terça-feira, 17 de novembro de 2020 15:04

16.5) 27) $F=(A,B,C)$, $G=(P,Q,R)$

Mostrar que: $\boxed{\operatorname{div}(F\times G)= G\cdot \operatorname{rot} F - F\cdot \operatorname{rot} G}$?

- $\operatorname{rot} F = (C_y - B_z, A_z - C_x, B_x - A_y)$
- $\operatorname{rot} G = (R_y - Q_z, P_z - R_x, Q_x - P_y)$
- $F \times G = (BR - CQ, CP - AR, AQ - BP)$

$$\Rightarrow \operatorname{div}(F\times G) = \frac{\partial}{\partial x}(BR-CQ) + \frac{\partial}{\partial y}(CP-AR) + \frac{\partial}{\partial z}(AQ-BP)$$

$$= (B_xR + BR_x - C_xQ - Q_xC) + (C_yP + P_yC - A_yR - R_yA)$$

P Q R
A B C

$$+ (A_zQ + Q_zA - B_zP - P_zB)$$

$$= P(C_y - B_z) + Q(A_z - C_x) + R(B_x - A_y) - \}$$ $(P,Q,R)\cdot \operatorname{rot} F$

$$A(R_y - Q_z) - B(P_z - R_x) - C(Q_x - P_y)$$

$$= G\cdot \operatorname{Rot} F - F\cdot \operatorname{Rot} G \;//$$

Teorema de Stokes: S é uma superfície suave; e C é o bordo dessa superfície $\Rightarrow \partial S : C$; C é uma curva fechada; F é diferenciável;

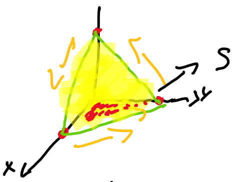
$$\boxed{\cdot \oint_C F\cdot ds = \iint_D \operatorname{rot} F\, ds}$$

Ex 1: $F=(x+y^2, y+z^2, z+x^2)$; ←

$C: (1,0,0) \to (0,1,0) \to (0,0,1) \to (1,0,0)$

→ Orientação Positiva

$$\oint_C F\, ds = \iint_D \operatorname{rot} F\cdot ds$$



$$\Rightarrow \operatorname{rot} F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y^2, & y+z^2, & z+x^2 \end{vmatrix} = (0-2z, -2x, -2y) = -2(z,x,y)$$

$$\oint_C F\, ds = \iint_D (-2z, -2x, -2y)\cdot ds$$

$\Rightarrow x + y + z = 1 \Rightarrow N = (1,1,1)$

$F\cdot N = (-2z, -2x, -2y)\cdot(1,1,1) = -2z - 2x - 2y$

$z = 1 - x - y$; $\Rightarrow F\cdot N = -2(1-x-y) - 2x - 2y$

$$= -2 + \cancel{2x} + \cancel{2y} - \cancel{2x} - \cancel{2y} = -2$$

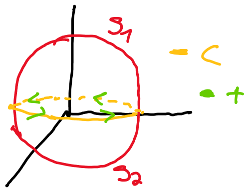
$$\Rightarrow \iint_D -2\, ds = -2\iint_D 1\, ds = -2A(D) = -2\cdot\frac{1}{2} = \boxed{-1}$$

Ex 2: S é uma esfera que satisfaz as condições para o Teorema de

Stokes; e $F = (F_1, F_2, F_3)$ também satisfaz

$\Rightarrow \iint_S \text{rot}\, F\, dS = 0$

$\iint_S \text{rot}\, F\, dS = \iint_{S_1} \text{rot}\, F\, dS + \iint_{S_2} \text{rot}\, F\, dS$



$= \oint_C F\, dS + \oint_{-C} F\, dS = \oint_C F\, dS - \int_C F\, dS = 0$

Obs: Se C não é fechada e $C \cup \gamma$ é fechada:

$\Rightarrow \int_C F = \int_{C\cup\gamma} F - \int_\gamma = \iint_D \text{rot}\, dS - \int_\gamma$

Teorema de Green: " " "

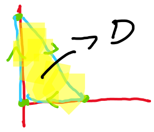
C é uma curva fechada:

$\Rightarrow \oint_C P dx + Q dy = \oint_C (P, Q)\, dS = \iint_D \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dx dy$

Ex3: $F = (y\cos x - xy\,\text{sen}\,x,\ xy + x\cos x)$

$C: (0,0) \to (0,4) \to (2,0) \to (0,0)$

$\oint_C F\, dS =$?



$F = (P, Q)$: $\frac{\partial Q}{\partial x} = y + \cancel{\cos x} - \cancel{x\,\text{sen}\,x}$

$\frac{\partial P}{\partial y} = \cancel{\cos x} - \cancel{x\,\text{sen}\,x}$

$\Rightarrow \oint_C P dx + Q dy = \iint_D (y)\, dy\, dx$; $0 \le x \le 2$, $0 \le y \le 4 - 2x$

$= \int_0^2 \int_0^{4-2x} y\, dy\, dx = \int_0^2 \left(\frac{16 - 16x + 4x^2}{2}\right) dx = \int_0^2 (8 - 8x + 2x^2)\, dx$

$= 8x - 4x^2 + \frac{2}{3}x^3 \Big|_0^2 = \cancel{16} - \cancel{16} + \frac{16}{3} = 16/3$

$\int_C x\, dy - y\, dx = x_1 y_2 - x_2 y_1$

$C: (x_1, y_1) \to (x_2, y_2)$

Ex4: $\frac{1}{2}\oint_C x\, dy - y\, dx$: é a área da região contida em C.

