bullet '*' f, p \rightarrow p \bullet '/' f, e \rightarrow p \bullet\})$
 $= \{p \rightarrow p \bullet '*' f, p \rightarrow p \bullet '/' f, e \rightarrow p \bullet\}.$
5. $s_4 := \text{goto}(s_0, e)$
 $= \text{closure}(\{s \rightarrow e \bullet \$, e \rightarrow e \bullet '+' p, e \rightarrow e \bullet '-' p\})$
 $= \{s \rightarrow e \bullet \$, e \rightarrow e \bullet '+' p, e \rightarrow e \bullet '-' p\}.$
6. $s_5 := \text{goto}(s_0, '(')$
 $= \text{closure}(\{f \rightarrow '(' \bullet e')'\})$
 $= \{f \rightarrow '(' \bullet e')'$
 $\quad e \rightarrow \bullet e '+' p, e \rightarrow \bullet e '-' p, e \rightarrow \bullet p,$
 $\quad p \rightarrow \bullet p '*' f, p \rightarrow \bullet p '/' f, p \rightarrow \bullet f,$
 $\quad f \rightarrow \bullet '(e')', f \rightarrow \bullet N$
 $\quad \}.$
7. $s_6 := \text{goto}(s_5, e)$
 $= \text{closure}(\{f \rightarrow '(' e \bullet ')', e \rightarrow e \bullet '+' p, e \rightarrow e \bullet '-' p\})$
 $= \{f \rightarrow '(' e \bullet ')', e \rightarrow e \bullet '+' p, e \rightarrow e \bullet '-' p\}.$
8. $s_7 := \text{goto}(s_6, ')')$
 $= \text{closure}(\{f \rightarrow '(' e')' \bullet\})$
 $= \{f \rightarrow '(' e')' \bullet\}.$
9. $s_8 := \text{goto}(s_4, '+')$
 $= \text{closure}(\{e \rightarrow e '+' \bullet p\})$
 $= \{e \rightarrow e '+' \bullet p$
 $\quad p \rightarrow \bullet p '*' f, p \rightarrow \bullet p '/' f, p \rightarrow \bullet f,$
 $\quad f \rightarrow \bullet '(e')', f \rightarrow \bullet N$
 $\quad \}.$
10. $s_9 := \text{goto}(s_4, '-')$
 $= \text{closure}(\{e \rightarrow e '-' \bullet p\})$
 $= \{e \rightarrow e '-' \bullet p$
 $\quad p \rightarrow \bullet p '*' f, p \rightarrow \bullet p '/' f, p \rightarrow \bullet f,$
 $\quad f \rightarrow \bullet '(e')', f \rightarrow \bullet N$
 $\quad \}.$
11. $s_{10} := \text{goto}(s_9, p)$
 $= \text{closure}(\{e \rightarrow e '-' p \bullet, p \rightarrow p \bullet '*' f, p \rightarrow p \bullet '/' f\})$
 $= \{e \rightarrow e '-' p \bullet, p \rightarrow p \bullet '*' f, p \rightarrow p \bullet '/' f\}.$
12. $s_{11} := \text{goto}(s_3, '/')$
 $= \text{closure}(\{p \rightarrow p '/' \bullet f\})$
 $= \{p \rightarrow p '/' \bullet f, f \rightarrow \bullet '(e')', f \rightarrow \bullet N\}.$
13. $s_{12} := \text{goto}(s_3, '*')$
 $= \text{closure}(\{p \rightarrow p '*' \bullet f\})$
 $= \{p \rightarrow p '*' \bullet f, f \rightarrow \bullet '(e')', f \rightarrow \bullet N\}.$
14. $s_{13} := \text{goto}(s_{12}, f)$
 $= \text{closure}(\{p \rightarrow p '*' f \bullet\})$
 $= \{p \rightarrow p '*' f \bullet\}.$
15. $s_{14} := \text{goto}(s_{11}, f)$
 $= \text{closure}(\{p \rightarrow p '/' f \bullet\})$
 $= \{p \rightarrow p '/' f \bullet\}.$

$$\begin{aligned}
16. \quad s_{15} &:= \text{goto}(s_8, p) \\
&= \text{closure}(\{e \rightarrow e '+' p\bullet, p \rightarrow p\bullet '*' f, p \rightarrow p\bullet '/' f\}) \\
&= \{e \rightarrow e '+' p\bullet, p \rightarrow p\bullet '*' f, p \rightarrow p\bullet '/' f\}.
\end{aligned}$$

Further calculations do not yield any new states. For instance, when we compute $\text{goto}(s_8, '(')$, we find:

$$\begin{aligned}
&\text{goto}(s_8, '(') \\
&= \text{closure}(\{f \rightarrow '(' \bullet e')'\}) \\
&= \{f \rightarrow '(' \bullet e')', \\
&\quad e \rightarrow \bullet e '+' p, e \rightarrow \bullet e '-' p, e \rightarrow \bullet p, \\
&\quad p \rightarrow \bullet p '*' f, p \rightarrow \bullet p '/' f, p \rightarrow \bullet f, \\
&\quad f \rightarrow \bullet '(' e')', f \rightarrow \bullet N\} \\
&= s_5.
\end{aligned}$$

Therefore, the set of states for the Shift-Reduce parser is given by

$$Q := \{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}\}$$

Next, we investigate whether there are any conflicts, and we look specifically at the set s_{15} . Due to the marked rule

$$p \rightarrow p\bullet '*' f$$

a shift must occur in state s_{15} if the next token is $'*'$. On the other hand, the state s_{15} includes the rule

$$e \rightarrow e '+' p\bullet.$$

This rule indicates that the symbol stack must be reduced with the grammar rule $e \rightarrow e '+' p$ if a character from the set $\text{Follow}(e)$ appears in the input. If $'*'$ were in $\text{Follow}(e)$, then we would have a Shift-Reduce conflict. However,

$$\text{Follow}(e) = \{ '+', '-', ')', '\$' \},$$

and therefore $'*' \notin \text{Follow}(e)$, so there is no Shift-Reduce conflict here. An examination of the other sets shows that there are also no Shift-Reduce or Reduce-Reduce conflicts in any of them.

Next, we calculate the *action* function. We consider two cases as examples.

1. First, we compute $\text{action}(s_1, '+')$. It holds that

$$\begin{aligned}
\text{action}(s_1, '+') &= \text{action}(\{p \rightarrow f\bullet\}, '+') \\
&= \langle \text{reduce}, p \rightarrow f \rangle,
\end{aligned}$$

since we have $'+' \in \text{Follow}(p)$.

2. Next, we compute $\text{action}(s_4, '+')$. It holds that

$$\begin{aligned}
\text{action}(s_4, '+') &= \text{action}(\{s \rightarrow e\bullet \$, e \rightarrow e\bullet '+' p, e \rightarrow e\bullet '-' p\}, '+') \\
&= \langle \text{shift}, \text{closure}(\{e \rightarrow e '+' \bullet p\}) \rangle \\
&= \langle \text{shift}, s_8 \rangle.
\end{aligned}$$

If we were to continue these calculations, we would obtain Table 8.1, as we have chosen the names of the states to match those of the corresponding states in Tables 8.1 and 8.2.

Exercise 26: Figure 8.7 shows a grammar for propositional formulas in conjunctive normal form.

- (a) Provide the sets $\text{First}(v)$ for all syntactic variables v .
- (b) Provide the sets $\text{Follow}(v)$ for all syntactic variables v .
- (c) Calculate the set of SLR states.
- (d) Specify the function *action*.
- (e) Specify the function *goto*.

Abbreviate the names of the syntactic variables and terminals with s , c , d , l , and I , where s stands for the newly introduced start symbol. \diamond

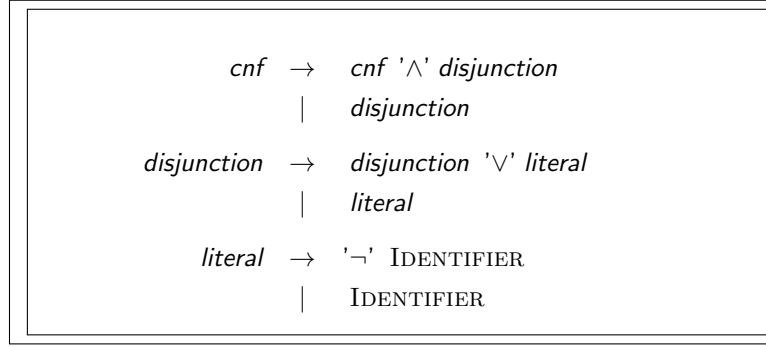


Figure 8.7: A grammar for Boolean expression in conjunctive normal form.

8.3.3 Shift/Reduce and Reduce/Reduce Conflicts

In this section, we examine shift-reduce and reduce-reduce conflicts in more detail and consider two examples. The first example shows a shift-reduce conflict. The grammar shown in Figure 8.8 is ambiguous because it does not specify whether the operator '+' binds stronger or weaker than the operator '*'. If we interpret the non-terminal N as an abbreviation for NUMBER, this grammar allows us to interpret the expression $1 + 2 * 3$ either as

$$(1 + 2) * 3 \quad \text{or as} \quad 1 + (2 * 3).$$

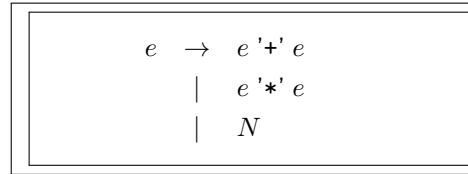


Figure 8.8: A grammar with shift/reduce conflicts.

We augment the grammar and calculate the start state s_0 :

$$\begin{aligned} s_0 &= \text{closure}(\{s \rightarrow \bullet e \$\}) \\ &= \{s \rightarrow \bullet e \$, e \rightarrow \bullet e' + e, e \rightarrow \bullet e' * e, e \rightarrow \bullet N\}. \end{aligned}$$

Next, we calculate $s_1 := \text{goto}(s_0, e)$:

$$\begin{aligned} s_1 &= \text{goto}(s_0, e) \\ &= \text{closure}(\{s \rightarrow e \bullet \$, e \rightarrow e \bullet + e, e \rightarrow e \bullet * e\}) \\ &= \{s \rightarrow e \bullet \$, e \rightarrow e \bullet + e, e \rightarrow e \bullet * e\} \end{aligned}$$

Now we calculate $s_2 := \text{goto}(s_1, '+')$:

$$\begin{aligned} s_2 &= \text{goto}(s_1, '+') \\ &= \text{closure}(\{e \rightarrow e' + \bullet e, \}) \\ &= \{e \rightarrow e' + \bullet e, e \rightarrow \bullet e' + e, e \rightarrow \bullet e' * e, e \rightarrow \bullet N\} \end{aligned}$$

Next, we calculate $s_3 := \text{goto}(s_2, e)$:

$$\begin{aligned} s_3 &= \text{goto}(s_2, e) \\ &= \text{closure}(\{e \rightarrow e' + e \bullet, e \rightarrow e \bullet + e, e \rightarrow e \bullet * e\}) \\ &= \{e \rightarrow e' + e \bullet, e \rightarrow e \bullet + e, e \rightarrow e \bullet * e\} \end{aligned}$$

Here, a shift-reduce conflict arises during the calculation of $\text{action}(s_3, '*')$, because on one hand, the marked rule

$$e \rightarrow e \bullet * e,$$

demand that the token '*' is pushed onto the stack, but on the other hand, we have

$$\text{Follow}(e) = \{ '+', '*', '\$' \},$$

so that, if the next token to be read has the value '*', the symbol stack should be reduced with the rule

$$e \rightarrow e' + e \bullet.$$

Remark: It is not surprising that we found a conflict in the grammar specified above, as this grammar is ambiguous. On the other hand, it can be shown that every SLR grammar must be unambiguous. Consequently, an ambiguous grammar is never an SLR grammar. However, the converse of this statement is not true, as we will see in the next example. \diamond

$$\begin{array}{lcl} s & \rightarrow & a 'x' a 'y' \\ & | & b 'y' b 'x' \\ a & \rightarrow & \lambda \\ b & \rightarrow & \lambda \end{array}$$

Figure 8.9: A grammar containing a reduce/reduce conflict.

Next, we examine a grammar that is not an SLR grammar because it contains reduce-reduce conflicts. We consider the grammar shown in Figure 8.9. This grammar is unambiguous, as we have

$$L(s) = \{ 'xy', 'yx' \}$$

and the string 'xy' can only be derived using the rule $s \rightarrow a 'x' a 'y'$, while the string 'yx' can only be generated using the rule $s \rightarrow b 'y' b 'x'$. To demonstrate that this grammar contains shift-reduce conflicts, we calculate the start state of an SLR parser for this grammar.

$$\begin{aligned} s_0 &= \text{closure}(\{\hat{s} \rightarrow \bullet s \$\}) \\ &= \{\hat{s} \rightarrow \bullet s \$, s \rightarrow \bullet a 'x' a 'y', s \rightarrow \bullet b 'y' b 'x', a \rightarrow \bullet \lambda, b \rightarrow \bullet \lambda\} \\ &= \{\hat{s} \rightarrow \bullet s \$, s \rightarrow \bullet a 'x' a 'y', s \rightarrow \bullet b 'y' b 'x', a \rightarrow \lambda \bullet, b \rightarrow \lambda \bullet\}, \end{aligned}$$

since $a \rightarrow \bullet \lambda$ is the same as $a \rightarrow \lambda \bullet$. In this state, there is a reduce-reduce conflict between the two marked rules

$$a \rightarrow \lambda \bullet \quad \text{and} \quad b \rightarrow \lambda \bullet.$$

This conflict arises when calculating

$$\text{action}(s_0, 'x')$$

because we have

$$\text{Follow}(a) = \{ 'x', 'y' \} = \text{Follow}(b).$$

Thus, it is unclear which of these rules the parser should use to reduce the input in state s_0 when the next token read has the value 'x', as this token is both an element of the set $\text{Follow}(a)$ and the set $\text{Follow}(b)$.

Remark: As part of the resources provided with this lecture, the file

[Formal-Languages/blob/master/Python/Chapter-9/SLR-Table-Generator.ipynb](https://github.com/robertwhitby/Formal-Languages/blob/master/Python/Chapter-9/SLR-Table-Generator.ipynb)

contains a *Python* program that checks whether a given grammar is an SLR grammar. This program computes the states as well as the action table of a given grammar. \diamond

Exercise 27: The github directory containing supplementary files for this lecture contains the grammar for the programming language C at

[Formal-Languages/blob/master/Python/Chapter-9/Examples/c-grammar.g](https://github.com/robertwhitby/Formal-Languages/blob/master/Python/Chapter-9/Examples/c-grammar.g)

This grammar is not an SLR-grammar. Use the SLR-table generator [SLR-Table-Generator.ipynb](https://github.com/robertwhitby/Formal-Languages/blob/master/Python/Chapter-9/SLR-Table-Generator.ipynb) to investigate the conflicts that arise. Transform the grammar into an SLR grammar by making a small number of changes. It is sufficient to remove three rules. After removing those rules, there will be one useless rule left which is of the form

$$a \rightarrow b.$$

This rule has to be removed, too. Furthermore, you have to rename the variable a occurring on the left hand side of this rule to b everywhere. \diamond

8.4 Canonical LR Parsers

The reduce-reduce conflict that occurs in the grammar shown in Figure 8.9 can be resolved as follows: In the state

$$\begin{aligned} s_0 &= \text{closure}(\{\hat{s} \rightarrow \bullet s \$\}) \\ &= \{\hat{s} \rightarrow \bullet s \$, s \rightarrow \bullet a 'x' a 'y', s \rightarrow \bullet b 'y' b 'x', a \rightarrow \lambda \bullet, b \rightarrow \lambda \bullet\} \end{aligned}$$

the marked rules $a \rightarrow \lambda \bullet$ and $b \rightarrow \lambda \bullet$ arise from the closure calculation of the rules

$$s \rightarrow \bullet a 'x' a 'y' \quad \text{and} \quad s \rightarrow \bullet b 'y' b 'x'.$$

For the first rule, it is clear that the first a must be followed by a ' x ', and for the second rule, we see that the first b must be followed by a ' y '. This information goes beyond what is contained in the sets $\text{Follow}(a)$ and $\text{Follow}(b)$, as we now consider the [context](#) in which the syntactic variable appears. This allows us to define the function $\text{action}(s_0, 'x')$ and $\text{action}(s_0, 'y')$ as follows:

$$\text{action}(s_0, 'x') = \langle \text{reduce}, a \rightarrow \lambda \rangle \quad \text{and} \quad \text{action}(s_0, 'y') = \langle \text{reduce}, b \rightarrow \lambda \rangle.$$

This definition resolves the reduce-reduce conflict. The central idea is to include the context in which a rule appears in the closure calculation. To do this, we expand the definition of a marked rule next.

Definition 27 (Extended Marked Rule)

An [extended marked rule](#) (abbreviated: [e.m.R.](#)) of a grammar $G = \langle V, T, R, s \rangle$ is a quadruple

$$\langle a, \beta, \gamma, L \rangle,$$

where:

1. $(a \rightarrow \beta\gamma) \in R$.
2. $L \subseteq T$.

We write the extended marked rule $\langle a, \beta, \gamma, L \rangle$ as

$$a \rightarrow \beta \bullet \gamma : L.$$

If L consists of only one element t , that is if $L = \{t\}$, we omit the set brackets and write the rule as

$$a \rightarrow \beta \bullet \gamma : t.$$

\diamond

Intuitively, we interpret the e.m.R. $a \rightarrow \beta \bullet \gamma : L$ as a state in which the following applies:

- (a) The parser attempts to recognize an a using the grammar rule $a \rightarrow \beta\gamma$.
- (b) So far, β has been recognized. For the rule $a \rightarrow \beta\gamma$ to be applied, γ must now be recognized.
- (c) Additionally, we know that a token from the set L must follow the syntactic variable a .

Therefore, we refer to the set L as the set of [Follow Tokens](#).

Working with extended marked rules is quite similar to working with marked rules, however, we need to modify the definitions of the functions *closure*, *goto*, and *action* slightly. We begin with the function *closure*.

Definition 28 ($\text{closure}(\mathcal{M})$) Let \mathcal{M} be a set of extended marked rules. We define the [closure](#) of \mathcal{M} as the smallest set \mathcal{K} of marked rules for which the following holds:

1. $\mathcal{M} \subseteq \mathcal{K}$,
thus the closure encompasses the original set of rules.

2. On one hand, if

$$a \rightarrow \beta \bullet c \delta : L$$

is an e.m.R. from the set \mathcal{K} , where c is a syntactic variable, and on the other hand, if

$$c \rightarrow \gamma$$

is a grammar rule of the underlying grammar G , then the e.m.R.

$$c \rightarrow \bullet \gamma : \bigcup \{ \text{First}(\delta t) \mid t \in L \}$$

is also an element of the set \mathcal{K} . The function $\text{First}(\alpha)$ computes for a string $\alpha \in (T \cup V)^*$ the set of all tokens t with which a string that has been derived from α can begin.

The uniquely determined set \mathcal{K} defined in this way is again denoted by $\text{closure}(\mathcal{M})$. \diamond

Remark: Compared to the old definition, only the computation of the set of follow tokens has been added. The context in which the variable c , to be recognized by the rule $c \rightarrow \gamma$, appears is initially given by the string δ that follows c in the rule $a \rightarrow \beta \bullet c \delta : L$. Now it is possible that δ derives the empty string λ . In this case, the follow tokens from the set L also play a role, because if $\delta \Rightarrow^* \lambda$ is true, then a follow token t from the set L can also follow the c . \diamond

For a given set of e.m.R.s \mathcal{M} , the computation of $\mathcal{K} := \text{closure}(\mathcal{M})$ can be done iteratively. Figure 8.10 shows the computation of $\text{closure}(\mathcal{M})$. The main difference compared to the earlier computation of $\text{closure}()$ is that in the e.m.R.s that we include for a variable c in $\text{closure}(\mathcal{M})$, we consider the context in which c appears in the set of Follow Tokens. This makes it possible to describe the states of the parser more precisely than with marked rules.

```

1  function closure( $\mathcal{M}$ ) {
2       $\mathcal{K} := \mathcal{M}$ ;
3       $\mathcal{K}^- := \{\}$ ;
4      while ( $\mathcal{K}^- \neq \mathcal{K}$ ) {
5           $\mathcal{K}^- := \mathcal{K}$ ;
6           $\mathcal{K} := \mathcal{K} \cup \{ (c \rightarrow \bullet \gamma : \bigcup \{ \text{First}(\delta t) \mid t \in L \}) \mid (a \rightarrow \beta \bullet c \delta : L) \in \mathcal{K} \wedge (c \rightarrow \gamma) \in R \}$ ;
7      }
8      return  $\mathcal{K}$ ;
9  }
```

Figure 8.10: The computation of $\text{closure}(\mathcal{M})$.

