Resolução A1

terça-feira, 29 de setembro de 2020 10:18

1) $g(x,y) = 8 - 2x^2 - 3y^2$, $f(x,y) = 4 - x^2 - y^2$

Planos tangentes Paralelos?

- $g_x = -4x$
- $g_y = -6y$

em $(1,2,-6)$: $g_x = -4$, $g_y = -12$

$\nabla g = \lambda \nabla h$ → Vetores Gradientes Paralelos

$f_x = -2x$
$f_y = -2y$

$\Rightarrow -2x = -4, \rightarrow x = 2$
$\Rightarrow -2y = -12, \rightarrow y = 6$

$\therefore P = (2, 6, -36)$

---

2) $r = (x, y, z)$, $p = \|r\|$; $\nabla\left(\frac{1}{p}\right) = \frac{-r}{p^3}$?

$p = \|r\| = \sqrt{x^2 + y^2 + z^2}$; Observe que sempre que derivar-mos, $\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{1}{p}$ aparece

$p_x = x \cdot \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{2}{2}$

$\vdots$

$\nabla p = (x, y, z) \cdot \frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{r}{p}$.

$\Rightarrow \nabla\left(\frac{1}{p}\right) = -\frac{1}{p^2} \cdot (\nabla p) = \frac{-r}{p^2 \cdot p} = -\frac{r}{p^3}$ //.

---

3) $u = g(x,y) = xy \cdot f\left(\frac{x+y}{xy}\right)$ ←

Primeiro vamos calcular $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$:

Seja $K = \frac{x+y}{xy}$: $\frac{\partial K}{\partial x} = \frac{xy - (x+y)y}{x^2y^2} = -\frac{1}{x^2}$

Analogamente: $\frac{\partial K}{\partial x} = -\frac{1}{y^2}$

(*) $u_x = y f(K) + xy f'(K) \cdot \left(\frac{-1}{x^2}\right) = y f(K) - \frac{y}{x} f'(K)$
$u_y = x f(K) + xy f'(K) \left(\frac{-1}{y^2}\right) = x f(K) - \frac{x}{y} f'(K)$

Regra da Cadeia

$\Rightarrow x^2 u_x = x^2 y f(K) - xy f'(K)$ (I)
$y^2 u_y = xy^2 f(K) - xy f'(K)$ (II)

$(I) - (II) = x^2 y f(K) - xy^2 f(K) = xy f(K)(x - y)$
$= g(x,y)(x-y)$

$\therefore G(x,y) = x - y$

(4) $f = \frac{x^3}{3} + \frac{y^3}{3} - \frac{x^2}{2} - \frac{5}{2}y^2 + 6y + 10.$

$f_x = x^2 - x = x(x-1) = 0 \Rightarrow x = \{0,1\}$
$f_y = y^2 - 5y + 6 = 0 \Rightarrow y = \{2,3\}$

$P_1 = (0,2)$
$P_2 = (0,3)$
$P_3 = (1,2)$
$P_4 = (1,3)$

$f_{xx} = 2x - 1$; $f_{xy} = 0$
$f_{yy} = 2y - 5$

(*) $D(P_1) = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 > 0$, $f_{xx} = -1$
↳ Máximo local

$D(P_2) = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 < 0$
↳ Sela

$D = \begin{vmatrix} 2x-1 & 0 \\ 0 & 2y-5 \end{vmatrix}$

$D(P_3) = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1 < 0$
↳ Sela

$D(P_4) = \begin{vmatrix} 1 & 0 \\ \ldots \end{vmatrix} = 1 > 0$, $f_{xx} = 1 > 0$

$$[\ldots] \; \to \text{Mínimo local.}$$

⑤ $f(x,y,z) = \ln x + \ln y + 3\ln z$, $g = x^2+y^2+z^2 = 5a^2$; $x,y,z > 0$.
↳ (i)

$$\nabla f = \left(\frac{1}{x}, \frac{1}{y}, \frac{3}{z}\right) = \lambda \cancel{2}(x, y, z) = \lambda \nabla g(x,y,z)$$

$\lambda \neq 0$ pois $1/x \neq 0$.
$\to \frac{1}{x} = \lambda x \to \lambda = \frac{1}{x^2}$

// $\frac{1}{y} = \frac{1}{x^2} \cdot y \Leftrightarrow y^2 = x^2$, por (i)
↳ $y = x$

$\Rightarrow \frac{3}{z} = \frac{1}{x^2} z \to z = x\sqrt{3}$

// Substituindo em g:
$x^2 + x^2 + 3x^2 = 5a^2$; $x = a$

logo; Seja P um ponto crítico:
$P = (a, a, \sqrt{3}a)$.

