Lecture 4: Beta-Representation of Returns and Factor Models

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Intro

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- All that matters for pricing is $\text{Proj}(m|\mathcal{R}) \implies$ many SDFs may have the same pricing implications
- Hansen and Jagannathan (1991) explored one SDF that is affine in returns, but it has lots of parameters
- Is there a more parsimonious representation for the SDF?

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- Hansen and Jagannathan (1991) explored one SDF that is affine in returns, but it has lots of parameters
- Is there a more parsimonious representation for the SDF?
- In industry and MBA classes, people talk a lot about the CAPM
- How is that related to all this SDF stuff we are talking?
- What about multi-factor models?

All Models Are Single-Factor Models

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Let $\beta_R \equiv \frac{\text{Cov}(m,R)}{\sigma^2(m)}$ and $\lambda \equiv -\frac{\sigma^2(m)}{\mathbb{E}[m]}$

$$\mathbb{E}[R - R_f] = \beta_R \cdot \lambda$$

Important Comments

- β_R is a characteristic of asset R only, while λ is a characteristic of the economy only
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- What's the value for λ if investors are risk-neutral?
- Why do assets have different risk premia? Because they have different betas with respect to the SDF
- If you use the conditional pricing equation, a conditional version holds:

$$\mathbb{E}_t[R_{t+1} - R_{f,t}] = \beta_{R,t} \cdot \lambda_t$$

The CAPM as a Special Case

- The CAPM is a special case of this general result
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$$m = a - bR_m$$

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Since m must price R_m and R_f, we have two equations in two unknowns (a, b):

$$\begin{cases} \mathbb{E}[mR_f] = 1 \\ \mathbb{E}[mR_m] = 1 \end{cases} \implies \begin{cases} a = \frac{1}{R_f} + b\mathbb{E}[R_m] \\ b = \frac{\mathbb{E}[R_m] - R_f}{\sigma^2(R_m)R_f} \end{cases}$$

Exercise: verify this!

The CAPM As A Special Case

Plugging back in, we have

$$m = \frac{1}{R_f}\left(1 - \frac{\mathbb{E}[R_m] - R_f}{\sigma^2(R_m)}(R_m - \mathbb{E}[R_m])\right)$$

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Calculating the beta, we have

$$\beta_R \equiv \frac{\mathsf{Cov}(a - bR_m, R)}{\sigma^2(a - bR_m)} = -\frac{\mathsf{Cov}(R_m, R)}{b \cdot \sigma^2(R_m)} \equiv -\frac{1}{b} \cdot \beta_R^{Market}$$

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and now the λ:

$$\lambda \equiv -\frac{\sigma^2(m)}{\mathbb{E}[m]} = -b^2 \cdot \frac{\sigma^2(R_m)}{\mathbb{E}[m]} = -b \cdot \mathbb{E}[R_m - R_f]$$

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Then:

$$\mathbb{E}[R - R_f] = -\frac{1}{b} \cdot \beta_R^{Market} \cdot \left(-b \cdot \mathbb{E}[R_m - R_f]\right) = \beta_R^{Market} \cdot \mathbb{E}[R_m - R_f]$$

SDFs and Return Representation

- Last slide: all we need to justify the CAPM is a way to justify $m = a - bR_m$
- Many ways to do this:
  - Quadratic utility in two periods
  - Exponential utility in two periods and log-normally distributed returns
  - Infinite horizon, quadratic utility, and i.i.d. returns...
- Most importantly: a theory about where returns come from implies a theory about the SDF
- The converse is obviously also true

The Arbitrage Pricing Theory from Ross (1976)

A bit of history:

- Early 1950s: Markowitz's seminar paper + early CAPM ideas
- 1960s: Sharpe, Lintner, Treynor formalize the CAPM using equilibrium arguments and strong assumptions
- Unpleasant stuff: tough conditions are needed to justify the CAPM in their framework (example: quadratic utility)
- 1970s: APT from Ross (1976) provides a more general way of thinking about expected returns;
- 1980s and later: very serious data work on returns and factor models

The Main Insight

- In the 1970s, the SDF-heavy approach to asset pricing was not yet mainstream
- Question 1: how can we justify several factors?
- Question 2: when can we write "expected returns $=$ something affine"?
- Ross (1976) provided an answer to both questions!
- Huberman (1982) provided a much nicer proof (which we will follow)
- Insight: if returns come from a factor model, there is no arbitrage, and there are many assets $\implies$ only systematic risk matters
- Great stuff: no utility functions, no equilibrium notion – lightweight environment

The Beta-Representation of Returns

- The CAPM implies that firms will have different returns because of different betas
- This idea actually generalized to more sources of risk

Definition (Beta-Representation of Returns)

Returns follow a *beta-representation* if there is $\gamma \in \mathbb{R}$ and $\lambda \in \mathbb{R}^K$ such that, for any return R_i, we can find $\beta_i \in \mathbb{R}^K$ ensuring that

$$\mathbb{E}[R_i] = \gamma + \beta_i' \lambda$$

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- Very important: only β_i depends on the asset i
- γ and λ are economy-wide parameters
- If the returns are excess returns, then $\gamma = 0$. Why?

