Exs. for An Introduction to Probability and Statistics by Rohatgi and Saleh (Third Edition)

Problems 8.2

(1) * T_n is a sequence of estimators for θ

* $E(T_n) \to \theta$ and $Var(T_n) \to 0$, $n \to \infty$

$\Rightarrow T_n$ is consistent for θ

* $T_n \xrightarrow{2} \theta$, se $\lim\limits_{n\to\infty} E[(T_n-\theta)^2] = 0$

$\boxed{\text{I}}$ T_n consistent for θ and $|T_n - \theta| \le A < \infty$

$$E[(T_n-\theta)^2] = \ldots = \underbrace{E[(T_n - E(T_n))^2]}_{\text{Variance}} + \underbrace{[E(T_n-\theta)]^2}_{\text{Bias}^2}$$

$$= Var(T_n) + [E(T_n-\theta)]^2$$

$$\lim_{n\to\infty} E[(T_n-\theta)^2] = \underbrace{\lim_{n\to\infty} Var(T_n)}_{\to 0} + \lim_{n\to\infty} [E(T_n-\theta)]^2$$

$$= \lim_{n\to\infty} [E(T_n) - E(\theta)]^2.$$

$$= \lim_{n\to\infty} [\underbrace{E(T_n) - E(\theta)}_{\to 0}] \cdot \lim_{n\to\infty} [\underbrace{E(T_n) - E(\theta)}_{\to 0}]$$

$$= 0$$

$\therefore T_n \xrightarrow{2} \theta$

?

Resolução do professor Raul:

$\boxed{\text{I}}$ T_n consistent for θ and $|T_n - \theta| \leq A < \infty$

Se Y for v.A não negativa,

$$EY = \int_0^\infty y \cdot f_\theta(y)\, dy = \int_0^\infty P(Y > \epsilon)\, d\epsilon$$

$$E|T_n - \theta|^2 \to 0, \quad n \to \infty$$

$$E|T_n - \theta|^2 = \int_0^\infty P(|T_n - \theta|^2 > \epsilon)\, d\epsilon$$

$$= \int_0^\infty P(|T_n - \theta| > \sqrt{\epsilon})\, d\epsilon$$

$\epsilon = \varepsilon^2$

$d\epsilon = 2\varepsilon\, d\varepsilon$

$$= 2 \cdot \int_0^\infty \varepsilon \cdot \underbrace{P(|T_n - \theta| > \varepsilon)} d\varepsilon$$

$T_n \xrightarrow{P} \theta$

$$\leq 2 \cdot A \cdot \int_0^A \underbrace{P(|T_n - \theta| > \varepsilon)}\, d\varepsilon \longrightarrow 0, \quad n \to \infty$$

$$\therefore \quad E|T_n - \theta|^2 \to 0, \; n \to \infty \implies T_n \xrightarrow{2} \theta$$

$\boxed{\text{II}}$ $|T_n - \theta| \leq A_n \leq \infty$

Para esse caso, temos:

$$E|T_n - \theta|^2 \leq 2 A_n \int_0^{A_n} P(|T_n - \theta| > \varepsilon)\, d\varepsilon$$

Uma vez que não sabemos o valor, no limite, por causa do fato de A_n estar indexado em n, não há garantia que $T_n \xrightarrow{2} \theta$.

② * $X_1, \ldots, X_n$ A.A. de $U[0, \theta]$, $\theta \in \Theta = (0, \infty)$

* $X_{(n)} = \max\{X_1, \dots, X_n\}$

* $X_{(n)} \xrightarrow{P} \theta$, se e somente se

$$P(|X_{(n)} - \theta| > \varepsilon) \to 0, \ n \to \infty, \forall \varepsilon > 0$$

o que é equivalente a.

$$P(|X_{(n)} - \theta| \leq \varepsilon) \to 1, \ n \to \infty, \forall \varepsilon > 0$$

- $$P(|X_{(n)} - \theta| \leq \varepsilon) = P(-\varepsilon \leq X_{(n)} - \theta \leq \varepsilon)$$

