Core Signatures and Inversions

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Abstract

The signature method allows transformations of the original data (either as a mathematical function or as irregularly sampled data) into a feature space with several interesting properties, allowing for models and controls to be developed into this feature space. Because the full signature (even truncated) is still redundant, it is usual to calculate the log signature, which has a lower size. In this paper we introduce core signatures, which are subsets of the original signatures with the same size as log signatures and a faster and easier calculation algorithm (systematic shuffle product leading to a sparse linear system). Because the core signatures are subsets of the original signatures, we can use Chen's Identity to invert a given core signature in order to easily find two (or three, where appropriate) line segments that, when concatenated, produce the original core signature, improving interpretability. We discuss briefly the use of time-ordered paths for term structure modeling in finance (to be discussed in depth in a subsequent paper). Mathematica code is available on GitHub at https://github.com/MarcosCarreira/CoreSignatures.

Keywords: Signatures; log signatures; inversion of signatures; term structure modeling

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1 Introduction

"E com cinco ou seis retas é fácil fazer um castelo"

Excerpt from the song Aquarela, by Antonio Pecci Filho (Toquinho), 1983

Signatures are one of the best ways to understand the behavior of paths.

What are paths? Mathematically, we can define paths as functions that map an interval (generally defined as $[a, b]$, which will later be reparametrized to $[0, 1]$ without problems) to the multidimensional space $\mathbb{R}^d$. Going forward we will assume that the paths are of bounded variation and piecewise differentiable. For a more complete definition and mathematical subtleties, the standard reference is [Lyons et al (2007)].

This signature representation is useful to analyze term structures because we compare the shapes of different curves at different points in time and scale, and some behaviors like inversions, humps, steepenings and others are quite common among different asset classes and have similar reasons. In feature space it is easier to compare the US Treasury yields (up to 30 years ahead) with Brazil's DIxPré Interest Rate Curve implied from futures (up to 10 years, but 20 years ago liquid up to 2 years).

We would like to share two practical results:

In the first result we show how to substitute log signatures (a more compressed representation of the signatures, but harder to calculate) with core signatures, which show the same compression; but a core signature is a subset of the signature, so it is easier to calculate using the shuffle product; the reconstruction of the full signature (now unnecessary) is also easier.

The second result is an inversion algorithm using Chen's Identity to find which concatenation of two (or, when necessary, three) signatures of line segments can generate the same core signature (up to a truncation level). We can either solve a more complex system involving the concatenation of a larger number of line segments or we apply the same (2/3) breakdown to find the indices of the breaking points (on time-indexed paths) and recalculate the estimated values given these indices.

This work relied heavily on: (i) [Chevryev and Kormilitzin (2016)], an excellent primer on signatures; we will provide the Mathematica code to some of the examples of this paper; (ii) [Fermanian (2021a)], a great and comprehensive PhD Thesis that collects several of Adeline Fermanian's papers on signatures, applications and inversion algorithms (we will quote the individual papers like [Fermanian (2021b)] and [Fermanian et al (2023)] when possible/applicable); (iii)[Reizenstein (2018)], [Reizenstein and Graham (2020)], [Reizenstein (2016)] and [Reizenstein (2015)], papers and documents that support Jeremy Reizenstein's work with the iisignature Python package.

Two papers ([Améndola, Friz and Sturmfels (2019)] and [Pfeffer, Seigal and Sturmfels (2018)]) discuss similar goals; [Améndola, Friz and Sturmfels (2019)] is focused on polynomial and piecewise linear paths, with the degree of prime ideals corresponding to the size of log signatures and core signatures; [Pfeffer, Seigal and Sturmfels (2018)] is focused on what can be learned from the third signature assuming polynomial paths, piecewise linear paths and a dictionary of paths. Although our results were derived in the same mathematical detail and completeness as those two papers, we think they are practical enough to be useful for signatures coming from more general paths, as we hope to show. Further work linking our results to these papers is probably worth pursuing.

