Lecture 5, September 29, 2025

Forecasting using ARIMA and ADL models

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In [1]:
```
#install.packages("quantmod")
#install.packages("fredr")
#install.packages("ggfortify")
#install.packages('urca')
#install.packages("tseries")
#install.packages("forecast")
#install.packages("dynlm")
#install.packages("stargazer")
#install.packages("pracma")
#install.packages("dLagM")
#install.packages("gets")
#install.packages("car")
#install.packages("lmtest")
#install.packages("astsa")
#install.packages("zoo")
#install.packages("conflicted")
#library(conflicted)
#conflict_prefer("lag", "stats")
options(warn=-1)
```

Introduction

Today we will focus on finding **ADL models** and forecasting using **ARIMA** and **ADL** models. For this exercise, we will work with two Canadian macroeconomic series:

1. **Canadian GDP** – the same quarterly series we introduced in Lecture 3.
2. **Weekly hours worked in the Canadian manufacturing sector** – the same quarterly series we introduced in Lecture 4.

Data Transformations

We begin by importing the data and transforming it to achieve **stationarity**, which is a prerequisite for applying ARIMA and/or ADL models.

- For **GDP**, we take the **first difference of the logarithm**.
  - This transformation allows us to interpret the series in terms of *quarterly growth rates*, while also stabilizing the variance and removing the stochastic trend.
- For **hours worked**, we take the **first difference of the level series**.
  - This transformation removes the unit root identified in the original series.

With these transformations, we now have two stationary series suitable for ARIMA and ADL modeling.

In [2]:
```
# Import the dataset from a CSV file downloaded from FRED
gdp =read.csv("NGDPRSAXDCCAQ.csv")
# Convert the second column of the dataset into a time series object
tsgdp = ts(gdp[, 2], frequency = 4, start = c(1961, 1))
# Transform the series into (approximate) percentage changes
dgdp = diff(log(tsgdp))
```

In [3]:
```
# Import the dataset from a CSV file downloaded from FRED
hours = read.csv("HOHWMN02CAM065N.csv")
# Convert the second column of the dataset into a time series object
tshours = ts(hours[, 2], frequency = 4, start = c(1960, 1))
# Transform the series into a change from the previous period
dhours = diff(tshours)
```

Exhaustive ADL Subset Search Based on Parameter Significance

In [41]:
```
##############################################
## Exhaustive ADL Subset Search based on parameter significance
## Goal: start from a general ADL(4,4) and iteratively drop the
##       least significant regressor until all remaining have
```

```r
##       |t| >= 1.96 (≈ 5% two-sided level), or none remain.
###############################################

data = ts.intersect(dgdp, dhours)  # Align (intersect) the two input ts objects to a common sample

library(gets)

## ---- 1) Create lagged variables -----------------------------------------
## Assume 'data' is a bivariate ts object:
##   - column 1: y (e.g., GDP growth)
##   - column 2: x (e.g., ΔHours worked)
## We build up to 4 lags for each series. Using negative 'k' in lag(z, -k)
## shifts the series BACK by k steps so that, e.g., lag(y,-1) corresponds to y_{t-1}.

y0 = data[, 1]                # y_t (dependent variable at time t)

y1 = lag(data[, 1], -1)       # y_{t-1}  (first own lag of y)
y2 = lag(data[, 1], -2)       # y_{t-2}
y3 = lag(data[, 1], -3)       # y_{t-3}
y4 = lag(data[, 1], -4)       # y_{t-4}

x1 = lag(data[, 2], -1)       # x_{t-1}  (first lag of x)
x2 = lag(data[, 2], -2)       # x_{t-2}
x3 = lag(data[, 2], -3)       # x_{t-3}
x4 = lag(data[, 2], -4)       # x_{t-4}

tempdata = ts.intersect(      # Align y_t and all lags so rows line up (drop initial NA rows from lags)
  y0,
  y1, y2, y3, y4,
  x1, x2, x3, x4
)

# Separate dependent and independent variables after alignment
depvar = tempdata[, 1]                      # y_t
indvar = tempdata[, 2:ncol(tempdata)]       # [y_{t-1..4}, x_{t-1..4}]

# Name columns explicitly for readability/tracking during elimination
colnames(indvar) = c("y1","y2","y3","y4","x1","x2","x3","x4")

# Initialize min |t| to trigger entry into the loop
tmin = 0

# Main backward-elimination loop based on individual t-statistics:
# Continue while the smallest absolute t-stat among non-constant coefficients is < 1.96.
while (tmin < 1.96) {

         # Estimate ARX with current regressor set 'indvar' and a constant (mc=TRUE).
         # try(..., silent=TRUE) avoids stopping on errors (e.g., singular design), and
         # suppressWarnings(...) silences routine warnings during re-estimation.
         model = try(suppressWarnings(arx(depvar, mxreg = indvar, mc = TRUE)), silent = TRUE)

         print(model)                # Print iteration’s model object (useful to monitor retained vars)

         b  = coef(model)             # Named vector of estimated coefficients (incl. constant 'mconst')
         V  = vcov(model)             # Variance–covariance matrix of estimates
         se = sqrt(diag(V))           # Standard errors from diagonal of V

         tstat = b / se              # Compute coefficient-wise t-statistics

         # Drop the constant's t-stat from the elimination rule:
         # We only compare *slope* coefficients (lags) to the threshold, not 'mconst'.
         tstat = tstat[setdiff(names(tstat), "mconst")]

         if (length(tstat) == 0) break         # If no slope coefficients are left, stop the loop

        tstat = abs(tstat)                     # Work with absolute t-values for two-sided testing
        tmin = min(tstat)                      # Record the smallest |t| among current regressors

        # Identify the variable with the minimum |t| and prepare a new regressor set that excludes it.
        # setdiff(colnames(indvar), names(which.min(tstat))) returns all columns except the one with min |t|.
        k = setdiff(colnames(indvar), names(which.min(tstat)))

        # If dropping that variable leaves no columns, set indvar to NULL
        # (intercept-only ARX on next iteration/exit).
        if (length(k) == 0) indvar = NULL

        # Otherwise, keep all remaining columns
        # (drop = FALSE preserves matrix structure when selecting a single column).
        if (length(k) != 0)  indvar = indvar[, k, drop = FALSE]


        # Loop continues until every remaining slope has |t| >= 1.96 (or no slopes remain).

}
print(model)
```

```
Date: Mon Sep 29 14:46:22 2025
Dependent var.: depvar
Method: Ordinary Least Squares (OLS)
Variance-Covariance: Ordinary
No. of observations (mean eq.): 246
Sample: 1962(2) to 2023(3)

Mean equation:

              coef   std.error  t-stat   p-value
mconst  0.00663890  0.00131206  5.0599 8.411e-07 ***
y1     -0.01557032  0.06819366 -0.2283   0.81959
y2      0.03205575  0.06793092  0.4719   0.63744
y3      0.01339174  0.06757077  0.1982   0.84307
y4      0.06839600  0.06566642  1.0416   0.29867
x1      0.00552215  0.00254866  2.1667   0.03126 *
x2     -0.00225605  0.00270668 -0.8335   0.40540
x3      0.00263759  0.00267562  0.9858   0.32524
x4      0.00073847  0.00255386  0.2892   0.77271
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Diagnostics and fit:

                      Chi-sq df   p-value
Ljung-Box AR(1)   1.2137e-05  1    0.9972
Ljung-Box ARCH(1) 4.5282e+01  1 1.706e-11 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

SE of regression   0.01252
R-squared          0.04534
Log-lik.(n=246)  732.93535

Date: Mon Sep 29 14:46:22 2025
Dependent var.: depvar
Method: Ordinary Least Squares (OLS)
Variance-Covariance: Ordinary
No. of observations (mean eq.): 246
Sample: 1962(2) to 2023(3)

Mean equation:

             coef  std.error  t-stat   p-value
mconst  0.0067450  0.0011953  5.6427 4.737e-08 ***
y1     -0.0149218  0.0679775 -0.2195   0.82644
y2      0.0316242  0.0677588  0.4667   0.64113
y4      0.0680705  0.0655133  1.0390   0.29984
x1      0.0055132  0.0025431  2.1679   0.03116 *
x2     -0.0022083  0.0026905 -0.8208   0.41260
x3      0.0027989  0.0025436  1.1004   0.27228
x4      0.0008553  0.0024799  0.3449   0.73048
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Diagnostics and fit:

                      Chi-sq df   p-value
Ljung-Box AR(1)   1.0435e-05  1    0.9974
Ljung-Box ARCH(1) 4.4983e+01  1 1.988e-11 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

SE of regression   0.01250
R-squared          0.04519
Log-lik.(n=246)  732.93286

Date: Mon Sep 29 14:46:22 2025
Dependent var.: depvar
Method: Ordinary Least Squares (OLS)
Variance-Covariance: Ordinary
No. of observations (mean eq.): 246
Sample: 1962(2) to 2023(3)

Mean equation:

              coef   std.error  t-stat   p-value
mconst  0.00662607  0.00106327  6.2318 2.066e-09 ***
y2      0.03231252  0.06755135  0.4783   0.63285
y4      0.06770278  0.06536129  1.0358   0.30133
x1      0.00533163  0.00239997  2.2215   0.02725 *
x2     -0.00235540  0.00260046 -0.9058   0.36597
x3      0.00276942  0.00253502  1.0925   0.27573
x4      0.00077061  0.00244479  0.3152   0.75288
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Diagnostics and fit:

                     Chi-sq df   p-value
Ljung-Box AR(1)    0.046023  1    0.8301
Ljung-Box ARCH(1) 46.959257  1 7.248e-12 ***
---
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

SE of regression   0.01247
R-squared          0.04499
Log-lik.(n=246)  732.92369

Date: Mon Sep 29 14:46:22 2025
Dependent var.: depvar
Method: Ordinary Least Squares (OLS)
Variance-Covariance: Ordinary
No. of observations (mean eq.): 246
Sample: 1962(2) to 2023(3)

Mean equation:

             coef  std.error  t-stat   p-value
mconst  0.0065696  0.0010461  6.2801 1.574e-09 ***
y2      0.0322440  0.0674241  0.4782   0.63292
y4      0.0725581  0.0634010  1.1444   0.25359
x1      0.0051510  0.0023262  2.2144   0.02774 *
x2     -0.0025801  0.0024961 -1.0337   0.30233
x3      0.0025007  0.0023829  1.0494   0.29503
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Diagnostics and fit:

                     Chi-sq df   p-value
Ljung-Box AR(1)    0.041735  1    0.8381
Ljung-Box ARCH(1) 46.579415  1 8.798e-12 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

SE of regression   0.01245
R-squared          0.04460
Log-lik.(n=246)  732.88614

Date: Mon Sep 29 14:46:22 2025
Dependent var.: depvar
Method: Ordinary Least Squares (OLS)
Variance-Covariance: Ordinary
No. of observations (mean eq.): 246
Sample: 1962(2) to 2023(3)

Mean equation:

              coef   std.error  t-stat   p-value
mconst  0.00680178  0.00092512  7.3523 3.026e-12 ***
y4      0.07524938  0.06304960  1.1935   0.23385
x1      0.00528289  0.00230611  2.2908   0.02284 *
x2     -0.00218588  0.00235226 -0.9293   0.35368
x3      0.00278639  0.00230311  1.2098   0.22753
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Diagnostics and fit:

                     Chi-sq df   p-value
Ljung-Box AR(1)    0.050922  1    0.8215
Ljung-Box ARCH(1) 45.269191  1 1.717e-11 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

SE of regression   0.01243
R-squared          0.04369
Log-lik.(n=246)  732.78042

Date: Mon Sep 29 14:46:22 2025
Dependent var.: depvar
Method: Ordinary Least Squares (OLS)
Variance-Covariance: Ordinary
No. of observations (mean eq.): 246
Sample: 1962(2) to 2023(3)

Mean equation:

             coef  std.error t-stat   p-value
mconst 0.00678932 0.00092477 7.3417 3.198e-12 ***
y4     0.08350118 0.06240354 1.3381  0.182125
x1     0.00591930 0.00220145 2.6888  0.007669 **
x3     0.00343547 0.00219402 1.5658  0.118695
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Diagnostics and fit:

                    Chi-sq df  p-value
Ljung-Box AR(1)    0.10175  1   0.7497
Ljung-Box ARCH(1) 47.45418  1 5.63e-12 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

SE of regression   0.01243
```

```
R-squared          0.04026
Log-lik.(n=246)  732.34979

Date: Mon Sep 29 14:46:22 2025
Dependent var.: depvar
Method: Ordinary Least Squares (OLS)
Variance-Covariance: Ordinary
No. of observations (mean eq.): 246
Sample: 1962(2) to 2023(3)

Mean equation:

             coef  std.error t-stat   p-value
mconst 0.00742438 0.00079498 9.3391 < 2.2e-16 ***
x1     0.00588612 0.00220488 2.6696  0.008107 **
x3     0.00339291 0.00219736 1.5441  0.123868
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Diagnostics and fit:

                    Chi-sq df   p-value
Ljung-Box AR(1)    0.04785  1    0.8268
Ljung-Box ARCH(1) 47.38444  1 5.834e-12 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

SE of regression   0.01245
R-squared          0.03316
Log-lik.(n=246)  731.45033

Date: Mon Sep 29 14:46:22 2025
Dependent var.: depvar
Method: Ordinary Least Squares (OLS)
Variance-Covariance: Ordinary
No. of observations (mean eq.): 246
Sample: 1962(2) to 2023(3)

Mean equation:

            coef std.error t-stat p-value
mconst 0.0073650 0.0007963 9.2490 < 2e-16 ***
x1     0.0052970 0.0021778 2.4323 0.01572 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Diagnostics and fit:

                     Chi-sq df   p-value
Ljung-Box AR(1)    0.043719  1    0.8344
Ljung-Box ARCH(1) 49.375064  1 2.114e-12 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

SE of regression   0.01248
R-squared          0.02367
Log-lik.(n=246)  730.25453

Date: Mon Sep 29 14:46:22 2025
Dependent var.: depvar
Method: Ordinary Least Squares (OLS)
Variance-Covariance: Ordinary
No. of observations (mean eq.): 246
Sample: 1962(2) to 2023(3)

Mean equation:

            coef std.error t-stat p-value
mconst 0.0073650 0.0007963 9.2490 < 2e-16 ***
x1     0.0052970 0.0021778 2.4323 0.01572 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Diagnostics and fit:

                     Chi-sq df   p-value
Ljung-Box AR(1)    0.043719  1    0.8344
Ljung-Box ARCH(1) 49.375064  1 2.114e-12 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

SE of regression   0.01248
R-squared          0.02367
Log-lik.(n=246)  730.25453
```

