Lecture 2: Welcome to Class, Mr. SDF

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Intro

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- Where do prices come from?

- If you ask your friend at an asset management company:

$$P_t = \sum_{i=1}^{\infty} \frac{\mathbb{E}[D_{t+i}|\mathcal{F}_t]}{(1 + r_t)^i}$$

- Key question: what's the right r_t?

- Cochrane (2011): the central question of asset pricing is what drives r_t;

- A theory about P_t is a theory about r_t, given cash-flow expectations;

Our agenda

- Let's do this in proper EPGE style: microfounded first principles;

- First: a two-period model to help us think about P_t, r_t;

- Second: let's study a contingent claims economy to understand prices better;

- Third: do we actually need all this to price an asset? Functional Analysis to the rescue!

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Food for thought:

- Why would you ever hold an asset that pays off in the future?;

- Why do you care about returns?

- Why do you care about uncertainty?

A Two-Period Model

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- She chooses how much to consumer at t and $t+1$;

- There is a single risky asset that pays a random payoff x_{t+1};

- At t: choose a savings (in ξ units of the asset) and a consumption plan;

- At $t+1$: consume endowment $+$ asset payoff;

The Consumer Problem

$$\max_{c_t, c_{t+1}, \xi} u(c_t) + \beta \mathbb{E}_t[u(c_{t+1})]$$

$$\text{s.t.}$$

$$c_t + P_t \xi = e_t$$

$$c_{t+1} = e_{t+1} + x_{t+1} \xi$$

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The FOC are:

$$u'(c_t) \cdot P_t = \beta \mathbb{E}_t[u'(c_{t+1})x_{t+1}] = \beta \mathbb{E}_t[u'(e_{t+1} + x_{t+1}\xi)x_{t+1}]$$
$$\downarrow$$
$$P_t = \mathbb{E}_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \cdot x_{t+1} \right]$$

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- Let $m_{t+1} \equiv \beta \frac{u'(c_{t+1})}{u'(c_t)}$. We call this guy the Stochastic Discount Factor (SDF);

- This notation implies the most important equation in Asset Pricing:

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This class is about m_{t+1}. Anything else is a sideshow.

Questions?

Pricing Returns

Another Version of the Fundamental Equation

- Define the gross return on the asset as $R_{t+1} \equiv \frac{x_{t+1}}{P_t}$;

- Example in the of a stock: $x_{t+1} = D_{t+1} + P_{t+1}$;

- Rearranging the fundamental equation:

$$\mathbb{E}_t[m_{t+1} R_{t+1}] = 1$$

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- It also holds that $\mathbb{E}[m_{t+1}R_t^f] = 1$;

- For any asset, $R_{t+1} - R_t^f$ is its *risk premium*;

- Rearranging the fundamental equation:

$$\mathbb{E}[m_{t+1}(R_{t+1} - R_t^f)] = 0$$

The Risk-Free Rate and the SDF

- From $\mathbb{E}[m_{t+1}R_t^f] = 1$, we have:

$$R_t^f = \frac{1}{\mathbb{E}_t[m_{t+1}]}$$

- In *any economy*, the risk-free rate is tightly connected to $\mathbb{E}_t[m_{t+1}]$;

- Importance for the literature: models for the risk-free interest rates are super important!

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- Challenge: what happens to the price P_t of the single risky asset if our consumer is risk-neutral?

- Tip: what would happen to his Bernoulli utility function?

When Do Assets Have High Returns?

- Recall: $\mathbb{E}_t[m_{t+1}(R_{t+1} - R_t^f)] = 0$ for any asset;

- Rearranging:

$$\mathbb{E}_t[R_{t+1} - R_t^f] = -\frac{\text{Cov}_t(m_{t+1}, R_{t+1})}{\mathbb{E}_t[m_{t+1}]} = -\text{Cov}_t(m_{t+1}, R_{t+1}) \cdot R_t^f$$

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- The most important object here: $\text{Cov}_t(m_{t+1}, R_{t+1})$;

- Recall: $m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$;

- When is m_{t+1} high? When c_{t+1} is low (why?);

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- When is m_{t+1} high? When c_{t+1} is low (why?);

- If $\mathsf{Cov}_t(m_{t+1}, R_{t+1}) < 0 \implies \mathbb{E}[R_{t+1} - R_t^f] > 0$. Why? Can you give an example?

- If $\mathsf{Cov}_t(m_{t+1}, R_{t+1}) > 0 \implies \mathbb{E}[R_{t+1} - R_t^f] < 0$. Why? Can you give an example?

When Do Assets Have High Prices?

- Recall: $P_t = \mathbb{E}_t[m_{t+1}x_{t+1}]$ for any asset;
- Rearranging:

$$P_t = \frac{\mathbb{E}_t[x_{t+1}]}{R_t^f} + \text{Cov}_t(m_{t+1}, x_{t+1})$$

- Given cash-flow expectations, assets have high prices when they have lower risk premium;
- This happens when $\text{Cov}_t(m_{t+1}, x_{t+1}) > 0$;
- How is this related to what your friend at the asset management company does?