$= \frac{1}{2}\iint_D (1 - (-1))\, dx dy = \frac{1}{2}\iint_D 2\, dx dy = \iint_D 1\, dx dy$

$= A(D)$

$F \cdot r'(t)$

$r: (x_1, y_1) \to (x_2, y_2)$; $(x_1, y_1) + t(x_2 - x_1, y_2 - y_1)$;
$0 \le t \le 1$;

$r(t) = (x_1 + t(x_2 - x_1),\ y_1 + t(y_2 - y_1))$, $\left.\begin{matrix} a = x_2 - x_1 \\ b = y_2 - y_1 \end{matrix}\right\}$

$= (x_1 + ta,\ y_1 + tb)$

$\to r'(t) = (a, b)$

$F(-y, x) = (-y_1 - tb,\ x_1 + t \cdot a)$

$\to F \cdot r' = x_1 y_2 - x_2 y_1$

$\Rightarrow \oint_C x\, dy - y\, dx = \int_0^1 F \cdot r'\, dt = \int_0^1 (x_1 y_2 - x_2 y_1)\, dt$

$= (x_1 y_2 - x_2 y_1) \cdot 1$

$\Rightarrow \boxed{\int_c x\,dy - y\,dx = x_1 y_2 - x_2 y_1} \leftarrow$

(I) $\oint_c x\,dy - y\,dx = A(D)$

(II) $\int_{C_i} x\,dy - y\,dx = x_{i+1}y_i - y_{i+1}x_i$

Seja P um polígono de n lados com vértices $(x_1,y_1),(x_2,y_2),\cdots,(x_n,y_n)$ e seja γ o bordo do polígono e c_i a i-ésima aresta, então a área de $P = \frac{1}{2}\int_\gamma x\,dy - y\,dx = \int_{C_1 \cup C_2 \cup \dots} F$

$= (x_2 y_1 - x_1 y_2) + (x_3 y_2 - x_2 y_3) + \dots + (x_n y_1 - x_1 y_n) = A(P)$

$\Rightarrow \frac{1}{2}\begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{vmatrix} = A(P)$

$\bigcup_{i=1}^{n} C_i = \gamma \to \gamma$ é fechado

$\to$ Aplico Green em γ

$\to \int_\gamma x\,dy - y\,dx = A(P)$

$= \sum_{i=1}^{n} (x_{i+1}y_i - x_i y_{i+1}) =$

Obs: $A_\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$

$\to$ Caso particular

$P: (0,0),(2,1),(1,3),(0,2),(-1,1)$

$\frac{1}{2}\begin{vmatrix} 0 & 2 & 1 & 0 & -1 \\ 0 & 1 & 3 & 2 & 1 \end{vmatrix} = \frac{0 + (6-1) + (2-0) + (0+2)}{2}$

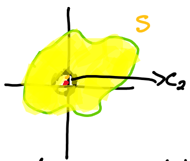
$= \boxed{9/2}$

• $F = (2xy, y^2 - x^2)/(\underbrace{x^2+y^2}_{})^2$; C é qualquer curva fechada que contém a origem

$F = (P,Q)$

$Q_x = P_y$

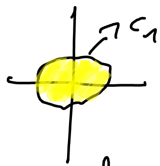
$\to \oint_c F\,ds = \iint_D (Q_x - P_y)\,ds = \iint_D 0\,ds = 0$



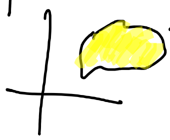
$\int_c F = \int_{C \cup C_2} F - \int_{C_2} F$

$= 0 - \int_{C_2} F$; C_2 é um círculo de raio "a", a pequeno o suficiente.

$= 0 - 0 = 0$



$\to C_1$ circunda a origem



$\to C_2$ ñ " "

$\oint_{C_2} F\,ds = \iint_D Q_x$

(6.5) 31) $R = (x,y,z)$; $r = |R| = \sqrt{x^2+y^2+z^2}$

a) $\nabla r = R/r$? ; $\nabla r \Rightarrow \frac{1}{\cancel{2}} \cdot \frac{1 \cdot \cancel{2}x}{r} = \frac{x}{r} = \frac{\partial R}{\partial x}$

$\to \frac{\partial R}{\partial y} = \frac{y}{r}$, $\frac{\partial R}{\partial z} = \frac{z}{r}$; $\nabla r = \frac{\partial R}{\partial x}i + \frac{\partial R}{\partial y}j + \frac{\partial R}{\partial z}k$

$= \frac{xi + yj + zk}{r} = (x,y,z)/r = R/r$

b) $\nabla \times R$; $\nabla \times (\nabla F) = 0$; $R = \nabla\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}\right) = \nabla f$

$\Rightarrow \nabla \times R = \nabla \times (\nabla f) = 0$

c) $\nabla\left(\frac{1}{r}\right) \Rightarrow \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}} \Rightarrow \frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{-1}{\cancel{2}} \cdot \frac{1}{r^3} \cdot \cancel{2}x$

$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \frac{-y}{r^3}$, $\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = \frac{-z}{r^3}$

$\Rightarrow \nabla\left(\frac{1}{r}\right) = -(x,y,z)/r^3$

d) $\nabla \ln r$; $\ln r = \frac{1}{2}\ln(x^2+y^2+z^2)$;

$\frac{\partial \ln r}{\partial x} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}x}{x^2+y^2+z^2} = \frac{x}{x^2+y^2+z^2} = \frac{x}{r^2}$

$\to \nabla \ln r = (x,y,z)/r^2$

$= R/r^2$