In [5]:

```
library(gets)
#https://search.r-project.org/CRAN/refmans/sysid/html/arx.html
#https://cran.r-project.org/web/packages/gets/gets.pdf

##GETS-modelling of mean specification:

#getsm(object, t.pval=0.05, wald.pval=t.pval, vcov.type=NULL,
#do.pet=TRUE, ar.LjungB=list(lag=NULL, pval=0.025),
```

```r
#arch.LjungB=list(lag=NULL, pval=0.025), normality.JarqueB=NULL,
#user.diagnostics=NULL, info.method=c("sc","aic","aicc", "hq"),
#gof.function=NULL, gof.method=NULL, keep=NULL, include.gum=FALSE,
#include.1cut=TRUE, include.empty=FALSE, max.paths=NULL, tol=1e-07,
#turbo=FALSE, print.searchinfo=TRUE, plot=NULL, alarm=FALSE)

# y-series: current value and its lags -------------------------------------------------

y0 = data[,1]                 # y_t : dependent variable at time t (first column of 'data')

y1 = lag(data[,1], -1)        # y_{t-1} : first lag of y
y2 = lag(data[,1], -2)        # y_{t-2} : second lag of y
y3 = lag(data[,1], -3)        # y_{t-3} : third lag of y
y4 = lag(data[,1], -4)        # y_{t-4} : fourth lag of y

# x-series: lags ----------------------------------------------------------------------

x1 = lag(data[,2], -1)        # x_{t-1} : first lag of x (second column of 'data')
x2 = lag(data[,2], -2)        # x_{t-2} : second lag of x
x3 = lag(data[,2], -3)        # x_{t-3} : third lag of x
x4 = lag(data[,2], -4)        # x_{t-4} : fourth lag of x

# Align all series on a common time index (drop rows with any NA from lagging) --------

tempdata = ts.intersect(
  y0,                         # y_t
  y1, y2, y3, y4,             # y lags: y_{t-1..4}
  x1, x2, x3, x4              # x lags: x_{t-1..4}
)

# Split into dependent and regressor blocks -----------------------------------------

depvar = tempdata[, 1]                    # dependent variable y_t after alignment
indvar = tempdata[, 2:ncol(tempdata)]     # matrix of regressors: (y_{t-1..4}, x_{t-1..4})

# Estimate general ADL with a constant, then run gets reduction ----------------------

model.temp = arx(depvar, mxreg = indvar, mc=TRUE)  # ARX/ADL: depvar ~ indvar + constant

getfit = getsm(
  model.temp,
  arch.Ljung = NULL,        # (diagnostics option) here set to NULL → skip ARCH Ljung-Box check
  print.searchinfo = FALSE, # suppress verbose search output
  t.pval = 0.05             # 5% threshold for t-tests during general-to-specific reduction
)

getfit                      # final selected (simplified) model object
```

```
Date: Mon Sep 29 13:30:11 2025
Dependent var.: depvar
Method: Ordinary Least Squares (OLS)
Variance-Covariance: Ordinary
No. of observations (mean eq.): 246
Sample: 1962(2) to 2023(3)

SPECIFIC mean equation:

            coef std.error t-stat p-value
mconst 0.0073650 0.0007963 9.2490 < 2e-16 ***
x1     0.0052970 0.0021778 2.4323 0.01572 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Diagnostics and fit:

                     Chi-sq df   p-value
Ljung-Box AR(1)    0.043719  1    0.8344
Ljung-Box ARCH(1) 49.375064  1 2.114e-12 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

SE of regression   0.01248
R-squared          0.02367
Log-lik.(n=246)  730.25453
```