Section 2 (Signatures) will describe signatures and provide plenty of examples; the idea is for the reader to become familiar with the structure of the signatures.

Section 3 (Core Signatures) will describe how to find which terms of the signature are independent (and therefore store the information about the path) and how the other terms depend on these core terms.

Section 4 will describe an inversion procedure where we approximate a path with linear segments which possess the same signature at increasing levels.

Section 5 will discuss an application of signatures - term structure modeling in finance.

2 Signatures

2.1 Definitions

2.1.1 Paths

Let's start by following [Chevryev and Kormilitzin (2016)] and showing the Mathematica code that can produce the plots and results.

Figure 1 shows two 2d smooth paths, defined by:

$$X_t = \left\{X_t^1, X_t^2\right\} = \left\{t, t^3\right\} \ , \ t \in [-2, +2] \tag{1}$$

And:

$$X_t = \left\{X_t^1, X_t^2\right\} = \left\{\cos(t), \sin(t)\right\} \ , \ t \in [0, +2\pi] \tag{2}$$

We can reproduce this with ParametricPlot (Figure 1):

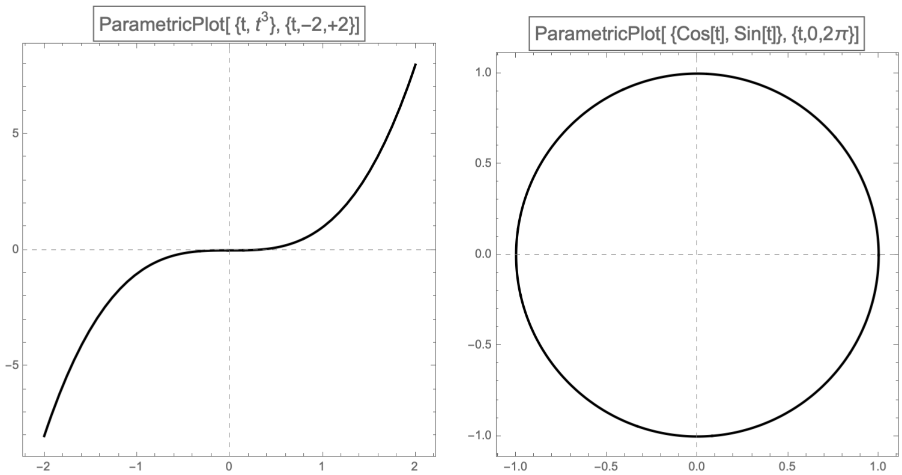

Figure 1: 2d smooth paths using ParametricPlot

And we can look at piecewise functions as well:

$$X_t = \left\{X_t^1, X_t^2\right\} = \begin{cases} \{t, t^2\} & t \in [-2, 0] \\ \{t, t^3\} & t \in (0, +2] \end{cases} \tag{3}$$

And piecewise linear functions (which will be used a lot); both are shown in Figure 2:

$$X_t = \{X_t^1, X_t^2\} = \begin{cases} \{t, 3+2t\} & t \in [-2,-1] \\ \{t, -t\} & t \in (-1,0] \\ \{t, 3t\} & t \in (0,+1] \\ \{t, 5-2t\} & t \in (+1,+2] \end{cases} \tag{4}$$

