Exs. for An Introduction to Probability and Statistics by Rohatgi and Saleh (Third Edition)

Problems 8.2

(1) * T_n is a sequence of estimators for θ

* $E(T_n) \to \theta$ and $Var(T_n) \to 0$, $n \to \infty$

$\Rightarrow T_n$ is consistent for θ

* $T_n \xrightarrow{2} \theta$, se $\lim_{n\to\infty} E[(T_n-\theta)^2] = 0$

$\boxed{I}$ T_n consistent for θ and $|T_n-\theta| \le A < \infty$

$$E[(T_n-\theta)^2] = \ldots = \underbrace{E[(T_n-E(T_n))^2]}_{Variance} + \underbrace{[E(T_n-\theta)]^2}_{Bias^2}$$

$$= Var(T_n) + [E(T_n-\theta)]^2$$

$$\lim_{n\to\infty} E[(T_n-\theta)^2] = \lim_{n\to\infty} Var(T_n) + \lim_{n\to\infty} [E(T_n-\theta)]^2$$

0

$$= \lim_{n\to\infty} [E(T_n) - E(\theta)]^2.$$

$$= \lim_{n\to\infty} [E(T_n) - E(\theta)] \cdot \lim_{n\to\infty} [E(T_n) - E(\theta)]$$

0 0

$= 0$

$\therefore T_n \xrightarrow{2} \theta$

$\boxed{II}$ $|T_n - \theta| \le A_n \le \infty$

?

(2) * $X_1, \dots, X_n$ A.A. de $U[0, \theta]$, $\theta \in \Theta = (0, \infty)$

* $X_{(n)} = \max\{X_1, \dots, X_n\}$

* $X_{(n)} \xrightarrow{P} \theta$, se e somente se

$$P(|X_{(n)} - \theta| > \varepsilon) \to 0, \ n \to \infty, \forall \varepsilon > 0$$

o que é equivalente a.

$$P(|X_{(n)} - \theta| \leq \varepsilon) \to 1, \ n \to \infty, \forall \varepsilon > 0$$

- $$P(|X_{(n)} - \theta| \leq \varepsilon) = P(-\varepsilon \leq X_{(n)} - \theta \leq \varepsilon)$$

$$= P(\theta - \varepsilon \leq X_{(n)} \leq \varepsilon + \theta)$$

$$= F_{X_{(n)}}(\varepsilon + \theta) - F_{X_{(n)}}(\theta - \varepsilon)$$

$$F_{X_{(n)}}(x) = P(X_{(n)} \leq x)$$

$$= P(\max\{X_1, \dots, X_n\} \leq x)$$

$$= P(\{X_1 \leq x\} \cap \dots \cap \{X_n \leq x\})$$

$$\overset{ind}{=} [P(X_1 \leq x)]^n$$

$$= \left(\frac{x}{\theta}\right)^n$$

$$= 1 - \left(\frac{\theta - \varepsilon}{\theta}\right)^n.$$

$$\to 1, \ n \to \infty \text{ e } 0 < \varepsilon < \theta.$$

$\therefore X_{(n)} \xrightarrow{P} \theta$

- $Y_n = 2\bar{X}$, $\bar{X} = \dfrac{X_1 + X_2 + \dots + X_n}{n}$

$$E(Y_n) = E(2\overline{X}) = 2 \cdot E(\overline{X}) = \frac{\cancel{2}}{\cancel{n}} \cdot \cancel{n} \frac{\theta}{\cancel{2}} = \theta$$

$$Var(Y_n) = Var(2\overline{X}) = 4 \cdot Var(\overline{X}) \overset{ind}{=} 4 \cdot \frac{1}{n^2} \, n \cdot \frac{\theta^2}{12}.$$

$$= \frac{4 \cdot \theta^2}{12 \cdot n} = \frac{\theta^2}{3n}$$

$$\to 0 \, , \, n \to \infty .$$

Pelo Teorema 2, uma vez que $E(Y_n) = \theta$ e $Var(Y_n) \to 0$, $n \to \infty$, Y_n é consistente para θ.

(3)

* $X_1, X_2, \dots, X_n$ i.i.d V.A's

* $E(X_i) = \mu$, $E(|X_i|^2) < \infty$

* $T(X_1, \dots, X_n) = \frac{2}{n(n+1)} \cdot \sum_{i=1}^{n} i \, X_i$

$$E[T(X_1, \dots X_n)] = \frac{2}{n(n+1)} \sum_{i=1}^{n} i \cdot E(X_i)$$

$$= \frac{2\mu}{n(n+1)} \sum_{i=1}^{n} \cdot i$$

$$= \frac{\cancel{2}\mu}{\cancel{n(n+1)}} \, \frac{\cancel{(1+n)} \cdot \cancel{n}}{\cancel{2}}$$

$$= \mu .$$

$$Var(T(X_1, \dots, X_n)) = \left[\frac{2}{n(n+1)}\right]^2 \cdot Var\left(\sum_{i=1}^{n} i \, X_i\right)$$