The manual loop performs a **backward-elimination search** starting from a general **ADL(4,4)** model. In each iteration, it estimates the current model with a constant, computes *t*-statistics for all non-constant coefficients, and identifies the regressor with the smallest absolute *t*-statistic. That weakest term is dropped, the model is re-estimated on the reduced set, and the process repeats.

The loop stops when either no regressors remain or all surviving slope coefficients satisfy $|t| \geq 1.96$, which corresponds to an approximate **5% two-sided significance threshold** under large-sample normality.

This greedy, nested-model approach is transparent and easy to follow, though it does not guarantee a **globally optimal subset** in terms of information criteria.

The `getsm()` routine from the **GETS** framework automates the same reduction while allowing for optional diagnostic checks.

Exercise 1

Use **iterative elimination of insignificant parameters** to find the best model for **Canadian GDP** as a function of **weekly hours worked in the manufacturing sector in Canada** and **personal consumption expenditures (PCE) in the US**. This approach begins from a general specification and gradually removes regressors that do not meet a chosen significance threshold, leaving a **parsimonious model** where all remaining coefficients are individually meaningful.

To transform the monthly data on PCE into quarterly frequency, as we practiced in **Lecture 3**, please run the code below. This step ensures that the frequency of all variables is aligned before estimating the model, which is crucial for meaningful inference.

In [6]:
```
library(zoo)

# Import the dataset from a CSV file downloaded from FRED --------------------
temp = read.csv("PCE.csv")                # reads the CSV into a data.frame 'temp'

# Parse monthly dates --------------------------------------------------------
# Convert the character/date column 'observation_date' to a monthly index.
# as.yearmon stores dates as Year-Month (e.g., 2020-01) without a specific day.
temp$Date = as.yearmon(temp$observation_date)

# Aggregate monthly to quarterly ---------------------------------------------
# Collapse the monthly series 'PCE' to calendar quarters using the mean of the
# three months in each quarter. Replace FUN as needed (sum, last, etc.).
pce = aggregate(PCE ~ as.yearqtr(Date), data = temp, FUN = mean)

head(pce)                     # quick peek at the first few quarterly observations
tail(pce)                     # quick peek at the most recent quarterly observations

# Convert the second column of the quarterly data to a ts object -------------
# Create a quarterly 'ts' starting in 1959 Q1. 'frequency = 4' indicates quarterly data.
# pce[, 2] is the numeric series (PCE levels) and the first column is the quarter index.
tspce = ts(pce[, 2], frequency = 4, start = c(1959, 1))
```

A data.frame: 6 × 2

	as.yearqtr(Date)	PCE
	<yearqtr>	<dbl>
1	1959 Q1	309.4667
2	1959 Q2	315.5000
3	1959 Q3	320.7333
4	1959 Q4	322.8667
5	1960 Q1	326.3667
6	1960 Q2	332.2000

A data.frame: 6 × 2

	as.yearqtr(Date)	PCE
	<yearqtr>	<dbl>
262	2024 Q2	19682.70
263	2024 Q3	19938.47
264	2024 Q4	20255.47
265	2025 Q1	20461.60
266	2025 Q2	20643.90
267	2025 Q3	20802.00

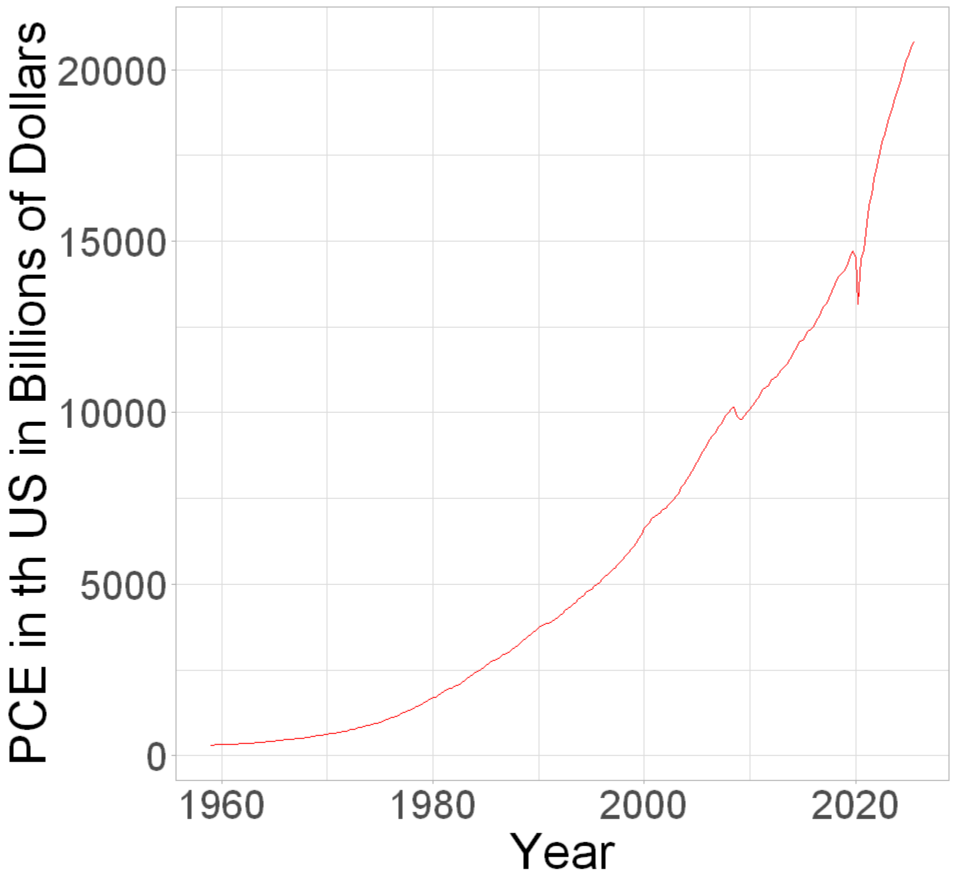
Solution to Exercise 1

In [7]:
```
# Calculate the number of observations in the time series object
nd = length(tspce)

# Print a message along with the number of observations
cat("There are", nd, "observations in our data set.")
```

```
There are 267 observations in our data set.
```

In [8]:
```
# Load required libraries
library(ggplot2)   # for plotting
library(ggfortify) # for autoplot of time series objects

# Set default figure size (width = 8, height = 8)
options(repr.plot.width = 8, repr.plot.height = 8)
```

```r
# Create an initial time series plot of tspce in red
fig = autoplot(tspce, colour = 'red')

# Customize the plot
fig = fig +
  theme(aspect.ratio = 1) +                         # make plot square-shaped
  theme_light() +                                   # use a light theme
  theme(aspect.ratio = 1) +                         # reinforce square aspect ratio
  theme(plot.margin = ggplot2::margin(0.2, 0.2, 0.2, 0.2, "cm")) + # adjust margins
  theme(text = element_text(size = 30)) +           # set large font size for readability
  labs(x = "Year") +                                # label x-axis
  labs(y = "PCE in th US in Billions of Dollars")# label y-axis

# Display the plot
fig
```



In [9]:
```r
library(urca)
summary(ur.df(tspce, type='trend', lags=8, selectlags="AIC"))
```

```
###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################

Test regression trend


Call:
lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)

Residuals:
     Min       1Q   Median       3Q      Max
-1527.58   -12.04    -0.07    16.55  1023.51

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 12.868353  25.266869   0.509   0.6110
z.lag.1      0.013244   0.005267   2.514   0.0125 *
tt          -0.058207   0.375801  -0.155   0.8770
z.diff.lag  -0.080146   0.063596  -1.260   0.2087
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 136.6 on 254 degrees of freedom
Multiple R-squared:   0.18,	Adjusted R-squared:  0.1703
F-statistic: 18.58 on 3 and 254 DF,  p-value: 6.27e-11


Value of test-statistic is: 2.5144 34.6555 25.5984

Critical values for test statistics:
      1pct  5pct 10pct
tau3 -3.98 -3.42 -3.13
phi2  6.15  4.71  4.05
phi3  8.34  6.30  5.36
```

Interpretation

We compare the reported test statistics (**2.5144**, **34.6555**, **25.5984**) to the critical values table.

