

Exs. for An Introduction to Probability and Statistics by
Rohatgi and Saleh (Third Edition)

Problems 8.2

- ①
 - * T_n is a sequence of estimators for θ
 - * $E(T_n) \rightarrow \theta$ and $\text{Var}(T_n) \rightarrow 0$, $n \rightarrow \infty$
 - $\Rightarrow T_n$ is consistent for θ
 - * $T_n \xrightarrow{z} \theta$, se $\lim_{n \rightarrow \infty} E[\bar{L}(T_n - \theta)^2] = 0$

| \bar{L} | T_n consistent for θ and $|T_n - \theta| \leq A < \infty$

$$\begin{aligned} E[\bar{L}(T_n - \theta)^2] &= \dots = E[(T_n - E(T_n))^2] + [E(T_n - \theta)]^2 \\ &\quad \underbrace{\hspace{1cm}}_{\text{Variance}} \quad \underbrace{\hspace{1cm}}_{\text{Bias}}^2 \\ &= \text{Var}(T_n) + [E(T_n - \theta)]^2 \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} E[\bar{L}(T_n - \theta)^2] &= \lim_{n \rightarrow \infty} \text{Var}(T_n) + \lim_{n \rightarrow \infty} [E(T_n - \theta)]^2 \\ &\quad \text{Red arrow from } 0 \text{ to } 0 \\ &= \lim_{n \rightarrow \infty} [E(T_n) - E(\theta)]^2 \\ &= \lim_{n \rightarrow \infty} [E(T_n) - E(\theta)] \cdot \lim_{n \rightarrow \infty} [E(T_n) - E(\theta)] \\ &= 0 \end{aligned}$$

?

Resoluções do professor Raul:

I T_n consistent for θ and $|T_n - \theta| \leq A < \infty$

Se Y for v.a não negativa,

$$\mathbb{E} Y = \int_0^\infty y \cdot f_\theta(y) dy = \int_0^\infty P(y > \epsilon) d\epsilon$$

$$\mathbb{E}|T_n - \theta|^2 \xrightarrow{\text{z}} 0, n \rightarrow \infty$$

$$\begin{aligned} \mathbb{E}|T_n - \theta|^2 &= \int_0^\infty P(|T_n - \theta|^2 > \epsilon) d\epsilon \\ &= \int_0^\infty P(|T_n - \theta| > \sqrt{\epsilon}) d\epsilon \end{aligned}$$

$$\begin{aligned} \epsilon &= \epsilon^2 \\ d\epsilon &= 2\epsilon d\epsilon \end{aligned}$$

$$\begin{aligned} &= 2 \cdot \int_0^\infty \epsilon \cdot P(|T_n - \theta| > \epsilon) d\epsilon \\ &\leq 2 \cdot A \cdot \int_0^A P(|T_n - \theta| > \epsilon) d\epsilon \xrightarrow{\text{z}} 0, n \rightarrow \infty \end{aligned}$$

$$\therefore \mathbb{E}|T_n - \theta|^2 \xrightarrow{\text{z}} 0, n \rightarrow \infty \Rightarrow T_n \xrightarrow{\text{z}} \theta$$

II $|T_n - \theta| \leq A_n < \infty$

Para este caso, temos:

$$\mathbb{E}|T_n - \theta|^2 \leq 2A_n \int_0^{A_n} P(|T_n - \theta| > \epsilon) d\epsilon$$

Uma vez que não sabemos o valor, no limite, por causa do fato de A_n estar indexado em n , não há garantia que $T_n \xrightarrow{\text{z}} \theta$.