Factor Models

- Main insight of factor models: a whole panel of variables is driven by few systematic sources of variation
- Very common in Macro: many time-series co-move over time
- Also true in Finance: many returns co-move over time

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Definition (K-Factor Model of Returns)

Returns follow a *K-factor model* if there are K factors $F_t \in \mathbb{R}^K$ such that, for any return $R_{i,t}$, we can write

$$R_{i,t} = \alpha_i + \beta_i' F_t + \epsilon_{i,t}$$

with $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}^K$, where $\epsilon_{i,t}$ is uncorrelated with F_t *and* $\mathbb{E}[\epsilon_{i,t}] = \mathbb{E}[\epsilon_{i,t}\epsilon_{j,t}] = 0$ for all $i \neq j$.

We can assume that $\mathbb{E}[F] = 0$ without loss of generality (otherwise redefine α_i accordingly).

Arbitrage in Large Markets

- Since everyone is exposed to F_t, there is no way to diversify that risk away
- The risk imposed by ϵ is easy to be diversified away since it's *uncorrelated* across assets
- The number of traded assets will go to ∞. We need another notion of "No-Arbitrage".

Definition (Arbitrage in Large Markets)

Consider a sequence of economies in which n returns $\{R_i\}_{i=1}^n$ are traded at the n-th economy. There exists an *arbitrage in large markets* if there is a sequence of portfolios $\left\{\theta^{(n)}\right\}_{n=1}^{\infty}$ with $\sum_{i=1}^{n} \theta_i^{(n)} = 0$, for all n, such that

$$\lim_{n\to\infty} \mathbb{E}\left[\sum_{i=1}^{n} \theta_i^{(n)} R_i\right] = \infty, \qquad \lim_{n\to\infty} \text{Var}\left(\sum_{i=1}^{n} \theta_i^{(n)} R_i\right) = 0$$

The APT Theorem

Theorem (Arbitrage Pricing Theory)

Assume returns follow a K-factor model, there is no arbitrage in large markets, and $\exists M$ *such that* $\mathbb{E}[\epsilon_{i,t}^2] \leq M$ *for all* i *and* t. *Then, we can find* $\gamma \in \mathbb{R}$ *and* $\lambda \in \mathbb{R}^K$ *such that*

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left(\mathbb{E}[R_i] - \gamma - \beta_i' \lambda\right)^2 \leq \infty$$

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- As $n \to \infty$, the expected returns are well-approximated by a beta-representation!
- In large markets with a factor structure, no arbitrage leads to a beta-representation of returns!
- Notice that γ has no index i. Otherwise the result would be trivial.
- Exercise: show that, if there is a risk-free asset, then $\gamma = R_f$.

Proof Part 1

- Take a fixed n. Let $E \equiv [\mathbb{E}[R_1], \dots, \mathbb{E}[R_n]]^\top$ and $epsilon \equiv [\epsilon_1, \dots, \epsilon_n]^\top$
- Let $e \equiv [1, \dots, 1]^\top$ and $\beta \equiv [\beta_1, \dots, \beta_n]^\top$
- Notice that we can write $R = E + \beta F + \epsilon$ where $R \equiv [R_1, \dots, R_n]^\top$
- We can always write $E = \gamma e + \beta\lambda + c$ with $c \in \mathbb{R}^n$ such that $e'c = 0$ and $\beta' c = 0$. Why?

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- We can always write $E = \gamma e + \beta \lambda + c$ with $c \in \mathbb{R}^n$ such that $e'c = 0$ and $\beta' c = 0$. Why?
- Let $\delta \in \mathbb{R}$ and consider the portfolio $\theta \equiv \delta \cdot c$
- The random return of θ is given by

$$R_\theta \equiv \sum_{i=1}^{n} \theta_i R_i = \delta \sum_{i=1}^{n} c_i R_i = \delta c^\top (E + \beta F + \epsilon) = \delta c^\top c + \delta c^\top \epsilon$$

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- The expected return of θ is given by $\mathbb{E}[R_\theta] = \delta c^\top c = \delta \|c\|^2$
- The variance of θ is given by $\text{Var}(R_\theta) = \delta^2 \mathbb{E}[(c^\top \epsilon)^2] \leq \delta^2 \|c\|^2 M$ since $\mathbb{E}[\epsilon_i^2] \leq M$ for all i.

