Homework 4 Exercice 2

Oscar Garnier

15 décembre 2022

Question 1

Set i as defined. We know there exists a clique of size $M = k - i + 1$ colored with colors $i, i+1, ..., k$, let us call this clique $K = \{u_i, ..., u_k\}$. Now suppose $i \neq 1$, which means there exists another color $c = i - 1$ not in the clique.

We know that $\forall u \in K$ is connected to some vertex colored c, because we applied the greedy algorithm, if u was not connected to a vertex colored $c \leq j$, it would have been colored c itself.

We also know there is no such c-colored vertex that is connected to all $u \in K$, by minimality of i.

Let $p = max\{|K \cap N(v)| for\ v \in V\ colored\ c\}$. Let $a \in V$ be such a max, (so a is colored c and has exactly p neighbors in K).
We know $p < M$, so $\exists u \in K$ s.t. $(a, u) \notin E$, and we know that $\exists b \in V$ colored c s.t. $(b, u) \in E$. b is not connected to every vertex in $N(a)$ by maximality of p, $\exists v \in N(a)$ s.t. $(v, b) \notin E$.
When reasoning on the subgraph induced by $F = \{a, b, u, v\}$, we have : $(a, v) \in E, (v, u) \in E$, because they are in a clique, $(u, b) \in E$. Aditionally, $(a, u) \notin E$ and $(v, b) \notin E$. This means the graph induced by F is P_4, which is absurd.
This means $i = 1$

Question 2

Let G be a *cograph*. Choose any ordering of the vertices, and color it greedily. Suppose you then get k colors. Since we have found such a coloring, we know $\Xi(G) \leq k$. According to question 1, there exists a click of cardnality k, with all k colors, which means $\Xi(G) \geq k$. Hence, $\Xi(G) = k$ and the coloring was optimal.