$$= P(\theta - \varepsilon \leq X_{(n)} \leq \varepsilon + \theta)$$

$$= F_{X_{(n)}}(\varepsilon + \theta) - F_{X_{(n)}}(\theta - \varepsilon)$$

$$F_{X_{(n)}}(x) = P(X_{(n)} \leq x)$$

$$= P(\max\{X_1, \dots, X_n\} \leq x)$$

$$= P(\{X_1 \leq x\} \cap \dots \cap \{X_n \leq x\})$$

$$\overset{ind}{=} [P(X_1 \leq x)]^n$$

$$= \left(\frac{x}{\theta}\right)^n$$

$$= 1 - \left(\frac{\theta - \varepsilon}{\theta}\right)^n.$$

$$\to 1, \ n \to \infty \text{ e } 0 < \varepsilon < \theta.$$

$\therefore X_{(n)} \xrightarrow{P} \theta$

- $Y_n = 2\bar{X}$, $\bar{X} = \dfrac{X_1 + X_2 + \dots + X_n}{n}$

$$E(Y_n) = E(2\bar{X}) = 2 \cdot E(\bar{X}) = \frac{\cancel{2}}{\cancel{n}} \cdot \cancel{n} \frac{\theta}{\cancel{2}} = \theta$$

$$\mathrm{Var}(Y_n) = \mathrm{Var}(2\bar{X}) = 4 \cdot \mathrm{Var}(\bar{X}) \overset{ind}{=} 4 \cdot \frac{1}{n^2}\, n \cdot \frac{\theta^2}{12}.$$

$$= \frac{4 \cdot \theta^2}{12 \cdot n} = \frac{\theta^2}{3n}$$

$$\to 0, \ n \to \infty.$$

Pelo Teorema 2, uma vez que $E(Y_n) = \theta$ e $\mathrm{Var}(Y_n) \to 0$, $n \to \infty$, Y_n é consistente para θ.

③

* $X_1, X_2, \dots, X_n$ i.i.d v.A's
* $E(X_i) = \mu$, $E(|X_i|^2) < \infty$
* $T(X_1, \dots, X_n) = \frac{2}{n(n+1)} \cdot \sum_{i=1}^{n} i\, X_i$

$$E[T(X_1, \dots X_n)] = \frac{2}{n(n+1)} \sum_{i=1}^{n} i \cdot E(X_i)$$

$$= \frac{2\mu}{n(n+1)} \sum_{i=1}^{n} i$$

$$= \frac{\cancel{2}\mu}{\cancel{n(n+1)}} \frac{\cancel{(1+n)} \cdot \cancel{n}}{\cancel{2}}$$

$$= \mu.$$

$$\mathrm{Var}(T(X_1, \dots, X_n)) = \left[\frac{2}{n(n+1)}\right]^2 \cdot \mathrm{Var}\left(\sum_{i=1}^{n} i\, X_i\right)$$

$$\overset{ind}{=} \left[\frac{2}{n(n+1)}\right]^2 \sum_{i=1}^{n} i^2 \cdot \mathrm{Var}(X_i)$$

$$= \left[\frac{2}{n(n+1)}\right]^2 \cdot \sigma_X^2 \cdot \sum_{i=1}^{n} i^2$$

$$= \left[\frac{2}{n(n+1)}\right]^2 \sigma_X^2 \cdot \frac{n \cdot (n+1) \cdot (2n+1)}{6}$$

$$= \frac{4}{n \cdot (n+1)} \cdot \frac{(2n+1)}{6} \cdot \sigma_X^2$$

$$= \left(\frac{2n+1}{n^2+n}\right) \cdot \frac{2}{3} \cdot \sigma^2_X.$$

$$\longrightarrow 0, \quad n \to \infty$$

Pelo Teorema 2, uma vez $E(T_n) \to \mu$ e $Var(T_n) \to 0, n \to \infty$,

T_n é consistente para μ.

(4) * $X_1, X_2, \dots, X_n$ amostra de $U[0,\theta]$

* $T_n = \left(\prod_{i=1}^{n} X_i\right)^{1/n}$

* $\psi(\theta) = \frac{\theta}{e}$

$$Y_n = \ln(T_n) \qquad (T_n = e^{Y_n})$$

$$= \frac{1}{n} \cdot \ln\left(\prod_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \ln(X_i)$$

$$E(\ln(X_i)) = \int_0^\theta \ln(x) \, \frac{1}{\theta} \cdot dx.$$

$$= \frac{1}{\theta} \int_0^\theta \ln(x) \, dx.$$

$$\int_0^\theta \ln(x)\, dx = \ln(x) \cdot x \Big|_0^\theta - \int_0^\theta x \cdot \frac{1}{x} \, dx$$

$u = \ln(x)$

$du = \frac{1}{x} dx$

$dv = dx$

$$= \ln(x) \cdot x \Big|_0^\theta - \int_0^\theta 1 \, dx$$

$$= \ln(\theta) \cdot \theta - \theta$$

$$= \frac{1}{\theta} \cdot \left[\ln(\theta) \cdot \theta - \theta \right]$$

$= \ln(\theta) - 1$

Pela Lei Forte dos Grandes Números.