Proof Part 2

Recall that $\|c\|^2 = \sum_{i=1}^{n} c_i^2$. Assume, for the sake of a contradiction, that $\lim_{n\to\infty} \|c\|^2 = \infty$.

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Then, we can choose $\delta \equiv \frac{1}{\|c\|}^{3/4}$ (exercise: what happens if $\|c\| = 0$?). This leads to

$$\mathbb{E}[R_\theta] = \|c\|^{1/4} \to \infty, \qquad \text{Var}(R_\theta) \leq \frac{M}{\|c\|^{1/2}} \to 0$$

which implies we have an arbitrage in large markets.

Hence, we must have $\lim\limits_{n\to\infty} \|c\|^2 < \infty$, which is equivalent to

$$\lim_{n\to\infty} \sum_{i=1}^{n} c_i^2 = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\mathbb{E}[R_i] - \gamma - \beta_i'\lambda\right)^2 \leq \infty$$

concluding the proof.

Comments on the APT Theorem

- Light environment: no utility functions, no equilibrium notion, correlation structure for factors is (almost) anything you want
- Major contribution: in large markets, the beta-representation is very general
- Super important insight: idiosyncratic risk cannot command a premium
- "*Oh, this CEO is really, really good. We should pay a premium for this stock*" $\implies$ BS
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Caveats:

- What are these factors? Are they the same over time?
- What if the β's change over time?
- What about transaction costs, liquidity, and other market frictions?

Questions?

Taking the APT to the Data

Strict Factor Structure

- Ross considered a *strict* factor structure
- It imposes structure on the covariance matrix of returns $\Sigma \equiv \mathbb{E}[(R - \mathbb{E}[R])(R - \mathbb{E}[R])^\top]$

$$\Sigma = \mathsf{Cov}(\beta F + \epsilon, \beta F + \epsilon) = \beta \mathsf{Cov}(F, F)\beta^\top + \mathsf{Cov}(\epsilon, \epsilon) = \beta \mathsf{Cov}(F, F)\beta^\top + D$$

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- Σ can be decomposed into a low-rank matrix plus a diagonal matrix
- What real-world situations would violate this?
- Notice that there is some rotation-invariance here!
- Bad news: interpreting factors can be super hard!

Approximate Factor Structure

- Chamberlain and Rothschild (1983) introduced a weaker notion: *approximate factor structure*

Definition (Approximate Factor Structure)

Returns follow an *approximate K-factor structure* if their covariance matrix Σ, for all n, can be written as

$$\Sigma = BB^\top + D$$

where B is an $n \times K$ matrix and D is a matrix whose largest eigenvalue is uniformly bounded.

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- They showed that the APT theorem still holds under this weaker assumption
- First cool result: if returns have an approximate factor structure, Σ will have K diverging eigenvalues and $n - K$ eigenvalues remain bounded as $n \to \infty$

Approximate Factor Structure

- Second cool result: if Σ has K diverging eigenvalues and $n - K$ bounded ones, then returns have an approximate factor structure
- β's can be recovered from the eigenvectors and eigenvalues of Σ!
- Write $\Sigma = Q\Lambda Q^\top$ with ordered eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, choose the first K eigenpairs with $\lambda_k \to \infty$ as $n \to \infty$.

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- Set $B \equiv Q_K \Lambda_K^{1/2}$ where Q_K stacks the top K eigenvectors and Λ_K is the diagonal matrix of the corresponding eigenvalues

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- Set $B \equiv Q_K \Lambda_K^{1/2}$ where Q_K stacks the top K eigenvectors and Λ_K is the diagonal matrix of the corresponding eigenvalues
- Then $D \equiv \Sigma - BB^\top$ has bounded eigenvalues
- The factor loadings for asset i are the rows of B (i.e., $\beta_i = B_{i\cdot}$)
- Factors are defined by projecting returns on the same eigenvectors: $F_t \equiv Q_K^\top(R_t - \mathbb{E}[R_t])$

Tests for Numbers of Factors

- Given data, how to estimate the number of factors K?
- No easy answer, it really depends on the context
- But tests typically rely on the eigenvalues of the sample covariance matrix $\hat{\Sigma}$ and the results above!