(2) * X_1, \dots, X_n A.A. de $TJ[\theta_1, \theta_2], \theta \in \mathbb{H} = (0, \infty)$

$$\star X_{(n)} = \max \{X_1, \dots, X_n\}$$

$\star X_{(n)} \xrightarrow{P} \theta$, se e somente se

$$P(|X_{(n)} - \theta| > \epsilon) \rightarrow 0, n \rightarrow \infty, \forall \epsilon > 0$$

e que é equivalente a.

$$P(|X_{(n)} - \theta| \leq \epsilon) \rightarrow 1, n \rightarrow \infty, \forall \epsilon > 0$$

- $P(|X_{(n)} - \theta| \leq \epsilon) = P(-\epsilon \leq X_{(n)} - \theta \leq \epsilon)$
 $= P(\theta - \epsilon \leq X_{(n)} \leq \epsilon + \theta)$
 $= F_{X_{(n)}}(\epsilon + \theta) - F_{X_{(n)}}(\theta - \epsilon)$

$$\begin{aligned}
 F_{X_{(n)}}(\infty) &= P(X_{(n)} \leq \infty) \\
 &= P(\max \{X_1, \dots, X_n\} \leq \infty) \\
 &= P(\{X_1 \leq \infty\} \cap \dots \cap \{X_n \leq \infty\}) \\
 &\stackrel{\text{ind}}{=} [P(X_1 \leq \infty)]^n \\
 &= \left(\frac{\infty}{\theta}\right)^n
 \end{aligned}$$

$$= 1 - \left(\frac{\theta - \epsilon}{\theta}\right)^n$$

$\rightarrow 1, n \rightarrow \infty \text{ e } 0 < \epsilon < \theta$.

$$\therefore X_{(n)} \xrightarrow{P} \theta$$

- $Y_n = z\bar{x}, \bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}$

$$E(Y_n) = E(z\bar{x}) = z \cdot E(\bar{x}) = \cancel{z} \cdot \cancel{n} \frac{\theta}{\cancel{z}} = \theta$$

$$\text{Var}(y_n) = \text{Var}(z\bar{x}) = 4 \cdot \text{Var}(\bar{x}) = 4 \cdot \frac{1}{m^2} m \cdot \frac{\sigma^2}{12} \\ = \frac{4 \cdot \sigma^2}{12 \cdot m} = \frac{\sigma^2}{3m}$$

$$\rightarrow 0, m \rightarrow \infty.$$

Pelo Teorema 2, uma vez que $E(y_n) = \theta$ e $\text{Var}(y_n) \rightarrow 0$,
 $n \rightarrow \infty$, y_n é consistente para θ .

(3)

- * X_1, X_2, \dots, X_m i.i.d V.A's
- * $E(X_i) = \mu, E(|X_i|^2) < \infty$
- * $T(X_1, \dots, X_m) = \frac{z}{m(m+1)} \cdot \sum_{i=1}^m i \cdot X_i$

$$E[T(X_1, \dots, X_m)] = \frac{z}{m(m+1)} \sum_{i=1}^m i \cdot E(X_i) \\ = \frac{z\mu}{m(m+1)} \sum_{i=1}^m i \\ = \frac{z\mu}{m(m+1)} \frac{(1+m) \cdot m}{2} \\ = \mu.$$

$$\text{Var}(T(X_1, \dots, X_m)) = \left[\frac{z}{m(m+1)} \right]^2 \cdot \text{Var}\left(\sum_{i=1}^m i \cdot X_i \right) \\ = \left[\frac{z}{m(m+1)} \right]^2 \sum_{i=1}^m i^2 \cdot \text{Var}(X_i) \\ = \left[\frac{z}{m(m+1)} \right]^2 \cdot \frac{\sigma^2}{X} \cdot \sum_{i=1}^m i^2 \\ = \left[\frac{z}{m(m+1)} \right]^2 \frac{\sigma^2}{X} \cdot \frac{m(m+1)(2m+1)}{6} \\ = \frac{4}{m(m+1)} \cdot \frac{(z m+1)}{6} \cdot \frac{\sigma^2}{X}$$

$$= \left(\frac{2m+1}{m^2+m} \right) \cdot \frac{2}{3} \cdot \sqrt{\frac{2}{x}}.$$

$$\rightarrow 0, \quad m \rightarrow \infty$$

Pelo Teorema 2, uma vez $E(T_m) \rightarrow \mu$ e $\text{Var}(T_m) \rightarrow 0, m \rightarrow \infty$,

T_m é consistente para μ .