Since the test statistic exceeds the relevant critical values, we **reject the null hypothesis** that the deterministic trend and constant are both zero.

This means that even after including a deterministic trend and constant, the series still behaves as if it has a **unit root** (i.e., it is **non-stationary**).

Because the data show **exponential growth** (a trending level effect), we need to first take the **logarithm** of the series. This stabilizes the trend by converting exponential growth into a *linear trend*. After logging, we then **difference the series** (log-differences correspond to growth rates) and retest for a unit root on the transformed series.

In [10]:
```
# Transform the PCE time series into growth rates ----------------------------
# 1. log(tspce): take natural logarithm of the level series (stabilizes exponential growth)
# 2. diff(...): compute first differences of the logged series (log-differences ≈ growth rates)
dpce = diff(log(tspce))
```

In [11]:
```
library(urca)
summary(ur.df(dpce, type='trend', lags=8, selectlags="AIC"))
```

```
###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################

Test regression trend


Call:
lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)

Residuals:
      Min        1Q    Median        3Q       Max 
-0.106581 -0.003592 -0.000344  0.003573  0.086624 

Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept)  1.679e-02  2.693e-03   6.236 1.88e-09 ***
z.lag.1     -7.276e-01  9.985e-02  -7.287 4.07e-12 ***
tt          -3.752e-05  1.063e-05  -3.531 0.000492 ***
z.diff.lag1 -2.774e-01  8.568e-02  -3.238 0.001365 ** 
z.diff.lag2 -1.410e-01  6.171e-02  -2.284 0.023206 *  
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.01131 on 252 degrees of freedom
Multiple R-squared:  0.5113,	Adjusted R-squared:  0.5036 
F-statistic: 65.92 on 4 and 252 DF,  p-value: < 2.2e-16


Value of test-statistic is: -7.2868 17.7311 26.5967 

Critical values for test statistics: 
      1pct  5pct 10pct
tau3 -3.98 -3.42 -3.13
phi2  6.15  4.71  4.05
phi3  8.34  6.30  5.36
```

Interpretation

The reported test statistic for the unit root test is **–7.2868**, which is far below the **1% critical value** of **–3.98**. This means we can clearly **reject the null hypothesis** of a unit root in the differenced (growth rate) series. In other words, there is **no second unit root**.

However, the presence of **significant deterministic components** is still indicated. This suggests that while the growth rate series is stationary in terms of stochastic trends, it still contains a **deterministic trend**.

Therefore, we need to **detrend the growth rates** to remove this deterministic trend and obtain a properly **stationary series** for further modeling.

In [34]:
```
# Load the pracma package (provides the detrend() function)
library(pracma)

# Remove the linear trend from the series "temp"
# as.numeric(temp) ensures pracma::detrend() works
# (expects a numeric vector, not a ts object)
# 'linear' specifies linear detrending
temp = detrend(as.numeric(dpce), 'linear')

# Define the start date for the time series
dpce = ts(temp, frequency = 4, start = c(1959, 2))
```

In [13]:
```
library(gets)
#https://search.r-project.org/CRAN/refmans/sysid/html/arx.html
#https://cran.r-project.org/web/packages/gets/gets.pdf
```

```r
##GETS-modelling of mean specification:

#getsm(object, t.pval=0.05, wald.pval=t.pval, vcov.type=NULL,
#do.pet=TRUE, ar.LjungB=list(lag=NULL, pval=0.025),
#arch.LjungB=list(lag=NULL, pval=0.025), normality.JarqueB=NULL,
#user.diagnostics=NULL, info.method=c("sc","aic","aicc", "hq"),
#gof.function=NULL, gof.method=NULL, keep=NULL, include.gum=FALSE,
#include.1cut=TRUE, include.empty=FALSE, max.paths=NULL, tol=1e-07,
#turbo=FALSE, print.searchinfo=TRUE, plot=NULL, alarm=FALSE)

# y-series: current value and its lags -------------------------------------------------

data = ts.intersect(dgdp, dhours, dpce)

y0 = data[,1]                  # y_t : dependent variable at time t
                               # (first column of 'data')

y1 = lag(data[,1], -1)        # y_{t-1} : first lag of y
y2 = lag(data[,1], -2)        # y_{t-2} : second lag of y
y3 = lag(data[,1], -3)        # y_{t-3} : third lag of y
y4 = lag(data[,1], -4)        # y_{t-4} : fourth lag of y

# x-series: lags -----------------------------------------------------------------------

x1 = lag(data[,2], -1)        # x_{t-1} : first lag of x (second column of 'data')
x2 = lag(data[,2], -2)        # x_{t-2} : second lag of x
x3 = lag(data[,2], -3)        # x_{t-3} : third lag of x
x4 = lag(data[,2], -4)        # x_{t-4} : fourth lag of x

# x-series: lags -----------------------------------------------------------------------

z1 = lag(data[,3], -1)        # z_{t-1} : first lag of z (third column of 'data')
z2 = lag(data[,3], -2)        # z_{t-2} : second lag of z
z3 = lag(data[,3], -3)        # z_{t-3} : third lag of z
z4 = lag(data[,3], -4)        # z_{t-4} : fourth lag of z

# Align all series on a common time index (drop rows with any NA from lagging) --------

tempdata = ts.intersect(
  y0,                          # y_t
  y1, y2, y3, y4,              # y lags: y_{t-1..4}
  x1, x2, x3, x4,              # x lags: x_{t-1..4}
  z1, z2, z3, z4               # z lags: z_{t-1..4}
)

# Split into dependent and regressor blocks -------------------------------------------

depvar = tempdata[, 1]                    # dependent variable y_t after alignment
indvar = tempdata[, 2:ncol(tempdata)]     # matrix of regressors:
                                          # (y_{t-1..4}, x_{t-1..4}, z_{t-1..4})

# Estimate general ADL with a constant, then run gets reduction ----------------------

model.temp = arx(depvar, mxreg = indvar, mc=TRUE)  # ARX/ADL: depvar ~ indvar + constant

getfit = getsm(
  model.temp,
  arch.Ljung = NULL,        # (diagnostics option) here set to NULL → skip ARCH Ljung-Box check
  print.searchinfo = FALSE,# suppress verbose search output
  t.pval = 0.05             # 5% threshold for t-tests during general-to-specific reduction
)

getfit                      # final selected (simplified) model object
```

```
Date: Mon Sep 29 13:30:12 2025
Dependent var.: depvar
Method: Ordinary Least Squares (OLS)
Variance-Covariance: Ordinary
No. of observations (mean eq.): 246
Sample: 1962(2) to 2023(3)

SPECIFIC mean equation:

            coef std.error t-stat p-value
mconst 0.0073650 0.0007963 9.2490 < 2e-16 ***
x1     0.0052970 0.0021778 2.4323 0.01572 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Diagnostics and fit:

                     Chi-sq df   p-value
Ljung-Box AR(1)    0.043719  1    0.8344
Ljung-Box ARCH(1) 49.375064  1 2.114e-12 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

SE of regression   0.01248
R-squared          0.02367
Log-lik.(n=246)  730.25453
```

Conclusion

Does US PCE help to model Canadian GDP?

From the **GETS** output for the specific mean equation, the final model retains only the **intercept** and one **non-PCE regressor (x1)**, while all lags of **US PCE** were eliminated by the reduction search because their *t*-statistics were below the **5% significance threshold**.

In other words, once we control for **weekly hours** (and the constant), the **US PCE terms** do not add incremental explanatory power for **Canadian GDP** — so they are not included in the **parsimonious specification**.

Forecasting ARIMA Models

AR(1) Model

Note: We will forecast **AR(1)** and **ARIMA(1,0,1)** models *for illustration*, so we can match R's built-in functions with our manual computations. As we saw last week, the **best model by BIC** was **white noise**, while the **best model by AIC** was a **subset ARIMA** with an AR term at lag 3 and MA terms at lags 3 and 4.

