Let $K^{'} = G(K, s)$ describe the law of motion for the aggregate capital. Capital, jointly with the exogenous state s, form $X = (K, s)$, the relevant aggregate state variable.

The household solves

$$V(a, X) = \max_{c, a'(X)} u(c) + \beta \sum_{X'} \Pi\left(X'|X\right) V\left(a^{'}\left(X^{'}\right), X^{'}\right)$$

s.t.

$$c + \sum_{X'} q(X'|X) a^{'}\left(X^{'}\right) \leq w(X) + a.$$

Notice that, in the above formulation a represents wealth measured in terms of consumption goods at the present state X, while $a^{'}\left(X^{'}\right)$ is wealth that will be contingently received at $X^{'}|X$. It is important to realize the implicit timing convention: a is measured **after** any returns on assets accrue.

$a^{'}$ it is chosen today, but given sequentially complete markets, it is a a random variable. That is why we treat it as chosen **realization-by-realization** by the household.

Let

$$a^{'}\left(X^{'}\right) = g_a\left(X^{'}|a, X\right)$$

denote the household's optimal policy function regarding contingent savings toward state $X^{'}$, when starting from state (a, X) .

A consistency requirement

Imagine a household that owned k pieces of capital at aggregate state $X = (K, s)$. It knows it can rent each unit of capital at $r_K(X) = F_K(\theta(s), K, L)$ and still sell remaining capital, after depreciation. That remaining capital is valued 1-to-1 against consumption goods.

So, if the household has k units of capital before production, its equivalent wealth, in terms of consumption goods, is

$$a = [1 + r_K(X) - \delta] k.$$

That is indeed the wealth of the representative household whenever $k = K$. That household chooses

$$a^{'}\left(X^{'}\right)=g_a\left(X^{'}|\left[1+r_K\left(X\right)-\delta\right]K,X\right).$$

But once again, $a^{'}$ is measured after returns, while capital is measured before any returns accrue. If that is the only household in the economy, it will need to own

$$K^{'}=G\left(K,s\right),$$

which in terms of consumption at state $X^{'}$ will be worth

$$a^{'}\left(X^{'}\right)=\left[1+r_K\underbrace{\left(X^{'}\right)}_{\left(G(K,s),s^{'}\right)}-\delta\right]G\left(K,s\right). \tag{1}$$

So, the correct consistency requirement, under this timing convention, is

$$g_a\left(X'|\left[1+r_K\left(X\right)-\delta\right]K,X\right)=\left[1+r_K\left(G\left(K,s\right),s^{'}\right)-\delta\right]G\left(K,s\right),$$

for all X' and X.

Another consistency requirement

Notice that we have assumed that the representative household holds all consumption claims backed by the capital share of output, plus the value of undepreciated capital stock. This is, indeed, a market-clearing condition for financial claims.[1]

We could, however, have allowed the household to hold both risky capital and contingent claims. In that case, we could rewrite its problem as

$$V\left(a,X\right)=\max_{c,a'(X)}u\left(c\right)+\beta\sum_{X'}\Pi\left(X^{'}|X\right)V\left(a^{'}\left(X^{'}\right),X^{'}\right) \tag{2}$$

s.t.

$$c+\sum q\left(X'|X\right)b^{'}\left(X^{'}\right)+k^{'}\leq w\left(X\right)+a$$

[1]It is analogous to conditions of the form $K=E\left[a^{'}\right]$ in the Aiyagari model (where aggregate shocks are absent, returns on capital are deterministic and we use the alternative timing convention in which wealth is measured before interest accrues).

$$a^{'}\left(X^{'}\right)=b^{'}\left(X^{'}\right)+\left(1+r_k\left(X^{'}\right)-\delta\right)k^{'}.$$

In this case the household can hold either capital units directly or financial claims (b), which are available in zero net-supply.

The FOCs wrt to $k^{'}$(+Envelope Condition) gives us

$$u^{'}\left(c\right)=\beta E\left[u^{'}\left(c^{'}\right)\left(1+r_k\left(X^{'}\right)-\delta\right)|X\right] \tag{3}$$

while wrt to $b\left(X^{'}\right)$, we get

$$u^{'}\left(c\right)q\left(X^{'}|X\right)=\beta\Pi\left(X^{'}|X\right)u^{'}\left(c^{'}\left(X^{'}\right)\right) \tag{4}$$

$$\Longrightarrow q\left(X^{'}|X\right)=\beta\Pi\left(X^{'}|X\right)\frac{u^{'}\left(c^{'}\left(X^{'}\right)\right)}{u^{'}\left(c\right)} \tag{5}$$

Notice that capital is a redundant asset, its payoff $\left(1+r_k\left(X^{'}\right)-\delta\right)$ can be replicating by purchasing a portfolio of financial claims so that

$$b^{'}\left(X^{'}\right)\propto\left(1+r_k\left(X^{'}\right)-\delta\right).$$

Combining eqs 4 and 3 (i.e., replacing state prices), we get the following no arbitrage condition

$$1=\sum_{X^{'}}q\left(X^{'}|X\right)\left(1+r_k\left(X^{'}\right)-\delta\right). \tag{6}$$

While we could have set-up the representative households problem as 2 with both financial claims and (redundant) capital, we can instead simply impose the no arbitrage condition (Eq. 6) and have financial claims be backed by capital earnings, as in Eq. 1.