$$Y_n = \frac{\ln(X_1) + \ldots + \ln(X_n)}{n} \xrightarrow{q.c} E(\ln(X_1)) = \ln(\theta) - 1$$

Pelo Teorema da Aplicação Contínua

$$Y_n \xrightarrow{q.c} \ln(\theta) - 1 \Rightarrow$$

$$T_n = e^{Y_n} \xrightarrow{q.c} \exp(\ln(\theta) - 1) = \theta \cdot e^{-1}$$

Uma vez que $T_n \xrightarrow{q.c} \psi(\theta) = \theta \cdot e^{-1}$, então $T_n \xrightarrow{P} \psi(\theta)$

$\therefore$ T_n é estimador consistente para $\psi(\theta) = \theta \cdot e^{-1}$

(5) $* \; T(\underset{\sim}{X}) = X_{(n)}$

$$F_{X_{(n)}}(x) = \begin{cases} 0, & x < 0 \\ \left(\frac{x}{\theta}\right)^n, & 0 \leq x < \theta \\ 1, & x \geq \theta \end{cases}$$

$\Rightarrow$

$$f_{X_{(n)}}(x) = \begin{cases} 0, & x \notin (0, \theta) \\ \frac{d}{dx}\left(\frac{x}{\theta}\right)^n = \frac{n \cdot x^{n-1}}{\theta^n}, & x \in (0, \theta) \end{cases}$$

$\theta \qquad n-1$

$$EX_{(n)} = \int_0^\theta x \cdot \frac{n \cdot x^{n-1}}{\theta^n} \cdot dx$$

$$= \frac{n}{\theta^n} \cdot \int_0^\theta \cdot x^n \, dx$$

$$= \frac{n}{\theta^n} \cdot \left[\frac{x^{n+1}}{n+1} \right] \Big|_0^\theta$$

$$= \frac{n}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1} = \left(\frac{n}{n+1} \right) \cdot \theta$$

Sabemos que $\{T_n\}$ é assintoticamente não viesado p/ θ, se.

$$\lim_{n \to \infty} E_\theta T_n(\underset{\sim}{X}) = \theta$$

OBS: O enunciado diz para mostrar que $T(\underset{\sim}{X}) = X_{(n)}$ é "asymptotically biased". Isso está errado, deveria dizer "asyphotically unbiased".

⑥ $* T(\underset{\sim}{X}) = c\, X_{(n)}, \quad c > 0$ (Há erro de impressão no enunciado)

$$* \; T_\theta(\underset{\sim}{X}) = \frac{(n+2)}{(n+1)} X_{(n)}$$

$$* \; MSE_\theta(T) = Var_\theta(T(\underset{\sim}{X})) + [bias(T, \theta)]^2.$$

$$MSE(cX_{(n)}, \theta) = c^2 \cdot Var(X_{(n)}) + [cE(X_{(n)}) - \theta]^2$$

$$E(X_{(n)}) = \left(\frac{n}{n+1} \right) \theta$$ (Calculado na questão 5)

$$E[X_{(n)}^2] = \int_0^\theta \frac{x^2 \cdot n \cdot x^{n-1}}{\theta^n} dx$$

$$= \int_0^\theta \frac{n\, x^{n+1}}{\theta^n} dx$$

$$= \int_0^{\theta} \frac{n\, x^{n+1}}{\theta^n}\, dx$$

$$= \frac{n}{\theta^n} \cdot \left[\frac{x^{n+2}}{(n+2)} \right] \Bigg|_0^{\theta}$$

$$= \frac{n \cdot \theta^{n+2}}{\theta^n \cdot (n+2)} = \frac{n}{n+2} \cdot \theta^2 .$$

$$\mathrm{Var}(X_{(n)}) = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2} \cdot \theta^2$$

$$= \left[\frac{n \cdot (n+1)^2 - n^2 \cdot (n+2)}{(n+2)(n+1)^2} \right] \theta^2 .$$

$$= \left[\frac{\cancel{n^3} + \cancel{2n^2} + n - \cancel{n^3} - \cancel{2n^2}}{(n+2)(n+1)^2} \right] \theta^2 .$$

$$= \left[\frac{n}{(n+2)(n+1)^2} \right] \cdot \theta^2 .$$

$$\mathrm{MSE}(c X_{(n)}, \theta) = c^2 \cdot \left[\frac{n}{(n+2)(n+1)^2} \right] \theta^2 + \left[c \frac{n}{n+1} - 1 \right]^2 \theta^2 .$$

Vamos escrever a função acima como uma função de c.