(4)

* X_1, X_2, \dots, X_n amostra de $\bar{U}[0, \theta]$

$$* T_m = \left(\prod_{i=1}^n X_i \right)^{1/m}$$

$$* \psi(\theta) = \frac{\theta}{e}$$

$$Y_m = \ln(T_m) \quad (T_m = e^{Y_m})$$

$$= \frac{1}{m} \cdot \ln \left(\prod_{i=1}^n X_i \right)$$

$$= \frac{1}{m} \sum_{i=1}^n \ln(X_i)$$

$$E(\ln(X_i)) = \int_0^\theta \ln(x) \cdot \frac{1}{\theta} \cdot dx.$$

$$= \frac{1}{\theta} \int_0^\theta \ln(x) dx.$$

$$\int_0^\theta \ln(x) dx = \ln(x) \cdot x \Big|_0^\theta - \int_0^\theta \frac{1}{x} dx$$

$$\begin{aligned} & \left. \begin{aligned} u &= \ln(x) \\ du &= \frac{1}{x} dx \\ dv &= dx \end{aligned} \right\} = \ln(x) \cdot x \Big|_0^\theta - \int_0^\theta 1 dx \\ & = \ln(\theta) \cdot \theta - \theta \end{aligned}$$

$$= \frac{1}{\theta} \cdot [\ln(\theta) \cdot \theta - \theta]$$

$$= \ln(\theta) - 1$$

Pela Lei da Grandeza dos Grandes Números.

$$Y_n = \frac{\ln(x_1) + \dots + \ln(x_n)}{n} \xrightarrow{q.c} E(\ln(x_1)) = \ln(\theta) - 1$$

Pelo Teorema da Aplicação Contínua

$$Y_n \xrightarrow{q.c} \ln(\theta) - 1 \Rightarrow$$

$$T_n = e^{Y_n} \xrightarrow{q.c} \exp(\ln(\theta) - 1) = \theta \cdot e^{-1}$$

Uma vez que $T_n \xrightarrow{q.c} \psi(\theta) = \theta \cdot e^{-1}$, então $T_n \xrightarrow{P} \psi(\theta)$

∴ T_n é estimador consistente para $\psi(\theta) = \theta \cdot e^{-1}$

$$\textcircled{5} * T(\tilde{x}) = X_{(n)}$$

$$F_{X_{(n)}}(x) = \begin{cases} 0, & x < 0 \\ \left(\frac{x}{\theta}\right)^n, & 0 \leq x < \theta \\ 1, & x \geq \theta \end{cases}$$

⇒

$$f_{X_{(n)}}(x) = \begin{cases} 0, & x \notin (0, \theta) \\ \frac{d}{dx} \left(\frac{x}{\theta}\right)^n = \frac{n \cdot x^{n-1}}{\theta^n}, & x \in (0, \theta) \end{cases}$$

$$\begin{aligned}
 E X_{(n)} &= \int_0^\Theta x \cdot \frac{n \cdot x^{n-1}}{\theta^n} \cdot dx \\
 &= \frac{n}{\theta^n} \cdot \int_0^\Theta x^n dx \\
 &= \frac{n}{\theta^n} \left[\frac{x^{n+1}}{n+1} \right] \Big|_0^\Theta \\
 &= \frac{n}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1} = \left(\frac{n}{n+1} \right) \cdot \theta
 \end{aligned}$$

Sabemos que $\{\bar{T}_n\}$ é assintoticamente não viésado p/ θ , se.

$$\lim_{n \rightarrow \infty} E_\theta \bar{T}_n(\tilde{x}) = \theta$$

OBS: O enunciado diz para mostrar que $\bar{T}(\tilde{x}) = X_{(n)}$ é "asymptotically biased". Isso está errado, deveria dizer "asymptotically unbiased".