Remark: The expression $\bigcup \{ \text{First}(\delta t) \mid t \in L \}$ looks more complicated than it actually is. To compute this expression, it is useful to make a distinction based on whether δ can derive the empty string λ or not, because it holds that

$$\bigcup \{ \text{First}(\delta t) \mid t \in L \} = \begin{cases} \text{First}(\delta) \cup L & \text{if } \delta \Rightarrow^* \lambda; \\ \text{First}(\delta) & \text{otherwise.} \end{cases}$$

The computation of $\text{goto}(\mathcal{M}, t)$ for a set \mathcal{M} of extended rules and a symbol x changes from the computation in the case of simple marked rules only by appending the set of **follow tokens**, which itself remains unchanged:

$$\text{goto}(\mathcal{M}, x) := \text{closure} \left(\{ a \rightarrow \beta x \bullet \delta : L \mid (a \rightarrow \beta \bullet x \delta : L) \in \mathcal{M} \} \right).$$

Similar to the theory of SLR parsers, we augment our grammar G by adding a new start variable \hat{s} to the set of variables and the new rule $\hat{s} \rightarrow s$ to the set of rules. Further, we add the symbol $\$$ to the tokens. Then the start state has the form

$$q_0 := \text{closure}(\{ \hat{s} \rightarrow \bullet s : \$ \}),$$

because the file end symbol “\$” must follow the start symbol. Finally, we show how the definition of the function $\text{action}()$ must be changed. We specify the computation of this function through the following conditional equations.

1. $(a \rightarrow \beta \bullet t \delta : L) \in \mathcal{M} \Rightarrow \text{action}(\mathcal{M}, t) := \langle \text{shift}, \text{goto}(\mathcal{M}, t) \rangle.$
2. $(a \rightarrow \beta \bullet : L) \in \mathcal{M} \wedge a \neq \hat{s} \wedge t \in L \Rightarrow \text{action}(\mathcal{M}, t) := \langle \text{reduce}, a \rightarrow \beta \rangle.$
3. $(\hat{s} \rightarrow s \bullet : \$) \in \mathcal{M} \Rightarrow \text{action}(\mathcal{M}, \$) := \text{accept}.$
4. Otherwise: $\text{action}(\mathcal{M}, t) := \text{error}.$

If a conflict occurs in these equations, because both the condition of the first equation and the condition of the second equation are met, then we speak again of a **shift-reduce conflict**. A shift-reduce conflict thus occurs in the computation of $\text{action}(\mathcal{M}, t)$ when there are two e.m.R.s

$$(a \rightarrow \beta \bullet t \delta : L_1) \in \mathcal{M} \quad \text{and} \quad (c \rightarrow \gamma \bullet : L_2) \in \mathcal{M} \quad \text{with} \quad t \in L_2$$

because then it is unclear whether in state \mathcal{M} the token t should be pushed onto the stack, or whether instead the symbol stack should be reduced with the rule $c \rightarrow \gamma$.

Remark: Compared to an SLR parser, the possibility of shift-reduce conflicts is reduced, because in an SLR parser, a shift-reduce conflict already occurs if $t \in \text{Follow}(c)$ and the set L_2 is usually smaller than the set $\text{Follow}(c)$. \diamond

A **reduce-reduce conflict** occurs when there are two e.m.R.s

$$(a \rightarrow \beta \bullet : L_1) \in \mathcal{M} \quad \text{and} \quad (c \rightarrow \delta \bullet : L_2) \in \mathcal{M} \quad \text{with} \quad L_1 \cap L_2 \neq \{\}$$

because then it is unclear which of these two rules should be used to reduce the symbol stack when the next token is an element of the intersection $L_1 \cap L_2$.

Remark: Compared to an SLR parser, the possibility of reduce-reduce conflicts is reduced, because in an SLR parser, a reduce-reduce conflict already occurs if there exists a t in the set $\text{Follow}(a) \cap \text{Follow}(c)$ and the Follow sets are often larger than the sets L_1 and L_2 . \diamond

Example: We revisit the example of the grammar shown in Figure 8.9 and first calculate the set of all states.

1. $s_0 := \text{closure}(\{\hat{s} \rightarrow \bullet s : \$\})$
 $= \{\hat{s} \rightarrow \bullet s : \$, s \rightarrow \bullet a'x'a'y' : \$, s \rightarrow \bullet b'y'b'x' : \$, a \rightarrow \bullet : 'x', b \rightarrow \bullet : 'y'\}.$
2. $s_1 := \text{goto}(s_0, a)$
 $= \text{closure}(\{s \rightarrow a \bullet 'x'a'y' : \$\})$
 $= \{s \rightarrow a \bullet 'x'a'y' : \$\}.$
3. $s_2 := \text{goto}(s_0, s)$
 $= \text{closure}(\{\hat{s} \rightarrow s \bullet : \$\})$
 $= \{\hat{s} \rightarrow s \bullet : \$\}.$
4. $s_3 := \text{goto}(s_0, b)$
 $= \text{closure}(\{s \rightarrow b \bullet 'y'b'x' : \$\})$
 $= \{s \rightarrow b \bullet 'y'b'x' : \$\}.$
5. $s_4 := \text{goto}(s_3, 'y')$
 $= \text{closure}(\{s \rightarrow b'y' \bullet b'x' : \$\})$
 $= \{s \rightarrow b'y' \bullet b'x' : \$, b \rightarrow \bullet : 'x'\}.$
6. $s_5 := \text{goto}(s_4, b)$
 $= \text{closure}(\{s \rightarrow b'y'b \bullet 'x' : \$\})$
 $= \{s \rightarrow b'y'b \bullet 'x' : \$\}.$
7. $s_6 := \text{goto}(s_5, 'x')$
 $= \text{closure}(\{s \rightarrow b'y'b'x' \bullet : \$\})$
 $= \{s \rightarrow b'y'b'x' \bullet : \$\}.$

8. $s_7 := \text{goto}(s_1, 'x')$
 $= \text{closure}(\{s \rightarrow a'x' \bullet a'y' : \$\})$
 $= \{s \rightarrow a'x' \bullet a'y' : \$, a \rightarrow \bullet : 'y'\}.$
9. $s_8 := \text{goto}(s_7, a)$
 $= \text{closure}(\{s \rightarrow a'x'a \bullet 'y' : \$\})$
 $= \{s \rightarrow a'x'a \bullet 'y' : \$\}.$
10. $s_9 := \text{goto}(s_8, 'y')$
 $= \text{closure}(\{s \rightarrow a'x'a'y' \bullet : \$\})$
 $= \{s \rightarrow a'x'a'y' \bullet : \$\}.$

Next, we examine whether there are any conflicts in the states. In the last section, we found a Reduce-Reduce conflict in the start state s_0 between the two rules $a \rightarrow \lambda$ and $b \rightarrow \lambda$, because

$$\text{Follow}(a) \cap \text{Follow}(b) = \{'x', 'y'\} \neq \{\}$$

holds. This conflict has now disappeared, as there is no conflict between the e.m.R.s

$$a \rightarrow \bullet : 'x' \quad \text{and} \quad b \rightarrow \bullet : 'y'$$

due to $'x' \neq 'y'$. It is easy to see that no conflicts arise in the other states either.

Exercise 28: Calculate the set of states for an LR parser for the following grammar:

$$\begin{array}{lcl} e & \rightarrow & e '+' p \\ & | & p \\ p & \rightarrow & p '*' f \\ & | & f \\ f & \rightarrow & '(' e ')' \\ & | & \text{Number} \end{array}$$

Additionally, investigate whether this grammar has shift-reduce conflicts or reduce-reduce conflicts.

Remark: As part of the resources provided with this lecture, the file

[ANTLR4-Python/LR-Parser-Generator/LR-Table-Generator.ipynb](#)

contains a *Python* program that checks whether a given grammar qualifies as a canonical LR grammar. This program computes the LR-states as well as the action table for a given grammar. \diamond

Remark: The theory of LR-parsing has been developed by Donald E. Knuth [Knu65]. His theory is described in the paper “[On the translation of languages from left to right](#)”. \diamond

8.5 LALR-Parser

The number of states in an LR parser is often significantly larger than the number of states in an SLR parser for the same grammar. For instance, an SLR parser for the [C grammar](#) manages with 349 states. However, since the language C is not an SLR language, creating an SLR parse table for C results in a number of [conflicts](#), so that an SLR parser for the language C does not work. In contrast, an LR parser for the language C has 1572 states, as you can see [here](#). In the seventies, when the available main memory of most computers was still modestly sized compared to today, LR parsers therefore had a size that was problematic in practice. A detailed analysis of the set of states of LR parsers showed that it is often possible to combine certain states. This can significantly reduce the number of states in most cases. We illustrate the concept with an example and consider the grammar shown in Figure 8.11, which I have taken from the [Dragon Book](#) [ASUL06]. (The “Dragon Book” is the standard work in the field of compiler construction.)

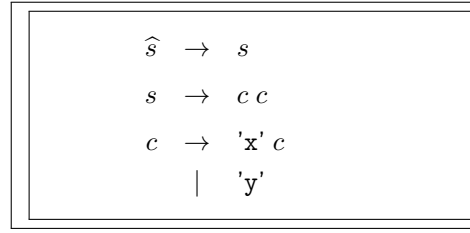


Figure 8.11: Eine Grammatik aus dem Drachenbuch.

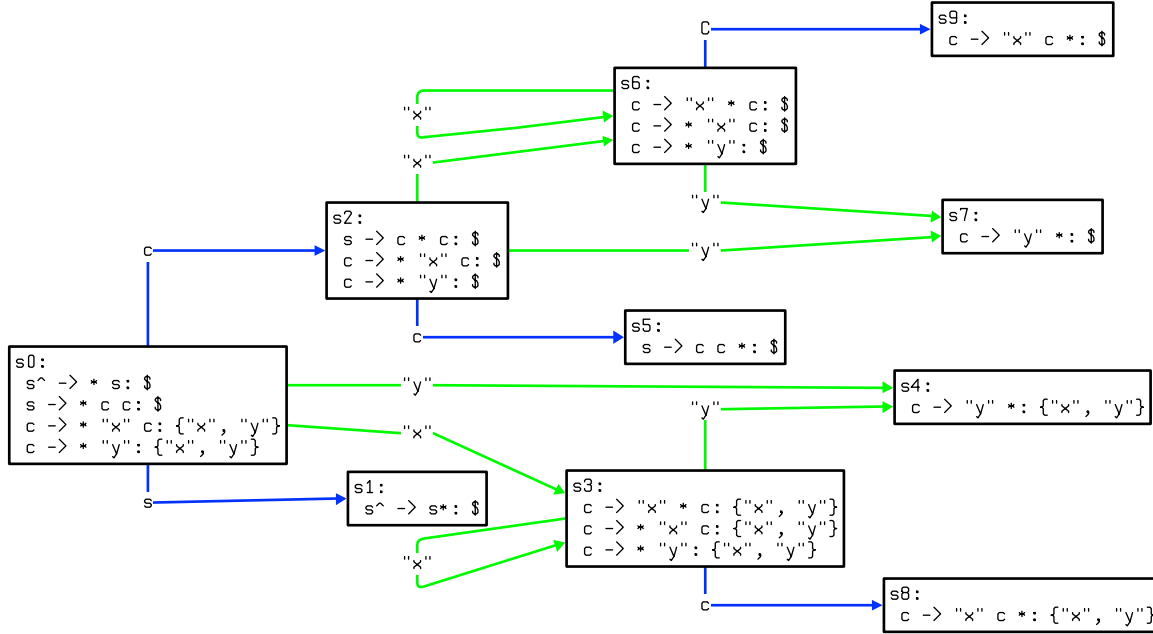


Figure 8.12: LR-goto-graph for the grammar of Figure 8.11.

Figure 8.12 shows the so-called **LR-goto-graph** for this grammar. The nodes of this graph are the states. Looking at the LR-goto-graph, we find that the states s_6 and s_3 differ only in the sets of follow tokens, as on one hand, we have

$$s_6 = \{s \rightarrow 'x' \bullet c : '\$', c \rightarrow \bullet 'x' c : '\$', c \rightarrow \bullet 'y' : '\$'\},$$

and on the other hand, we have

$$s_3 = \{s \rightarrow 'x' \bullet c : \{'x', 'y'\}, c \rightarrow \bullet 'x' c : \{'x', 'y'\}, c \rightarrow \bullet 'y' : \{'x', 'y'\}\}.$$

Clearly, the set s_3 is derived from the set s_6 by replacing everywhere '\$' with the set $\{'x', 'y'\}$. Similarly, the set s_7 can be transformed into s_4 , and s_9 into s_8 . The crucial realization is that the function `goto()` is invariant under this kind of transformation, because in the definition of this function, the set of follow tokens is irrelevant. For example, we see that

$$\text{goto}(s_3, c) = s_8 \quad \text{and} \quad \text{goto}(s_6, c) = s_9$$

hold and that, on the other hand, the state s_9 changes into the state s_8 when we replace everywhere in s_9 the terminal '\$' with the set $\{'x', 'y'\}$. If we define the **core** of a set of extended marked rules by omitting the set of follow tokens in each rule, and then combine states with the same core, we obtain the goto-graph shown in Figure 8.13.

To generalize and formalize the observations made while examining the grammar shown in Figure 8.11, we define

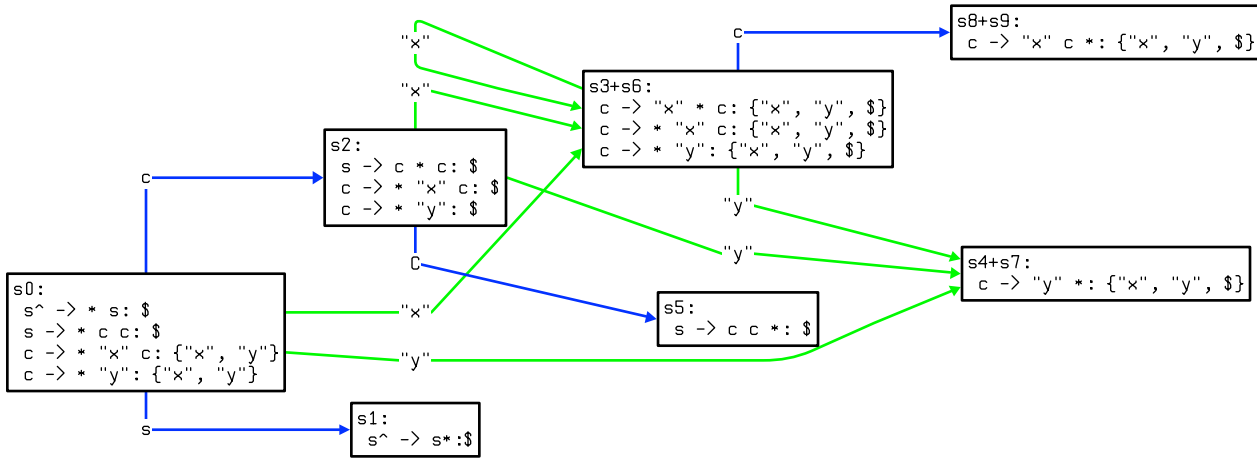


Figure 8.13: The LALR-goto-graph for the grammar from Figure 8.11.

a function $core()$, which calculates the core of a set of e.m.R.s (extended marked rules) and thus transforms this set into a set of marked rules:

$$core(\mathcal{M}) := \{a \rightarrow \beta \bullet \gamma \mid (a \rightarrow \beta \bullet \gamma : L) \in \mathcal{M}\}.$$

Thus, the function $core()$ simply removes the set of follow tokens from the e.m.R.s. We had defined the function $goto()$ for a set \mathcal{M} of extended marked rules and a symbol x as

$$goto(\mathcal{M}, x) := closure\left(\{a \rightarrow \beta x \bullet \gamma : L \mid (a \rightarrow \beta \bullet x \gamma : L) \in \mathcal{M}\}\right).$$

Clearly, the set of follow tokens plays no role in the calculation of $goto(\mathcal{M}, x)$, formally for two e.m.R.-sets \mathcal{M}_1 and \mathcal{M}_2 and a symbol x , the formula holds:

$$core(\mathcal{M}_1) = core(\mathcal{M}_2) \Rightarrow core(goto(\mathcal{M}_1, x)) = core(goto(\mathcal{M}_2, x)).$$

For two sets of e.m.R.s (extended marked rules) \mathcal{M} and \mathcal{N} that have the same core, we define the **extended union** $\mathcal{M} \uplus \mathcal{N}$ of \mathcal{M} and \mathcal{N} as

$$\mathcal{M} \uplus \mathcal{N} := \{a \rightarrow \beta \bullet \gamma : K \cup L \mid (a \rightarrow \beta \bullet \gamma : K) \in \mathcal{M} \wedge (a \rightarrow \beta \bullet \gamma : L) \in \mathcal{N}\}.$$

We generalize this definition to an operation \uplus , defined on a set of sets of e.m.R.s: If \mathcal{I} is a set of sets of e.m.R.s, all having the same core, i.e.,

$$\mathcal{I} = \{\mathcal{M}_1, \dots, \mathcal{M}_k\} \quad \text{with} \quad core(\mathcal{M}_i) = core(\mathcal{M}_j) \quad \text{for all } i, j \in \{1, \dots, k\},$$

then we define

$$\biguplus \mathcal{I} := \mathcal{M}_1 \uplus \dots \uplus \mathcal{M}_k.$$

Let Δ be the set of all states of an LR parser. Then, the set of states of the corresponding LALR parser is given by the extended union of all subsets of Δ whose elements have the same core:

$$\Omega := \left\{ \biguplus \mathcal{I} \mid \mathcal{I} \in 2^\Delta \wedge \forall \mathcal{M}, \mathcal{N} \in \mathcal{I} : core(\mathcal{M}) = core(\mathcal{N}) \wedge \text{and } \mathcal{I} \text{ is maximal} \right\}.$$

The requirement “ \mathcal{I} maximal” in the above definition expresses that in \mathcal{I} , indeed all sets from Δ are combined that have the same core. The thus defined set Ω is the set of LALR states.

Next, we consider how the computation of $goto(\mathcal{M}, X)$ must change if \mathcal{M} is an element of the set Ω of LALR states. To compute $goto(\mathcal{M}, X)$, we first calculate the set

$$closure\left(\{A \rightarrow \alpha X \bullet \beta : L \mid (A \rightarrow \alpha \bullet X \beta : L) \in \mathcal{M}\}\right).$$

The problem is that this set is generally not an element of the set Ω , as the states in Ω arise from the combination of

several LR states. However, the states that are combined in the computation of Ω all have the same core. Therefore, the set

$$\left\{ q \in \Omega \mid \text{core}(q) = \text{core}\left(\text{closure}\left(\{a \rightarrow \beta x \bullet \gamma : L \mid (a \rightarrow \beta \bullet x \gamma : L) \in \mathcal{M}\}\right)\right) \right\}$$

contains exactly one element and this element is the value of $\text{goto}(\mathcal{M}, X)$. Thus, we can set

$$\text{goto}(\mathcal{M}, X) := \text{arb}\left(\left\{ q \in \Omega \mid \text{core}(q) = \text{core}\left(\text{closure}\left(\{a \rightarrow \beta X \bullet \gamma : L \mid (a \rightarrow \beta \bullet X \gamma : L) \in \mathcal{M}\}\right)\right) \right\}\right)$$

The function $\text{arb}()$ used here serves to extract an arbitrary element from a set. Since the set from which the element is extracted contains exactly one element, $\text{goto}(\mathcal{M}, x)$ is well-defined. The computation of the expression $\text{action}(\mathcal{M}, t)$ does not change compared to the computation for an LR parser.

8.6 Comparison of SLR, LR, and LALR Parsers

We now want to compare the different methods with which we have constructed Shift-Reduce parsers in this chapter. We call a language \mathcal{L} an **SLR language** if \mathcal{L} can be recognized by an SLR parser. The terms **canonical LR language** and **LALR language** are defined analogously. The following relationships exist between these languages:

$$\text{SLR language} \subsetneq \text{LALR language} \subsetneq \text{canonical LR language} \quad (\star)$$

These inclusions are easy to understand: In defining the LR parsers, we added sets of follow tokens to the marked rules. This made it possible to avoid Shift-Reduce and Reduce-Reduce conflicts in certain cases. As the state sets of canonical LR parsers can become very large, we then combined sets of extended marked rules that have the same core. This is how we obtained the LALR parsers. However, by combining rule sets, we can in some cases introduce Reduce-Reduce conflicts, so the set of LALR languages is a subset of the canonical LR languages.

We will show in the following subsections that the inclusions in (\star) are proper.