$$\overset{ind}{=} \left[\frac{2}{n(n+1)}\right]^2 \sum_{i=1}^{n} \cdot i^2 \cdot Var(X_i)$$

$$= \left[\frac{2}{n(n+1)}\right]^2 \cdot \sigma_X^2 \cdot \sum_{i=1}^{n} i^2$$

$$= \left[\frac{2}{n(n+1)}\right]^2 \sigma_X^2 \cdot \frac{n \cdot (n+1) \cdot (2n+1)}{6}$$

$$= \frac{4}{n \cdot (n+1)} \cdot \frac{(2n+1)}{6} \cdot \sigma_X^2$$

$$= \left(\frac{2n+1}{n^2+n}\right) \cdot \frac{2}{3} \cdot \sigma_X^2 .$$

$$\longrightarrow 0, \quad n \to \infty$$

Pelo Teorema 2, uma vez $E(T_n) \to \mu$ e $Var(T_n) \to 0, n \to \infty$,

T_n é consistente para μ.

(4)

* $X_1, X_2, \ldots, X_n$ amostra de $U[0, \theta]$
* $T_n = \left(\prod_{i=1}^{n} X_i\right)^{1/n}$
* $\psi(\theta) = \frac{\theta}{e}$

$$Y_n = \ln(T_n) \qquad (T_n = e^{Y_n})$$

$$= \frac{1}{n} \cdot \ln\left(\prod_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \ln(X_i)$$

$$E(\ln(X_i)) = \int_0^\theta \ln(x) \cdot \frac{1}{\theta} \cdot dx .$$

$$= \frac{1}{\theta} \int_0^\theta \ln(x)\, dx .$$

$$\int_0^\theta \ln(x)\, dx = \ln(x) \cdot x \Big|_0^\theta - \int_0^\theta x \cdot \frac{1}{x}\, dx$$

$u = \ln(x)$

$du = \frac{1}{x} dx$

$dv = dx$

$$= \ln(x) \cdot x \Big|_0^\theta - \int_0^\theta 1\, dx$$

$$= \ln(\theta) \cdot \theta - \theta$$

$$= \frac{1}{\theta} \cdot \left[\ln(\theta) \cdot \theta - \theta \right]$$

$$= \ln(\theta) - 1$$

Pela Lei Forte dos Grandes Números.

$$Y_n = \frac{\ln(X_1) + \dots + \ln(X_n)}{n} \xrightarrow{q.c} E(\ln(X_1)) = \ln(\theta) - 1$$

Pelo Teorema da Aplicação Contínua

$$Y_n \xrightarrow{q.c} \ln(\theta) - 1 \Rightarrow$$

$$T_n = e^{Y_n} \xrightarrow{q.c} \exp(\ln(\theta) - 1) = \theta \cdot e^{-1}$$

Uma vez que $T_n \xrightarrow{q.c} \psi(\theta) = \theta \cdot e^{-1}$, então $T_n \xrightarrow{P} \psi(\theta)$

$\therefore$ T_n é estimador consistente para $\psi(\theta) = \theta \cdot e^{-1}$

Problemas 8.3

1

(a) $* X \sim B(\alpha, \beta)$

(i) α desconhecido, β conhecido

$$f(x_1, \dots, x_n) = \prod_{i=1}^{n} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x_i^{\alpha-1} (1-x_i)^{\beta-1}$$

$$= \frac{1}{[\Gamma(\beta)]^n} \frac{\prod (1-x_i)^{\beta}}{\prod x_i (1-x_i)} \cdot \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \right]^n \left(\prod x_i \right)^{\alpha}$$

$$\underbrace{\qquad\qquad}_{h(x_1, x_2, \dots, x_n)} \quad \underbrace{\qquad\qquad}_{g_\alpha(\prod x_i)}.$$

$\therefore \prod x_i$ é estatística suficiente para α.

(ii) α conhecido, β desconhecido

$$f(x_1, \dots, x_n) = \underbrace{\frac{1}{[\Gamma(\alpha)]^n} \cdot \frac{\prod x_i^{\alpha}}{\prod x_i(1-x_i)}}_{h(x_1, \dots, x_n)} \cdot \underbrace{\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^n \left(\prod(1-x_i)\right)^\beta}_{g_\beta(\prod(1-x_i))}$$

$\therefore \prod(1-x_i)$ é estatística suficiente para β.

(iii) α, β desconhecidos.

$\theta = (\alpha, \beta)$

$$f(x_1, \dots, x_n) = \underbrace{\prod \frac{1}{x_i(1-x_i)}}_{h(x_1, \dots, x_n)} \cdot \underbrace{\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right]^n \left(\prod x_i\right)^\alpha \left(\prod(1-x_i)\right)^\beta}_{g_\theta(\prod x_i, \prod(1-x_i))}$$

$\therefore \prod x_i$ e $\prod(1-x_i)$ são estatísticas suficientes para α e β.