$$⑥ * T(\tilde{x}) = c X_{(n)}, \quad c > 0 \quad (\text{Há erro de impressão no enunciado})$$

$$* \bar{T}_\theta(\tilde{x}) = \frac{(n+2)}{(n+1)} X_{(n)}$$

$$* MSE_\theta(T) = \text{Var}_\theta(\bar{T}(\tilde{x})) + [\text{bias}(\bar{T}, \theta)]^2$$

$$MSE(c X_{(n)}, \theta) = c^2 \text{Var}(X_{(n)}) + [c E(X_{(n)}) - \theta]^2$$

$$E(X_{(n)}) = \left(\frac{n}{n+1} \right) \theta \quad (\text{Calculado na questão 5})$$

$$\begin{aligned}
 E[X_{(n)}^2] &= \int_0^\Theta \frac{x^2 \cdot n \cdot x^{n-1}}{\theta^n} dx \\
 &= \int_0^\Theta \frac{n x^{n+1}}{\theta^n} dx
 \end{aligned}$$

$$\begin{aligned}
&= \int_0^\theta \frac{n x^{n+1}}{\theta^n} dx \\
&= \frac{1}{\theta^n} \cdot \left[\frac{x^{n+2}}{(n+2)} \right] \Big|_0^\theta \\
&= \frac{n \cdot \theta^{n+2}}{\theta^n \cdot (n+2)} = \frac{n}{n+2} \cdot \theta^2
\end{aligned}$$

$$\begin{aligned}
\text{Var}(X_{(n)}) &= \frac{n}{n+2} \theta^2 - \frac{n^2}{(n+1)^2} \cdot \theta^2 \\
&= \left[\frac{n(n+1)^2 - n^2 \cdot (n+2)}{(n+2)(n+1)^2} \right] \theta^2 \\
&= \left[\frac{n^3 + n^2 - n^3 - 2n^2}{(n+2)(n+1)^2} \right] \theta^2 \\
&= \left[\frac{n}{(n+2)(n+1)^2} \right] \theta^2
\end{aligned}$$

$$\text{MSE}(c X_{(n)}, \theta) = c^2 \cdot \left[\frac{n}{(n+2)(n+1)^2} \right] \theta^2 + \left[c \frac{n}{n+1} - 1 \right]^2 \theta^2$$

Vamos resolver a função acima como uma função de c .

$$\text{MSE}(c) = c^2 \cdot \left[\frac{n}{(n+2)(n+1)^2} \right] \theta^2 + \left[c \frac{n}{n+1} - 1 \right]^2 \theta^2 \Leftrightarrow$$

$$\begin{aligned}
\frac{\text{MSE}(c)}{\theta^2} &= c^2 \left[\frac{n}{(n+2)(n+1)^2} \right] + \left(c \cdot \frac{n}{n+1} - 1 \right)^2 \\
&= c^2 \left[\frac{n}{(n+2)(n+1)^2} \right] + \frac{c^2 \cdot n^2}{(n+1)^2} \frac{-2 \cdot c \cdot n}{(n+1)} + 1
\end{aligned}$$

$$\text{MSE}^*(c) = \frac{\text{MSE}(c)}{\theta^2}$$

$$\frac{d}{dc} \text{MSE}^*(c) = 2c \cdot \left[\frac{n}{(n+2)(n+1)^2} \right] + \frac{n^2}{(n+1)^2} \cdot 2c - \frac{-2n}{(n+1)}$$

$$d \text{MSE}^*(c) = 0 \Rightarrow$$

$$\frac{d}{dc} \text{MSE}^*(c) = 0 \iff$$

$$2c \left[\frac{n}{(n+2)(n+1)^2} \right] + \frac{n^2}{(n+1)^2} \cdot 2c - \frac{2n}{(n+1)} = 0$$

\iff

$$n \cdot c + n^2 \cdot (n+2)c - n \cdot (n+1) \cdot (n+2) = 0$$

\iff

$$c + n \cdot (n+2)c - (n+1)(n+2) = 0$$

\iff

$$c \cdot [1 + n \cdot (n+2)] = (n+1)(n+2)$$

$$c [1 + n^2 + 2n] = (n+1)(n+2)$$

$$c [(n+1)(n+2)] = (n+1)(n+2)$$

$$c = \frac{n+2}{n+1}$$

$\therefore T_\theta(\tilde{x}) = \left(\frac{n+2}{n+1}\right) X_{(n)}$ na classe dos estimadores $T(\tilde{x}) = c X_{(n)}$ e

o de menor erro quadrático médio.

