Lecture 3: The Geometry of No-Arbitrage

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Intro

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- We introduced the notion of a Stochastic Discount Factor (SDF);
- Very connected to the idea of marginal utility and state prices;
- But do we really need utility functions and notions of equilibrium to have an SDF?

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- We introduced the notion of a Stochastic Discount Factor (SDF);
- Very connected to the idea of marginal utility and state prices;
- But do we really need utility functions and notions of equilibrium to have an SDF?
- The answer is a sharp **No**
- The cornerstone assumption is **No Arbitrage**;
- This will guarantee the existence of *one* SDF that prices all assets correctly;
- Uniqueness will be more subtle;

Our Agenda

- **No-Arbitrage implies state prices exist**: very weak assumptions guarantee SDF existence;
- **Uniqueness of SDFs depends on market completeness**;
- **Hansen-Jagannathan bound**: observable Sharpe ratios constrain SDF volatility;
- **Minimum-variance SDF** and its role in asset pricing;

Setup

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- There are S possible states of the world, with probabilities $\pi_s > 0$ for $s = 1, \dots, S$;
- There are only two dates;
- There are N assets, with time-1 payoffs $D_i = (d_{i1}, \dots, d_{iS})$;
- Collect these payoffs in a $N \times S$ matrix $\mathbf{D}$;
- The prices of these securities are q_i for $i = 1, \dots, N$;

Definition

A *portfolio* is a vector $\theta = (\theta_1, \dots, \theta_N)$ indicating the number of units held of each asset. The cost of the portfolio is given by $q^\top \theta \in \mathbb{R}$ and the payoffs are given by $\mathbf{D}^\top \theta \in \mathbb{R}^S$.

Absence of Arbitrage

Definition

An *arbitrage* is a portfolio θ such that one of the two conditions hold:

1. $q^\top \theta < 0$ and $\mathbf{D}^\top \theta \geq 0$;
2. $q^\top \theta \leq 0$ and $\mathbf{D}^\top \theta > 0$;

If no such portfolio exists, we say that the market satisfies the *No Arbitrage* (NA) condition.

- You cannot get money and never pay anything;

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Definition

A *state-price vector* is a vector $\psi \in \mathbb{R}^S_{++}$ with $q = \mathbf{D}\psi$.

- $\mathbf{D}$ is like a basis for payoffs;
- ψ ensures that the Law of One Price holds given this basis;

No-Abitrage is Equivalent to the Existence of a State-Price Vector

Proposition

The market satisfies the No Arbitrage condition if, and only if, there exists a state-price vector.

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- We could have a continuum of states, and time could be continuous too;
- We could also have several periods;

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- This theorem holds in way more general setups;
- We could have a continuum of states, and time could be continuous too;
- We could also have several periods;
- Tools for the general version: Hahn-Banach, Separating Hyperplane Theorem, Riesz Representation Theorem;
- See Hansen and Richard (1987) for all intricate details;
- Here: we will need a Separating Hyperplane Theorem for cones;

Main Proof

A Useful Theorem

Definition

A set $C \subseteq \mathbb{R}^k$ is a *cone* if for any $x \in C$ and any $\lambda \geq 0$, we have that $\lambda x \in C$.

- Is every cone convex?
- Is every cone open? Is every cone closed?

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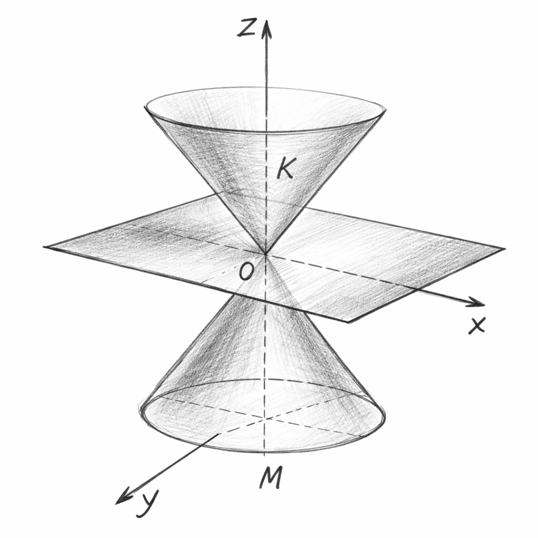
Theorem (Separating Hyperplane Theorem for Cones)