8.6.1 SLR Language \subsetneq LALR Language

The states of an LALR parser, as compared to those of an SLR parser, include sets of follow tokens. This makes LALR parsers at least as powerful as SLR parsers. We will now demonstrate that LALR parsers are indeed more powerful than SLR parsers. To support this claim, we present a grammar for which there exists an LALR parser, but no SLR parser. We had seen on page 110 that the grammar

$$\begin{aligned} s &\rightarrow a'x'a'y' \mid b'y'b'x' \\ a &\rightarrow \lambda \\ b &\rightarrow \lambda \end{aligned}$$

is not an SLR grammar. Later, we saw that this grammar can be parsed by a canonical LR parser. We will now show that this grammar can also be parsed by an LALR parser. To do this, we calculate the set of LALR states. This requires first calculating the set of canonical LR states. We had already carried out this calculation earlier and obtained the following states:

1. $s_0 = \{\widehat{s} \rightarrow \bullet s : \$, s \rightarrow \bullet a'x'a'y' : \$, s \rightarrow \bullet b'y'b'x' : \$, a \rightarrow \bullet : 'x', b \rightarrow \bullet : 'y'\},$
2. $s_1 = \{s \rightarrow a \bullet 'x'a'y' : \$\},$
3. $s_2 = \{\widehat{s} \rightarrow s \bullet : \$\},$
4. $s_3 = \{s \rightarrow b \bullet 'y'b'x' : \$\},$
5. $s_4 = \{s \rightarrow b'y' \bullet b'x' : \$, b \rightarrow \bullet : 'x'\},$
6. $s_5 = \{s \rightarrow b'y'b \bullet 'x' : \$\},$
7. $s_6 = \{s \rightarrow b'y'b'x' \bullet : \$\},$
8. $s_7 = \{s \rightarrow a'x' \bullet a'y' : \$, a \rightarrow \bullet : 'y'\},$

$$9. s_8 = \{s \rightarrow a'x'a \bullet 'y' : \$\},$$

$$10. s_9 = \{s \rightarrow a'x'a'y' \bullet : \$\}.$$

We observe that the cores of all the states listed here are different. Therefore, for this grammar, the set of states of the LALR parser coincides with the set of states of the canonical LR parser. This implies that there are no conflicts in the LALR states, because in the transition from canonical LR parsers to LALR parsers, we have only combined states with the same core, and the definition of the functions *goto()* and *action()* remained unchanged.

8.6.2 LALR Language \subsetneq Canonical LR Language

We defined LALR parsers by combining different states of a canonical LR parser. Therefore, it is clear that canonical LR parsers are at least as powerful as LALR parsers. To demonstrate that canonical LR parsers are indeed more powerful than LALR parsers, we need a grammar for which a canonical LR parser can be generated, but not an LALR parser. Figure 8.14 shows such a grammar, which I have taken from the Dragon Book.

s	\rightarrow	$'v' a 'y'$
	$ $	$'w' b 'y'$
	$ $	$'v' b 'z'$
	$ $	$'w' a 'z'$
a	\rightarrow	$'x'$
b	\rightarrow	$'x'$

Figure 8.14: A canonical LR grammar that is not an LALR grammar.

We first calculate the set of states for a canonical LR parser for this grammar. We obtain the following sets of extended marked rules:

1. $s_0 = \text{closure}(\hat{s} \rightarrow \bullet s : \$) = \{ \hat{s} \rightarrow \bullet s : \$, \\ s \rightarrow \bullet 'v' a 'y' : \$, \\ s \rightarrow \bullet 'v' b 'z' : \$, \\ s \rightarrow \bullet 'w' a 'z' : \$, \\ s \rightarrow \bullet 'w' b 'y' : \$ \},$
2. $s_1 = \text{goto}(s_0, s) = \{ \hat{s} \rightarrow s \bullet : \$ \}$
3. $s_2 = \text{goto}(s_0, 'v') = \{ s \rightarrow 'v' \bullet b 'z' : \$, \\ s \rightarrow 'v' \bullet a 'y' : \$, \\ a \rightarrow \bullet 'x' : 'y', \\ b \rightarrow \bullet 'x' : 'z' \},$
4. $s_3 = \text{goto}(s_0, 'w') = \{ s \rightarrow 'w' \bullet a 'z' : \$, \\ s \rightarrow 'w' \bullet b 'y' : \$, \\ a \rightarrow \bullet 'x' : 'z', \\ b \rightarrow \bullet 'x' : 'y' \},$
5. $s_4 = \text{goto}(s_2, 'x') = \{ a \rightarrow 'x' \bullet : 'y', b \rightarrow 'x' \bullet : 'z' \},$
6. $s_5 = \text{goto}(s_3, 'x') = \{ a \rightarrow 'x' \bullet : 'z', b \rightarrow 'x' \bullet : 'y' \},$
7. $s_6 = \text{goto}(s_2, a) = \{ s \rightarrow 'v' a \bullet 'y' : \$ \},$
8. $s_7 = \text{goto}(s_6, 'y') = \{ s \rightarrow 'v' a 'y' \bullet : \$ \},$

9. $s_8 = \text{goto}(s_2, b) = \{s \rightarrow 'v'b \bullet 'z' : \$\},$
10. $s_9 = \text{goto}(s_8, 'z') = \{s \rightarrow 'v'b'z' \bullet : \$\},$
11. $s_{10} = \text{goto}(s_3, a) = \{s \rightarrow 'w'a \bullet 'z' : \$\},$
12. $s_{11} = \text{goto}(s_{10}, 'z') = \{s \rightarrow 'w'a'z' \bullet : \$\},$
13. $s_{12} = \text{goto}(s_3, b) = \{s \rightarrow 'w'b \bullet 'y' : \$\},$
14. $s_{13} = \text{goto}(s_{12}, 'y') = \{s \rightarrow 'w'b'y' \bullet : \$\}.$

The only states where conflicts could occur are the sets s_4 and s_5 , as in these states, reductions with both rules

$$a \rightarrow 'x' \quad \text{and} \quad b \rightarrow 'x'$$

are potentially possible. However, since the sets of follow tokens have an empty intersection, there is actually no conflict, and the grammar is a canonical LR grammar.

Next, we calculate the LALR states for the grammar mentioned above. The only states that have a common core are s_4 and s_5 , as

$$\text{core}(s_4) = \{a \rightarrow 'x' \bullet, b \rightarrow 'x' \bullet\} = \text{core}(s_5).$$

In calculating the LALR states, these two states are combined into one state $s_{\{4,5\}}$. This new state takes the form

$$s_{\{4,5\}} = \{A \rightarrow 'x' \bullet : \{'y', 'z'\}, B \rightarrow 'x' \bullet : \{'y', 'z'\}\}.$$

Here, there is obviously a reduce-reduce conflict, as on one hand we have

$$\text{action}(s_{\{4,5\}}, 'y') = \langle \text{reduce}, A \rightarrow 'x' \rangle,$$

but on the other hand, it also holds that

$$\text{action}(s_{\{4,5\}}, 'y') = \langle \text{reduce}, B \rightarrow 'x' \rangle.$$

Exercise 29: Let $G = \langle V, T, R, s \rangle$ be an LR grammar and \mathcal{N} be the set of LALR states of the grammar. Explain why there can be no shift-reduce conflicts in the set \mathcal{N} . \diamond

Historical Notes The theory of LALR parsing is due to Franklin L. DeRemer [DeR71]. At the time of its invention, the space savings of LALR parsing in comparison to LR parsing were crucial.

8.6.3 Evaluation of the Different Methods

In practice, SLR parsers are not sufficient, as there are a number of practically relevant language constructs for which no SLR parser can be generated. Canonical LR parsers are much more powerful but often require significantly more states. LALR parsers represent a compromise here: on the one hand, LALR languages are almost as expressive as canonical LR languages, but on the other hand, the memory requirements of LALR parsers are in the same order of magnitude as those of SLR parsers. For example, the SLR parse table for the C language has a total of 349 states, the corresponding LR parse table has 1572 states, while the LALR parser manages with 350 states, thus only one state more than the SLR parser. In the main memories typically available today, however, canonical LR parsers can usually be accommodated without difficulty, so there is actually no compelling reason to use an LALR parser instead of an LR parser.

On the other hand, no one will want to program an LALR parser or a canonical LR parser by hand. Instead, you will later use a parser generator like *Bison* or *JavaCup*, which generates a parser for you. *Bison* is a parser generator for C, C++ and also offers, albeit still experimental, support for *Java*, while *JavaCup* is limited to the language *Java*. If you use *JavaCup*, you have no choice, as this tool always generates an LALR parser. With *Bison*, from version 3.0, it is also possible to generate an LR parser.

8.7 Check your Understanding

(a) What is the definition of a [shift-reduce parser](#) and how is the set *Action* defined?

(b) How does a shift-reduce parser work?

If you are unsure about this question, you should run the notebook

<https://github.com/karlstroetmann/Formal-Languages/blob/master/Python/Chapter-09/Shift-Reduce-Parser.ipynb>

and inspect its output.

(c) How is the concept of a [marked rule](#) defined? How is a marked rule interpreted?

(d) Can you compute the closure of a set of marked rules?

(e) Are you able to define the functions `First` and `Follow` for an SLR-parser?

(f) Are you able to compute the states of an SLR-parser?

(g) Given the set of all states of an SLR-parser, can you compute both the action table and the goto table?

(h) How is the concept of an [extended marked rule](#) defined?

(i) Can you compute the closure of a set of extended marked rules?

(j) Why are LR-parser-generators more powerful than SLR-parser-generators?

Chapter 9

Using Ply as a Parser Generator

Most¹ modern programming languages can be parsed using an LALR-Parser. As this lesson is based on the programming language *Python*, this chapter discusses how the parser generator [PLY](#) can be used to generate a parser for any language that has an LALR grammar. In [Chapter 3](#) we have already seen how PLY can be used to generate a scanner. This chapter focuses on the parser-generating aspect of PLY. If you haven't done so already, you can install PLY via *anaconda* as follows:

```
conda install -c conda-forge ply
```

9.1 A Simple Example

[Figure 9.1](#) on [page 121](#) shows the grammar of a simple [symbolic calculator](#). This grammar is similar to the grammar shown in [Figure ??](#) on [page ??](#) in [Chapter ??](#).

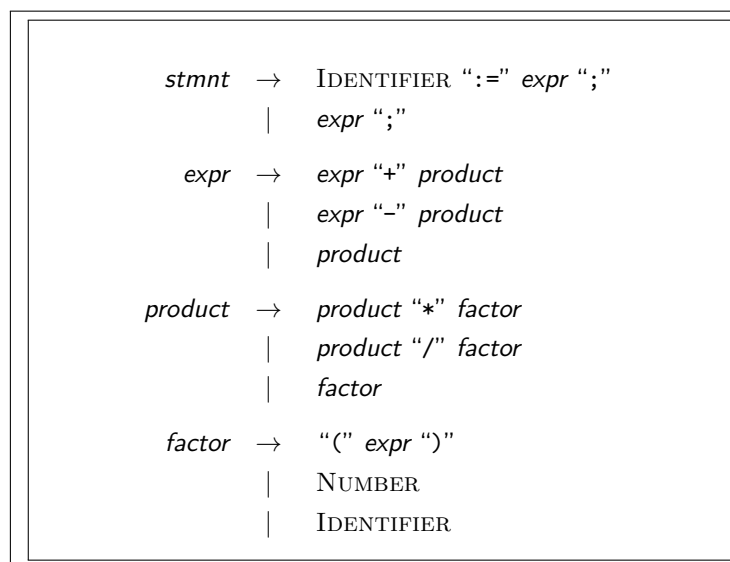


Figure 9.1: A grammar for a symbolic calculator.

In order to generate a symbolic calculator that is based on this grammar we first need to implement a scanner. [Figure 9.2](#) shows how to specify an appropriate scanner with PLY. As we have discussed scanner generation with PLY at length in [Chapter 3](#) there is no need for further discussions here.

¹The programming language C++ is a notable exception.

```

1  import ply.lex as lex
2
3  tokens = [ 'NUMBER', 'IDENTIFIER', 'ASSIGN_OP' ]
4
5  def t_NUMBER(t):
6      r'0|[1-9][0-9]*(\.[0-9]+)?([eE][+-]?([1-9][0-9]*)?)?'
7      t.value = float(t.value)
8      return t
9
10 def t_IDENTIFIER(t):
11     r'[a-zA-Z][a-zA-Z0-9_]*'
12     return t
13
14 def t_ASSIGN_OP(t):
15     r':='
16     return t
17
18 literals = ['+', '-', '*', '/', '(', ')', ';']
19
20 t_ignore = ' \t'
21
22 def t_newline(t):
23     r'\n+'
24     t.lexer.lineno += t.value.count('\n')
25
26 def t_error(t):
27     c = t.value[0]
28     n = t.lexer.lexpos
29     l = t.lexer.lineno
30     print(f"Illegal character '{c}' at character number {n} in line {l}.")
31     t.lexer.skip(1)
32
33 lexer = lex.lex()

```

Figure 9.2: A scanner for the symbolic calculator.

Figure 9.3 on page 123 shows how the grammar is implemented in PLY. We discuss it line by line.

1. Line 1 imports the module `ply.yacc`. This module contains the function `ply.yacc.yacc` which is responsible for computing the parse table. The name `yacc` is a homage to the Unix tool `YACC`, which is a popular parser generator for the language C and, furthermore, is part of the standard utilities of the Unix operating system.
2. Line 3 specifies that the syntactical variable `stmt` is the [start symbol](#) of the grammar.
3. Line 5 – 7 define the function `p_stmt_assign` which implements the grammar rule

$$stmt \rightarrow IDENTIFIER \text{ ":" } expr.$$

Note that this grammar rule itself is represented by the [document string](#) of the function `p_stmt_assign`. In general, if

$$v \rightarrow \alpha$$

is a grammar rule, then this grammar rule is represented by a function that has the name `p_v_s`. Here, the

```
1  import ply.yacc as yacc
2
3  start = 'stmnt'
4
5  def p_stmnt_assign(p):
6      "stmnt : IDENTIFIER ASSIGN_OP expr ';' "
7      Names2Values[p[1]] = p[3]
8
9  def p_stmnt_expr(p):
10     "stmnt : expr ';' "
11     print(p[1])
12
13  def p_expr_plus(p):
14     "expr : expr '+' prod"
15     p[0] = p[1] + p[3]
16
17  def p_expr_minus(p):
18     "expr : expr '-' prod"
19     p[0] = p[1] - p[3]
20
21  def p_expr_prod(p):
22     "expr : prod"
23     p[0] = p[1]
24
25  def p_prod_mult(p):
26     "prod : prod '*' factor"
27     p[0] = p[1] * p[3]
28
29  def p_prod_div(p):
30     "prod : prod '/' factor"
31     p[0] = p[1] / p[3]
32
33  def p_prod_factor(p):
34     "prod : factor"
35     p[0] = p[1]
36
37  def p_factor_group(p):
38     "factor : '(' expr ')'"
39     p[0] = p[2]
40
41  def p_factor_number(p):
42     "factor : NUMBER"
43     p[0] = p[1]
44
45  def p_factor_id(p):
46     "factor : IDENTIFIER"
47     p[0] = Names2Values.get(p[1], float('nan'))
```

Figure 9.3: A scanner for the symbolic calculator, part 1.

prefix “p_” specifies that the function implements a grammar rule (the p is short for `parser`), v should² be the name of the variable defined by this grammar rule, and s is a string chosen by the user to distinguish between different grammar rules for the same variable. Of course, s has to be chosen in a way such that the string `p_v.s` is a legal *Python* identifier.

The function always takes one argument `p`. This argument is a sequence of objects that can be indexed with array notation. If the grammar rule defining v has the form

$$v \rightarrow X_1 \cdots X_n,$$

then this sequence has a length of $n + 1$. If X_i is a token, then `p[i]` is the property with name `value` that is associated with this token. Often, this value is just a string, but it can also be a number. If X_i is a variable, then `p[i]` is the value that is returned when X_i is recognized. The value associated with the variable v is stored in the location `p[0]`. In the grammar rule shown in line 5–7, we have not assigned any value to `p[0]` and therefore there is no value associated with the syntactical variable `stmt` that is defined by this grammar rule.

Note: Line 6 shows how a grammar rule is represented for `PLY`. A grammar rule of the form

$$v \rightarrow X_1 \cdots X_n$$

is represented as the string:

$$"v : X_1 \cdots X_n"$$

It is very **important** that the character “:” is surrounded by space characters. Otherwise, the parser generator does not work but rather generates error messages that are difficult to understand.

The function `p_stmt_assign` has the task of evaluating the expression that is on the right hand side of the assignment operator “:=”. The result of this evaluation is then stored in the dictionary `Names2Values`. The key that is used is the name of the identifier to the left of the assignment operator.

4. The function in line 9 – 11 implements the grammar rule

$$stmt \rightarrow expr ";"$$

The rule is implemented by evaluating the expression and then printing it.

5. The function `p_expr_plus` implements the grammar rule

$$expr \rightarrow expr "+" prod.$$

It is implemented by evaluating the expression to the left of the operator “+”, which is stored in `p[1]`, and the product to the right of this operator, which is stored in `p[3]`, and then adding the corresponding values. Finally, the resulting sum is stored in `p[0]` so that it is available later as the value of the expression that has been parsed.

The remaining functions are similar to the ones that are discussed above.

²This is just a convention. Technically, v can be any string that is a valid *Python* identifier.

```

1  def p_error(p):
2      if p:
3          print(f'Syntax error at {p.value} in line {p.lexer.lineno}.')
4      else:
5          print('Syntax error at end of input.')
6
7  parser = yacc.yacc(write_tables=False, debug=True)
8
9  Names2Values = {}
10
11 def main():
12     while True:
13         s = input('calc > ')
14         if s == '':
15             break
16         yacc.parse(s)

```

Figure 9.4: A scanner for the symbolic calculator, part 2.

Figure 9.4 on page 125 is discussed next.

1. Line 1 – 5 shows the function `p_error` which is used to print error messages in the case that the input can not be parsed because of a syntax error. The argument `p` is the token t that caused the entry $\text{action}(s, t)$ in the action table to be undefined. If the syntax error happens at the end of the input, `p` has the value `None`.
2. Line 7 generates the parser.
 - (a) The first argument `write_tables` has to be set to `False` to prevent an obscure bug.
 - (b) The argument `debug` has to be set to `True` if we want to dump the parse table to the disk. The parse table is then written to the file `parser.out`.
3. Line 9 initializes the dictionary `Names2Values`. For every identifier x defined interactively, `Names2Values[x]` is the value associated with x .
4. The function `main` is used as a driver for the parser. It reads a string `s` from the command line and tries to parse `s` using the function `yacc.parse`. The function `yacc.parse` is generated behind the scenes when the function `yacc.yacc` is invoked in line 7.

9.2 Shift-Reduce and Reduce-Reduce Conflicts

In this section we show how shift-reduce and reduce-reduce conflicts are dealt with in PLY. Figure 9.5 on page 125 shows a grammar for arithmetical expressions that is ambiguous because it does not specify the precedence of the different arithmetical operators.

```

1  expr : expr '+' expr
2      | expr '*' expr
3      | NUMBER

```

Figure 9.5: An ambiguous grammar for arithmetical expressions.

This grammar does not specify whether the string

“1 + 2 * 3” is interpreted as “(1 + 2) * 3” or as “1 + (2 * 3)”.

Since every LALR is unambiguous, but the grammar shown in Figure 9.5 is ambiguous, it has to have shift-reduce or reduce-reduce conflicts. This grammar is part of the jupyter notebook

[Formal-Languages/tree/master/Ply/Conflicts.ipynb](https://formal-languages.github.io/tree/master/Ply/Conflicts.ipynb).

```

1  state 5
2
3      (1) expr -> expr + expr .
4      (1) expr -> expr . + expr
5      (2) expr -> expr . * expr
6
7      ! shift/reduce conflict for + resolved as shift
8      ! shift/reduce conflict for * resolved as shift
9      $end          reduce using rule 1 (expr -> expr + expr .)
10     +             shift and go to state 3
11     *             shift and go to state 4
12
13     ! +           [ reduce using rule 1 (expr -> expr + expr .) ]
14     ! *           [ reduce using rule 1 (expr -> expr + expr .) ]

```

Figure 9.6: An excerpt from the file parse.out.

When we try to generate a parser for this grammar using PLYs yacc command we get the message

```
WARNING: 4 shift/reduce conflicts.
```

The file parser.out that is generated by PLY shows how these conflicts are resolved. This file contains the LALR states created by the parser generator and for every state the possible actions are shown. Given the grammar shown above, PLY creates 6 different states. There are conflicts in two of these states. Figure 9.6 on page 126 shows state number 5 and its actions. We see that there are 2 shift-reduce conflicts in this state. Unfortunately, PLY only prints the marked rules defining these states, but it does not show the follow sets of these rules. We see that PLY resolves all conflicts in favour of shifting. The exclamation marks in the beginning of the line 13 and 14 are to be interpreted as negations and show those reduce actions that would have been possible in state 5, but are discarded in favour of the shift actions shown in line 10 and 11. Of course, in this example the shift action in line 10 is wrong because then the string

1 * 2 + 3

is interpreted as 1 * (2 + 3) and not as (1 * 2) + 3.