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- But tests typically rely on the eigenvalues of the sample covariance matrix $\hat{\Sigma}$ and the results above!
- Directly related to testing the number of factors with an approximate structure: Connor and Korajczyk (1993)
- Classic in Macro using large N and large T: Bai and Ng (2002)
- For factors in continuous time driven by jump-diffusion processes: Li, Todorov, and Tauchen (2019)
- For "weak factors" that are nonetheless pervasive: Lettau and Pelger (2020)
- Empirical evidence: 1-6 factors for the US equity market.

Applications of APT

- *Many* tests of APT, in different shapes and forms. See Connor and Korajczyk (1995)
- One important caveat: it's *impossible* to test the APT directly, since it relies on $n \to \infty$
- Any test of APT is also a test of *extra assumptions* needed to make $c_i = 0$ for all i. See Chen and Ingersoll (1983) for an early example.
- One cool application: performance evaluation of fund managers
- *You should only pay your manager if she generates* α! If APT holds, you should pay no one.
- Veeeery long literature on this. "Consensus": some skill exists, but it's small, hard to find, and may not survive trading costs. See Fama and French (2010).
- Super cool recent paper about it: DeMiguel et al. (2023)

Questions?

Towards an Equivalence Result

Towards an Equivalence Result

- Question: is there a connection between potential factors driving returns and the SDF?
- Answer: Yes! Beta-representations derived from a factor model imply the existence of an affine SDF!

Theorem

Given the model

$$m = a + b'F, \qquad \mathbb{E}[mR_i] = 1, \forall i \tag{1}$$

one can find $\gamma \in \mathbb{R}$ *and* $\lambda \in \mathbb{R}^K$ *such that*

$$\mathbb{E}[R_i] = \gamma + \beta_i' \lambda$$

where β_i *are the regression coefficients of* R_i *on* F *(and a constant). Conversely, given* γ *and* λ *satisfying the beta-representation, one can find* $a \in \mathbb{R}$ *and* $b \in \mathbb{R}^K$ *such that the SDF in (1) prices all assets.*

Proof - Part 1

We can assume without any loss of generality that $\mathbb{E}[F] = 0$ (otherwise redefine a accordingly).

Let's use the pricing equation in the following form:

$$\mathbb{E}[R_i] = \frac{1}{\mathbb{E}[m]} - \frac{\text{Cov}(m, R_i)}{\mathbb{E}[m]}$$

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Notice that $\mathbb{E}[m] = a$. Also,

$$\mathsf{Cov}(m, R_i) = \mathsf{Cov}(a + b'F, R_i) = b'\mathsf{Cov}(F, R_i) = b'\mathsf{Cov}[F, F]\mathsf{Cov}[F, F]^{-1}\mathsf{Cov}(F, R_i)$$

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If we define $\beta_i \equiv \text{Cov}[F, F]^{-1}\text{Cov}(F, R_i)$, we have

$$\mathbb{E}[R_i] = \frac{1}{a} - \frac{b'\text{Cov}[F, F]\beta_i}{a} = \gamma + \beta_i'\lambda$$

with $\gamma \equiv \frac{1}{a}$ and $\lambda \equiv -\frac{\text{Cov}[F, F]b}{a}$.

Proof - Part 2

Conversely, assume a beta-representation. Then, we can just define

$$a \equiv \frac{1}{\gamma}, \qquad b \equiv -\text{Cov}[F, F]^{-1} \frac{\lambda}{\gamma}$$

which concludes the proof.

Important:

- We said nothing about uniqueness. This is not the unique SDF if markets aren't complete.
- The beta-representation is not necessarily unique either.

Proof - Part 2

Conversely, assume a beta-representation. Then, we can just define

$$a \equiv \frac{1}{\gamma}, \qquad b \equiv -\mathsf{Cov}[F, F]^{-1}\frac{\lambda}{\gamma}$$

which concludes the proof.

Important:

- We said nothing about uniqueness. This is not the unique SDF if markets aren't complete.
- The beta-representation is not necessarily unique either.

Even more important:

$$\lambda = -\gamma \cdot \mathbb{E}[mF]$$

What's the intuition? λ is usually called the "price of risk".

AI-Generated Summary

- SDFs imply a general beta representation: $\mathbb{E}[R - R_f] = \beta_R \lambda$
- CAPM is a special case with an affine SDF in R_m
- APT: factor structure $+$ no arbitrage in large markets $\Rightarrow$ approximate beta representation
- Approximate factor structure links eigenvalues of Σ to recover betas/factors
- Affine SDF $\Longleftrightarrow$ beta representation (non-unique in incomplete markets)

- (C): Chapters 6, 7, and 9

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