2

* $\underset{\sim}{X} = (X_1, \dots, X_n)$ amostra de $N(\alpha\sigma, \sigma^2)$, $\alpha \in \mathbb{R}$ conhecido

* $T(\underset{\sim}{X}) = (\sum X_i, \sum X_i^2)$

$\boxed{\text{I}}$ Demonstrar que $T(\underset{\sim}{X})$ é suficiente para σ

$$f_\sigma(x_1, \dots, x_n) = \frac{1}{(\sqrt{2\pi})^n} \cdot \frac{1}{\sigma^n} \cdot \exp\left(-\frac{\sum(x_i - \alpha\sigma)^2}{2\sigma^2}\right)$$

$$= \frac{1}{(\sqrt{2\pi})^n} \cdot \frac{1}{\sigma^n} \cdot \exp\left(\frac{-\sum x_i^2}{2\sigma^2} + \frac{\alpha\cancel{\sigma}\sum x_i}{\sigma^{\cancel{2}}} - \frac{n\cdot\alpha^2\cancel{\sigma^2}}{2\cdot\cancel{\sigma^2}}\right)$$

$$= \frac{1}{(\sqrt{2\pi})^n} \cdot \frac{1}{\sigma^n} \cdot \exp\left(\frac{-\sum x_i^2}{2\sigma^2} + \frac{\alpha \cancel{\sigma} \sum x_i}{\sigma^{\cancel{2}}} - \frac{n \cdot \alpha^2 \cancel{\sigma^2}}{2 \cancel{\sigma^2}} \right)$$

$$= \underbrace{\frac{1}{(\sqrt{2\pi})^n}}_{h(x_1,\ldots,x_n)} \cdot \underbrace{\frac{1}{\sigma^n} \exp\left(\frac{-\sum x_i^2}{2\sigma^2} + \frac{\alpha \sum x_i}{\sigma} - \frac{n\alpha^2}{2} \right)}_{g_\sigma\left(\sum x_i, \sum x_i^2\right)}$$

$\therefore\ T(\underset{\sim}{X}) = (\underbrace{\sum X_i}_{T_1}, \underbrace{\sum X_i^2}_{T_2})$ é estatística suficiente para σ

$\boxed{\text{II}}$ Demonstrar que a família de distribuições $T(\underset{\sim}{X})$ é não completa.

$$E_\sigma \left\{ 2\left(\sum X_i\right)^2 - (n+1) \sum X_i^2 \right\} = 0, \forall \sigma$$

Vamos supor que $P(\{2T_1^2 - (n+1) T_2\} = 0) = 1$.

$2T_1^2 - (n+1)T_2 = 0 \Rightarrow T_2 = \frac{2}{(n+1)} T_1^2$

?

(12) * X_1, X_2 amostra de $P(\lambda)$

* $T(X_1, X_2) = X_1 + \alpha X_2$, $\alpha > 1$ inteiro

Sabe-se que $T = T(\underset{\sim}{X})$ é suficiente para θ se, e somente se, a distribuição condicional de X, dado $T = t$ não depende de θ.

$$P(X_1 = 0, X_2 = 1 \mid X_1 + \alpha X_2 = \alpha) = \frac{P(X_1 = 0, X_2 = 1)}{P(X_1 + \alpha X_2 = \alpha)}$$

$\boxed{I}$ Calcular numerador.

$$P(X_1 = 0, X_2 = 1) = e^{-\lambda} \cdot e^{-\lambda} \lambda$$
$$= \lambda \cdot e^{-2\lambda}$$

$\boxed{II}$ Calcular denominador

$$P(X_1 + \alpha X_2 = \alpha) = P(X_1 + \alpha X_2 = \alpha, X_2 = 1) + P(X_1 + \alpha X_2 = \alpha, X_2 = 0)$$
$$= P(\{X_1 + \alpha X_2 = \alpha\} \mid X_2 = 1) \cdot P(X_2 = 1) +$$
$$P(\{X_1 + \alpha X_2 = \alpha\} \mid X_2 = 0) \cdot P(X_2 = 0)$$
$$= P(X_1 = 0) \cdot P(X_2 = 1) + P(X_1 = \alpha) \cdot P(X_2 = 0)$$
$$= \lambda \cdot e^{-2\lambda} + \frac{e^{-\lambda} \cdot \lambda^{\alpha}}{\alpha!} \cdot e^{-\lambda}$$
$$= \lambda e^{-2\lambda} \left(1 + \frac{\lambda^{\alpha - 1}}{\alpha!} \right)$$

$\boxed{III}$ Cálculo final

$$P(X_1 = 0, X_2 = 1 \mid X_1 + \alpha X_2 = \alpha) = \left[1 + \frac{\lambda^{\alpha-1}}{\alpha!} \right]^{-1}$$

$\therefore$ $X_1 + \alpha X_2$ não é estatística suficiente para λ