9.3 Operator Precedence Declarations

It is possible to resolve shift-reduce conflicts using [operator precedence declarations](#). For example, for the grammar for arithmetical expressions shown in Figure 9.7 on page 127 we can use the following [operator precedence declarations](#):

```
precedence = ( ('left', '+', '-'),
               ('left', '*', '/'),
               ('right', '^')
             )
```

This declaration specifies that the operators “+” and “-” have a lower precedence than the operators “*” and “/”. Furthermore, it specifies that all these operators associate to the left. The operator “^” has the highest precedence

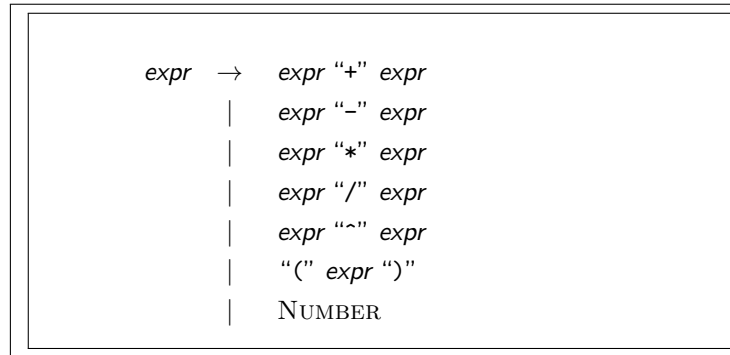


Figure 9.7: A grammar for a arithmetical expressions.

and associates to the right as specified by the keyword "right". The jupyter notebook

[Formal-Languages/tree/master/Ply/Conflicts-Resolved.ipynb](#).

shows this grammar. When we run this notebook, PLY doesn't give us a warning about any conflicts. If we inspect the generated file `parse.out`, the action table for the state number 11 has the form shown in Figure 9.8 on page 127.

```

1  state 11
2
3      (2) expr -> expr - expr .
4      (1) expr -> expr . + expr
5      (2) expr -> expr . - expr
6      (3) expr -> expr . * expr
7      (4) expr -> expr . / expr
8      (5) expr -> expr . ^ expr
9
10     +          reduce using rule 2 (expr -> expr - expr .)
11     -          reduce using rule 2 (expr -> expr - expr .)
12     $end       reduce using rule 2 (expr -> expr - expr .)
13     )          reduce using rule 2 (expr -> expr - expr .)
14     *          shift and go to state 6
15     /          shift and go to state 7
16     ^          shift and go to state 8
17
18     ! *        [ reduce using rule 2 (expr -> expr - expr .) ]
19     ! /        [ reduce using rule 2 (expr -> expr - expr .) ]
20     ! ^        [ reduce using rule 2 (expr -> expr - expr .) ]
21     ! +        [ shift and go to state 4 ]
22     ! -        [ shift and go to state 5 ]

```

Figure 9.8: An excerpt from the file `parse.out` when conflicts are resolved.

1. Since the operators "+" and "-" have the same precedence, we have

$$\text{action}(\text{state11}, "+") = \langle \text{reduce}, \text{expr} \rightarrow \text{expr} "-" \text{expr} \rangle$$

This way, the expression $1 - 2 + 3$ is parsed as $(1 - 2) + 3$ and not as $1 - (2 + 3)$ as it would if we would shift the operator "+" instead.

2. Since the operator “-” is left associative, we have

$$\text{action}(\text{state11}, "-") = \langle \text{reduce}, \text{expr} \rightarrow \text{expr} "-" \text{expr} \rangle$$

This way, the expression $1 - 2 - 3$ is parsed as $(1 - 2) - 3$.

3. Since the precedence of the operator “*” is higher than the precedence of the operator “-”, we have

$$\text{action}(\text{state11}, "*") = \langle \text{shift}, \text{state6} \rangle$$

This way, the expression $1 - 2 * 3$ is parsed as $1 - (2 * 3)$.

4. Since the precedence of the operator “/” is higher than the precedence of the operator “-”, we have

$$\text{action}(\text{state11}, "/") = \langle \text{shift}, \text{state7} \rangle$$

This way, the expression $1 - 2/3$ is parsed as $1 - (2/3)$.

Next, we explain in detail how PLY uses operator precedence relations to resolve shift-reduce conflicts.

1. First, PLY assigns a precedence level to every grammar rule. This precedence level is the precedence level of the last operator symbol occurring in the grammar rule. Most of the times, there is just one operator that determines the precedence of the grammar rule. In the grammar at hand the precedences of the rules would be as shown in the table below.

rule	precedence
$\text{expr} \rightarrow \text{expr} "+" \text{expr}$	1
$\text{expr} \rightarrow \text{expr} "-" \text{expr}$	1
$\text{expr} \rightarrow \text{expr} "*" \text{expr}$	2
$\text{expr} \rightarrow \text{expr} "/" \text{expr}$	2
$\text{expr} \rightarrow \text{expr} "^" \text{expr}$	3
$\text{expr} \rightarrow "(" \text{expr} ")"$	—
$\text{expr} \rightarrow \text{NUMBER}$	—

If a grammar rule does not contain an operator that has been given a precedence, then the precedence of the grammar rule remains undefined.

2. If s is a state that contains two e.m.R.s r_1 and r_2 such that

$$r_1 = (a \rightarrow \beta \bullet o \delta : L_1) \quad \text{and} \quad r_2 = (c \rightarrow \gamma \bullet : L_2) \quad \text{where} \quad o \in L_2,$$

then there is a shift-reduce conflict when

$$\text{action}(s, o)$$

is computed. Let us assume that the precedence of the operator o is $p(o)$ and the precedence of the rule r_2 is $p(r_2)$. Then there are six cases that depend on the relative values of $p(o)$ and $p(r_2)$ and on the associativity of the operator o .

- (a) $p(o) > p(r_2)$.

In this case the precedence of the operator o is higher than the precedence of the rule r_2 . Therefore the operator o is shifted:

$$\text{action}(s, o) = \langle \text{shift}, \text{goto}(s, o) \rangle.$$

To understand this rule we just have to watch what happens when we parse

$$1+2*3$$

using the grammar given above. After the part “1+2” has been read and the next token is the operator “*”, the parser is in the following state:

$$\left\{ \begin{array}{l} \text{expr} \rightarrow \text{expr} \bullet "+" \text{expr} : \{\$, "+", "-", "*", "/", "\wedge"\}, \\ \text{expr} \rightarrow \text{expr} \bullet "-" \text{expr} : \{\$, "+", "-", "*", "/", "\wedge"\}, \\ \text{expr} \rightarrow \text{expr} \bullet "*" \text{expr} : \{\$, "+", "-", "*", "/", "\wedge"\}, \\ \text{expr} \rightarrow \text{expr} \bullet "/" \text{expr} : \{\$, "+", "-", "*", "/", "\wedge"\}, \\ \text{expr} \rightarrow \text{expr} \bullet "\wedge" \text{expr} : \{\$, "+", "-", "*", "/", "\wedge"\}, \\ \text{expr} \rightarrow \text{expr} "+" \text{expr} \bullet : \{\$, "+", "-", "*", "/", "\wedge"\} \end{array} \right\}.$$

When the parser next sees the token `"*"`, then it must not reduce the symbol stack using the rule $\text{expr} \rightarrow \text{expr} "+" \text{expr}$, because it has to multiply the numbers 2 and 3 first. Therefore, the token `"*"` has to be shifted.

(b) $p(o) < p(r_2)$.

Now the precedence of the operator that occurs in the rule r_2 is higher than the precedence of the operator o . Therefore the correct action is to reduce with the rule r_2 :

$$\text{action}(s, o) = \langle \text{reduce}, r_2 \rangle.$$

To see that this makes sense we discuss the parsing of the expression

$$1 * 2 + 3$$

with the grammar given previously. Assume the string `"1*2"` has already been read and the next token that is processed is the token `"+"`. Then the state of the parser is as follows:

$$\left\{ \begin{array}{l} \text{expr} \rightarrow \text{expr} \bullet "+" \text{expr} : \{\$, "+", "-", "*", "/", "\wedge"\}, \\ \text{expr} \rightarrow \text{expr} \bullet "-" \text{expr} : \{\$, "+", "-", "*", "/", "\wedge"\}, \\ \text{expr} \rightarrow \text{expr} \bullet "*" \text{expr} : \{\$, "+", "-", "*", "/", "\wedge"\}, \\ \text{expr} \rightarrow \text{expr} \bullet "/" \text{expr} : \{\$, "+", "-", "*", "/", "\wedge"\}, \\ \text{expr} \rightarrow \text{expr} \bullet "\wedge" \text{expr} : \{\$, "+", "-", "*", "/", "\wedge"\}, \\ \text{expr} \rightarrow \text{expr} "*" \text{expr} \bullet : \{\$, "+", "-", "*", "/", "\wedge"\} \end{array} \right\}.$$

When the parser now sees the operator `"+"`, it has to reduce the string `"1*2"` using the rule

$$\text{expr} \rightarrow \text{expr} "*" \text{expr},$$

as it has to multiply the numbers 1 and 2.

(c) $p(o) = p(r_2)$ and the operator o is left associative.

Then we reduce the symbol stack with the rule r_2 , we have

$$\text{action}(s, o) = \langle \text{reduce}, r_2 \rangle.$$

To convince yourself that this is the right thing to do, inspect what happens when the string

$$1 - 2 - 3$$

is parsed with the grammar discussed previously. Assume that the string `"1-2"` has already be read and the next token is the operator `"-"`. Then the state of the parser is as follows:

$$\left\{ \begin{array}{l} \text{expr} \rightarrow \text{expr} \bullet "+" \text{expr} : \{\$, "+", "-", "*", "/", "\wedge"\}, \\ \text{expr} \rightarrow \text{expr} \bullet "-" \text{expr} : \{\$, "+", "-", "*", "/", "\wedge"\}, \\ \text{expr} \rightarrow \text{expr} \bullet "*" \text{expr} : \{\$, "+", "-", "*", "/", "\wedge"\}, \\ \text{expr} \rightarrow \text{expr} \bullet "/" \text{expr} : \{\$, "+", "-", "*", "/", "\wedge"\}, \\ \text{expr} \rightarrow \text{expr} \bullet "\wedge" \text{expr} : \{\$, "+", "-", "*", "/", "\wedge"\}, \\ \text{expr} \rightarrow \text{expr} "-" \text{expr} \bullet : \{\$, "+", "-", "*", "/", "\wedge"\} \end{array} \right\}.$$

If the next token is the operator `"-"`, then the parser has to reduce the symbol stack using the rule $\text{expr} \rightarrow \text{expr} "-" \text{expr}$ as it has to subtract 2 from 1. If it would shift instead it would compute $1 - (2 - 3)$ instead of computing $(1 - 2) - 3$.

(d) $p(o) = p(r_2)$ and the operator o associates to the right.

In this case the operator o is shifted

$$\text{action}(s, o) = \langle \text{shift}, \text{goto}(s, o) \rangle.$$

In order to understand this case, parse the string

$$2^3^4$$

with the grammar rules

$$expr \rightarrow expr \wedge expr \mid \text{NUMBER}.$$

Consider the situation when the string “1^2” has already been read and the next token is the exponentiation operator “^”. The state of the parser is then as follows:

$$\left\{ \begin{array}{l} expr \rightarrow expr \bullet “+” expr : \{ \$, “+”, “-” “*”, “/”, “^” \}, \\ expr \rightarrow expr \bullet “-” expr : \{ \$, “+”, “-” “*”, “/”, “^” \}, \\ expr \rightarrow expr \bullet “*” expr : \{ \$, “+”, “-” “*”, “/”, “^” \}, \\ expr \rightarrow expr \bullet “/” expr : \{ \$, “+”, “-” “*”, “/”, “^” \}, \\ expr \rightarrow expr \bullet “^” expr : \{ \$, “+”, “-” “*”, “/”, “^” \}, \\ expr \rightarrow expr “^” expr \bullet : \{ \$, “+”, “-” “*”, “/”, “^” \} \end{array} \right\}.$$

Here the token “^” has to be shifted since we first have to compute the expression “3^4”.

- (e) $p(o) = p(r_2)$ and the operator o has been declared to be non-associative.

In this case we have a syntax error:

$$action(s, o) = error.$$

To understand this case, try to parse a string of the form

$$1 < 1 < 1$$

using the grammar rules

$$expr \rightarrow expr “<” expr \mid expr “+” expr \mid \text{NUMBER}.$$

Once the string “1 < 1” has been read and the next token is the operator “<” the parser recognizes that there is an error. Therefore, PLY will resolve this shift-reduce conflict by putting an error entry into the action table.

- (f) $p(o)$ is undefined or $p(r_2)$ is undefined.

In this case there is a shift-reduce conflict and PLY prints a warning message when generating the parser. The conflict is then resolved in favour of shifting.

9.4 Resolving Shift-Reduce and Reduce-Reduce Conflicts

We start our discussion by categorizing conflicts with respect to their origin.

1. *Mehrdeutigkeits-Konflikte* sind Konflikte, die ihre Ursache in einer Mehrdeutigkeit der zu Grunde liegenden Grammatik haben. Solche Konflikte weisen damit auf ein tatsächliches Problem der Grammatik hin. Wir hatten ein Beispiel für solche Konflikte gesehen, als wir in [Abbildung 9.5](#) versucht hatten, die Syntax arithmetischer Ausdrücke ohne die syntaktischen Kategorien *product* und *factor* zu beschreiben.

Wir hatten damals bereits gesehen, dass wir das Problem durch die Einführung von Operator-Präzedenzen lösen können. Falls dies nicht möglich ist, dann bleibt nur das Umschreiben der Grammatik.

2. *Look-Ahead-Konflikte* sind Reduce-Reduce-Konflikte, bei denen die Grammatik zwar einerseits eindeutig ist, für die aber andererseits ein Look-Ahead von einem Token nicht ausreichend ist um den Konflikt zu lösen.
3. *Mysteriöse Konflikte* entstehen erst beim Übergang von den LR-Zuständen zu den LALR-Zuständen durch das Zusammenfassen von Zuständen mit dem gleichen Kern. Diese Konflikte treten also genau dann auf, wenn das Konzept einer LALR-Grammatik nicht ausreichend ist um die Syntax der zu parsenden Sprache zu beschreiben.

Wir betrachten die letzten beiden Fälle nun im Detail und zeigen Wege auf, wie die Konflikte gelöst werden können.

9.4.1 Look-Ahead-Konflikte

Ein Look-Ahead-Konflikt liegt dann vor, wenn die Grammatik zwar eindeutig ist, aber ein Look-Ahead von einem Token nicht ausreicht um zu entscheiden, mit welcher Regel reduziert werden soll. Abbildung 9.9 zeigt die Grammatik [Look-Ahead.ipynb](#)³, die zwar eindeutig ist, aber nicht die LR(1)-Eigenschaft hat und damit erst recht keine LALR(1) Grammatik ist.

```

1  a : b 'U' 'V'
2    | c 'U' 'W'
3  b : 'X'
4  c : 'X'

```

Figure 9.9: Eine eindeutige Grammatik ohne die LR(1)-Eigenschaft.

Berechnen wir die LR-Zustände dieser Grammatik, so finden wir unter anderem den folgenden Zustand:

$$\{b \rightarrow "X" \bullet : "U", c \rightarrow "X" \bullet : "U" \}.$$

Da die Menge der Folge-Token für beide Regeln gleich sind, haben wir hier einen Reduce-Reduce-Konflikt. Dieser Konflikt hat seine Ursache darin, dass der Parser mit einem Look-Ahead von nur einem Token nicht entscheiden kann, ob ein "X" als ein *b* oder als ein *c* zu interpretieren ist, denn dies entscheidet sich erst, wenn das auf "U" folgende Zeichen gelesen wird: Handelt es sich hierbei um ein "V", so wird insgesamt die Regel

$$a \rightarrow b "U" "V"$$

verwendet werden und folglich ist das "X" als ein *b* zu interpretieren. Ist das zweite Token hinter dem "X" hingegen ein "W", so ist die zu verwendende Regel

$$a \rightarrow c "U" "W"$$

und folglich ist das "X" als *c* zu lesen.

```

1  a : b 'V'
2    | c 'W'
3  b : 'X' 'U'
4  c : 'X' 'U'

```

Figure 9.10: Eine zu Abbildung 9.9 äquivalente LR(1)-Grammatik.

Das Problem bei dieser Grammatik ist, dass sie versucht, abhängig vom Kontext ein "X" wahlweise als ein *b* oder als ein *c* zu interpretieren. Es ist offensichtlich, wie das Problem gelöst werden kann: Wenn der Kontext "U", der sowohl auf *b* als auch auf *c* folgt, mit in die Regeln für *b* und *c* aufgenommen wird, dann verschwindet der Konflikt, denn dann hat der Zustand, in dem früher der Konflikt auftrat, die Form

$$\{b \rightarrow "X" "U" \bullet : "V", c \rightarrow "X" "U" \bullet : "W" \}.$$

Hier entscheidet sich nun anhand des nächsten Tokens, mit welcher Regel wir in diesem Zustand reduzieren müssen: Ist das nächste Token ein "V", so reduzieren wir mit der Regel

$$b \rightarrow "X" "U",$$

ist das nächste Token hingegen der Buchstabe "W", so nehmen wir stattdessen die Regel

$$c \rightarrow "X" "U" \bullet : "W".$$

³Diese Grammatik habe ich im Netz auf der Seite von Pete Jinks unter der Adresse

<http://www.cs.man.ac.uk/~pjj/cs212/ho/node19.html>

gefunden.

Abbildung 9.10 zeigt die entsprechend modifizierte Grammatik, die Sie unter

github.com/karlstroetmann/Formal-Languages/tree/master/Ply/Look-Ahead-Solved.ipynb

im Netz finden.

9.4.2 Mysterious Reduce-Reduce Conflicts

A conflict is called a [mysterious reduce-reduce conflict](#) if the conflict results from the merger of states that happens when we go from an LR parsing table to an LALR parsing table. The grammar in Figure 9.11 on page 132 is the same as the grammar shown in Figure 8.14 on page 118 in the previous chapter. Then we had seen that this grammar is an LR grammar, but not an LALR grammar. Let us see what happens if we use PLY to generate the states for this grammar.

```

1  s : 'v' a 'y'
2    | 'w' b 'y'
3    | 'v' b 'z'
4    | 'w' a 'z'
5
6  a : X
7
8  b : X

```

Figure 9.11: A grammar that generates a mysterious reduce-reduce conflict.

When we run PLY to produce the parsing table, we get the states shown in Figure 9.12 on page 132. This Figure only shows two states, state 6 and state 9. I have taken the liberty to annotate the extended marked rules occurring in these states with their follow sets. Taken by itself, none of these two states has a conflict since the follow sets of the respective rules are disjoint. However, it is obvious that these two states have the same core and should have been merged. The resulting state would have the form

$$\{a \rightarrow X\bullet : \{'y', 'z'\}, b \rightarrow X\bullet : \{'y', 'z'\}\}$$

and obviously has a reduce-reduce conflict if the next token is either 'y' or 'z'.

```

1  state 6
2
3      (5) a -> X . : 'y'
4      (6) b -> X . : 'z'
5
6      y                reduce using rule 5 (a -> X .)
7      z                reduce using rule 6 (b -> X .)
8
9  state 9
10
11     (6) b -> X . : 'y'
12     (5) a -> X . : 'z'
13
14     y                reduce using rule 6 (b -> X .)
15     z                reduce using rule 5 (a -> X .)

```

Figure 9.12: A grammar that generates a mysterious reduce-reduce conflict.

Interestingly, PLY does not merge these states and is therefore able to generate a parse table without conflicts. On the other hand, PLY claims to generate LALR tables. Therefore, I have written an email to [David Beazley](#) asking whether this behaviour is a feature or a bug. He has classified this example as an “*interesting curiosity*”.

Exercise 30: Use PLY to implement a *Python* parser that is able to evaluate formulas from propositional logic. The language should support the operators “ \leftrightarrow ” (equivalence), “ \rightarrow ” (implication), “ \vee ” (disjunction), “ \wedge ” (conjunction), and “ \neg ” (negation). The operator “ \neg ” should bind stronger than the operator “ \wedge ”, which in turn binds stronger than the operator “ \vee ”. The operators “ \vee ” and “ \wedge ” are both left associative. The operator “ \rightarrow ” associates to the right and binds weaker than the operator “ \vee ”. The operator “ \leftrightarrow ” is not associative and is the weakest of all operators in terms of binding strength. Furthermore, the language should support parenthesis, variables, and the assignment operator “ $:=$ ”. \diamond

Chapter 10

Assembler

A compiler translates programs written in a high level language like C or *Java* into some low level representation. This low level representation can be either machine code or some form of assembler code. For the programming language *Java*, the command `javac` compiles a program written in *Java* into *Java* byte code. This byte code is then executed using the *Java virtual machine* (JVM).