$$\mathrm{MSE}(c) = c^2 \cdot \left[\frac{n}{(n+2)(n+1)^2} \right] \theta^2 + \left[c \frac{n}{n+1} - 1 \right]^2 \theta^2 \quad \Leftrightarrow$$

$$\frac{\mathrm{MSE}(c)}{\theta^2} = c^2 \left[\frac{n}{(n+2)(n+1)^2} \right] + \left(c \cdot \frac{n}{n+1} - 1 \right)^2 .$$

$$= c^2 \left[\frac{n}{(n+2)(n+1)^2} \right] + \frac{c^2 \cdot n^2}{(n+1)^2} \; \frac{-2 \cdot c \cdot n}{(n+1)} + 1 .$$

$$\mathrm{MSE}^*(c) = \frac{\mathrm{MSE}(c)}{\theta^2}$$

$$\frac{d}{dc} \mathrm{MSE}^*(c) = 2c \cdot \left[\frac{n}{(n+2)(n+1)^2} \right] + \frac{n^2}{(n+1)^2} \cdot 2c \; \frac{-2n}{(n+1)}$$

Como provar que $T_\theta(X) = \frac{n+2}{n+1} X_{(n)}$ possui o menor MSE?

Problema 8.3

1

(a) * $X \sim B(\alpha, \beta)$

(i) α desconhecido, β conhecido

$$f(x_1, \dots, x_n) = \prod_{i=1}^{n} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x_i^{\alpha-1} (1-x_i)^{\beta-1}$$

$$= \underbrace{\frac{1}{[\Gamma(\beta)]^n} \frac{\prod(1-x_i)^{\beta}}{\prod x_i (1-x_i)}}_{h(x_1, x_2, \dots, x_n)} \cdot \underbrace{\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\right]^n \left(\prod x_i\right)^{\alpha}}_{g_\alpha(\prod x_i)}$$

$\therefore \prod x_i$ é estatística suficiente para α.

(ii) α conhecido, β desconhecido

$$f(x_1, \dots, x_n) = \frac{1}{[\Gamma(\alpha)]^n} \cdot \frac{\prod x_i^{\alpha}}{\prod x_i (1-x_i)} \cdot \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^n \left(\prod (1-x_i)\right)^{\beta}$$

$$\underbrace{\qquad\qquad}_{h(x_1,\dots,x_n)} \quad \underbrace{\qquad\qquad}_{g_\beta\left(\prod(1-x_i)\right)}$$

$\therefore \prod(1-x_i)$ é estatística suficiente para β.

(iii) α, β desconhecidos.

$\theta = (\alpha, \beta)$

$$f(x_1,\dots,x_n) = \underbrace{\prod \frac{1}{x_i(1-x_i)}}_{h(x_1,\dots,x_n)} \cdot \underbrace{\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right]^n \left(\prod x_i\right)^\alpha \left(\prod(1-x_i)\right)^\beta}_{g_\theta\left(\prod x_i, \prod(1-x_i)\right)}$$

$\therefore \prod x_i$ e $\prod(1-x_i)$ são estatísticas suficientes para α e β.

2

* $\underset{\sim}{X} = (X_1,\dots,X_n)$ amostra de $N(\alpha\sigma, \sigma^2)$, $\alpha \in \mathbb{R}$ conhecido
* $T(\underset{\sim}{X}) = \left(\sum X_i, \sum X_i^2\right)$

$\boxed{\text{I}}$ Demonstrar que $T(\underset{\sim}{X})$ é suficiente para σ

$$f_\sigma(x_1,\dots,x_n) = \frac{1}{(\sqrt{2\pi})^n} \cdot \frac{1}{\sigma^n} \cdot \exp\left(-\frac{\sum(x_i-\alpha\sigma)^2}{2\sigma^2}\right)$$

$$= \frac{1}{(\sqrt{2\pi})^n} \cdot \frac{1}{\sigma^n} \cdot \exp\left(\frac{-\sum x_i^2}{2\sigma^2} + \frac{\alpha\cancel{\sigma}\sum x_i}{\sigma^{\cancel{2}}} - \frac{n\cdot\alpha^2\cancel{\sigma^2}}{2\cancel{\sigma^2}}\right)$$

$$= \underbrace{\frac{1}{(\sqrt{2\pi})^n}}_{h(x_1,\dots,x_n)} \cdot \underbrace{\frac{1}{\sigma^n} \exp\left(\frac{-\sum x_i^2}{2\sigma^2} + \frac{\alpha\sum x_i}{\sigma} - \frac{n\alpha^2}{2}\right)}_{g_\sigma\left(\sum x_i, \sum x_i^2\right)}$$

$\therefore T(\underset{\sim}{X}) = \left(\underbrace{\sum X_i}_{T_1}, \underbrace{\sum X_i^2}_{T_2}\right)$ é estatística suficiente para σ

III Demonstrar que a família de distribuições $T(\underset{\sim}{X})$ é não completa.

$$E_\sigma \{2(\sum X_i)^2 - (n+1) \sum X_i^2 \} = 0, \forall \sigma$$

Vamos supor que $P(\{2T_1^2 - (n+1) T_2\} = 0) = 1$.

$$2T_1^2 - (n+1) T_2 = 0 \Rightarrow T_2 = \frac{2}{(n+1)} T_1^2$$

?