A Quick Look Into Infinite Periods

- Assume that the asset pays a stream of dividends d_t for $t = 0, 1, 2, ...$;

- The consumer seeks to maximize $\mathbb{E}_t \left[\sum\limits_{j=0}^{\infty} u(c_{t+j}) \right]$;

- The payoff at $t + 1$ is now $x_{t+1} = d_{t+1} + P_{t+1}$;

- Your job: show at home that

$$P_t = \mathbb{E}_t \left[m_{t+1} \left(P_{t+1} + d_{t+1} \right) \right] = \mathbb{E}_t \left[\sum_{j=1}^{\infty} \left(\underbrace{\prod_{k=1}^{j} m_{t+k}}_{m_{t,t+j}} \right) d_{t+j} \right] = \mathbb{E}_t \left[\sum_{j=1}^{\infty} m_{t,t+j} d_{t+j} \right]$$

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- Exercise: show that, under a certain limiting condition, we have
$P_t = \sum\limits_{j=1}^{\infty} \frac{\mathbb{E}[d_{t+j}]}{R_{t,t+j}^f} + \sum\limits_{j=1}^{\infty} \mathsf{Cov}\left(d_{t+j}, m_{t,t+j} \right)$

Real Returns vs Nominal Returns

- The basic equation $\mathbb{E}[m_{t+1}R_{t+1} = 1]$ holds for both real and nominal returns;

- The main difference is the definition of m_{t+1};

- There is a *real* SDF (denominated in goods) and a *nominal* SDF (denominated in dollars);

- They differ due to inflation!

- If you assumed that P_t and x_{t+1} were nominal variables, you can write

$$\frac{P_t}{\Pi_t} = \mathbb{E}_t\left[m_{t+1}\frac{x_{t+1}}{\Pi_{t+1}}\right] \implies P_t = \mathbb{E}_t\left[m_{t+1}\frac{\Pi_t}{\Pi_{t+1}}x_{t+1}\right]$$

  where Π_t is the price level and $m_{t+1}\frac{\Pi_t}{\Pi_{t+1}}$ is the nominal SDF;

Questions?

A Contingent Claims Economy

Intro

- We don't actually need utility functions to have an SDF;

- We don't need an equilibrium notion either;

- Next lecture: what are the minimal conditions for an SDF to exist?

- For now: what is the SDF with contingent claims?

Primitives

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- Each state s occurs with probability $\pi_s > 0$;
- There are S Arrow-Debreu securities, or contingent claims;
- Security j pays 1 unit of account if state j occurs, 0 otherwise;
- Let q_j be the price of security j;
- We assume these prices exist and they clear markets;
- Crucial assumption: the Law of One Price holds;

Pricing Any Asset

- Let's price an arbitrary asset that pays x_s units of account in state s;

- By the Law of One Price, we can replicate this asset with Arrow-Debreu securities;

- The price of the asset is:

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- We can rewrite this as:

$$P = \sum_{s=1}^{S} \underbrace{\frac{q_s}{\pi_s}}_{\equiv m_s} \cdot \pi_s \cdot x_s = \sum_{s=1}^{S} \pi_s \cdot m_s \cdot x_s = \mathbb{E}[mx]$$

- We have found an SDF: $m_s = \frac{q_s}{\pi_s}$;

- Exercise: show at home that this SDF is unique, given prices q_s and probabilities π_s.

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- What's the return of this asset?

$$R^* \equiv \frac{1}{P^*} = \frac{1}{\mathbb{E}[m]} > 0$$

- This asset is risk-free by construction! So let's write $R^* = R^f$;

Risk-Neutral Measure

- Consider another probability distribution $\{\pi_1^*, \dots, \pi_S^*\}$ such that:

$$\pi_s^* \equiv R^f \cdot m_s \cdot \pi_s = R^f q_s = \frac{q_s}{\sum\limits_{j=1}^{S} q_j}$$

- Notice that $\pi_s^* > 0$ and $\sum\limits_{s=1}^{S} \pi_s^* = 1$;

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- Importantly, this probability measure prices assets *as if* everyone were risk-neutral:

$$P = \sum_{s=1}^{S} q_s x_s = \sum_{s=1}^{S} \pi_s^* \cdot \frac{x_s}{R^f} = \mathbb{E}^* \left[\frac{x}{R^f} \right]$$

where $\mathbb{E}^*$ is the expectation under the risk-neutral measure;

Intuition for the Risk-Neutral Measure

- Important: this is a *made up* measure! *We* created it;

- It depends on two things:
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- The risk-neutral measure shifts mass from good states to bad states;

- Pay a lot of attention to states that either very likely (high π_s) *or* really bad (high m_s);

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- Exercise: show that $q_s = \beta \pi_s \frac{u'(c_{t+1}(s))}{u'(c_t)}$;

- Using our definitions: $m(s) = \beta \frac{u'(c_{t+1}(s))}{u'(c_t)} \implies \frac{m(s_1)}{m(s_2)} = \frac{u'(c_{t+1}(s_1))}{u'(c_{t+1}(s_2))}$. Intuition?

Questions?

Readings and References

(C): Chapters 1 and 3;

Cochrane, John H. 2011. "Presidential Address: Discount Rates." *The Journal of Finance* 66 (4): 1047–1108. https://doi.org/10.1111/j.1540-6261.2011.01671.x.