The compiler that we are going to develop in the next chapter generates a particular form of assembler code known as *JVM assembler code*. This assembler code can be translated directly into *Java byte code*, which is also the byte code generated by the program `javac`. The program for translating JVM assembler into bytecode is called *Jasmin*. You can download *Jasmin* at

<http://sourceforge.net/projects/jasmin>.

Jasmin takes an byte code produced by *Jasmin* can be executed using the command `java` just like any other “.class”-file. This chapter will discuss the syntax and semantics of *Jasmin* assembler code.

10.1 Introduction into Jasmin Assembler

```
1  .class public Hello
2  .super java/lang/Object
3
4  .method public <init>()V
5      aload 0
6      invokenonvirtual java/lang/Object/<init>()V
7      return
8  .end method
9
10 .method public static main([Ljava/lang/String;)V
11     .limit locals 1
12     .limit stack 2
13     getstatic      java/lang/System/out Ljava/io/PrintStream;
14     ldc            "Hello World!"
15     invokevirtual  java/io/PrintStream/println(Ljava/lang/String;)V
16     return
17 .end method
```

Figure 10.1: An assembler program to print “Hello World!”.

To get used to the syntax of *Jasmin* assembler, we start with a small program that prints the string

"Hello World!"

on the standard output stream. Figure 10.1 on page 134 shows the program `Hello.jas`. We discuss this program line by line.

1. Line 1 uses the directive `".class"` to define the name of the class file that is to be produced by the assembler. In this case, the class name is `Hello`. Therefore, *Jasmin* will translate this file into the class file `"Hello.class"`.
2. Line 2 uses the directive `".super"` to specify the super class of the class `Hello`. In our examples, the super class will always be the class `Object`. Since this class resides in the package `"java.lang"`, the super class has to be specified as

`java/lang/Object.`

Observe that the character `"."` in the class name `"java.lang.Object"` has to be replaced by the character `"/"`. This is true even if a *Windows* operating system is used.

3. Line 4 to 8 initialize the program. This code is always the same and corresponds to a constructor for the class `Hello`. As this code is copied verbatim to the beginning of every class file, we will not discuss it further.
4. Line 10 – 17 defines the method `main` that does the actual work.

- (a) Line 10 uses the directive `".method"` to declare the name of the method and its signature. The string

`main([Ljava/lang/String;)V`

specifies the signature:

- i. The string `"main"` is the name of the method that is defined.
 - ii. The character `"["` specifies that the first argument is an array.
 - iii. The character `"L"` specifies that this array consists of objects.
 - iv. The string `"java/lang/String;"` specifies that these objects are objects of the class `"java.lang.String"`.
 - v. Finally, the character `"V"` specifies that the return type of the method `main` is `"void"`.
- (b) Line 11 uses the directive `".limit locals"` to specify the number of local variables used by the method `main`. In this case, there is just one local variable. This variable corresponds to the parameter of the method `main`. The assembler file shown in Figure 10.1 on page 134 corresponds to the *Java* code shown in Figure 10.2 below. The method `main` has one local variable, which is the parameter `args`. The information on the number of local variables is needed by the *Java* virtual machine in order to allocate memory for these variables on the stack.

```

1  public class Hello {
2      public static void main(String[] args) {
3          System.out.println("Hello World!");
4      }
5  }

```

Figure 10.2: Printing Hello world in *Java*.

- (c) For the purpose of the following discussion, we basically assume that there exist two types of processors: Those that store the objects they work upon in registers and those that store these objects on a stack residing in main memory. The *Java* virtual machine is of the second type. Hence, *Jasmin* assembler programs do not refer to registers but rather refer to this stack¹. Line 12 uses the directive

`".limit stack"`

¹In reality, all real processors make use of registers. However, it is possible to simulate a stack machine using a real processor and that is what is done in the *Java* virtual machine.

to specify the maximal height of the stack. In this case, the stack is allowed to contain a maximum of two objects. It is easy to see that we indeed do never place more than two objects onto the stack, since line 13 pushes the object

```
java.lang.System.out
```

onto the stack. This is an object of class “`java.io.PrintStream`”. Then, line 14 pushes a reference to the string “Hello World” onto the stack. The other instructions do not push anything onto the stack.

- (d) Line 15 calls the method `println`, which is a method of the class “`java.io.PrintStream`”. It also specifies that `println` takes one argument of type `java.lang.String` and returns nothing.

In reality, `println` needs a second argument. This argument is the object `java.lang.System.out` that we had previously pushed on the stack using the method `getstatic`.

- (e) Line 16 returns from the method `main`.
- (f) In line 17 the directive “`.end`” marks the end of the code corresponding to the method `main`.

Before we proceed, let us assume that we are working on a Unix operating system and that there is an executable file called `jasmin` somewhere in our path that contains the following code:

```
#!/bin/bash
java -jar /usr/local/lib/jasmin.jar $@
```

Of course, for this to work the directory `/usr/local/lib/` has to contain the file “`jasmin.jar`”. If we were working on a Windows operating system, we would have a file called `jasmin.bat` somewhere in our `PATH`. This file would contain the following line:

```
java -jar %USERPROFILE%/Dropbox/Software/jasmin-2.4/jasmin.jar %*
```

Of course, for this to work the directory `%USERPROFILE%/Dropbox/Software/jasmin-2.4` has to contain the file “`jasmin.jar`”.

In order to execute the assembler program discussed above, we first have to translate the assembler program into a `class`-file. This is done using the command

```
jasmin Hello.jas
```

Executing this command creates the file “`Hello.class`”. This class file can then be executed just like any class file generated from `javac` by typing

```
java Hello
```

in the command line, provided the environment variable `CLASSPATH` contains the current directory, i.e. the `CLASSPATH` has to contain the directory “`.`”.

We will proceed to discuss the different assembler commands in more detail later. To this end, we first have to discuss some background: One of the design goal of the programming language *Java* was compatibility. The idea was that it should be possible to execute *Java* class files on any computer. Therefore, the *Java* designers decided to create a so called *virtual machine*. A virtual machine is a computer architecture that, instead of being implemented in silicon, is simulated. Programs written in *Java* are first compiled into so called *class files*. These class files correspond to the machine code of the *Java* virtual machine (JVM). The architecture of the virtual machine is a *stack machine*. A stack machine does not have any registers to store variables. Instead, there is a stack and all variables reside on the stack. Every command takes its arguments from the top of the stack and replaces these arguments with the result of the operation performed by the command. For example, if we want to add two values, then we first have to push both values onto the stack. Next, performing the add operation will pop these values from the stack and then push their sum onto the stack.

10.2 Assembler Instructions

We proceed to discuss some of the JVM instructions. Since there are more than 160 JVM instructions, we can only discuss a subset of all instructions. We restrict ourselves to those instructions that deal with integers: For example,

there is an instruction called `iadd` that adds two 32 bit integers. There are also instructions like `fadd` that adds two floating point numbers and `dadd` that adds two double precision floating point numbers but, since our time is limited, we won't discuss these instructions. Before we are able to discuss the different instructions we have to discuss how the main memory is organized in the JVM: In the JVM, the memory is split into four parts:

1. The **program memory** contains the program code as a sequence of bytes.
2. The operands of the different machine instructions are put onto the **stack**. Furthermore, the stack contains the arguments and the local variables of a procedure. However, in the context of the JVM the procedures are called *methods* instead of procedures.

The register `SP` points to the top of the stack. If a method is called, the arguments of the method are placed on the stack. The register `LV` (**local variables**) points to the first argument of the current method. On top of the arguments, the local variables of the method are put on the stack. Both the arguments and the local variables can be accessed via the register `LV` by specifying their offset from the first argument. We will discuss the register `LV` in more detail when we discuss the invocation of methods.

3. The **heap** is used for dynamically allocated memory. Newly created objects are located in the heap.
4. The **constant pool** contains the definitions of constants and also the addresses of methods in program memory.

In the following, we will be mostly concerned with the stack and the heap. We proceed to discuss some of the assembler instructions.

Arithmetical and Logical Instructions

1. The instruction "`iadd`" adds those values that are on top of the stack and replaces these values by their sum. Figure 10.3 on page 137 show how this command works. The left part of the figure shows the stack as it is before the command `iadd` is executed, while the right part of the figure depicts the situation after the execution of this command. Note that after the command is executed, the stack pointer points to the position where formerly the first argument had been stored.

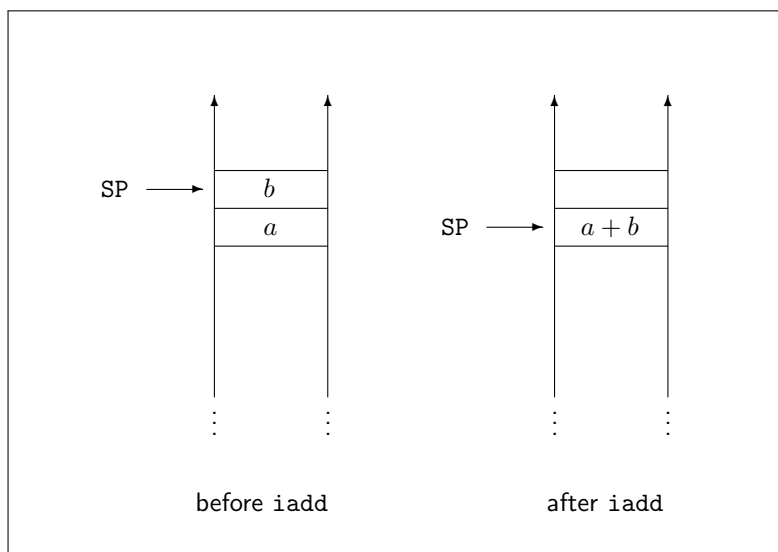


Figure 10.3: The effect of `iadd`.

2. The instruction "`isub`" subtracts the integer value on top of the stack from the value that is found on the position next to the top of the stack. Figure 10.4 on page 138 pictures this command. The left part of the figure shows the stack as it is before the command `isub` is executed, while the right part of the figure depicts the situation after the execution of this command. Note that after the command is executed, the stack pointer points to the position where formerly the first argument had been stored.

Figure 10.4: The effect of `isub`.

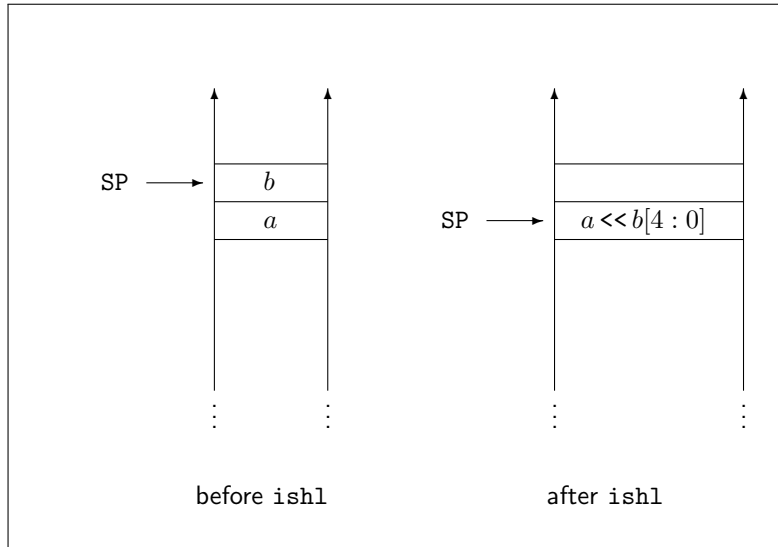
3. The instruction “`imul`” multiplies the two integer values which are on top of the stack. This instruction works similar to the instruction `iadd`. If the product does not fit in 32 bits, only the lowest 32 bits of the result are written onto the stack.
4. The instruction “`idiv`” divides the integer value that is found on the position next to the top of the stack by the value on top of the stack.
5. The instruction “`irem`” computes the remainder $a \% b$ of the division of a by b where a and b are integer values found on top of the stack.
6. The instruction “`iand`” computes the bitwise “and” of the values on top of the stack.
7. The instruction “`ior`” computes the bitwise “or” of the integer values that are on top of the stack.
8. The instruction “`ixor`” computes the bitwise “*exclusive or*” of the integer values that are on top of the stack.

Shift Instructions

1. The instruction “`ishl`” shifts the value a to the left by $b[4:0]$ bits. Here, a and b are assumed to be the two values on top of the stack: b is the value on top of the stack and a is the value below b . $b[4:0]$ denotes the natural number that results from the 5 lowest bits of b . Figure 10.5 on page 139 pictures this command.
2. The instruction “`ishr`” shifts the value a to the right by $b[4:0]$ bits. Here, a and b are assumed to be the two values on top of the stack: b is the value on top of the stack and a is the value below b . $b[4:0]$ denotes the natural number that results from the 5 lowest bits of b . Note that this instruction performs an *arithmetic shift*, i.e. the sign bit is preserved.

10.2.1 Instructions to Manipulate the Stack

1. The instruction “`dup`” duplicates the value that is on top of the stack. Figure 10.6 on page 139 pictures this command.
2. The instruction “`pop`” removes the value that is on top of the stack. Figure 10.7 on page 140 pictures this command. The value is not actually erased from memory, only the stack pointer is decremented. The next instruction that puts a new value onto the stack will therefore overwrite the old value.

Figure 10.5: The effect of `ishl`.Figure 10.6: The effect of `dup`.

3. The instruction “`nop`” does nothing. The name is short for “no operation”.
4. The instruction “`bipush b`” pushes the byte `b` that is given as argument onto the stack. Figure 10.8 on page 140 pictures this command.
5. The instruction “`getstatic v c`” takes two parameters: `v` is the name of a static variable and `c` is the type of this variable. For example,

```
getstatic java/lang/System/out Ljava/io/PrintStream;
```

pushes a reference to the `PrintStream` that is known as

```
java.lang.System.out
```

onto the stack.

6. The instruction “`iload v`” reads the local variable with index `v` and pushes it on top of the stack. Figure 10.9

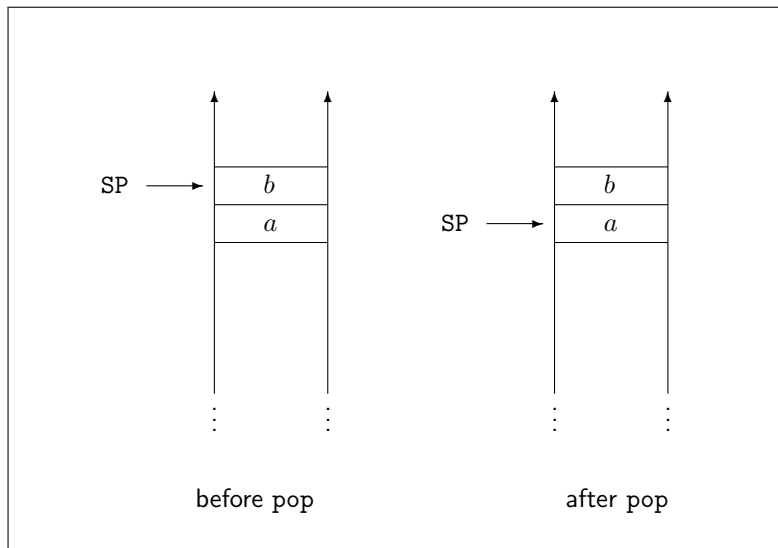
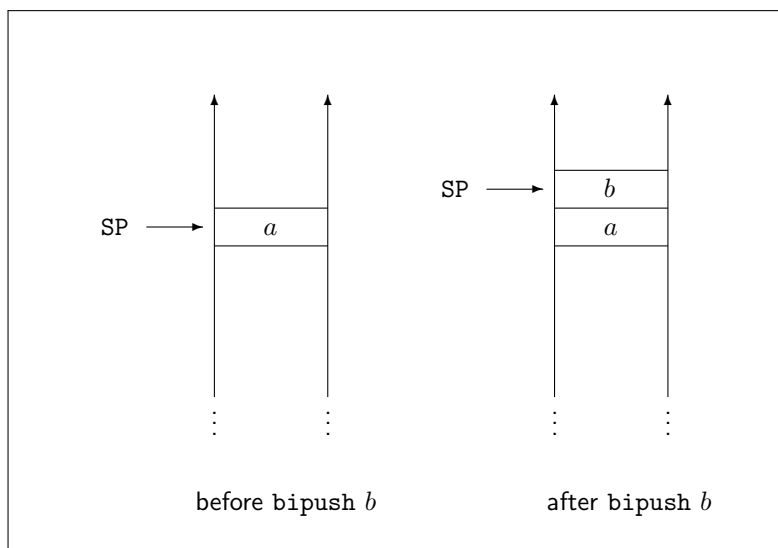
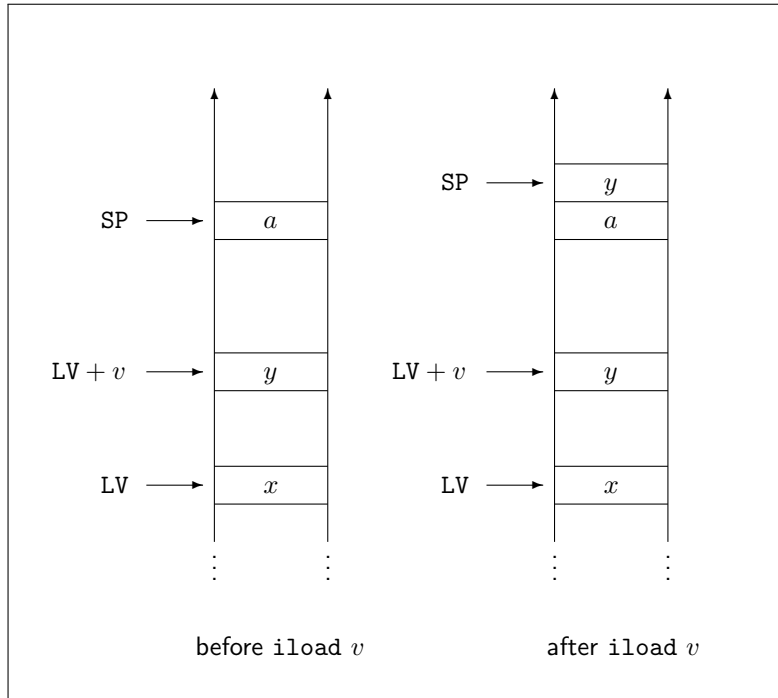


Figure 10.7: The effect of pop.

Figure 10.8: The effect of bipush *b*.

on page 141 pictures this command. Note that LV denotes the register pointing to the beginning of the local variables of a method.

7. The instruction “istore *v*” removes the value which is on top of the stack and stores this value at the location for the local variable with number *v*. Hence, *v* is interpreted as an index into the local variable table of the method. Figure 10.10 on page 141 pictures this command.
8. The instruction “ldc *c*” pushes the constant *c* onto the stack. This constant can be an integer, a single precision floating point number, or a (pointer to) a string. If *c* is a string, this string is actually stored in the so called *constant pool* and in this case the command “ldc *c*” will only push a pointer to the string onto the stack.

Figure 10.9: The effect of `iload v`.Figure 10.10: The effect of `istore v`.

Branching Commands

In this subsection we discuss those commands that change the control flow.

1. The instruction “goto *l*” jumps to the [label](#) *l*. Here the label *l* is a label name that has to be declared inside

the method containing this `goto` command. A label with name *target* is declared using the syntax

target:

The next section presents an example assembler program that demonstrates this command.

2. The instruction “`if_icmpeq l`” checks whether the value on top of the stack is the same as the value preceding it. If this is the case, the program will branch to the label *l*. Otherwise, the control flow is not changed. Observe that both values that are compared are removed from the stack.
3. The instruction “`if_icmpne l`” checks whether the value on top of the stack is different from the value preceding it. If this is the case, the program will branch to the label *l*. Otherwise, the control flow is not changed. Observe that both values that are compared are removed from the stack.
4. The instruction “`if_icmplt l`” checks whether the value that is below the top of the stack is less than the value on top of the stack. If this is the case, the program will branch to the label *l*. Otherwise, the control flow is not changed. Observe that both values that are compared are removed from the stack.
5. The instruction “`if_icmple l`” checks whether the value that is below the top of the stack is less or equal than the value on top of the stack. If this is the case, the program will branch to the label *l*. Otherwise, the control flow is not changed. Observe that both values that are compared are removed from the stack.

There are similar commands called `if_icmpgt` and `if_icmpge`.

6. The instruction “`ifeq l`” checks whether the value on top of the stack is zero. If this is the case, the program will branch to the label *l*. Otherwise, the control flow is not changed. Observe that the value that is tested is removed from the stack.
7. The instruction “`iflt l`” checks whether the value on top of the stack is less than zero. If this is the case, the program will branch to the label *l*. Otherwise, the control flow is not changed. Observe that the value that is tested is removed from the stack.
8. The instruction “`invokevirtual m`” is used to call the method *m*. Here *m* has to specify the full name of the method. For example, in order to invoke the method `println` of the class `java.io.PrintStream` we have to write

```
invokevirtual java/io/PrintStream/println(I)V
```

Before the command `invokevirtual` is executed, we have to put the arguments of the method onto the stack. For example, if we want to invoke the method `println` that takes an integer argument, we first have to push an object of type `PrintStream` onto the stack. Furthermore, we need to push an integer onto the stack.