R: Forecasting GDP growth rate using AR(1), 1-step ahead:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \epsilon_t$$

In [35]:

```
library(stargazer)
# Fit an AR(1) to already-stationary GDP growth (dgdp)
model.a = arima(dgdp, order = c(1, 0, 0))

# Print the model
stargazer(model.a, type = "text")
```

```
=============================================
                      Dependent variable:
                  ---------------------------
                             dgdp
---------------------------------------------
ar1                          0.027
                            (0.063)

intercept                  0.007***
                            (0.001)

---------------------------------------------
Observations                  256
Log Likelihood              758.550
sigma2                      0.0002
Akaike Inf. Crit.         -1,511.100
=============================================
Note:              *p<0.1; **p<0.05; ***p<0.01
```

In [36]:

```
# Load forecasting
library(forecast)

# One-step-ahead forecast from the fitted ARIMA model
# h = 1 → forecast the next period only
# level sets the prediction interval(s); add/remove as you like
f.a = forecast::forecast(model.a, h = 1, level = c(80, 95))

# Print a tidy summary (point forecast + intervals)
f.a

# Access pieces programmatically
f.a$mean      # point forecast
f.a$lower     # lower bounds for levels specified
f.a$upper     # upper bounds for levels specified
```

```
        Point Forecast        Lo 80      Hi 80       Lo 95      Hi 95
2025 Q2    0.007406762 -0.008612644 0.02342617 -0.01709281 0.03190633
```

A Time Series: 1 × 1

	Qtr2
2025	0.007406762

A Time Series: 1 × 2

	80%	95%
2025 Q2	-0.008612644	-0.01709281

A Time Series: 1 × 2

	80%	95%
2025 Q2	0.02342617	0.03190633

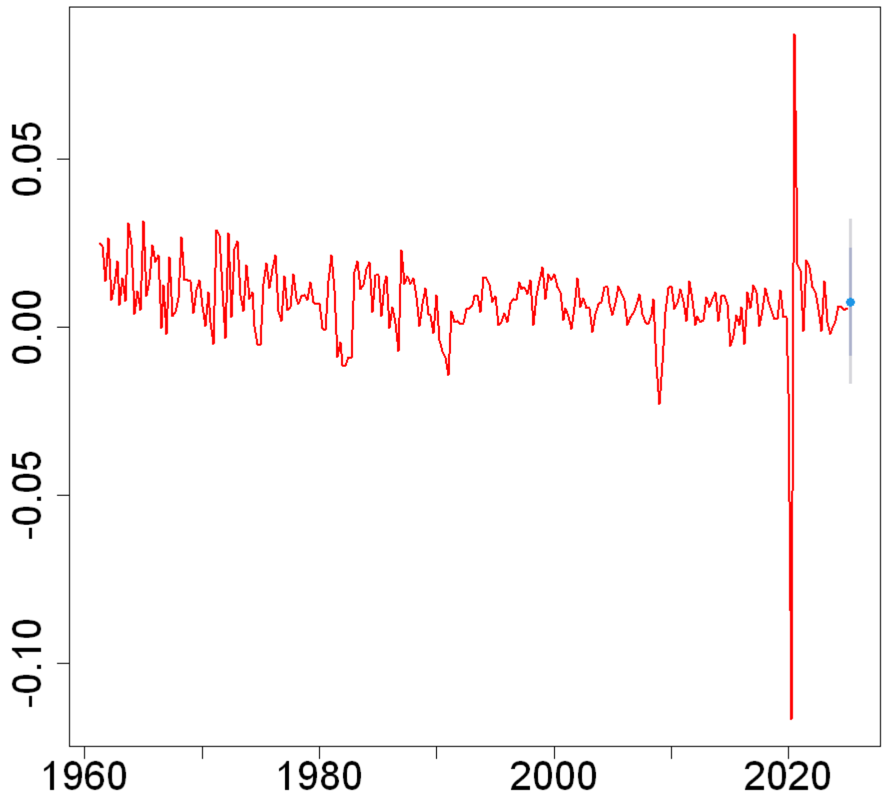
In [16]:

```
# --- Manual calculations for 1-step-ahead AR(1) forecast from model.a ---
# Extract estimated coefficients from the fitted ARIMA model
```

```
coef.a = coef(model.a)

# Reparameterize: for an AR(1) with mean μ, intercept (c) = μ*(1 - φ)
alpha0 = coef.a["intercept"] * (1 - coef.a["ar1"])
alpha1 = coef.a["ar1"]

 # Extract last observed value of the stationary series y_T
yT = tail(dgdp, n = 1)

# Compute the conditional mean for T+1
yT1.a = alpha0 + alpha1 * yT

# Express as ~percent (dgdp is Δlog ⇒ approx growth rate), rounded to two decimal places
yT1.a = round(yT1.a * 100, 2)

# 1-step-ahead point forecast:
cat("AR(1) generated 1-step-ahead forcast of the growth rate of the Canadian GDP is", as.numeric(yT1.a),"%.")
```

```
AR(1) generated 1-step-ahead forcast of the growth rate of the Canadian GDP is 0.74 %.
```

In [17]:
```
# Set Jupyter/IRkernel figure size in inches for the output cell
options(repr.plot.width = 8, repr.plot.height = 8)

# Enlarge base R graphics text: axis labels, tick labels, and (potential) main title
par(cex.lab = 2, cex.axis = 2, cex.main = 2)

# Plot the 1-step-ahead forecast object from the 'forecast' package.
plot(f.a, aspect.ratio = 1, col = "red", lwd = 2, main = "")
```



Forecasting ARIMA Models

ARIMA(1,1) Model

R: Forecasting GDP growth rate using ARIMA(1,1), 1-step ahead:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \delta_1 \epsilon_{t-1} + \epsilon_t$$

In [18]:
```
library(stargazer)
# Fit an ARIMA(1,1) to already-stationary GDP growth (dgdp)
model.b = arima(dgdp, order = c(1, 0, 1))

# Print the model
stargazer(model.b, type = "text")
```

```
==============================================
                      Dependent variable:
                  ---------------------------
                             dgdp
----------------------------------------------
ar1                        0.990***
                            (0.015)

ma1                        -0.964***
                            (0.024)

intercept                  0.008***
                            (0.002)

----------------------------------------------
Observations                  256
Log Likelihood              760.936
sigma2                       0.0002
Akaike Inf. Crit.         -1,513.873
==============================================
Note:              *p<0.1; **p<0.05; ***p<0.01
```

In [19]:

```
# Load forecasting
library(forecast)

# One-step-ahead forecast from the fitted ARIMA model
# h = 1 → forecast the next period only
# level sets the prediction interval(s); add/remove as you like
f.b = forecast::forecast(model.b, h = 1, level = c(80, 95))

# Print the output
f.b

# Access pieces programmatically
f.b$mean      # point forecast
f.b$lower     # lower bounds for levels specified
f.b$upper     # upper bounds for levels specified
```

```
        Point Forecast       Lo 80      Hi 80       Lo 95      Hi 95
2025 Q2    0.005713515 -0.01014489 0.02157192 -0.01853982 0.02996685
```