Suppose M and K are closed convex cones in $\mathbb{R}^k$ with $M \cap K = \{0\}$. If K does not contain a linear subspace other than $\{0\}$, than there is a nonzero linear functional F such that $F(x) < F(y)$ for each $x \in M$ and $y \in K$, with $y \neq 0$.

Visualizing the Theorem (Thanks, Chat GPT!)



Proof of the Main Theorem

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Define the following sets:

- $M = \{(-q^\top \theta, \mathbf{D}^\top \theta) : \theta \in \mathbb{R}^N\}$
- $K = \mathbb{R}_+ \times \mathbb{R}_+^S$ and $L = \mathbb{R} \times \mathbb{R}^S$;

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- $K = \mathbb{R}_+ \times \mathbb{R}_+^S$ and $L = \mathbb{R} \times \mathbb{R}^S$;
- Notice that K and M are closed convex cones;
- Moreover, notice that K does not contain any linear subspace other than $\{0\}$;

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- $K = \mathbb{R}_+ \times \mathbb{R}_+^S$ and $L = \mathbb{R} \times \mathbb{R}^S$;
- Notice that K and M are closed convex cones;
- Moreover, notice that K does not contain any linear subspace other than $\{0\}$;
- No arbitrage is **equivalent** to $M \cap K = \{0\}$;

Proof of the Main Theorem

Let's start easy. Assume there exists a state-price vector ψ;

- Then, the cost of any portfolio θ is given by $q^\top \theta = \psi^\top \mathbf{D}^\top \theta$;
- If $\mathbf{D}^\top \theta \geq 0$, then $q^\top \theta = \psi^\top \mathbf{D}^\top \theta \geq 0$ since $\psi \in \mathbb{R}^S_{++}$;
- If a portfolio generates non-negative payoffs, its cost cannot be negative;
- Thus, no arbitrage;

Proof of the Main Theorem

Now, assume No-Arbitrage holds;

- Then, $M \cap K = \{0\}$;
- By the Separating Hyperplane Theorem for Cones, there exists a non-zero linear functional F such that $F(x) < F(y)$ for each $x \in M$ and $y \in K$, with $y \neq 0$;

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- Notice that we have $F(x) \leq 0, \forall x \in M$; (Why?)
- Assume $F(x) < 0$ for some $x \in M$;
- Then, for any $y \in K$, we have that $F(-x) > 0$ and $F(-x) > 0$;
- This is a contradiction since $-x \in M$ (M is a linear subspace!);
- Thus, $F(x) = 0, \forall x \in M$;

Proof of the Main Theorem

Since, we are working in $\mathbb{R}^{S+1}$, for any $(c, p) \in \mathbb{R} \times \mathbb{R}^S$, we can write:

$$F(c, p) = \alpha \cdot c + \beta^\top p, \text{ for some } \alpha \in \mathbb{R}, \beta \in \mathbb{R}^S.$$

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- Thus, for any portfolio θ, we have that:

$$F(-q^\top \theta, \mathbf{D}^\top \theta) = -\alpha q^\top \theta + \beta^\top \mathbf{D}^\top \theta = 0, \forall \theta \in \mathbb{R}^N.$$

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- This implies that $\alpha q = \mathbf{D}\beta$;
- Notice that $e_1 \equiv (1, 0, \dots, 0) \in K \implies \alpha > 0$;
- Also, for any $p \in \mathbb{R}^S_+$, we have that $(0, p) \in K \implies \beta \in \mathbb{R}^S_{++}$;

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- Also, for any $p \in \mathbb{R}^S_+$, we have that $(0,p) \in K \implies \beta \in \mathbb{R}^S_{++}$;
- We define $\psi \equiv \frac{\beta}{\alpha} \in \mathbb{R}^S_{++}$ and conclude that $q = \mathbf{D}\psi$. We are done!

Questions?

Uniqueness of State-Price Vectors

- Can we say anything about uniqueness of State-Price Vectors?
- No-Arbitrage ensures the existence of a positive one;
- What if there are several ones?

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- This requires at least $N \geq S$ assets;
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- If $N > S$, $\text{rank}(\mathbf{D}) = S \iff (\mathbf{D}^\top \mathbf{D})^{-1}$ exists;
- In that case, we can write $\psi \equiv \left(\mathbf{D}^\top \mathbf{D}\right)^{-1} \mathbf{D}^\top q$ is unique;

Comments

- We need at least S linearly independent assets to span the state space;
- With incomplete markets, there are many SDFs that price assets correctly;
- They *will* disagree about the payoffs we do not trade;
- That makes sense: how would we price by no-arbitrage something we cannot *replicate*?

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- Good and bad: generality vs lack of discipline;

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- No-Arbitrage is very weak: there is no utility function, no equilibrium notion...
- Good and bad: generality vs lack of discipline;
- From state prices to SDFs: just re-scale by the physical probabilities;
- State-prices and the SDF encode the same fundamental information!

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Recall that for any risk premium $R - R_f$, we can write:

$$\mathbb{E}[m(R - R_f)] = 0 \implies \mathbb{E}[R - R_f] = -\text{Cov}(m, R)$$

- Let $\rho(m, R - R_f) \equiv \frac{\text{Cov}(m, R - R_f)}{\sigma(m)\sigma(R - R_f)}$ be the correlation between m and $R - R_f$;

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- Then, we can write:

$$\mathbb{E}[R - R_f] = -\rho(m, R - R_f) \cdot \sigma(m) \cdot \sigma(R - R_f) \cdot \frac{1}{\mathbb{E}[m]}$$

The Hansen-Jagannathan Bound

- Rearranging, we get:

$$\frac{\mathbb{E}[R - R_f]}{\sigma(R - R_f)} = -\rho(m, R - R_f) \cdot \sigma(m) \cdot \frac{1}{\mathbb{E}[m]}$$

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- Using that $|\rho(\cdot, \cdot)| \leq 1$, we get the **Hansen-Jagannathan** bound:

$$\underbrace{\left|\frac{\mathbb{E}[R - R_f]}{\sigma(R - R_f)}\right|}_{\text{absolute Sharpe Ratio}} \leq \sigma(m) \cdot \frac{1}{\mathbb{E}[m]}, \forall \text{ returns } R \text{ and any sdf } m$$

- Stare at it again and reflect how powerful this is.
- First derived in Hansen and Jagannathan (1991);

Maximal Sharpe Ratio Portfolio

- We can take the $\sup$ over all returns R and the $\inf$ over all SDFs m to get:

$$\sup_R \left| \frac{\mathbb{E}[R - R_f]}{\sigma(R - R_f)} \right| \leq \inf_{m:\mathbb{E}[mR]=1} \left[\frac{\sigma(m)}{\mathbb{E}[m]} \right]$$

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- The maximum Sharpe Ratio attained by a portfolio is bounded the volatility of the SDF;
- This is very general: we only assumed No-Arbitrage!
- Important: to mimick the high Sharpe Ratios observed in the data, the SDF must be very volatile;
- The return process with the highest (absolute) Sharpe Ratio is such that $\rho(m, R - R_f) = \pm 1 \implies R - R_f = a + b \cdot m +$ noise;
- What's the intuition?

Questions?

What's the least volatile SDF?

- Consider N assets with random payoffs $R = (R_1, \dots, R_N)^\top$ and a risk-free rate R_f;
- Consider finding the SDF with the smallest variance that prices these assets correctly;
- This is *the* SDF that *attains* the HJ bound;

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- Let $\mathcal{R}$ be the linear span of $(1, R^\top)^\top$;
- Notice the following for any SDF m

$$m = \text{Proj}(m|\mathcal{R}) + e$$

  where e is orthogonal to $\mathcal{R}$;

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- Crucial question: can we find and SDF in $\mathcal{R}$?

An SDF in $\mathscr{R}$

- Any element of $\mathscr{R}$ can be written as:

$$m(a,b) \equiv a + b^\top \left(R - R_f \mathbf{1}\right)$$

An SDF in $\mathcal{R}$

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and $\mathbb{E}[m(a,b)] = 1/R_f$

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- Great news! This is a system of $N+1$ equations with $N+1$ unknowns (a,b)!
- Exercise: show that the solution is given by:

$$a^* = \frac{1}{R_f}, \qquad b^* = -\frac{1}{R_f} \cdot \mu^\top \Sigma^{-1}$$

where $\mu \equiv \mathbb{E}[R - \mathbb{E}[R]]$ and $\Sigma \equiv \mathbb{E}\left[(R - \mathbb{E}[R])(R - \mathbb{E}[R])^\top\right]$.

The Hansen-Jagannathan SDF

- The SDF with the smallest variance that prices the N risky assets and the risk-free rate correctly is given by:

$$m^* = \frac{1}{R_f} - \frac{1}{R_f} \cdot \mu^\top \Sigma^{-1} \left(R - R_f \mathbf{1} \right)$$

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- Very important lesson: only Proj$(m, \mathcal{R})$ matters for pricing assets;
- If two SDFs have the same projection into $\mathcal{R}$, they price all assets the same;

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- Very important lesson: only $\text{Proj}(m, \mathcal{R})$ matters for pricing assets;
- If two SDFs have the same projection into $\mathcal{R}$, they price all assets the same;
- Exercise: show that the highest Sharpe Ratio allowed by this SDF is given by:

$$\text{SR}_{\max} = \sqrt{\mu^\top \Sigma^{-1} \mu}$$

Comments

- This SDF prices all included assets *perfectly*;
- But it made no promises on pricing *other* assets;
- What would be challenges to its implementation?
- Does it need to be positive? Does that imply it's possible to find an arbitrage?
- Please read Hansen and Jagannathan (1991)!

Questions?

AI-Generated Summary and Key Takeaways

I asked Chat GPT to summarize this lecture. Here is the unfiltered output:

- **SDF exists** $\iff$ **No-Arbitrage** (no preferences/equilibrium needed)
- **NA** $\iff \exists$ **state prices** $\psi \in \mathbb{R}^S_{++}$ s.t. $q = D\psi$ (pricing = a positive linear functional)
- **Uniqueness** $\iff$ **completeness**: $\text{col}(D) = \mathbb{R}^S$; incomplete markets   many SDFs (disagree off-span)
- **Hansen–Jagannathan bound**:

$$\left| \frac{\mathbb{E}[R - R_f]}{\sigma(R - R_f)} \right| \leq \frac{\sigma(m)}{\mathbb{E}[m]}$$

  High Sharpe ratios $\implies$ **volatile** SDF

- **Minimum-variance SDF** lies in $\mathcal{R} = \text{span}(1, R)$; only $\text{Proj}(m \mid \mathcal{R})$ matters for pricing

Readings and Rereferences

- (C): Chapters 1, 4, and 5;
- (D): Chapter 1;

Hansen, Lars Peter, and Ravi Jagannathan. 1991. "Implications of Security Market Data for Models of Dynamic Economies." *Journal of Political Economy* 99 (2): 225–62. https://doi.org/10.1086/261749.

Hansen, Lars Peter, and Scott F. Richard. 1987. "The Role of Conditioning Information in Deducing Testable Restrictions Implied by Dynamic Asset Pricing Models." *Econometrica* 55 (3): 587. https://doi.org/10.2307/1913601.