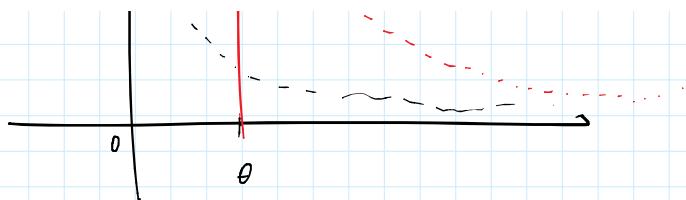
(7)

* X_1, \dots, X_n be i.i.d with PDF $f_\theta(x) = \exp\{-(x-\theta)\}$, $x > \theta$.

* $T(\tilde{x}) = X_{(1)} + b$, $b \in \mathbb{R}$.

* $T(\tilde{x}) = X_{(1)} - 1/n$ (possui menor EQM)





$$\mathbb{E}(X) = 1 + \theta$$

$$\text{Var}(X) = 1.$$

* Professor não calculou na mão. Apenas argumentou, de modo intuitivo, que era isso

$$F_X(x) = \begin{cases} 1 - e^{-(x-\theta)} & , x \geq \theta \\ 0 & , \text{c.c.} \end{cases}$$

$$P(X_{(1)} \leq x) = 1 - P(X_{(1)} \geq x)$$

$$= 1 - P(X_{(1)} \geq x, X_{(2)} \geq x, \dots, X_{(n)} \geq x)$$

$$= 1 - P(X_1 \geq x, X_2 \geq x, \dots, X_n \geq x)$$

$$= 1 - e^{-n(x-\theta)}$$

$$\mathbb{E}(X_{(1)}) = \frac{1}{n} + \theta \Rightarrow$$

$$\mathbb{E}\left[X_{(1)} - \frac{1}{n}\right] = \theta.$$

Assim, $\bar{T}(\bar{X}) = X_{(1)} - 1/n$ é estimador não viésado.

$$\text{MSE}(X_{(1)} + b, \theta) = \text{Var}(X_{(1)}) + [\mathbb{E}(X_{(1)} + b) - \theta]^2$$

$$= \frac{1}{n^2} + \left[\frac{1}{n} + \theta + b - \theta \right]^2$$

$$\begin{aligned}
 &= \frac{1}{n^2} + \left(\frac{1}{n} + b \right)^2 \\
 &= \frac{1}{n^2} + \frac{1}{n^2} + \frac{2b}{n} + b^2 \\
 &= \frac{2}{n^2} + \frac{2b}{n} + b^2
 \end{aligned}$$

$$\frac{\partial}{\partial b} \text{MSE}(X_{(1)} + b, \theta) = \frac{2}{n} + 2b$$

$$\frac{\partial}{\partial b} \text{MSE}(X_{(1)} + b, \theta) = 0 \quad \Leftrightarrow .$$

$$2b = -\frac{2}{n}$$

$$\Leftrightarrow \\ b = -1/n \cdot 1/2$$

$\therefore T_\theta(\underline{x}) = X_{(1)} - \frac{1}{n}$ na classe das estimadoras $T(\underline{x}) = X_{(1)} + b$ e
 o de menor erro quadrático médio.

Problems 8.3

(1)

$$(a) * X \sim B(\alpha, \beta)$$

(i) α desconhecido, β conhecido

$$\begin{aligned} f(x_1, \dots, x_n) &= \prod_{i=1}^n \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} \cdot x_i^{\alpha-1} (1-x_i)^{\beta-1} \\ &= \frac{1}{[\Gamma(\beta)]^n \prod x_i (1-x_i)} \cdot \left[\frac{\prod (\alpha + \beta)}{\prod \alpha} \right]^n (\prod x_i)^{\alpha} \\ &\quad \underbrace{h(x_1, x_2, \dots, x_n)}_{\text{ }} \quad \underbrace{g_{\alpha}(\prod x_i)}_{\text{ }}. \end{aligned}$$

$\therefore \prod x_i$ é estatística suficiente para α .