9. The instruction “`invokestatic m`” is used to call the method *m*. Here *m* has to specify the full name of the method. Furthermore, *m* needs to be a static method. For example, in order to invoke a method called `sum` that resides in the class `Sum` and that takes one integer argument we have to write

```
invokestatic Sum/sum(I)I
```

In the type specification “`sum(I)I`” the first “`I`” specifies that `sum` takes one integer argument, while the second “`I`” specifies that the method `sum` returns an integer.

Before the command `invokestatic` is executed, we have to put the arguments of the method onto the stack. For example, if we want to invoke the method `sum` described above, then we have to push an integer onto the stack.

10. The instruction “`ireturn`” returns from the method that is currently invoked. This method also returns a value to the calling procedure. In order for `ireturn` to be able to return a value *v*, this value *v* has to be pushed onto the stack before the command `ireturn` is executed.

In general, if a method taking *n* arguments a_1, \dots, a_n is to be called, then first the arguments a_1, \dots, a_n have to be pushed onto the stack. When the method *m* is done and has computed its result *r*, the arguments a_1, \dots, a_n will have been replaced with the single value *r*.

11. The instruction “return” returns from the method that is currently invoked. However, in contrast to the command `ireturn`, this command is used if the method that has been invoked does not return a result.

10.3 An Example Program

Figure 10.11 shows the C program `sum.c` that computes the sum

$$\sum_{i=1}^{6^2} i.$$

The function `sum(n)` computes the sum $\sum_{i=1}^n i$ and the function `main` calls this function with the argument $6 \cdot 6$. Figure 10.12 on page 144 shows how this program can be translated into the assembler program `Sum.jas`. We discuss the implementation of this program next.

```

1  #import "stdio.h"
2
3  int sum(int n) {
4      int s;
5      s = 0;
6      while (n != 0) {
7          s = s + n;
8          n = n - 1;
9      };
10     return s;
11 }
12 int main() {
13     printf("%d\n", sum(6*6));
14     return 0;
15 }

```

Figure 10.11: A C function to compute $\sum_{i=1}^{36} i$.

1. Line 1 specifies the name of the generated class which is to be `Sum`.
2. Line 2 specifies that the class `Sum` is a subclass of the class `java.lang.Object`.
3. Lines 4 – 8 initialize the class. The code used here is the same as in the example printing “Hello World!”.
4. Line 10 declares the method `main`.
5. Line 11 specifies that there is just one local variable.
6. Line 12 specifies that the stack will contain at most 3 temporary values.
7. Line 13 pushes the object `java.lang.out` onto the stack. We need this object later in order to invoke `println`.
8. Line 14 pushes the number 6 onto the stack.
9. Line 15 duplicates the value 6. Therefore, after line 15 is executed, the stack contains three elements: The object `java.lang.out`, the number 6, and again the number 6.
10. Line 16 multiplies the two values on top of the stack and replaces them with their product, which happens to be 36.

```

1  .class public Sum
2  .super java/lang/Object
3
4  .method public <init>()V
5      aload 0
6      invokevirtual java/lang/Object/<init>()V
7      return
8  .end method
9
10 .method public static main([Ljava/lang/String;)V
11     .limit locals 1
12     .limit stack 3
13     getstatic      java/lang/System/out Ljava/io/PrintStream;
14     ldc           6
15     dup
16     imul
17     invokestatic   Sum/sum(I)I
18     invokevirtual  java/io/PrintStream/println(I)V
19     return
20 .end method
21
22 .method public static sum(I)I
23     .limit locals 2
24     .limit stack 2
25     ldc          0
26     istore 1      ; s = 0;
27 loop:
28     iload 0      ; n
29     ifeq finish ; if (n == 0) goto finish;
30     iload 1      ; s
31     iload 0      ; n
32     iadd
33     istore 1      ; s = s + n;
34     iload 0      ; n
35     ldc          1
36     isub
37     istore 0      ; n = n - 1;
38     goto loop
39 finish:
40     iload 1
41     ireturn      ; return s;
42 .end method

```

Figure 10.12: An assembler program to compute the sum $\sum_{i=1}^{36} i$.

11. Line 17 calls the method `sum` defined below. After this call has finished, the number 36 on top of the stack is replaced with the value `sum(36)`.
12. Line 18 prints the value that is on top of the stack.
13. Line 22 declares the method `sum`. This method takes one integer argument and returns an integer as result.

14. Line 23 specifies that the method `sum` has two local variables: The first local variable is the parameter `n` and the second local variable corresponds to the variable `s` in the C program.
15. The effect of lines 25 and 26 is to initialize this variable `s` with the value 0.
16. Line 28 pushes the local variable `n` on the stack so that line 29 is able to test whether `n` is already 0. If `n = 0`, the program branches to the label `finish` in line 39, pushes the result `s` onto the stack (line 40) and returns. If `n` is not yet 0, the execution proceeds normally to line 30.
17. Line 30 and line 31 push the sum `s` and the variable `n` onto the stack. These values are then added and the result is written to the local variable `s` in line 33. The combined effect of these instructions is therefore to perform the assignment

$$s = s + n;$$
18. The instructions in line 34 up to line 37 implement the assignment

$$n = n - 1;$$
19. In line 38 we jump back to the beginning of the while loop and test whether `n` has become zero.
20. The declaration in line 42 terminates the definition of the method `sum`.

Exercise 31: Implement an assembler program that computes the factorial function. Test your program by printing $n!$ for $n = 1, \dots, 10$.

10.4 Disassembler*

Sometimes it is useful to transform a file consisting of *Java* byte code back into something that looks like assembler code. After all, a *Java* class file is a binary file and can therefore only be viewed via commands like `od` that produce an *octal* or *hexadecimal dump* of the given file. The command `javap` is a *disassembler*, i.e. it takes a *Java* byte code file and transforms it in something that looks similar to *Jasmin* assembler. For example, Figure 10.13 shows the class *Java* class `Sum` to compute the sum

$$\sum_{i=1}^{36} i$$

written in *Java*. If this program is stored in a file called “Sum.java”, we can compile it via the following command:

```
javac Sum.java
```

This will produce a class file with the name “Sum.class” containing the byte code.

Next, in order to *decompile* the “.class” file, we run the command

```
javap -c Sum.class
```

The output of this command is shown in Figure 10.14. We see that the syntax used by `javap` differs a bit from the syntax used by *Jasmin*. It is rather unfortunate that the company developing *Java* has decided not to make their assembler public. Still, we can see that the output produced by `javap` is quite similar to the input accepted by *jasmin*.

```
1 public class Sum {  
2     public static void main(String[] args) {  
3         System.out.println(sum(6 * 6));  
4     }  
5  
6     static int sum(int n) {  
7         int s = 0;  
8         for (int i = 0; i <= n; ++i) {  
9             s += i;  
10        }  
11        return s;  
12    }  
13 }
```

Figure 10.13: A *Java* program computing $\sum_{i=0}^{6^2} i$.

```

1  Compiled from "Sum.java"
2  public class Sum {
3      public Sum();
4      Code:
5          0: aload_0
6          1: invokespecial #1          // Method java/lang/Object."<init>":()V
7          4: return
8
9      public static void main(java.lang.String[]);
10     Code:
11         0: getstatic     #2          // Field java/lang/System.out:Ljava/io/PrintStream;
12         3: bipush       36
13         5: invokestatic  #3          // Method sum:(I)I
14         8: invokevirtual #4          // Method java/io/PrintStream.println:(I)V
15        11: return
16
17     static int sum(int);
18     Code:
19         0: iconst_0
20         1: istore_1
21         2: iconst_0
22         3: istore_2
23         4: iload_2
24         5: iload_0
25         6: if_icmpgt    19
26         9: iload_1
27        10: iload_2
28        11: iadd
29        12: istore_1
30        13: iinc         2, 1
31        16: goto         4
32        19: iload_1
33        20: ireturn
34 }

```

Figure 10.14: The output of “javap -c Sum.class”.

Chapter 11

Development of a Simple Compiler

In this chapter, we construct a compiler that translates a fragment of the `C` language into *Java* assembler. The fragment of the `C` language translated by the compiler is referred to as *Integer-C*, because only the data type `int` is available. Although it would be feasible to support additional data types, the extra effort would no longer be proportionate to the educational benefit of the example.

A compiler essentially consists of the following components:

1. The scanner reads the file to be translated and breaks it down into a sequence of tokens. We will develop our scanner using the `PLY` tool.
2. The parser reads the sequence of tokens and produces an abstract syntax tree as a result. We will also use `PLY` for the creation of the parser.
3. The type-checker checks the abstract syntax tree for type errors. Since the language we translate contains only a single data type, this phase is not needed for our compiler.
4. The code generator then translates the parse tree into a sequence of assembler commands that conform to the syntax of *Jasmin*.
5. This can be followed by an *optimization phase*. Due to time constraints we cannot cover this topic.
6. The resulting assembler program can be translated into Java bytecode using *Jasmin*.
7. The Java bytecode can be executed using the `java` command.

In compilers whose target code is a RISC assembler program, a code similar to JVM code is usually generated first. Such a code is referred to as *intermediate code*. It then remains the task of a so-called *backend* to generate an assembler program for a given processor architecture. The most challenging task here is to find a register allocation for the used variables, such that as many variables as possible can be held in registers. Due to time constraints, we cannot address the topic of register allocation in this lecture.

11.1 The Programming Language Integer-C

We now introduce the language *Integer-C*, which our compiler is intended to translate. In this context, we also refer to it as the *source language* of our compiler. Figure 11.1 on page 149 shows the grammar of the source language in Extended Backus-Naur Form (EBNF). The grammar for *Integer-C* uses the following terminals:

1. `ID` is an abbreviation for *Identifier* and represents a sequence of digits, letters, and underscores that begins with a letter. An `ID` designates either a variable or the name of a function.
2. `NUMBER` stands for a sequence of digits interpreted as a decimal number.
3. In addition, there are a number of operators such as `+`, `-`, etc., as well as keywords like `if`, `else`, and `while`.

```

program → function+

function → "int" ID "(" paramList ")" "{" decl+ stmt+ "}"

paramList → ("int" ID ("," "int" ID)*)?

decl → "int" ID ";";

stmt → "{" stmt* "}"
      | ID "=" expr ";";
      | "if" "(" boolExpr ")" stmt
      | "if" "(" boolExpr ")" stmt "else" stmt
      | "while" "(" boolExpr ")" stmt
      | "return" expr ";";
      | expr ";";

boolExpr → expr ("==" | "!=" | "<=" | ">=" | "<" | ">") expr
          | "!" boolExpr
          | boolExpr ("&&" | "||") boolExpr

expr → expr ("+" | "-" | "*" | "/" | "%") expr
      | "(" expr ")"
      | NUMBER
      | ID "(" (expr ("," expr)*)? ")"?

```

Figure 11.1: An EBNF grammar for *Integer-C*

According to the grammar provided above, a program is a list of functions. A function consists of the declaration of its signature, followed by a list of declarations (*decl*) and commands (*stmt*) enclosed in curly braces. The structure of the individual commands is then similar to that of the language SL, for which we have developed an interpreter in Chapter 6.4. The grammar shown in Figure 11.1 is ambiguous:

1. The grammar has the *Dangling-Else Problem*. We have already discussed this problem in Chapter 8 as part of an exercise. We will solve this problem via operator precedences.
2. For the operators used in arithmetic and Boolean expressions, precedences must be set to resolve ambiguities in the interpretation of these expressions.

Figure 11.2 shows a simple INTEGER-C program. The function *sum*(*n*) calculates the sum

$$\sum_{i=1}^n i$$

and the function *main*() calls the function *sum* with the argument 6 · 6. The function *println* used in the program outputs its argument followed by a newline. We assume that this function is predefined.

11.2 Developing the Scanner and the Parser

We construct both the scanner and the parser using *Ply*. Figure 11.3 shows the scanner. The scanner recognizes the operators, keywords, identifiers, and numbers. As the scanner is mostly similar to those scanners that we have already seen before, there are only a few points worth mentioning.

```

1  int sum(int n) {
2      int s;
3      s = 0;
4      while (n != 0) {
5          s = s + n;
6          n = n - 1;
7      };
8      return s;
9  }
10 int main() {
11     int n;
12     n = 6 * 6;
13     println(sum(n));
14 }

```

Figure 11.2: A simple INTEGER-C program to compute the sum $\sum_{i=1}^{6-6} i$.

1. If a token does not have to be manipulated by the scanner, then it can be defined by a simple equation of the form

$$t_name = regexp$$

where *name* is the name of the token and *regexp* is the regular expression defining the token. We have used this shortcut to define most of the tokens recognized by our scanner.

2. While we do not need to declare tokens for those operators that consist of just one token, we have to declare those tokens that consist of two or more characters. For example, the operator “==” is represented by the token ‘EQ’ and is defined in line 9.
3. Our programming language supports single line comments that start with the string “//” and extend to the end of the line. These comments are implemented via the token COMMENT that is defined in line 16–18. Note that the function `t.COMMENT` does not return a value. Therefore, these comments are simply discarded. This is also the reason that we have to use a function to define this token.
4. The most interesting aspect is the implementation of keywords like “while” or “return”. Note that we have defined separate tokens for all of the keywords but we have not defined any of these tokens. For example, we have not defined `t.return`. The reason is that, syntactically, all of the keywords are identifiers and are recognized by the regular expression

$$r'[a-zA-Z][a-zA-Z0-9_]*'$$

that is used for recognizing identifiers in line 28. However, the function `t.ID` that recognizes identifiers does not immediately return the token ‘ID’ when it has scanned the name of an identifier. Rather, it first checks whether this name is a predefined keyword. The predefined keywords are the key of the dictionary `Keywords` that is defined in line 38–43. The values of the keywords in this dictionary are the token types. Therefore, the function `t.ID` sets the *token type* of the token that is returned to the value stored in the dictionary `Keywords`. In case a name is not defined in this dictionary, the token type is set to ‘ID’.

5. The scanner implements the function `t.newline` in line 6. This function is needed to update the attribute `lineno` of the scanner. This attribute stores the line number and is needed for error messages.

Now we are ready to present the specification of our parser. For reasons of space, we have split the grammar into six parts. We start with Figure 11.4 on page 152.

1. Line 3 declares the start symbol `program`.

```

1  import ply.lex as lex
2
3  tokens = [ 'NUMBER', 'ID', 'EQ', 'NE', 'LE', 'GE', 'AND', 'OR',
4            'INT', 'IF', 'ELSE', 'WHILE', 'RETURN'
5            ]
6
7  t_NUMBER = r'0|[1-9][0-9]*'
8
9  t_EQ  = r'=='
10 t_NE  = r'!='
11 t_LE  = r'<='
12 t_GE  = r'>='
13 t_AND = r'&&'
14 t_OR  = r'\|\|'
15
16 def t_COMMENT(t):
17     r'//[^\n]*'
18     pass
19
20 Keywords = { 'int'      : 'INT',
21             'if'       : 'IF',
22             'else'     : 'ELSE',
23             'while'    : 'WHILE',
24             'return'   : 'RETURN'
25             }
26
27 def t_ID(t):
28     r'[a-zA-Z][a-zA-Z0-9]*'
29     t.type = Keywords.get(t.value, 'ID')
30     return t
31
32 literals = ['+', '-', '*', '/', '%', '(', ')', '{', '}', ';', '=', '<', '>', '!', ',', '']
33
34 t_ignore = ' \t\r'
35
36 def t_newline(t):
37     r'\n+'
38     t.lexer.lineno += t.value.count('\n')
39
40 def t_error(t):
41     print(f"Illegal character '{t.value[0]}' in line {t.lineno}.")
42     t.lexer.skip(1)
43
44 __file__ = 'main'
45
46 lexer = lex.lex()

```

Figure 11.3: The scanner for *Integer-C*.

2. Line 5–12 declare the operator precedences.

(a) The operator “`||`” that represents **logical or** has the lowest precedence and associates to the left.

```

1  import ply.yacc as yacc
2
3  start = 'program'
4
5  precedence = (
6      ('left', 'OR'),
7      ('left', 'AND'),
8      ('right', '!'),
9      ('nonassoc', 'EQ', 'NE', 'LE', 'GE', '<', '>'),
10     ('left', '+', '-'),
11     ('left', '*', '/', '%')
12 )
13
14 def p_program_one(p):
15     "program : function"
16     p[0] = ('program', p[1])
17
18 def p_program_more(p):
19     "program : function program"
20     p[0] = ('program', p[1]) + p[2][1:]
21
22 def p_function(p):
23     "function : INT ID '(' param_list ')' '{' decl_list stmt_list '}'"
24     p[0] = ('fct', p[2], p[4], p[7], p[8])

```

Figure 11.4: Grammar for Integer-C, part 1.

- (b) The operator “&&” that represents **logical and** binds stronger than “||” but not as strong as the negation operator “!”.
 - (c) The negation operator “!” is right associative because we want to interpret an expression of the form `!!a` as `!(!a)`.
 - (d) The comparison operators “==”, “!=”, “<=”, “>=”, “<”, and “>” are non-associative, since our programming languages disallows the chaining of comparison operators, i.e. an expression of the form $x < y < z$ is syntactically invalid.
 - (e) The arithmetical operators “+” and “-” have a lower precedence than the arithmetical operators “*” and “/”.
3. A **program** is a non-empty list of **function** definitions. Therefore, it is either a single function definition (line 14–16) or it is a function definition followed by more function definitions (line 18–20).

The purpose of the parser is to turn the given program into an abstract syntax tree. This abstract syntax tree is represented as a **nested tuple**. A program consisting of the functions f_1, \dots, f_n is represented as the nested tuple.

(“program”, f_1, \dots, f_n).

If in line 19 a **program** consisting of a **function** and another **program** is parsed, the expression `p[2]` in line 20 refers to the abstract syntax tree representing the **program** that follows the **function**. The expression `p[2][1:]` discards the keyword “program” that is at the start of this nested tuple. Hence, `p[2][1:]` is just a tuple of the functions following the first **function**. Therefore, the assignment to `p[0]` creates a nested tuple that starts with the keyword “program” followed by all the functions defined in this program.

4. A **function** definition starts with the keyword “int” that is followed by the name of the function, an opening

parenthesis, a list of the parameters of the function, a closing parenthesis, an opening brace, a list of declarations, a list of statements, and finally a closing brace.

```

25 def p_param_list_empty(p):
26     "param_list :"
27     p[0] = ('.', )
28
29 def p_param_list_one(p):
30     "param_list : INT ID"
31     p[0] = ('.', p[2])
32
33 def p_param_list_more(p):
34     "param_list : INT ID ',' ne_param_list"
35     p[0] = ('.', p[2]) + p[4][1:]
36
37 def p_ne_param_list_one(p):
38     "ne_param_list : INT ID"
39     p[0] = ('.', p[2])
40
41 def p_ne_param_list_more(p):
42     "ne_param_list : INT ID ',' ne_param_list"
43     p[0] = ('.', p[2]) + p[4][1:]
44
45 def p_decl_list_one(p):
46     "decl_list :"
47     p[0] = ('.',)
48
49 def p_decl_list_more(p):
50     "decl_list : INT ID ';' decl_list"
51     p[0] = ('.', p[2]) + p[4][1:]
52
53 def p_stmt_list_one(p):
54     "stmt_list :"
55     p[0] = ('.',)
56
57 def p_stmt_list_more(p):
58     "stmt_list : stmt stmt_list"
59     p[0] = ('.', p[1]) + p[2][1:]

```

Figure 11.5: Grammar for Integer-C, part 2.

Figure 11.5 shows the definition of parameter lists, declaration lists, and statement lists.

1. The definition of a list of parameters is quite involved, because parameter lists may be empty and, furthermore, different parameters have to be separated by “,”. Therefore, we have to introduce the additional syntactical variable `ne_param_list`, which represents a non-empty parameter list. The grammar rules for `param_list` are as follows:

```

param_list
: /* epsilon */
| 'int' ID
| 'int' ID ',' ne_param_list

```

```

;
ne_param_list
: 'int' ID
| 'int' ID ',' ne_param_list
;

```

We use the “.” as a key to represent lists of any kind as nested tuples.

2. The grammar rules for a list of declarations are as follows:

```

decl_list
: /* epsilon */
| 'int' ID ';' decl_list
;

```

Observe that with this definition a list of declarations may be empty. The grammar rules for a list of declarations are simpler than the grammar rules for parameter lists because every variable declaration is ended with a semicolon, while parameters are separated by commas and the last parameter must not be followed by a comma.