A Time Series: 1 × 1

	Qtr2
2025	0.005713515

A Time Series: 1 × 2

	80%	95%
2025 Q2	-0.01014489	-0.01853982

A Time Series: 1 × 2

	80%	95%
2025 Q2	0.02157192	0.02996685

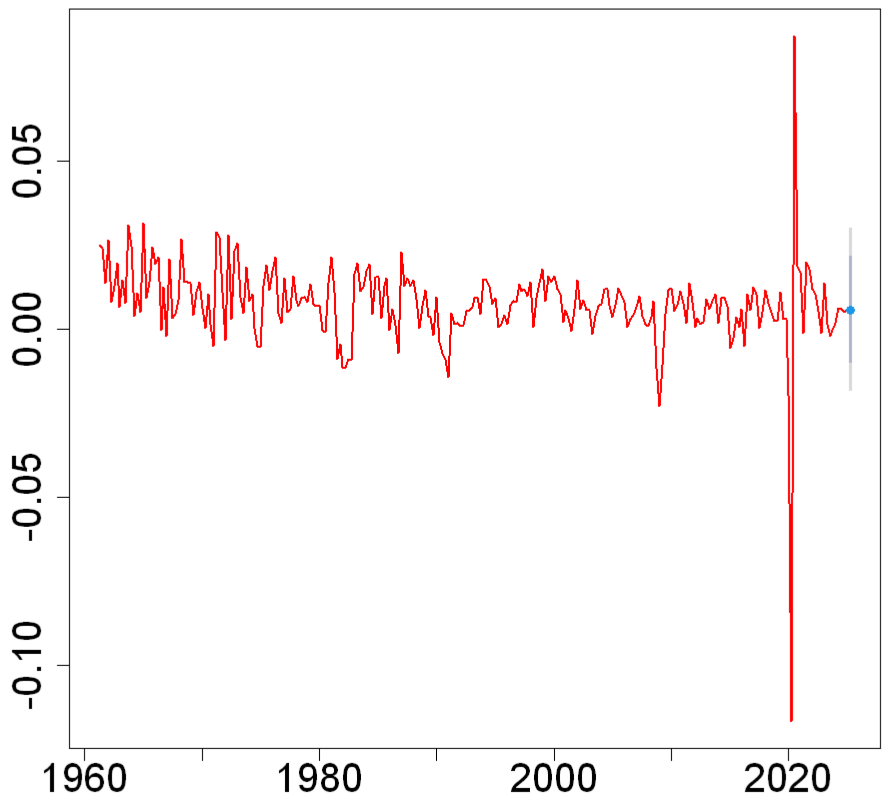
In [20]:

```
# Manual calcualtions
coef.b = coef(model.b)

alpha0 = coef.b["intercept"] * (1 - coef.b["ar1"])
alpha1 = coef.b["ar1"]
delta1 = coef.b["ma1"]

# Extract last observed value of the stationary series y_T and the correspoding residual e_T
yT = tail(dgdp, n = 1)
eT = tail(residuals(model.b), n = 1)

# Compute the conditional mean for T+1
yT1.b = alpha0 + alpha1 * yT + delta1 * eT

# Express as ~percent (dgdp is Δlog ⇒ approx growth rate), rounded to two decimal places
yT1.b = round(yT1.b * 100, 2)

# 1-step-ahead point forecast:
cat("ARIMA(1,1) generated 1-step-ahead forcast of the growth rate of the Canadian GDP is", as.numeric(yT1.b), "%.")
```

```
ARIMA(1,1) generated 1-step-ahead forcast of the growth rate of the Canadian GDP is 0.57 %.
```

In [40]:

```
# Set Jupyter/IRkernel figure size in inches for the output cell
options(repr.plot.width = 8, repr.plot.height = 8)

# Enlarge base R graphics text: axis labels, tick labels, and (potential) main title
par(cex.lab = 2, cex.axis = 2, cex.main = 2)

# Plot the 1-step-ahead forecast object from the 'forecast' package.
plot(f.b, aspect.ratio = 1, col = "red", lwd = 2, main = "")
```



Recursive Out-of-Sample Forecasts for ARIMA Models

What This Loop Is Doing (and Why)

We want to evaluate how an **AR(1)** forecasting model performs **out-of-sample**. To do this fairly, we mimic "real time":

1. **Fix an initial estimation window** of size `Tstar`.
   Reason: we need a reasonably long sample to estimate the AR(1) parameters reliably before we start judging forecasts.
2. **Expanding window (a.k.a. recursive estimation):**
   At iteration `i`, we estimate the model using `dgdp[1 : (Tstar + i - 1)]`.
   This mimics being at time `Tstar + i - 1` in real time, only using data **available then**.
3. **Forecast one step ahead (*h* = 1):**
   Using the model estimated at iteration `i`, we forecast the next observation, `dgdp[Tstar + i]`.
4. **Record results:**
   We store the point forecast, its 95% interval, the realized value, and the absolute error.
5. **Repeat** until the end of the sample.
   The number of iterations is `Tmax = T - Tstar`.
   This produces a sequence of **genuine pseudo-real-time forecasts** for a fair accuracy assessment.

Why not use the full sample each time? Because that would leak future information and overstate forecast performance.
Why not keep the window rolling with a fixed length? That's a different design (rolling/slider window). Here we prefer the **expanding window** to use all past information accumulated up to each forecast origin, which often stabilizes AR parameter estimates.

Recursive Out-of-Sample Forecasts for AR(1) model

In [22]:
```
# https://www.rdocumentation.org/packages/forecast/versions/8.16/topics/forecast
library(forecast)

# Length of sample
T = length(dgdp)

# Container for results
results = data.frame()

# Initial estimation window
Tstar = 200

# Number of iterations
Tmax = T - Tstar

cat("Our data has", T, "observations.\n")
cat("We will keep the first", Tstar, "observations fixed for stability (initial estimation window).\n")
cat("From there, we add one observation at a time and forecast one step ahead at each iteration.\n")
cat("We will produce", Tmax, "out-of-sample 1-step-ahead forecasts, ending at observation", T, ".\n\n")


for (i in 1:Tmax) {
```

```r
  # Expand window estimation: use data up to time Tstar + i - 1
  model = arima(dgdp[1:(Tstar + i - 1)], order = c(1, 0, 0))

  # One-step-ahead forecast with a single 95% PI
  f = forecast::forecast(model, h = 1, level = 95)

  # Extract components (coerce to numeric to avoid ts attributes in arithmetic)
  f.mean = as.numeric(f$mean[1])              # point forecast y_{T+1|T}
  f.lower95 = as.numeric(f$lower[1])          # 95% lower bound
  f.upper95 = as.numeric(f$upper[1])          # 95% upper bound
  obs.val = as.numeric(dgdp[Tstar + i])       # actual y_{T+1}

  # Absolute forecast error
  f.error = abs(f.mean - obs.val)

  # Row to append
  temp = c(f.mean, f.lower95, f.upper95, obs.val, f.error)
  results = rbind(results, temp)
}

# Name the columns
names(results) = c("Mean forecast", "Lower bound 95%", "Upper bound 95%", "Actual value", "Absolute error")

# Quick peek at the first 6 interations
head(results)

# Quick peek at the last 6 iterations
tail(results)

# Save AR(1) rolling results
results.AR1 = results

# Mean Absolute Error (MAE)
mae.AR1 = mean(results.AR1[, "Absolute error"])

# Root Mean Squared Forecast Error (RMSFE)
rmsfe.AR1 = sqrt(mean((results.AR1[, "Mean forecast"] - results.AR1[, "Actual value"])^2))

cat("The MAE is", round(mae.AR1, 5), "and the RMSFE is", round(rmsfe.AR1, 5))
```

```
Our data has 256 observations.
We will keep the first 200 observations fixed for stability (initial estimation window).
From there, we add one observation at a time and forecast one step ahead at each iteration.
We will produce 56 out-of-sample 1-step-ahead forecasts, ending at observation 256 .
```

A data.frame: 6 × 5

	Mean forecast	Lower bound 95%	Upper bound 95%	Actual value	Absolute error
	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
1	0.008011533	-0.007919723	0.02394279	0.0018136394	0.006197894
2	0.005725205	-0.010189328	0.02163974	0.0135896277	0.007864423
3	0.010346690	-0.005565246	0.02625863	0.0079159116	0.002430778
4	0.008136847	-0.007739376	0.02401307	0.0006311038	0.007505743
5	0.005283030	-0.010587519	0.02115358	0.0032454382	0.002037592
6	0.006279366	-0.009554827	0.02211356	0.0013675060	0.004911860

A data.frame: 6 × 5

	Mean forecast	Lower bound 95%	Upper bound 95%	Actual value	Absolute error
	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
51	0.007309076	-0.01745116	0.03206931	-0.0003829651	0.007692042
52	0.007308181	-0.01742088	0.03203724	0.0017659246	0.005542256
53	0.007335084	-0.01735429	0.03202446	0.0062945465	0.001040538
54	0.007451791	-0.01718907	0.03209266	0.0059438355	0.001507956
55	0.007436371	-0.01715664	0.03202938	0.0050857420	0.002350629
56	0.007403866	-0.01714256	0.03195029	0.0054794362	0.001924430

```
The MAE is 0.00988 and the RMSFE is 0.02609
```

Recursive out-of-sample forecasts for ARIMA(1,1) model

Note that the only difference between the two recursive loops (AR(1) and ARIMA(1,1)) is the order argument of the arima function.