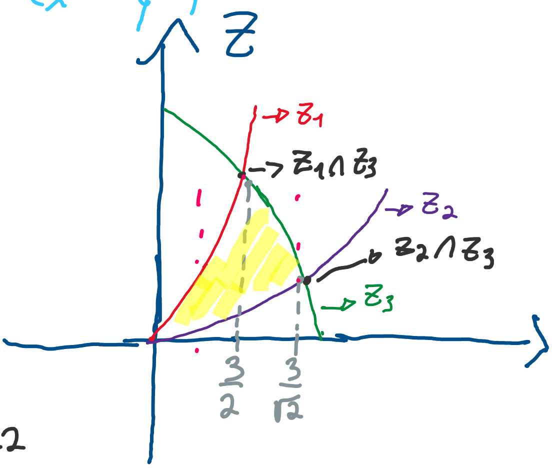
Em f: $f(a, a, \sqrt{3}a) = \boxed{\ln a + \ln a + 3\ln(a\sqrt{3})}$

⑥ $z_1 = 3(x^2+y^2)$, $z_2 = (x^2+y^2)$, $z_3 = 9-(x^2+y^2)$

Utilizando Coordenadas Polares:

$z_1 \cap z_3 : 3r^2 = 9 - r^2 \Leftrightarrow r^2 = \frac{9}{4} \Leftrightarrow r = \frac{3}{2}$

$z_2 \cap z_3 : r^2 = 9 - r^2 \Leftrightarrow r^2 = \frac{9}{2} \Leftrightarrow r = \frac{3}{\sqrt{2}}$



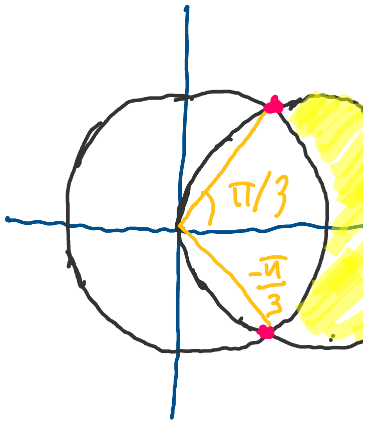
(I) $r \leq \frac{3}{2} \Rightarrow$ Precisamos integrar
$z_1 - z_2 = 3r^2 - r^2 = 2r^2$

(II) $\frac{3}{2} \leq r \leq \frac{3}{\sqrt{2}} \Rightarrow$ integramos:
$z_3 - z_2 = 9 - r^2 - r^2 = 9 - 2r^2$

$$\therefore V = \int_0^{2\pi}\int_0^{3/2} r(2r^2)\,dr\,d\theta + \int_0^{2\pi}\int_{3/2}^{3/\sqrt{2}} r(9-2r^2)\,dr\,d\theta$$

$$\therefore V = \frac{567\pi}{32} \;///$$

(7) (a) $(x-1)^2+y^2 \leq 1,\quad x^2+y^2 \geq 1$



Coordenadas Polares:

$x^2-2x+y^2 \leq 0 \iff r^2 \leq 2r\cos\theta \iff r \leq 2\cos\theta$

$x^2+y^2 \geq 1 \iff r \geq 1$ // Como $r \geq 1$, $2\cos\theta \geq 1$

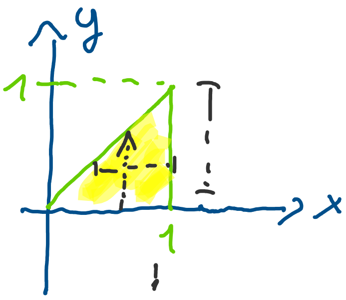
logo: $-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$

(*) $\int_{-\pi/3}^{\pi/3}\int_1^{2\cos\theta} r\,dr\,d\theta$

$\hookrightarrow = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(4\cos^2\theta - 1)\,dr\,d\theta$

$$= \int_{-\pi/3}^{\pi/3}(1+\cos 2\theta)\,d\theta - \frac{\pi}{3} = \frac{\pi}{3} + \frac{1}{2}\left(\underbrace{\operatorname{sen} 2\theta}_{\to\, 2\sqrt{3}}\Big|_{-\pi/3}^{\pi/3}\right) = \frac{\pi}{3} +$$

(b) $\int_0^1\int_y^1 e^{x^2}\,dx\,dy$;



$0 \leq y \leq 1,\; y \leq x \leq 1$

$\iff 0 \leq x \leq 1,\; 0 \leq y \leq x$

$\Rightarrow \int_0^1\int_0^x e^{x^2}\,dx = \int_0^1 x e^{x^2}\,dx$; $u = x^2$, $\frac{du}{2} = x\,dx$

$= \int_0^1 \frac{1}{2} e^u\,du = \frac{1}{2}\left(e^u\Big|_0^1\right) = \frac{1}{2}(e-1) \;///$