3. The grammar rules for a list of statements are as follows:

```

stmtnt_list
: /* epsilon */
| stmtnt stmtnt_list
;

```

Figure 11.6 on page 155 shows how the variable `stmtnt` is defined. The grammar rules for statements are as follows:

```

stmtnt : '{' stmtnt_list '}'
| ID '=' expr ';'
| 'if' '(' bool_expr ')' stmtnt
| 'if' '(' bool_expr ')' stmtnt 'else' stmtnt
| 'while' '(' bool_expr ')' stmtnt
| 'return' expr ';'
| expr ';'
;

```

Figure 11.7 on page 156 shows how Boolean expressions are defined. The grammar rules are as follows:

```

bool_expr : bool_expr '||' bool_expr
| bool_expr '&&' bool_expr
| '!' bool_expr
| '(' bool_expr ')'
| expr '==' expr
| expr '!=' expr
| expr '<=' expr
| expr '>=' expr
| expr '<' expr
| expr '>' expr
;

```

Figure 11.8 on page 157 shows how arithmetic expressions are defined. The grammar rules are as follows:

```

60 def p_stmtnt_block(p):
61     "stmtnt : '{' stmtnt_list '}'"
62     p[0] = p[2]
63
64 def p_stmtnt_assign(p):
65     "stmtnt : ID '=' expr ';' "
66     p[0] = ('=', p[1], p[3])
67
68 def p_stmtnt_if(p):
69     "stmtnt : IF '(' bool_expr ')' stmtnt"
70     p[0] = ('if', p[3], p[5])
71
72 def p_stmtnt_if_else(p):
73     "stmtnt : IF '(' bool_expr ')' stmtnt ELSE stmtnt"
74     p[0] = ('if-else', p[3], p[5], p[7])
75
76 def p_stmtnt_while(p):
77     "stmtnt : WHILE '(' bool_expr ')' stmtnt"
78     p[0] = ('while', p[3], p[5])
79
80 def p_stmtnt_return(p):
81     "stmtnt : RETURN expr ';' "
82     p[0] = ('return', p[2])
83
84 def p_stmtnt_expr(p):
85     "stmtnt : expr ';' "
86     p[0] = p[1]

```

Figure 11.6: Grammar for Integer-C, part 3.

```

expr : expr '+' expr
     | expr '-' expr
     | expr '*' expr
     | expr '/' expr
     | expr '%' expr
     | '(' expr ')'
     | NUMBER
     | ID
     | ID '(' expr_list ')'

```

Finally, Figure 11.9 on page 158 shows how `expr_list` is defined.

```

expr_list : /* epsilon */
           | expr
           | expr ',' ne_expr_list
           ;

ne_expr_list : expr
             | expr ',' ne_expr_list
             ;

```

Furthermore, this function shows the definition of the function `p_error`, which is called if PLY detects a syntax error. PLY will detect a syntax error in the case that the state of the generated shift/reduce parser has no action for the

```

87 def p_bool_expr_or(p):
88     "bool_expr : bool_expr OR bool_expr"
89     p[0] = ('||', p[1], p[3])
90
91 def p_bool_expr_and(p):
92     "bool_expr : bool_expr AND bool_expr"
93     p[0] = ('&&', p[1], p[3])
94
95 def p_bool_expr_neg(p):
96     "bool_expr : '!' bool_expr"
97     p[0] = ('!', p[2])
98
99 def p_bool_expr_paren(p):
100    "bool_expr : '(' bool_expr '"
101    p[0] = p[2]
102
103 def p_bool_expr_eq(p):
104    "bool_expr : expr EQ expr"
105    p[0] = ('==', p[1], p[3])
106
107 def p_bool_expr_ne(p):
108    "bool_expr : expr NE expr"
109    p[0] = ('!=', p[1], p[3])
110
111 def p_bool_expr_le(p):
112    "bool_expr : expr LE expr"
113    p[0] = ('<=', p[1], p[3])
114
115 def p_bool_expr_ge(p):
116    "bool_expr : expr GE expr"
117    p[0] = ('>=', p[1], p[3])
118
119 def p_bool_expr_lt(p):
120    "bool_expr : expr '<' expr"
121    p[0] = ('<', p[1], p[3])
122
123 def p_bool_expr_gt(p):
124    "bool_expr : expr '>' expr"
125    p[0] = ('>', p[1], p[3])

```

Figure 11.7: Grammar for Integer-C, part 4.

next input token. The argument `p` to the function `p_error` is the first token for which the action of the shift/reduce parser is undefined. In this case `p.value` is the part of the input string corresponding to this token.

Exercise 32: Extend the parser that is available at

<https://github.com/karlstroetmann/Formal-Languages/blob/master/Ply/Compiler.ipynb>

so that it supports the ternary C-operator for conditional expressions. For example, in C, the following assignment using the ternary operator can be used to assign to the variable `max` the maximum of the values of the variables `x` and `y`:

```
max = x > y ? x : y;
```

```
126 def p_expr_plus(p):
127     "expr : expr '+' expr"
128     p[0] = ('+', p[1], p[3])
129
130 def p_expr_minus(p):
131     "expr : expr '-' expr"
132     p[0] = ('-', p[1], p[3])
133
134 def p_expr_times(p):
135     "expr : expr '*' expr"
136     p[0] = ('*', p[1], p[3])
137
138 def p_expr_divide(p):
139     "expr : expr '/' expr"
140     p[0] = ('/', p[1], p[3])
141
142 def p_expr_modulo(p):
143     "expr : expr '%' expr"
144     p[0] = ('%', p[1], p[3])
145
146 def p_expr_group(p):
147     "expr : '(' expr ')'"
148     p[0] = p[2]
149
150 def p_expr_number(p):
151     "expr : NUMBER"
152     p[0] = ('Number', p[1])
153
154 def p_expr_id(p):
155     "expr : ID"
156     p[0] = p[1]
157
158 def p_expr_fct_call(p):
159     "expr : ID '(' expr_list ')'"
160     p[0] = ('call', p[1]) + p[3][1:]
```

Figure 11.8: Grammar for Integer-C, part 5.

Extend the parser so it can process the assignment shown above.

◇

```
161 def p_expr_list_empty(p):
162     "expr_list : "
163     p[0] = ('.',)
164
165 def p_expr_list_one(p):
166     "expr_list : expr"
167     p[0] = ('.', p[1])
168
169 def p_expr_list_more(p):
170     "expr_list : expr ',' ne_expr_list"
171     p[0] = ('.', p[1]) + p[3][1:]
172
173 def p_ne_expr_list_one(p):
174     "ne_expr_list : expr"
175     p[0] = ('.', p[1])
176
177 def p_ne_expr_list_more(p):
178     "ne_expr_list : expr ',' ne_expr_list"
179     p[0] = ('.', p[1]) + p[3][1:]
180
181 def p_error(p):
182     if p:
183         print(f'Syntax error at token "{p.value}" in line {p.lineno}.')
184     else:
185         print('Syntax error at end of input.')
186
187 parser = yacc.yacc(write_tables=False, debug=True)
```

Figure 11.9: Grammar for Integer-C, part 6.

11.3 Code Generation

Next, we discuss the generation of code. We structure our representation by discussing the code generation for arithmetic expressions, Boolean expression, statements, and function definitions separately.

11.3.1 Translation of Arithmetic Expressions

Given an arithmetic expression e , the translation of e is supposed to generate some code that, when executed, places the result of evaluating the expression e onto the stack. To this end we define a function `compile_expr` that has the following signature:

$$\text{compile_expr} : \text{Expr} \times \text{SymbolTable} \times \text{ClassName} \rightarrow \langle \text{List}[\text{AsmCmd}], \mathbb{N} \rangle.$$

A call of this function has the form

```
compile_expr(expr, st, name).
```

The interpretation of the arguments is as follows:

- (a) `expr` is the arithmetic expression that is to be translated into assembler code.
- (b) `st` is the [symbol table](#): Concretely, this is a dictionary that maps the variable names to their position in the local variable frame. For example, if “x” is a variable that occupies the third slot in the local variable frame, then we have

$$\text{st}['x'] = 2,$$

because the first variable in the local variable frame has index 0. We will discuss later how the positions of the variables in the local variable frame is assigned.

- (c) `name` is the name of the class that is to be used by our compiler.

As we generate *Java* assembler and in *Java* every function has to be a part of a class, all functions that we create have to be static functions that are defined inside the *Java* `.class` file that is generated. The parameter `name` specifies the name of this class.

The argument `name` is only needed when function calls are translated.

The function `compile_expr` returns a pair.

- (a) The first component of this pair is a list of *Java* assembler commands that adhere to the syntax recognized by *Jasmin*. In general, when the expression that is translated is complex, the execution of these assembler commands might need considerable room on the stack. However, it has to be guaranteed that when the execution of these commands finishes, the stack is back to its original height plus one because the net effect of executing these commands must be to put the value of the expression on the stack.
- (b) The second component of the return value of `compile_expr` is a natural number. This natural number tells us how much the stack might grow when `expr` is evaluated. This information is needed because the *Java* virtual machine needs this information in advance: In *Java*, every function has to declare how much space it might use on the stack. This declaration is done using the pseudo assembler command

```
.limit stack sz
```

where `sz` is the maximum size of the stack. Controlling the maximum height of the stack is a security feature of *Java* that prevents those exploits that utilize stack overflows.

In the following, we discuss the evaluation of the different arithmetic expressions one by one.

Translation of a Variable

If the expression that is to be compiled is a variable v , we have to load this variable onto the stack using the command `iload`. This command has one parameter which is the index of the variable on the local variable frame. This index

is stored in the symbol table `st`. Since we just need one entry on the stack to store the variable, we have

$$\text{compile_expr}(v, \text{st}, \text{name}) = \langle [\text{ iload } \text{st}[v]], 1 \rangle \quad \text{if } v \text{ is a variable.}$$

The code for translating a variable is shown in Figure 11.10 on page 160.

```

1  def compile_expr(expr, st, class_name):
2      if isinstance(expr, str):
3          return [f'ildc {st[expr]}'], 1
4      ...

```

Figure 11.10: Translation of a variable.

Translation of a Constant

We can load a constant c onto the stack using the assembler command

`ldc c .`

As we only need the room to store c on the stack, we have

$$\text{compile_expr}(c, \text{st}, \text{name}) = \langle [\text{ ldc } c], 1 \rangle \quad \text{if } c \text{ is a constant.}$$

The code for translating a constant is shown in Figure 11.11 on page 160.

```

5  def compile_expr(expr, st, class_name):
6      ...
7      elif expr[0] == 'Number':
8          _, n = expr
9          return [f'ldc {n}'], 1
10     ...

```

Figure 11.11: Translation of a variable.

Translation of an Arithmetic Operator

In order to translate an expression of the form

$lhs \text{ "+" } rhs$

into assembler code, we first have to recursively translate the expressions lhs and rhs into assembler code. Later, when this code is executed the values of the expressions lhs and rhs are placed on the stack. We add these values using the command `iadd`. If the evaluation of lhs needs s_1 words on the stack and the evaluation of rhs needs s_2 words on the stack, then the evaluation of $lhs + rhs$ needs

$$\max(s_1, 1 + s_2)$$

words on the stack, because when rhs is evaluated, the value of lhs occupies already one position on the stack and hence the evaluation of rhs can only use the memory cells that are above the position where lhs is stored. Therefore, the compilation of $lhs + rhs$ can be specified as follows:

$$\begin{aligned}
& \text{compile_expr}(lhs, st, name) = \langle L_1, s_1 \rangle \\
\wedge \quad & \text{compile_expr}(rhs, st, name) = \langle L_2, s_2 \rangle \\
\rightarrow \quad & \text{compile_expr}(lhs + rhs, st, name) = \langle L_1 + L_2 + [iadd], \max(s_1, 1 + s_2) \rangle.
\end{aligned}$$

Figure 11.12 on page 161 shows how the translation of arithmetic operators is implemented. The translation of subtraction, multiplication, and division is similar.

```

11  def compile_expr(expr, st, class_name):
12      ...
13      elif expr[0] in ['+', '-', '*', '/', '%']:
14          op, lhs, rhs = expr
15          L1, sz1 = compile_expr(lhs, st, class_name)
16          L2, sz2 = compile_expr(rhs, st, class_name)
17          OpToCmd = { '+': 'iadd', '-': 'isub', '*': 'imul', '/': 'idiv', '%': 'irem' }
18          Cmd      = indent(OpToCmd[op])
19          return L1 + L2 + [Cmd], max(sz1, 1 + sz2)
20      ...

```

Figure 11.12: Translation of the addition of expressions.

Translation of a Function Call

In order to translate a function call of the form $f(a_1, \dots, a_n)$ we first have to translate the arguments a_1, \dots, a_n . Then there are two cases:

1. f is a user defined function. In this case, the byte code treats f as a static function of the class `class_name`. This static function can be called using the assembler command `invokestatic`. In order to calculate the stack size that needs to be reserved for the evaluation of $f(a_1, \dots, a_n)$, let us assume that we have

$$\text{compile_expr}(a_i, st, \text{class_name}) = \langle L_i, s_i \rangle,$$

i.e. evaluation of the i^{th} argument is done by the list of assembler commands L_i and needs a stack size of s_i . As we are evaluating the arguments a_i in the order from a_1 to a_n , the evaluation of a_i needs a stack size of $i - 1 + s_i$, since the results from the evaluation of the arguments a_1, \dots, a_{i-1} are already placed on the stack. Therefore, if we define

$$s := \max(s_1, 1 + s_2, \dots, i - 1 + s_i, \dots, n - 1 + s_n),$$

then s is the total amount of stack size needed to evaluate the arguments. Furthermore, let us define the **signature** `fs` of the function f as the string

$$fs := \text{class_name}/f(\underbrace{I \dots I}_n)I.$$

Then we can define the value of $\text{compile_expr}(f(a_1, \dots, a_n), st, \text{class_name})$ as follows:

$$\text{compile_expr}(f(a_1, \dots, a_n), st, \text{class_name}) := \langle L_1 + \dots + L_n + [invokestatic\ fs], \max(s, 1) \rangle.$$

2. If f is the function `println`, then *Jasmin* treats the function f as a method of the predefined *Java* object `java.lang.System.out`. This method can be invoked using the assembler command `invokevirtual`. As before, let us assume that we have

$$\text{compile_expr}(a_i, st, \text{class_name}) = \langle L_i, s_i \rangle.$$

This time, we have to start by putting the object `java.lang.System.out` onto the stack before we can evaluate any of the arguments. Therefore, in order to calculate the stack size we now define

$$s := \max(1 + s_1, 2 + s_2, \dots, i - 1 + s_i, \dots, n + s_n).$$

In this case we have to define the **signature** `fs` of `f` as the string

$$fs := \text{java/io/PrintStream/println}(\underbrace{I \dots I}_n)V.$$

Furthermore, we define the command `gs` for putting the object `java.lang.System.out` onto the stack as

$$gs := \text{getstatic java/lang/System/out Ljava/io/PrintStream};.$$

Then we can define the value of `compile_expr(f(a1, ..., an), st, class_name)` as follows:

$$\text{compile_expr}(f(a_1, \dots, a_n), st, class_name) := \\ \langle [gs] + L_1 + \dots + L_n + [invokestatic fs], \max(s, 1) \rangle.$$

Figure 11.12 on page 161 shows how the translation of function calls is implemented.

```

21  def compile_expr(expr, st, class_name):
22      ...
23      elif expr[0] == 'call' and expr[1] == 'println':
24          _, _, *args = expr
25          CmdLst = ['getstatic java/lang/System/out Ljava/io/PrintStream;']
26          stck_size = 0
27          cnt = 0
28          for arg in args:
29              L, sz_arg = compile_expr(arg, st, class_name)
30              stck_size = max(stck_size, cnt + 1 + sz_arg)
31              CmdLst += L
32              cnt += 1
33          CmdLst += [f'invokevirtual java/io/PrintStream/println({"I"*cnt})V']
34          return CmdLst, stck_size
35      elif expr[0] == 'call' and expr[1] != 'println':
36          _, f, *args = expr
37          CmdLst = []
38          stck_size = 0
39          cnt = 0
40          for arg in args:
41              L, sz_arg = compile_expr(arg, st, class_name)
42              stck_size = max(stck_size, cnt + sz_arg)
43              CmdLst += L
44              cnt += 1
45          CmdLst += [f'invokestatic {class_name}/{f}({"I"*cnt})I']
46          return CmdLst, max(stck_size, 1)
47      ...

```

Figure 11.13: Translation of function calls.

11.3.2 Translation of Boolean Expressions

Boolean expressions are build from equations and inequations using the logical operators “!” (logical not), “&&” (logical and), and “||” (logical or). To compile Boolean expressions we define a function `compile_bool` that has the following signature:

$\text{compile_bool} : \text{BoolExpr} \times \text{SymbolTable} \times \text{ClassName} \rightarrow \langle \text{List}[\text{AsmCmd}], \mathbb{N} \rangle$.

The interpretation of the arguments is similar to the interpretation of the arguments of the function `compile_expr`. When we execute the list of assembler commands that result from the compilation of a Boolean expression, we expect that the execution of these commands either puts the number 1 (representing `True`) or the number 0 (representing `False`) onto the stack. We start our discussion of the function `compile_bool` with the translation of equations.

Translation of Equations

In the *Java* virtual machine, the logical values `True` and `False` are represented by the integers 1 and 0, respectively. Therefore, the code produced from translating an equation of the form

$lhs == rhs$

should either place the number 1 or 0 onto the stack: If the value of lhs is equal to the value of rhs , the number 1 has to be put on the stack, otherwise the value 0 has to be put onto the stack. Let us assume that we have

$\text{compile_expr}(lhs, st, class_name) = \langle L_1, s_1 \rangle$ and $\text{compile_expr}(rhs, st, class_name) = \langle L_2, s_2 \rangle$.

Then the stack size needed for evaluating the equation $lhs == rhs$ is given by the expression

$\max(s_1, 1 + s_2)$

and the function `compile_bool` can be specified as follows:

$$\begin{aligned} \text{compile_bool}(lhs == rhs, st, class_name) = \langle & L_1 \\ & + L_2 \\ & + [\text{if_icmpeq } true] \\ & + [\text{bipush } 0] \\ & + [\text{goto } next] \\ & + [true:] \\ & + [\text{bipush } 1] \\ & + [next:], s \rangle \end{aligned}$$

Here, `true` and `next` have to be new labels that do not occur elsewhere in the code for the function that is compiled. Let me explain this equation in detail:

1. L_1 and L_2 are the lists of assembler commands that evaluate lhs and rhs , respectively.
2. Executing L_1 and L_2 leaves two values on the stack. These values are then compared using the assembler command `if_icmpeq`. If these value are the same, execution proceeds at the label `true`.
3. Otherwise execution commences with the next instruction and hence the value 0 is pushed onto the stack.
4. Next, the control flow jumps to the label `next`, which is at the end of the generated list of assembler commands. Hence, in this case the execution of equation $lhs == rhs$ is finished.
5. If the values of lhs and rhs are the same, the number 1 is pushed onto the stack.

The translation of a negated equation of the form

$lhs != rhs$

is similar to the translation of an equation: We only have to replace the command `if_icmpeq` with the command `if_icmpne`. Similarly, if the inequation has the form

$lhs <= rhs$

we have to replace the command `if_icmpeq` with the command `if_icmple`. If the inequation has the form

$lhs >= rhs$

we have to replace the command `if_icmpeq` with the command `if_icmpge`. If the inequation has the form

$$lhs < rhs$$

we have to replace the command `if_icmpeq` with the command `if_icmplt`. If the inequation has the form

$$lhs > rhs$$

we have to replace the command `if_icmpeq` with the command `if_icmpgt`. Figure 11.14 on page 164 shows how the translation of equations and inequations is implemented.

```

1  def compile_bool(expr, st, class_name):
2      if expr[0] in ['==', '!=', '<=', '>=', '<', '>']:
3          OpToCmd = { '==': 'if_icmpeq',
4                     '!=': 'if_icmpne',
5                     '<=': 'if_icmple',
6                     '>=': 'if_icmpge',
7                     '<' : 'if_icmplt',
8                     '>' : 'if_icmpgt'
9                 }
10         op, lhs, rhs = expr
11         L1, sz1 = compile_expr(lhs, st, class_name)
12         L2, sz2 = compile_expr(rhs, st, class_name)
13         true_label = new_label()
14         next_label = new_label()
15         CmdLst = L1 + L2
16         cmd = OpToCmd[op]
17         CmdLst += [indent(cmd + ' ' + true_label)]
18         CmdLst += [indent('bipush 0')]
19         CmdLst += [indent('goto ' + next_label)]
20         CmdLst += [' ' * 4 + true_label + ':']
21         CmdLst += [indent('bipush 1')]
22         CmdLst += [' ' * 4 + next_label + ':']
23         return CmdLst, max(sz1, 1 + sz2)
24         ...

```

Figure 11.14: Translation of equations.