In [23]:
```r
# https://www.rdocumentation.org/packages/forecast/versions/8.16/topics/forecast
library(forecast)

# Length of sample
T = length(dgdp)

# Container for results
results = data.frame()
```

```r
# Initial estimation window
Tstar = 200

# Number of iterations
Tmax = T - Tstar

cat("Our data has", T, "observations.\n")
cat("We will keep the first", Tstar, "observations fixed for stability (initial estimation window).\n")
cat("From there, we add one observation at a time and forecast one step ahead at each iteration.\n")
cat("We will produce", Tmax, "out-of-sample 1-step-ahead forecasts, ending at observation", T, ".\n\n")


for (i in 1:Tmax) {
  # Expand window estimation: use data up to time Tstar + i - 1
  model = arima(dgdp[1:(Tstar + i - 1)], order = c(1, 0, 1))

  # One-step-ahead forecast with a single 95% PI
  f = forecast::forecast(model, h = 1, level = 95)

  # Extract components (coerce to numeric to avoid ts attributes in arithmetic)
  f.mean = as.numeric(f$mean[1])             # point forecast y_{T+1|T}
  f.lower95 = as.numeric(f$lower[1])         # 95% lower bound
  f.upper95 = as.numeric(f$upper[1])         # 95% upper bound
  obs.val = as.numeric(dgdp[Tstar + i])      # actual y_{T+1}

  # Absolute forecast error
  f.error = abs(f.mean - obs.val)

  # Row to append
  temp = c(f.mean, f.lower95, f.upper95, obs.val, f.error)
  results = rbind(results, temp)
}

# Name the columns
names(results) = c("Mean forecast", "Lower bound 95%", "Upper bound 95%", "Actual value", "Absolute error")

# Quick peek at the first 6 interations
head(results)

# Quick peek at the last 6 iterations
tail(results)

# Save ARMA(1,1) rolling results
results.ARMA11 = results

# Mean Absolute Error (MAE)
mae.ARMA11 = mean(results.ARMA11[, "Absolute error"])

# Root Mean Squared Forecast Error (RMSFE)
rmsfe.ARMA11 = sqrt(mean((results.ARMA11[, "Mean forecast"] - results.ARMA11[, "Actual value"])^2))

cat("The MAE is", round(mae.ARMA11, 5), "and the RMSFE is", round(rmsfe.ARMA11, 5))
```

```
Our data has 256 observations.
We will keep the first 200 observations fixed for stability (initial estimation window).
From there, we add one observation at a time and forecast one step ahead at each iteration.
We will produce 56 out-of-sample 1-step-ahead forecasts, ending at observation 256 .
```

A data.frame: 6 × 5

	Mean forecast	Lower bound 95%	Upper bound 95%	Actual value	Absolute error
	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
1	0.008304613	-0.007545557	0.02415478	0.0018136394	0.006490974
2	0.006067570	-0.009768451	0.02190359	0.0135896277	0.007522058
3	0.009450258	-0.006380321	0.02528084	0.0079159116	0.001534347
4	0.008524317	-0.007268620	0.02431725	0.0006311038	0.007893213
5	0.005759379	-0.010031834	0.02155059	0.0032454382	0.002513941
6	0.005805135	-0.009951210	0.02156148	0.0013675060	0.004437629

A data.frame: 6 × 5

	Mean forecast	Lower bound 95%	Upper bound 95%	Actual value	Absolute error
	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
51	0.005975644	-0.01855346	0.03050475	-0.0003829651	0.0063586096
52	0.005740100	-0.01874974	0.03022994	0.0017659246	0.0039741757
53	0.005631166	-0.01881470	0.03007703	0.0062945465	0.0006633802
54	0.005654663	-0.01874172	0.03005105	0.0059438355	0.0002891724
55	0.005706781	-0.01864266	0.03005622	0.0050857420	0.0006210393
56	0.005694128	-0.01860657	0.02999483	0.0054794362	0.0002146917

```
The MAE is 0.00984 and the RMSFE is 0.02605
```

Conclusion

Both AR(1) and ARIMA(1,1) models perform virtually identical when comes to forecasting growth rate of the Canadian GDP one period ahead.

Forecasting ADL Models

ADL(0,1) Model

Note: We will forecast **ADL(0,1)** and **ADL(1,1)** models *for illustration*, so we can match R's built-in functions with our manual computations. As we saw last week, the **best model by BIC** was **ADL(0,1)**, while the **best model by AIC** was a **subset ADL** with the first and third lag of the change in weekly hours worked.

R: Forecasting GDP growth rate using the lagged value of the change in hours worked, 1-step ahead:

$$y_t = \alpha_0 + \beta_1 x_{t-1} + \varepsilon_t$$

In [24]:
```
# Align series on overlapping dates (common time index)
data = ts.intersect(dgdp, dhours)

# ADL(0,1): dgdp_t on lagged dhours_{t-1}
library(dynlm)
model.c = dynlm(dgdp ~ L(dhours, 1), data)

# Print a compact regression table
library(stargazer)
stargazer(model.c, type = "text")
```

```
===============================================
                        Dependent variable:
                    ---------------------------
                               dgdp
-----------------------------------------------
L(dhours, 1)                 0.005**
                             (0.002)

Constant                    0.008***
                             (0.001)

-----------------------------------------------
Observations                   249
R2                            0.022
Adjusted R2                   0.018
Residual Std. Error      0.013 (df = 247)
F Statistic            5.617** (df = 1; 247)
===============================================
Note:               *p<0.1; **p<0.05; ***p<0.01
```

In [25]:
```
# 1-step ahead forecast
newdata = data.frame("dhours" = tail(data[,"dhours"], n=1))
# 95% prediction interval (future outcome uncertainty)
f.c = predict(model.c, newdata = newdata, interval = "prediction", level = 0.95)
f.c
```

A matrix: 1 × 3 of type dbl

	fit	lwr	upr
1	0.008041573	-0.01667723	0.03276038

In [26]:
```
# --- Manual calculations for 1-step-ahead ADL(0,1) forecast from model.c ---
# https://www.econometrics-with-r.org/14.5-apatadlm.html
coef.c = coef(model.c)

alpha0 = coef.c["(Intercept)"]
beta1  = coef.c["L(dhours, 1)"]

#Use the most recent dhours (time T) to predict dgdp at T+1
T = nrow(data)
yT1.c = alpha0 + beta1 * data[T, "dhours"]

# Express as ~percent (dgdp is Δlog ⇒ approx growth rate), rounded to two decimal places
yT1.c = round(yT1.c * 100, 7)

# 1-step-ahead point forecast:
cat("ADL(0,1) generated 1-step-ahead forcast of the growth rate of the Canadian GDP is", as.numeric(yT1.c),"%.")
```

```
ADL(0,1) generated 1-step-ahead forcast of the growth rate of the Canadian GDP is 0.8041573 %.
```

Forecasting ADL Models

ADL(1,1) Model

R: Forecasting GDP growth rate using the **ADL(1,1)** model, 1-step ahead:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \beta_1 x_{t-1} + \varepsilon_t$$

In [27]:
```
# Align series on overlapping dates (common time index)
data = ts.intersect(dgdp, dhours)

# ADL(1,1): dgdp_t on lagged dgdp_{t-1} and lagged dhours_{t-1}
library(dynlm)
model.d = dynlm(dgdp ~ L(dgdp, 1) + L(dhours, 1), data)

# Print a compact regression table
library(stargazer)
stargazer(model.d, type = "text")
```

```
===============================================
                        Dependent variable:
                    ---------------------------
                               dgdp
-----------------------------------------------
L(dgdp, 1)                    -0.016
                              (0.065)

L(dhours, 1)                  0.005**
                              (0.002)

Constant                     0.008***
                              (0.001)

-----------------------------------------------
Observations                    249
R2                             0.022
Adjusted R2                    0.015
Residual Std. Error      0.013 (df = 246)
F Statistic            2.828* (df = 2; 246)
===============================================
Note:               *p<0.1; **p<0.05; ***p<0.01
```

In [28]:
```
# 1-step ahead forecast
newdata = data.frame("dgdp" = tail(data[,"dgdp"], n=1), "dhours" = tail(data[,"dhours"], n=1))
# 95% prediction interval (future outcome uncertainty)
f.d = predict(model.d, newdata = newdata, interval = "prediction", level = 0.95)
f.d
```

A matrix: 1 × 3 of type dbl

	fit	lwr	upr
1	0.008216719	-0.01658865	0.03302209

In [29]:
```
# --- Manual calculations for 1-step-ahead ADL(0,1) forecast from model.c ---
# https://www.econometrics-with-r.org/14.5-apatadlm.html
coef.d = coef(model.d)

alpha0 = coef.d["(Intercept)"]
beta1  = coef.d["L(dhours, 1)"]
alpha1  = coef.d["L(dgdp, 1)"]

#Use the most recent dgdp and the most recent dhours (time T) to predict dgdp at T+1
T = nrow(data)
yT1.d = alpha0 + alpha1 * data[T, "dgdp"] + beta1 * data[T, "dhours"]

# Express as ~percent (dgdp is Δlog ⇒ approx growth rate), rounded to two decimal places
yT1.d = round(yT1.d * 100, 7)

# 1-step-ahead point forecast:
cat("ADL(1,1) generated 1-step-ahead forcast of the growth rate of the Canadian GDP is", as.numeric(yT1.d),"%.")
```

```
ADL(1,1) generated 1-step-ahead forcast of the growth rate of the Canadian GDP is 0.8216719 %.
```