(3)

* $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$
* $\underset{\sim}{X} = (X_1, \ldots, X_n)$ clearly sufficient for $N(\mu, \sigma^2), \mu \in \mathbb{R}, \sigma > 0$.

(4)

* $X_1, X_2, \ldots, X_n$ sample from $U(\theta - 1/2, \theta + 1/2), \theta \in \mathbb{R}$.
* $T(X_1, \ldots, X_n) = (\min X_i, \max X_i)$

- $f_\theta(x) = \begin{cases} 1, & x \in (\theta - 1/2, \theta + 1/2) \\ 0, & \text{c.c.} \end{cases}$

$f_\theta(x_1, \ldots, x_n) = \mathbb{1}_A(x_1, \ldots, x_n)$, em que

$A = \left\{ (x_1, \ldots, x_n) : \theta - \frac{1}{2} \leq \min x_i \leq \max x_i \leq \theta + \frac{1}{2} \right\}$

$\Rightarrow$ Pelo Critério da Fatoração, $T(\underset{\sim}{X}) = (\min X_i, \max X_i)$ é suficiente.

- Note que $\max X_i - \min X_i$ é estatística ancilar para θ.

Pelo Teorema de Basu: se $T(\underset{\sim}{X})$ for estatística suficiente e completa e se $R(\underset{\sim}{X})$ for uma estatística ancilar, então $T(\underset{\sim}{X})$ e $R(\underset{\sim}{X})$ são independentes.

Caso $T(\underset{\sim}{X}) = (\min X_i, \max X_i)$ fosse completa, seria independente de $R(\underset{\sim}{X}) = \max X_i - \min X_i$. Contudo, isso é uma contradição pois $(\min X_i, \max X_i)$ determina completamente $\max X_i - \min X_i$.

12 * X_1, X_2 amostra de $P(\lambda)$

* $T(X_1, X_2) = X_1 + \alpha X_2$, $\alpha > 1$ inteiro

Sabe-se que $T = T(\underset{\sim}{X})$ é suficiente para θ se, e somente se, a distribuição condicional de X, dado $T = t$ não depende de θ.

$$P(X_1 = 0, X_2 = 1 \mid X_1 + \alpha X_2 = \alpha) = \frac{P(X_1 = 0, X_2 = 1)}{P(X_1 + \alpha X_2 = \alpha)}$$

$\boxed{\text{I}}$ Calcular numerador.

$$P(X_1 = 0, X_2 = 1) = e^{-\lambda} \cdot e^{-\lambda} \lambda$$
$$= \lambda \cdot e^{-2\lambda}$$

$\boxed{\text{II}}$ Calcular denominador

$$P(X_1 + \alpha X_2 = \alpha) = P(X_1 + \alpha X_2 = \alpha, X_2 = 1) + P(X_1 + \alpha X_2 = \alpha, X_2 = 0)$$

Lei da Probabilidade Total

$$= P(\{X_1 + \alpha X_2 = \alpha\} \mid X_2 = 1) \cdot P(X_2 = 1) + P(\{X_1 + \alpha X_2 = \alpha\} \mid X_2 = 0) \cdot P(X_2 = 0)$$

$$= P(X_1 = 0) P(X_2 = 1) + P(X_1 = \alpha) P(X_2 = 0)$$

$$= \lambda \cdot e^{-2\lambda} + \frac{e^{-\lambda} \cdot \lambda^{\alpha}}{\alpha!} \cdot e^{-\lambda}$$

$$= \lambda e^{-2\lambda} \left(1 + \frac{\lambda^{\alpha - 1}}{\alpha!} \right)$$

$\boxed{\text{III}}$ Cálculo final

$$P(X_1 = 0, X_2 = 1 \mid X_1 + \alpha X_2 = \alpha) = \left[1 + \frac{\lambda^{\alpha-1}}{\alpha!} \right]^{-1}$$

$\therefore$ $X_1 + \alpha X_2$ não é estatística suficiente para λ