Translation of Binary Boolean Operators

In order to translate an expression of the form

$$lhs \&\& rhs$$

into assembler code, we first have to recursively translate the expressions *lhs* and *rhs* into assembler code. Later, when this code is executed the values of the expressions *lhs* and *rhs* are placed on the stack. We combine these values using the command `iand`. If the evaluation of *lhs* needs s_1 words on the stack and the evaluation of *rhs* needs s_2 words on the stack, then the evaluation of $lhs + rhs$ needs

$$\max(s_1, 1 + s_2)$$

words on the stack, because when *rhs* is evaluated, the value of *lhs* occupies already one position on the stack and hence the evaluation of *rhs* can only use the memory cells that are above the position where *lhs* is stored. Therefore,

the compilation of $lhs \ \&\& \ rhs$ can be specified as follows:

$$\begin{aligned} & \text{compile_bool}(lhs, st, name) = \langle L_1, s_1 \rangle \\ \wedge \quad & \text{compile_bool}(rhs, st, name) = \langle L_2, s_2 \rangle \\ \rightarrow \quad & \text{compile_bool}(lhs + rhs, st, name) = \langle L_1 + L_2 + [\text{iand}], \max(s_1, 1 + s_2) \rangle. \end{aligned}$$

The translation of an expression of the form

$lhs \ || \ rhs$

is similar: Instead of using the command `iand` we have to use the command `ior` instead. Figure 11.15 on page 165 shows how the translation of binary logical operators is implemented.

```

25  def compile_expr(expr, st, class_name):
26      ...
27      elif expr[0] in ['&&', '||']:
28          op, lhs, rhs = expr
29          OpToCmd       = { '&&': iand, '||': ior }
30          L1, sz1       = compile_bool(lhs, st, class_name)
31          L2, sz2       = compile_bool(rhs, st, class_name)
32          cmd           = OpToCmd[op]
33          CmdLst        = L1 + L2 + [indent(cmd)]
34          return CmdLst, max(sz1, 1 + sz2)
35      ...

```

Figure 11.15: Translation of binary logical operators.

It should be noted that our translation of the binary logical operators is different from what happens in the language `C`. In `C`, if an expression of the form

$lhs \ \&\& \ rhs$

is evaluated, the evaluation is stopped as soon the evaluation of lhs returns 0 because then there is no point to evaluate rhs , since if the value of lhs is 0, the result of the expression $lhs \ \&\& \ rhs$ is 0, independent from the value of the expression rhs .

Translation of Negations

The translation of an expression of the form $!expr$ is not as straightforward as the translation of conjunctions and disjunctions. The reason is that *Jasmin* has no assembler command of the form `inot` that negates a logical value. But there is another way: Since we represent the truth values as 0 and 1, we can specify negation arithmetically as follows:

$$!x = 1 - x.$$

Therefore, if we have

$$\text{compile_bool}(arg, st, class_name) = \langle L, s \rangle$$

we can define:

$$\text{compile_bool}(!arg, st, class_name) = \langle [\text{bipush } 1] + L + [\text{isub}], 1 + s \rangle.$$

Figure 11.16 on page 166 shows how the translation of the negation operator is implemented.

```

36 def compile_expr(expr, st, class_name):
37     ...
38     elif expr[0] == '!':
39         _, arg = expr
40         L, sz = compile_expr(arg, st, class_name)
41         CmdLst = ['bipush 1'] + L + ['isub']
42         return CmdLst, max(sz1, sz + 1)
43     ...

```

Figure 11.16: Translation of the negation operator.

11.3.3 Translation of Statements

Next, we show how statements are compiled. First of all, we agree that the execution of a statement must not change the size of the stack: The size of stack before the execution of a statement must be the same as the size of the stack after the statement has been executed. Of course, during the execution of the statement the stack may well grow and shrink. But once the execution of the statement has finished, the stack has to be cleaned from all intermediate values that have been put on the stack during the execution of the statement.

To compile statements we define a function `compile_stmtnt` that has the following signature:

$$\text{compile_stmtnt} : \text{Stmtnt} \times \text{SymbolTable} \times \text{ClassName} \rightarrow \langle \text{List}[\text{AsmCmd}], \mathbb{N} \rangle.$$

The interpretation of the arguments is similar to the interpretation of the arguments of the function `compile_expr`. The only difference is that the first argument now is an abstract syntax tree that represents a statement.

Translation of Assignments

To begin with, we show how an assignment of the form

$$x = \text{expr}$$

is translated. The idea is to evaluate `expr`. As a consequence of this evaluation the value of `expr` remains on the stack, from where it can be stored in the variable `x` using the assembler command `istore`. Therefore, if we have

$$\text{compile_expr}(\text{expr}, \text{st}, \text{class_name}) = \langle L, s \rangle$$

we can define:

$$\text{compile_stmtnt}(x = \text{expr}, \text{st}, \text{class_name}) = \langle L + [\text{istore st}[x]], s \rangle.$$

Figure 11.17 on page 166 shows how the translation of an assignment is implemented.

```

1 def compile_stmtnt(stmtnt, st, class_name):
2     if stmtnt[0] == '=':
3         _, var, expr = stmtnt
4         CmdLst, sz = compile_expr(expr, st, class_name)
5         CmdLst += [f'istore {st[var]}']
6         return CmdLst, sz
7     ...

```

Figure 11.17: Translation of an assignment statement.

The Translation of Branching Commands

Next, we show how to translate a branching command of the form

`if (bool_expr) stmt.`

Obviously, we first have to translate the Boolean expression *bool_expr*. If we have

`compile_expr(bool_expr, st, class_name) = $\langle L_1, s_1 \rangle$`

and, furthermore,

`compile_stmt(stmt, st, class_name) = $\langle L_2, s_2 \rangle$,`

then we can define:

$$\begin{aligned} \text{compile_stmt}(\text{if } (expr) \text{ stmt}, st, class_name) = & \langle \\ & + L_1 \\ & + [\text{ifeq } else] \\ & + L_2 \\ & + [else:], \quad \max(s_1, s_2) \rangle \end{aligned}$$

This works because executing the list of assembler commands L_1 leaves either a 0 or a 1 on the stack. If *expr* is false, the top of the stack stores the number 0. In this case, the branching command `ifeq` branches to the label *else* and the assembler commands in the list L_2 are not executed. If *expr* is true, the top of the stack stores the number 1. Therefore, the branching command `ifeq` does not branch and the assembler commands in the list L_2 are executed, i.e. *stmt* is executed.

The translation of a branching command of the form

`if (bool_expr) stmt1 else stmt2`

is similar. Let us assume that we have the following:

1. `compile_expr(bool_expr, st, class_name) = $\langle L_1, s_1 \rangle$`
2. `compile_stmt(stmt1, st, class_name) = $\langle L_2, s_2 \rangle$,` and
3. `compile_stmt(stmt2, st, class_name) = $\langle L_3, s_3 \rangle$.`

Then we define:

$$\begin{aligned} \text{compile_stmt}(\text{if } (bool_expr) \text{ stmt}_1 \text{ else stmt}_2, st, class_name) = & \langle \\ & + L_1 \\ & + [\text{ifeq } else] \\ & + L_2 \\ & + [\text{goto next}] \\ & + [else:] \\ & + L_3 \\ & + [next:], \quad \max(s_1, s_2, s_3) \rangle \end{aligned}$$

Figure 11.18 on page 168 shows how the translation of branching statements is implemented.

Translation of a Loop

If we want to translate a `while` loop of the form

`while (cond) stmt`

we recursively compute

`compile_expr(cond, st, class_name) = $\langle L_1, s_1 \rangle$`

and

`compile_stmt(stmt, st, class_name) = $\langle L_2, s_2 \rangle$,`


```

1  def compile_stmt(stmt, st, class_name):
2      ...
3      elif stmt[0] == 'if':
4          _, expr, sub_stmt = stmt
5          L1, sz1 = compile_bool(expr, st, class_name)
6          L2, sz2 = compile_stmt(sub_stmt, st, class_name)
7          else_label = new_label()
8          CmdLst = L1 + [f'ifeq {else_label}'] + L2 + [else_label + ':']
9          return CmdLst, max(sz1, sz2)
10     elif stmt[0] == 'if-else':
11         _, expr, then_stmt, else_stmt = stmt
12         L1, sz1 = compile_bool(expr, st, class_name)
13         L2, sz2 = compile_stmt(then_stmt, st, class_name)
14         L3, sz3 = compile_stmt(else_stmt, st, class_name)
15         else_label = new_label()
16         next_label = new_label()
17         if_stmt = f'ifeq {else_label}'
18         else_stmt = else_label + ':'
19         next_stmt = next_label + ':'
20         goto_stmt = f'goto {next_label}'
21         CmdLst = L1 + [if_stmt] + L2 + [goto_stmt, else_stmt] + L3 + [next_stmt]
22         return CmdLst, max(sz1, sz2, sz3)
23     ...

```

Figure 11.18: Translation of branching statements.

Then we can define

$$\begin{aligned}
 \text{compile_stmt}(\text{while } (cond) \text{ stmt}, st, class_name) = \langle & [loop:] \\
 & + L_1 \\
 & + [\text{ifeq next}] \\
 & + L_2 \\
 & + [\text{goto loop}] \\
 & + [\text{next: }], \quad \max(s_1, s_2) \rangle
 \end{aligned}$$

Figure 11.19 on page 169 shows how the translation of a loop is implemented.

Translation of a Return Statement

In order to translate a return statement of the form

```
return expr;
```

we first have to translate the expression *expr*. Assume that

$$\text{compile_expr}(expr, st, class_name) = \langle L, s \rangle.$$

Then we can define

$$\text{compile_stmt}(\text{return } expr;, st, class_name) = \langle L + [\text{ireturn}], s \rangle.$$

Figure 11.20 on page 169 shows how a return statement is translated.

```

1  def compile_stmt(stmt, st, class_name):
2      ...
3      elif stmt[0] == 'while':
4          _, expr, body_stmt = stmt
5          L1, sz1 = compile_bool(expr, st, class_name)
6          L2, sz2 = compile_stmt(body_stmt, st, class_name)
7          loop_label = new_label()
8          next_label = new_label()
9          if_stmt = f'ifeq {next_label}'
10         loop_stmt = loop_label + ':'
11         next_stmt = next_label + ':'
12         goto_stmt = f'goto {loop_label}'
13         CmdLst = [loop_stmt] + L1 + [if_stmt] + L2 + [goto_stmt, next_stmt]
14         return CmdLst, max(sz1, sz2)
15     ...

```

Figure 11.19: Translation of a loop.

```

1  def compile_stmt(stmt, st, class_name):
2      ...
3      elif stmt[0] == 'return':
4          _, expr = stmt
5          CmdLst, sz = compile_expr(expr, st, class_name)
6          CmdLst += ['ireturn']
7          return CmdLst, sz
8      ...

```

Figure 11.20: Translation of a return statement.

Translating a Block Statement

The translation of block statement of the form

$$\{stmt_1; \dots stmt_n;\}$$

proceeds by translating the statements $stmt_i$ separately. Assume that

$$compile_stmt(stmt_i, st, class_name) = \langle L_i, s_i \rangle \quad \text{for } i = 1, \dots, n.$$

Then we define

$$compile_stmt(\{stmt_1; \dots stmt_n;\}, st, class_name) := \langle L_1 + \dots + L_n, \max(s_1, \dots, s_n) \rangle.$$

Figure 11.21 on page 170 shows how the translation of a block statement is implemented.

Translation of Expression Statements

When we translate an expression statement of the form

$$expr ;$$

we assume that the evaluation of the expression does not leave a value on the stack. At present, the only expression

```

1  def compile_stmtnt(stmtnt, st, class_name):
2      ...
3      elif stmtnt[0] == '.':
4          _, *stmtnt_lst = stmtnt
5          CmdLst = []
6          size = 0
7          for s in stmtnt_lst:
8              L, sz = compile_stmtnt(s, st, class_name)
9              CmdLst += L
10             size = max(size, sz)
11         return CmdLst, size

```

Figure 11.21: Translation of an expression statement.

whose evaluation does not create a value is an invocation of the function `println`. Then in order to evaluate the expression statement, we just have to evaluate the expression. Therefore, we have

```
compile_stmtnt(expr, st, class_name) = compile_expr(expr, st, class_name).
```

Figure 11.22 on page 170 shows how the translation of an expression statement is implemented.

```

1  def compile_stmtnt(stmtnt, st, class_name):
2      ...
3      else: # it must be an expression statement
4          CmdLst, sz = compile_expr(stmtnt, st, class_name)
5          return CmdLst, sz

```

Figure 11.22: Translation of an expression statement.

11.3.4 Translation of a Function Definition

Figure 11.23 on page 171 shows how a function definition is translated. In order to understand how this works we have to understand what type of code the function `compile` has to generate. Depending on whether the function that is translated is the function `main` or not there are two cases.

1. If the function that is to be compiled has the name “main”, the code that has to be generated has the following form:

```

1      .method public static main([Ljava/lang/String;)V
2      .limit locals l
3      .limit stack s
4      s1
5      ⋮
6      sn
7      return
8      .end method

```

```

1  def compile_fct(fct_def, class_name):
2      global label_counter
3      label_counter = 0
4      _, name, parameters, variables, stmnts = fct_def
5      _, *parameters = parameters
6      _, *variables = variables
7      _, *stmnts = stmnts
8      m = len(parameters)
9      n = len(variables)
10     st = {}
11     cnt = 0
12     for var in parameters:
13         st[var] = cnt
14         cnt += 1
15     for var in variables:
16         st[var] = cnt
17         cnt += 1
18     CmdLst = []
19     size = 0
20     for stmt in stmnts:
21         L, sz = compile_stmt(stmt, st, class_name)
22         CmdLst += L
23         size = max(size, sz)
24     limit_locals = f'.limit locals {m+n}'
25     limit_stack = f'.limit stack {size}'
26     if name != 'main':
27         method = f'.method public static {name}({{"I"*m})I'
28         CmdLst = [method, limit_locals, limit_stack] + CmdLst + ['.end method']
29         return CmdLst, sz
30     else:
31         method = '.method public static main([Ljava/lang/String;)V'
32         CmdLst = [method, limit_locals, limit_stack] + CmdLst + \
33             ['.return', '.end method']
34     return CmdLst, sz

```

Figure 11.23: Translation of a function definition.

Here l is the number of all variables used by the function `main`, while s is the maximum height of the stack. Finally, s_1, \dots, s_n are the assembler commands that result from translating the block inside the function.

2. Otherwise, the following code is to be generated:

```

1      .method public static f(I...I)I
2      .limit locals l
3      .limit stack s
4      s1
5      ⋮
6      sn
7      .end method

```

Here f is the name of the function. The signature of f is a string of the form

$$f(\underbrace{I \cdots I}_m)I$$

where m is the number of the parameters. Furthermore, l is the number of all local variables and s is the maximum height of the stack. Finally, s_1, \dots, s_n are the assembler commands that result from translating the block inside the function.

We proceed to discuss the implementation shown in Figure 11.23 line by line.

1. We initialize the global variable `label_counter` to 0. Every time a new label is generated, this counter will be incremented by the function `new_label` that creates labels of the form `l1`, `l2`, etc.

2. The nested tuple that is produced by the parser from a function definition has the form

$$('fct', name, parameters, variables, stmnts).$$

(a) *name* is the name of the function.

(b) *parameters* is the abstract syntax tree representing the list of parameters.

(c) *variables* is the abstract syntax tree representing the list of all local variables declared in the function.

(d) *stmnts* is the the abstract syntax tree representing the list of all statements in the body of the function.

3. Initially, the parameters, variables, and statements are stored as nested tuples. In line 5-7 these nested tuples are transformed into proper lists.

4. m is the number of parameters.

5. n is the number of local variables.

6. The `for` loops in line 12–17 initialize the symbol table `st`. Given a parameter or local variable x , the expression `st[x]` returns the position in the local variable frame where the parameter or variable is stored.

7. The `for` loop in line 20–23 translates the statements in the body of the function one by one. Furthermore, it computes the size of the stack that is needed.

8. Line 24 declares the size of the local variable frame. The pseudo-command `.limit locals` reserves memory for the parameters and the local variables.

9. Line 25 declares the size of the stack needed by the function.

10. Depending on whether the function that is compiled is the function `main` or not, the correct code is returned.

```

1  import os
2
3  def compile_program(file_name):
4      directory = os.path.dirname(file_name)
5      base      = os.path.basename(file_name)
6      base      = base[:-2]
7      with open(file_name, 'r') as handle:
8          program = handle.read()
9      ast = yacc.parse(program)
10     _, *fct_lst = ast
11     CmdLst      = []
12     for fct in fct_lst:
13         L, _ = compile_fct(fct, base)
14         CmdLst += L + ['\n']
15     with open(directory + '/' + base + '.jas', 'w') as handle:
16         handle.write('.class public ' + base + '\n');
17         handle.write('.super java/lang/Object\n\n');
18         handle.write('.method public <init>()V\n');
19         handle.write('    aload 0\n');
20         handle.write('    invokenonvirtual java/lang/Object/<init>()V\n');
21         handle.write('    return\n');
22         handle.write('.end method\n\n');
23     for cmd in CmdLst:
24         handle.write(cmd + '\n')

```

Figure 11.24: Translation of a program.

11.3.5 Compiling a Program

We conclude this chapter by presenting the function `compile_program` that is shown in Figure 11.24 on page 173. This function takes the name of a file as its argument. This file is expected to contain a program written in the subset of the programming language `C` that has been presented in this chapter. If the name of this file is, for example, `Test.c`, then the function `compile_program` generates a file with the name `Test.jas`, that contains an assembler program that adheres to the rules of *Jasmin*. We proceed to discuss the function `compile_program` line by line.

1. In line 1 we import the module `os`. This module contains the submodule `path` which contains the functions `dirname` and `basename`.

- (a) The function `dirname` takes a string specifying a *file path*, e.g. something like

```
'~/Dropbox/Formal-Languages/Ply/Examples/Test.c'
```

and returns the name of the directory. In the example given above it would return the string

```
'~/Dropbox/Formal-Languages/Ply/Examples'.
```

- (b) The function `basename` takes a *file path* and returns the name of the filename of the directory. In the example given previously it would return the string `'Test.c'`.

The module `os` is aware of the differences in naming directories that exist in different operating systems.

2. After computing the `directory` of the file and its `base` in line 4 and 5, we remove the file extension `'.c'` in line 6.
3. Line 7 and 8 read the file and store its content into the string `program`.

4. In line 9 our parser converts the program into an abstract syntax tree that is represented as a nested tuple.
5. `fct_lst` is a list of all function definitions occurring in the given program.
6. `CmdLst` is a list of all assembler commands that will be generated.
7. The `for` loop in line 12–14 translates all function definitions into assembler code and concatenates the assembler statements into the list `CmdLst`.
8. In line 15 the output file is opened for writing.
9. Line 16–22 write the initialization code to the output file.
10. The `for` loop in line 23–24 writes the generated assembler commands to the output file.

Figure 11.25 on page 174 shows an integer C program for computing the sum $\sum_{i=1}^{6^2} i$. When we translate this program using the compiler developed in this chapter, we get the program that is shown in Figure 11.26 on page 175.

```
1  int sum(int n) {  
2      int s;  
3      s = 0;  
4      while (n != 0) {  
5          s = s + n;  
6          n = n - 1;  
7      }  
8      return s;  
9  }  
10  
11 int main() {  
12     int n;  
13     n = 6 * 6;  
14     println(sum(n));  
15 }
```

Figure 11.25: A C program for computing the sum $\sum_{i=1}^{6^2} i$.

```
1  .class public MySum
2  .super java/lang/Object
3
4  .method public <init>()V
5      aload 0
6      invokenonvirtual java/lang/Object/<init>()V
7      return
8  .end method
9
10 .method public static sum(I)I
11 .limit locals 2
12 .limit stack 2
13     ldc 0
14     istore 1
15     l3: iload 0
16     ldc 0
17     if_icmpne l1
18     bipush 0
19     goto l2
20     l1: bipush 1
21     l2: ifeq l4
22     iload 1
23     iload 0
24     iadd
25     istore 1
26     iload 0
27     ldc 1
28     isub
29     istore 0
30     goto l3
31     l4: iload 1
32     ireturn
33 .end method
34
35 .method public static main([Ljava/lang/String;)V
36 .limit locals 1
37 .limit stack 2
38     ldc 6
39     ldc 6
40     imul
41     istore 0
42     getstatic java/lang/System/out Ljava/io/PrintStream;
43     iload 0
44     invokestatic MySum/sum(I)I
45     invokevirtual java/io/PrintStream/println(I)V
46     return
47 .end method
```

Figure 11.26: Assembler program generated by our compiler.

Exercise 33: Extend the compiler that is presented in this chapter and is available at

<https://github.com/karlstroetmann/Formal-Languages/blob/master/Ply/Compiler.ipynb>

so that a new kind of `for`-loops is supported. Figure 11.27 on page 177 shows an example of the use of this kind of `for` loop. ◇

```
1  int sum(int n) {  
2      int s;  
3      int i;  
4      s = 0;  
5      for (i = 1 .. n) {  
6          s = s + i;  
7      }  
8      return s;  
9  }
```

Figure 11.27: A function for computing the sum $\sum_{i=1}^n i$.

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