Recursive Of-of-Sample Forecasts for ADL(0,1) Model

In [30]:
```
# Align series on overlapping dates (common time index)
data = ts.intersect(dgdp, dhours)

# Length of sample
T = nrow(data)

# Container for results
results = data.frame()

# Initial estimation window
Tstar = 200

# Number of iterations
```

```
Tmax = T - Tstar

for (i in 1: (T - Tstar)) {
    model = dynlm(dgdp ~  L(dhours, 1), data[1: (Tstar + i - 1), ])
    newdata = data.frame("dhours" = data[(Tstar + i - 1), "dhours"])

    # One-step-ahead forecast with a single 95% PI
    f = predict(model, newdata = newdata, interval = "prediction", level = 0.95)

    # Extract components (coerce to numeric to avoid ts attributes in arithmetic)
    f.mean = as.numeric(f[1,1])
    f.lower95 = as.numeric(f[1,2])
    f.upper95 = as.numeric(f[1,3])

    # Realized value at time (Tstar + i) to compare against the forecast
    obs.val = as.numeric(data[(Tstar + i), "dgdp"])

    # Absolute forecast error
    f.error = abs(f.mean - obs.val)

    # Row to append
    temp = c(f.mean, f.lower95, f.upper95, obs.val, f.error)
    results = rbind(results, temp)
}
names(results)=c("Mean forecast", "Lower bound 95%", "Upper bound 95%", "Actual value", "Absolute error")

# Quick peek at the first 6 iterations
head(results)

# Quick peek at the last 6 iterations
tail(results)

# Save ADL(0,1) rolling results
results.ADL01 = results

# Mean Absolute Error (MAE)
mae.ADL01 = mean(results.ADL01[, "Absolute error"])

# Root Mean Squared Forecast Error (RMSFE)
rmsfe.ADL01 = sqrt(mean((results.ADL01[, "Mean forecast"] - results.ADL01[, "Actual value"])^2))

cat("The MAE is", round(mae.ADL01, 5), "and the RMSFE is", round(rmsfe.ADL01, 5))
```

A data.frame: 6 × 5

	Mean forecast	Lower bound 95%	Upper bound 95%	Actual value	Absolute error
	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
1	0.011759182	-0.01319617	0.03671454	0.0018136394	0.009945543
2	0.002835183	-0.02217004	0.02784041	0.0135896277	0.010754445
3	0.009134477	-0.01561688	0.03388583	0.0079159116	0.001218565
4	0.010184359	-0.01462690	0.03499561	0.0006311038	0.009553255
5	0.008609535	-0.01612281	0.03334188	0.0032454382	0.005364097
6	0.005984830	-0.01876233	0.03073199	0.0013675060	0.004617324

A data.frame: 6 × 5

	Mean forecast	Lower bound 95%	Upper bound 95%	Actual value	Absolute error
	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
45	0.008609535	-0.01612281	0.03334188	0.010229149	0.001619613
46	0.007034712	-0.01768454	0.03175396	0.005138201	0.001896511
47	0.008609535	-0.01612281	0.03334188	-0.001370987	0.009980522
48	0.007034712	-0.01768454	0.03175396	0.013454326	0.006419614
49	0.009659418	-0.01511825	0.03443709	0.001425423	0.008233995
50	0.008609535	-0.01612281	0.03334188	-0.002258095	0.010867631

```
The MAE is 0.00962 and the RMSFE is 0.02192
```

Recursive Out-of-Sample Dorecasts for ADL(1,1) Model

In [31]:

```
# Align series on overlapping dates (common time index)
data = ts.intersect(dgdp, dhours)

# Length of sample
T = nrow(data)

# Container for results
results = data.frame()

# Initial estimation window
Tstar = 200
```

```r
# Number of iterations
Tmax = T - Tstar

for (i in 1: (T - Tstar)) {
    model = dynlm(dgdp ~  L(dgdp, 1) + L(dhours, 1), data[1: (Tstar + i - 1), ])
    newdata = data.frame("dgdp" = data[(Tstar + i - 1), "dgdp"],
                         "dhours" = data[(Tstar + i - 1), "dhours"])

    # One-step-ahead forecast with a single 95% PI
    f = predict(model, newdata = newdata, interval = "prediction", level = 0.95)

    # Extract components (coerce to numeric to avoid ts attributes in arithmetic)
    f.mean = as.numeric(f[1,1])
    f.lower95 = as.numeric(f[1,2])
    f.upper95 = as.numeric(f[1,3])

    # Realized value at time (Tstar + i) to compare against the forecast
    obs.val = as.numeric(data[(Tstar + i), "dgdp"])

    # Absolute forecast error
    f.error = abs(f.mean - obs.val)

    # Row to append
    temp = c(f.mean, f.lower95, f.upper95, obs.val, f.error)
    results = rbind(results, temp)
}
names(results)=c("Mean forecast", "Lower bound 95%", "Upper bound 95%", "Actual value", "Absolute error")

# Quick peek at the first 6 iterations
head(results)

# Quick peek at the last 6 iterations
tail(results)

# Save ADL(1,1) rolling results
results.ADL11 = results

# Mean Absolute Error (MAE)
mae.ADL11 = mean(results.ADL11[, "Absolute error"])

# Root Mean Squared Forecast Error (RMSFE)
rmsfe.ADL11 = sqrt(mean((results.ADL11[, "Mean forecast"] - results.ADL11[, "Actual value"])^2))

cat("The MAE is", round(mae.ADL11, 5), "and the RMSFE is", round(rmsfe.ADL11, 5))
```

A data.frame: 6 × 5

	Mean forecast	Lower bound 95%	Upper bound 95%	Actual value	Absolute error
	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
1	0.011674797	-0.01331729	0.03666688	0.0018136394	0.009861158
2	0.002872213	-0.02215299	0.02789742	0.0135896277	0.010717414
3	0.008980548	-0.01579286	0.03375396	0.0079159116	0.001064637
4	0.010102980	-0.01473300	0.03493896	0.0006311038	0.009471876
5	0.008635228	-0.01614105	0.03341151	0.0032454382	0.005389789
6	0.005987289	-0.01877879	0.03075337	0.0013675060	0.004619783

A data.frame: 6 × 5

	Mean forecast	Lower bound 95%	Upper bound 95%	Actual value	Absolute error
	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
45	0.008482817	-0.01626973	0.03323536	0.010229149	0.001746331
46	0.006936716	-0.01780460	0.03167803	0.005138201	0.001798516
47	0.008573624	-0.01618317	0.03333042	-0.001370987	0.009944610
48	0.007095270	-0.01766379	0.03185433	0.013454326	0.006359057
49	0.009504839	-0.01529319	0.03430287	0.001425423	0.008079416
50	0.008624371	-0.01614750	0.03339624	-0.002258095	0.010882466

```
The MAE is 0.00961 and the RMSFE is 0.02184
```

Comparison Across the Models

AR(1), ARMA(1,1), ADL(0,1), ADL(1,1)

In [32]:

```r
data.frame("AR(1)" = mae.AR1, "ARMA(1,1)"  = mae.ARMA11, "ADL(0,1)" = mae.ADL01, "ADL(1,1)"  = mae.ADL11)
data.frame("AR(1)" = rmsfe.AR1, "ARMA(1,1)"  = rmsfe.ARMA11, "ADL(0,1)" = rmsfe.ADL01, "ADL(1,1)"  = rmsfe.ADL11)
```

A data.frame: 1 × 4

AR.1.	ARMA.1.1.	ADL.0.1.	ADL.1.1.
<dbl>	<dbl>	<dbl>	<dbl>
0.009880319	0.009840262	0.009619095	0.00961235

A data.frame: 1 × 4

AR.1.	ARMA.1.1.	ADL.0.1.	ADL.1.1.
<dbl>	<dbl>	<dbl>	<dbl>
0.02608751	0.02605203	0.02192396	0.02184136

Our tables shows the lowest MAE and RMSFE for ADL(1,1), slightly better than ADL(0,1) and clearly better than AR(1)/ARMA(1,1). That means ADL(1,1) is your best of the four on this sample.

In [ ]: