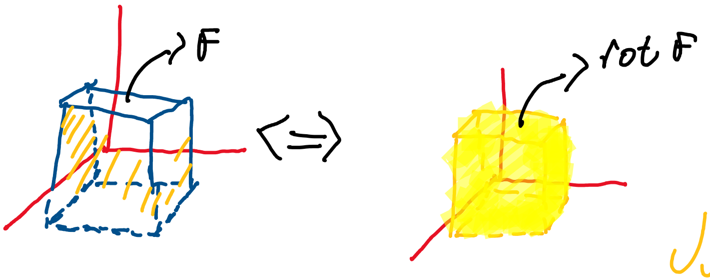
Campos Vetoriais

terça-feira, 24 de novembro de 2020 15:26

Teorema de Gauss: E é uma região sólida, é simult. dos tipos I, II, III. F é um campo vet. e S é o bordo de E; S é aberta

$$\boxed{\iint_{\partial E} F\cdot dS = \iiint_E \operatorname{div} F\, dV}$$



$\iint F.dS$

Ex: $F=(z,y,x)$, $E: x^2+y^2+z^2\leq 16$

Sol 1: Usando coordenadas polares, vamos parametrizar

$\partial E: \quad r(\theta,\phi)=4(\cos\theta \operatorname{sen}\phi, \operatorname{sen}\theta \operatorname{sen}\phi, \cos\phi)$

$\to N = 16 s\phi (\cos\theta \operatorname{sen}\phi, \operatorname{sen}\theta \operatorname{sen}\phi, \cos\phi)$

$F(r) = 4(\cos\phi, \operatorname{sen}\theta . \operatorname{sen}\phi, \operatorname{sen}\phi \cos\theta)$

$\Rightarrow F.N = 64(\operatorname{sen}^3\phi \operatorname{sen}^2\theta + 2\cos^2\phi \cos\theta) = 64P$

$\operatorname{div} F = (0+1+0) = 1$

$\Rightarrow \iint_{\partial E} F.dS = \int_0^{2\pi}\int_0^{\pi} 64P\, d\phi\, d\theta = \frac{256\pi}{3}$ ///

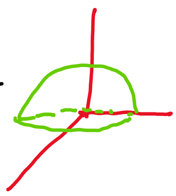
Sol 2: $\operatorname{div} F = 0+1+0=1 \Rightarrow \iint_{\partial E} F.dS = \boxed{\iiint_E 1.\, dV}$

$= Vol(E) = \frac{4}{3}\pi.4^3 = \frac{256\pi}{3}$ ///

17) $F = (z^2x, \frac{y^3}{3}+tg\, z, x^2z+y^2)$; $S: x^2+y^2+z^2=1$, $z\geq 0$

$S_1: x^2+y^2\leq 1, \quad z=0$

$S\cup S_1$ é bordo da parte superior da bola de raio 1.



$\Rightarrow \iint_S F = \iint_{S\cup S_1} F - \iint_{S_1} F$

$\Rightarrow S_1: \sigma(r,\theta) = (r\cos\theta, r\operatorname{sen}\theta, 0)$

$\sigma_r = (\cos\theta, \operatorname{sen}\theta, 0)$; $\sigma_\theta = r(-\operatorname{sen}\theta, \cos\theta, 0)$

$$\sigma_r \times \sigma_\theta : \quad \begin{array}{cccc} \operatorname{sen}\theta & 0 & \cos\theta & \operatorname{sen}\theta \\ r\cos\theta & 0 & -r\operatorname{sen}\theta & r\cos\theta \end{array}$$

$= (0, 0, 1)$

$z(x,y) = 0 \Rightarrow N = (0, 0, 1)$

$F(\sigma) = (0, y^3/3, y^2)$; $F \cdot N = 0 + 0 + y^2$

$\Rightarrow \int_0^{2\pi}\int_0^1 r^2 \operatorname{sen}^2\theta \, dr \, d\theta = -\frac{\pi}{4}$

$\iint_{S \cup S_1} F \overset{\text{Gauss}}{=} \iiint_E r^2 \cdot dV \overset{\text{C.E.}}{=} \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 r^2 \cdot r^2 \operatorname{sen}\phi \, dr \, d\phi \, d\theta$

$= \frac{1}{5} \cdot 2\pi \int_0^{\pi/2} \operatorname{sen}\phi \, d\phi = \frac{2\pi}{5}\left(-\cos\phi \Big|_0^{\pi/2}\right) = \frac{2\pi}{5}$

$\Rightarrow \iint_S F \cdot dS = \iint_{S \cup S_1} F - \iint_{S_1} F = \frac{2\pi}{5} - \left(-\frac{\pi}{4}\right) = \frac{8\pi + 5\pi}{20} = \frac{13\pi}{20}$

Ex: $\iint_{\partial E} \underbrace{(\overset{f}{x^2 + y + z})}_{\text{Escalar}} dS$; $E: x^2 + y^2 + z^2 \leq 1$

$n: (x, y, z)$;