(ii) α conhecido, β desconhecido

$$\begin{aligned} f(x_1, \dots, x_n) &= \frac{1}{[\Gamma(\alpha)]^n} \cdot \frac{\prod x_i^{\alpha}}{\prod x_i (1-x_i)} \cdot \left[\frac{\prod (\alpha + \beta)}{\prod \beta} \right]^n (\prod (1-x_i))^{\beta} \\ &\quad \underbrace{h(x_1, \dots, x_n)}_{\text{ }} \quad \underbrace{g_{\beta}(\prod (1-x_i))}_{\text{ }}. \end{aligned}$$

$\therefore \prod (1-x_i)$ é estatística suficiente para β .

(iii) α, β desconhecidos.

$$\Theta = (\alpha, \beta)$$

$$f(x_1, \dots, x_n) = \prod \frac{1}{x_i (1-x_i)} \cdot \left[\frac{\prod (\alpha + \beta)}{\prod \alpha \prod \beta} \right]^n (\prod x_i)^{\alpha} (\prod (1-x_i))^{\beta}$$

$$h(x_1, \dots, x_n) \quad g_{\theta}(\pi_{x_i}, \pi_{(1-x_i)})$$

$\therefore \pi_{x_i}$ e $\pi_{(1-x_i)}$ são estatísticas suficientes para α e β .

(2) * $X = (X_1, \dots, X_n)$ amostra de $N(\alpha\sigma, \sigma^2)$, $\alpha \in \mathbb{R}$ conhecido

$$* T(X) = (\sum x_i, \sum x_i^2)$$

I \underline{I} Demonstrar que $T(X)$ é suficiente para σ

$$\begin{aligned} f_{\sigma}(x_1, \dots, x_n) &= \frac{1}{(\sqrt{2\pi})^n} \cdot \frac{1}{\sigma^n} \cdot \exp\left(-\frac{\sum (x_i - \alpha\sigma)^2}{2\sigma^2}\right) \\ &= \frac{1}{(\sqrt{2\pi})^n} \cdot \frac{1}{\sigma^n} \cdot \exp\left(-\frac{\sum x_i^2}{2\sigma^2} + \cancel{\alpha} \frac{\sum x_i}{\sigma} - \frac{n \cdot \alpha^2 \cancel{\sigma^2}}{2 \cancel{\sigma^2}}\right) \\ &= \underbrace{\frac{1}{(\sqrt{2\pi})^n} \cdot \frac{1}{\sigma^n} \cdot \exp\left(-\frac{\sum x_i^2}{2\sigma^2}\right)}_{h(x_1, \dots, x_n)} \underbrace{\cancel{\alpha} \frac{\sum x_i}{\sigma} - \frac{n \cdot \alpha^2 \cancel{\sigma^2}}{2 \cancel{\sigma^2}}}_{g_{\sigma}(\sum x_i, \sum x_i^2)} \end{aligned}$$

$\therefore T(X) = (\sum x_i, \sum x_i^2)$ é estatística suficiente para σ

I \underline{II} Demonstrar que a família de distribuições $T(X)$ é não completa.

$$(i) \bar{X} = \frac{\sum_{i=1}^n x_i}{n} \longrightarrow E(X_i) = \alpha\sigma$$

$$(ii) S^2 = \frac{\sum_{i=1}^{n-1} (x_i - \bar{X})^2}{n-1} \longrightarrow \sigma^2$$

O professor argumenta que a propriedade de completude está diretamente associada com a existência de ceras

um estimador possível para estimar o parâmetro. Uma vez que há mais de um estimador possível (i) e (ii), a família de distribuições $T(X)$ é não completa.