F, $F \cdot n = x^2 + y + z$

$(x, 1, 1) = F \Rightarrow \operatorname{div} F = 1 + 0 + 0 = 1$

$z = \pm\sqrt{1 - x^2 - y^2}$; $z(x,y)$

$n = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial x}, 1\right) = \left(\frac{x}{z}, \frac{y}{z}, 1\right)$

$\frac{\partial z}{\partial x} = \frac{1}{\cancel{2}} \cdot \frac{-\cancel{2}x}{\sqrt{1 - x^2 - y^2}} = -\frac{x}{z}$; $n \cdot z \perp S$

$n \cdot z = (x, y, z)$

$\frac{\partial z}{\partial y} = -\frac{y}{z}$;

$\Rightarrow \iint_S (x^2 + y + z) \, dS = \iint_{\partial E} F \cdot n \, dS = \iiint_E \operatorname{div} F \, dv$

$= \iiint_E 1 \, dv = \operatorname{Vol}(E)$

$= \boxed{\frac{4\pi}{3}}$

Ex: f uma função harmônica ; $\nabla^2 f = 0$,

$\int_C f_y \, dx - f_x \, dy$ é independente do caminho numa região simples

$\star \int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y) \, dS$

$\nabla^2 f = 0 = f_{xx} + f_{yy} = 0 \Leftrightarrow f_{xx} = -f_{yy}$

$\Rightarrow \int_C \underset{P}{f_y} \, dx - \underset{Q}{f_x} \, dy = \iint_D \underbrace{(\overset{-f_{yy}}{-f_{xx}} - f_{yy})}_{=0} \, dS = \iint_D 0 \, dS$

Como S é uma região simples:

Geo: f é ind. do caminho $\Leftrightarrow \oint_C F\,ds = 0$

$\rightarrow$ f é ind. do Caminho

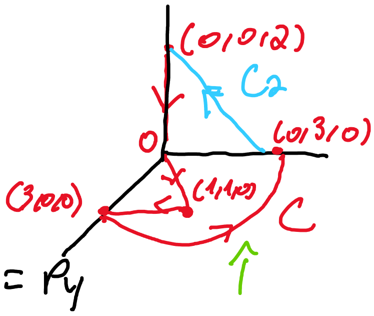
$\nabla \cdot (\text{rot}\, F) = 0$; $\text{rot}(\nabla f) = 0$

Mostre que ñ existe G t.q. $\text{rot}\, G = (2x, 3yz, -xy^2) = F$

Se $F = \text{rot}\, G \Rightarrow \nabla \cdot F = 0 = 2 + 3z - 0) \neq 0$; Absurdo.

Ex $F = (3x^2yz - 3y,\ x^3z - 3x,\ x^3y + 2z) = (P, Q, R)$

(I) Vamos verificar que F é conservativo



F é cons. $\Rightarrow P_y = Q_x$

$P_y = 3x^2z - 3$; $Q_x = 3x^2z - 3 = P_y$

$f \mid \nabla f = F$

$\Rightarrow \int P\,dx = f = \int 3x^2yz - 3y\,dx = x^3yz - 3xy + h(y,z)$

$f = x^3yz - 3xy + h(y,z)$

$f_y = Q = \cancel{x^3z} - \cancel{3x} + h_y = \cancel{x^3z} - \cancel{3x} \Rightarrow h_y = 0$

$\Rightarrow h(z)$

$f_z = R = \cancel{x^3y} + 2z = \cancel{x^3y} + h_z \Rightarrow h = z^2 + \cancel{C}$

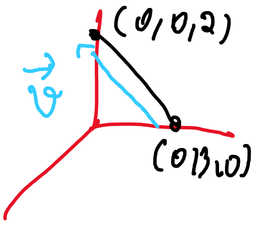
$\Rightarrow f = x^3yz - 3xy + z^2$

$\Rightarrow \int_C F\,ds = f(b) - f(a) = f(0,3,0) - f(0,0,2)$

$= 0 - 4 = \boxed{-4}$

(II) $\text{rot}\, F = (0,0,0) = 0$

$\int_C F = \int_{C \cup C_1} F - \int_{C_1} F \overset{\text{Stokes}}{=} \iint_S 0\,ds - \int_{C_1} F\,ds = 0 - \boxed{\int_{C_1} F\,ds}$



$c(t) = (0,3,0) + t \cdot \vec{v} = (0,3,0) + t[(0,0,2) - (0,3,0)]$

$= (0, 3-3t, 2t)$

$F(c(t)) = (-3(3-3t), 0, 4t)$

$c'(t) = (0,-3,2) \Rightarrow F \cdot c' = 0 + 0 + 2 \cdot 2t = 8t$

$\Rightarrow \int F\,dt = \int_0^1 8t\,dt = 4$

$\Rightarrow \int_{C_1} \dots dt = \int_0 \dots t \, dt = \dots$

$\Rightarrow \int_C F \, ds = -\int_{C_1} F \, ds' = \boxed{-4}$