O professor utiliza a definição de completude, com um caso que refuta a completude, mas não entendi.

$$(3) * X_1, \dots, X_n \sim N(\mu, \sigma^2)$$

* $\mathbf{X} = (X_1, \dots, X_n)$ clearly sufficient for $N(\mu, \sigma^2)$, $\mu \in \mathbb{R}, \sigma > 0$.

$$(4) * X_1, X_2, \dots, X_m \text{ sample from } T\left(\theta - \frac{1}{2}, \theta + \frac{1}{2}\right), \theta \in \mathbb{R}.$$

$$* T(X_1, \dots, X_m) = (\min X_i, \max X_i)$$

$$\bullet f_{\theta}(x) = \begin{cases} 1, & x \in \left(\theta - \frac{1}{2}, \theta + \frac{1}{2}\right) \\ 0, & \text{c.c.} \end{cases}$$

$$f_{\theta}(x_1, \dots, x_m) = I_A(x_1, \dots, x_m), \text{ em que}$$

$$A = \left\{ (x_1, \dots, x_m) : \theta - \frac{1}{2} \leq \min x_i \leq \max x_i \leq \theta + \frac{1}{2} \right\}$$

\Rightarrow Pelo critério da Fatou, $T(\mathbf{X}) = (\min X_i, \max X_i)$ é suficiente.

- Note que $\max X_i - \min X_i$ é estatística auxiliar para θ .

Pelo Teorema de Basu: se $T(\tilde{X})$ for estatística suficiente e completa e se $R(\tilde{X})$ for uma estatística auxiliar, então $T(\tilde{X})$ e $R(\tilde{X})$ são independentes.

Caso $T(\tilde{X}) = (\min X_i, \max X_i)$ fosse completa, seria independente de $R(\tilde{X}) = \max X_i - \min X_i$. Contudo, isso é uma contradição pois $(\min X_i, \max X_i)$ determina $\max X_i - \min X_i$.

$$(5) * T = g(V) \text{ e } T \text{ é suficiente}$$

$$\begin{aligned} f_{\theta}(x_1, \dots, x_n) &= g_{\theta}(T) h(x) \\ &= (g_{\theta} \circ g(u)) h(x) \\ &= m_{\theta}(u) \cdot h(x) \end{aligned}$$

$\therefore T$ é suficiente.

12

* X_1, X_2 amostra de $P(\lambda)$

* $T(X_1, X_2) = X_1 + \alpha X_2$, $\alpha > 1$ inteiro

Sabe-se que $T = T(X)$ é suficiente para θ se, e somente se, a distribuição condicional de X , dado $T=t$ não depende de θ .

$$P(X_1=0, X_2=1 \mid X_1 + \alpha X_2 = \alpha) = \frac{P(X_1=0, X_2=1)}{P(X_1 + \alpha X_2 = \alpha)}$$

I Calcular numerador.

$$\begin{aligned} P(X_1=0, X_2=1) &= e^{-\lambda} \cdot e^{-\lambda} \lambda \\ &= \lambda \cdot e^{-2\lambda} \end{aligned}$$

II Calcular denominador

$$\begin{aligned} P(X_1 + \alpha X_2 = \alpha) &= P(X_1 + \alpha X_2 = \alpha, X_2=1) + P(X_1 + \alpha X_2 = \alpha, X_2=0) \\ &= P(\{X_1 + \alpha X_2 = \alpha\} \mid X_2=1) \cdot P(X_2=1) + \\ &\quad P(\{X_1 + \alpha X_2 = \alpha\} \mid X_2=0) \cdot P(X_2=0) \end{aligned}$$

$$\begin{aligned}
 &= P(X_1=0) \cdot P(X_2=1) + P(X_1=1) \cdot P(X_2=0) \\
 &= \lambda \cdot e^{-\lambda} + \frac{\lambda^1 \cdot 1^\alpha}{1!} \cdot e^{-\lambda} \\
 &= \lambda e^{-\lambda} \left(1 + \frac{\lambda^{\alpha-1}}{\alpha!} \right)
 \end{aligned}$$

III) Cálculo final

$$P(X_1=0, X_2=1 \mid X_1+\alpha X_2=\alpha) = \left[1 + \frac{\lambda^{\alpha-1}}{\alpha!} \right]^{-1}$$

$\therefore X_1 + \alpha X_2$ não é estatística suficiente para λ