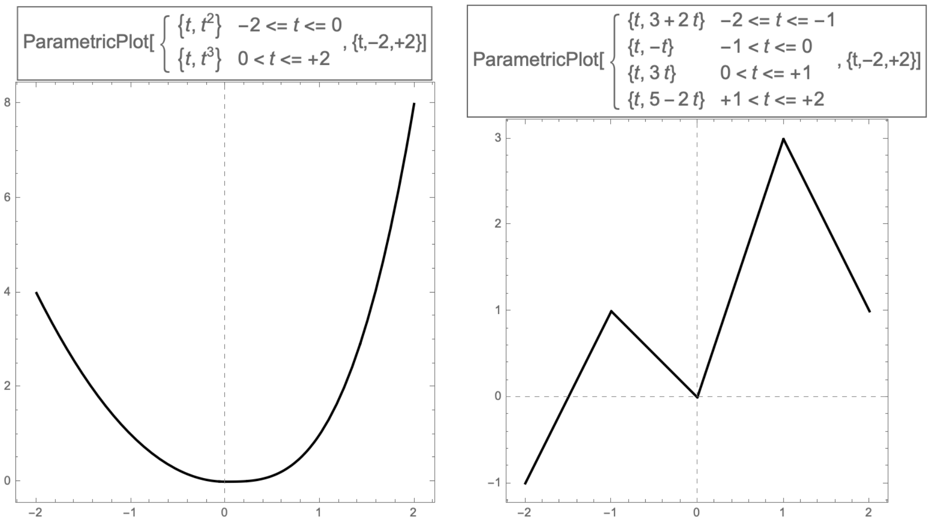

Figure 2: 2d piecewise smooth paths using ParametricPlot

2.1.2 Iterated Integrals

Following again [Chevryev and Kormilitzin (2016)], looking at the Example 3 in 1.1.2 (Path integrals):

$$X_t = \{X_t^1, X_t^2\} = \{t^2, t^3\} \ \ , \ t \in [0,+1]$$

We can also calculate the iterated integral:

$$\int_0^1 X_t^1 \mathrm{d}X_t^2 = \int_0^1 t^2 3t^2 \mathrm{d}t = \frac{3}{5}$$

As:

$$\text{LineIntegrate}\left[t^2, \{x\} \in \text{ParametricRegion}\left[\left\{t^3\right\}, \{\{t,0,1\}\}\right]\right] = \frac{3}{5}$$

We will automate the iterated integrals with:

```
iterinteg[functions_, limits_, assumptions_]:= Fold[
  Integrate[#1 * #2[[1]], #2[[2]],
  Assumptions->assumptions]&, 1, Transpose[{functions, limits
     }]];
```

With the signature defined as an infinite collection of iterated integrals:

$$
\begin{aligned}
S(X)_{a,b} = \{1, & \\
& S(X)^{1}_{a,b}, \ldots, S(X)^{d}_{a,b}, \\
& S(X)^{1,1}_{a,b}, S(X)^{1,2}_{a,b}, \ldots, S(X)^{d,d}_{a,b}, \\
& \ldots\}
\end{aligned} \tag{5}
$$

With each term $S(X)^{i_1,\ldots,i_k}_{a,b}$ is defined by the iterated integrals:

$$
S(X)^{i_1,\ldots,i_k}_{a,t} = \int_{\substack{a \\ t_k}}^{t} \ldots \int_{\substack{a \\ t_1}}^{t_2} \mathrm{d}X^{i_1}_{t_1} \ldots \mathrm{d}X^{i_k}_{t_k} \tag{6}
$$

It might be useful to view them grouped by level and folded as tensors, as shown in [Reizenstein (2016)]:

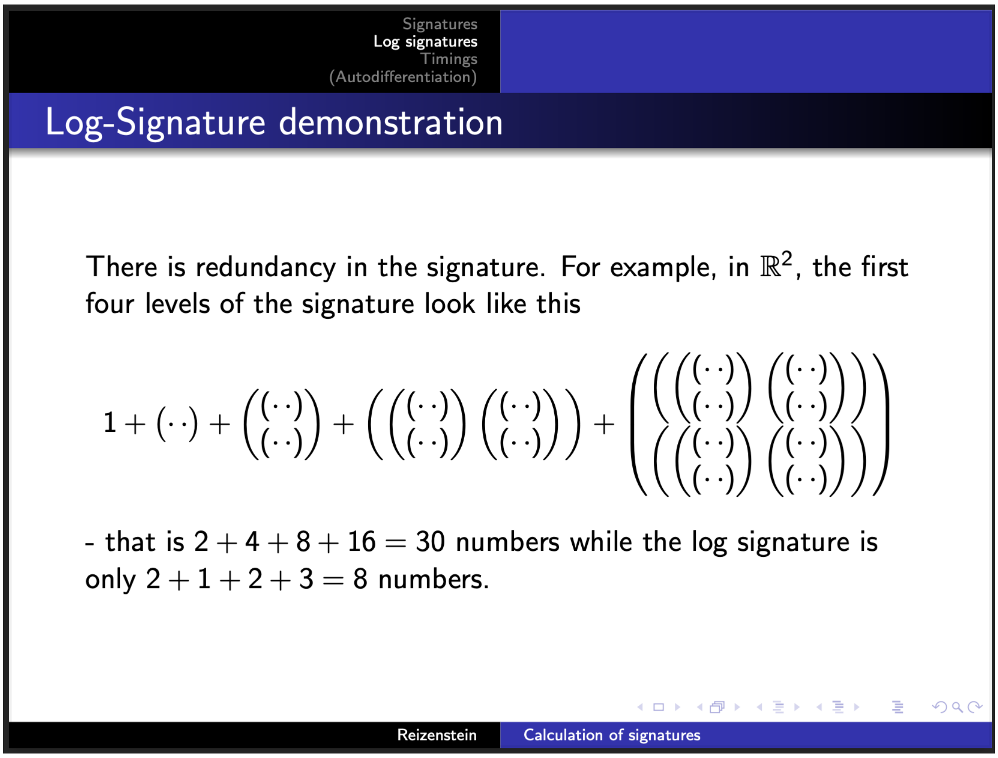

Figure 3: Signatures with d=2

Where the set W of all multi-indexes is (mirroring Figure 3):

1	2

11	12
21	22

$$\begin{matrix} 111 & 112 & 211 & 212 \\ 121 & 122 & 221 & 222 \end{matrix}$$

$$\begin{matrix} 1111 & 1112 & 1211 & 1212 \\ 1121 & 1122 & 1221 & 1222 \\ 2111 & 2112 & 2211 & 2212 \\ 2121 & 2122 & 2221 & 2222 \end{matrix}$$

And the total number of terms in the signature of dimension d up to level k (excluding the 1 at level 0) is equal to:

$$\begin{cases} \sum_{k=0}^{K} d^k = \frac{d^{K+1}-d}{d-1} & d > 1 \\ K & d = 1 \end{cases} \tag{7}$$

So with a helper function to "fold" each level in the correct tensor shape:

```
tensorfold[list_, dim_:2]:= If[Length[list]>=dim,
  ArrayReshape[list, ConstantArray[dim, Log[dim, Length[list
     ]]]],
  list];
```

We define the SIGNAT function in 1, which takes as inputs a list of *paths*, the domain (a, b) and the optional value *level*, which is the level at which we truncate the signature (going forward we will always work with the truncated signatures, and we might refer to them as signatures):

Algorithm 1 The SIGNAT function

```
signat[paths_, limits_, level_:3]:= Module[
  {dpaths, dpathsf, integrands, ilimits, iassump, results},
  dpaths= D[paths,t];
  dpathsf= Map[Function[t, #]&, dpaths, {2}];
  integrands= Table[Map[Tuples,Transpose[
    Table[Map[#[Subscript[t,j]]&,dpathsf,{2}],{j,1,k}]]],
    {k,1,level}];
  ilimits= Table[Table[{Subscript[t,j],limits[[1]],
    If[j==k,limits[[2]],Subscript[t,j+1]]},{j,1,k}],
    {k,1,level}];
  iassump= Fold[(#1 && (limits[[1]]<=Subscript[t,#2]<=limits
     [[2]]))&,
    (limits[[1]]<=Subscript[t,1]<=limits[[2]]), Range[2,level]];
  results= Table[Map[iterinteg[#,ilimits[[k]],iassump]&,
     integrands[[k]],{2}],
    {k,1,level}];
  Map[tensorfold,Map[Prepend[#,{1}]&,Transpose[results]],{2}]];
```

And look at Example 6 at 1.2.2 in [Chevryev and Kormilitzin (2016)]:

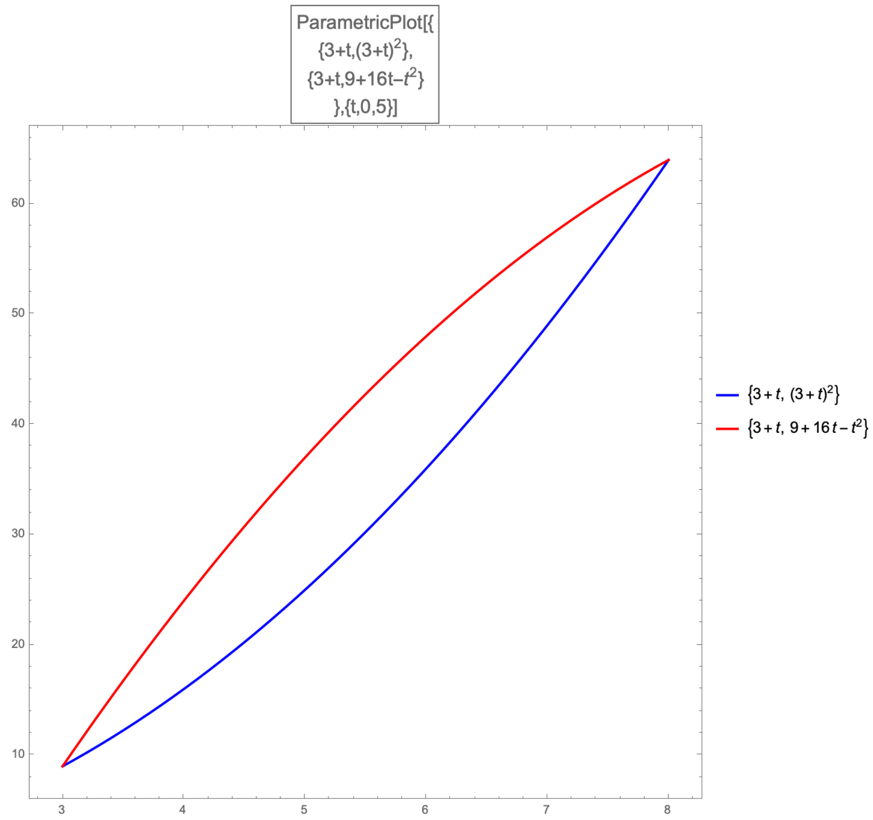

Figure 4: Two 2d paths with the same chord

We can see their truncated signatures up to level 3 on Table 1, where path 1 is the original $\left\{3+t,(3+t)^2\right\}$ path from [Chevryev and Kormilitzin (2016)] and path 2 is the "inverted" $\{3+t, 9+16\cdot t-t^2\}$:

Level	Path	Signature
1	1	$(5, 55)$
1	2	$(5, 55)$
2	1	$\begin{pmatrix} 25/2 & 475/3 \\ 350/3 & 3025/2 \end{pmatrix}$
2	2	$\begin{pmatrix} 25/2 & 350/3 \\ 475/3 & 3025/2 \end{pmatrix}$
3	1	$\begin{pmatrix} 125/6 & 1125/4 \\ 1375/6 & 18875/6 \end{pmatrix} \begin{pmatrix} 2125/12 & 7250/3 \\ 2000 & 166375/6 \end{pmatrix}$
3	2	$\begin{pmatrix} 125/6 & 2125/12 \\ 1375/6 & 2000 \end{pmatrix} \begin{pmatrix} 1125/4 & 7250/3 \\ 18875/6 & 166375/6 \end{pmatrix}$

Table 1: Signatures of the 2d paths up to level 3

We can see how level 2 is flipped around the diagonal ($12 \leftrightarrow 21$), and if we think of the tensors on level 3 as a cube, we can see this flip on a diagonal also ($1x2 \leftrightarrow 2x1$).

We can calculate the signature of other functions, like $\sin(k\cdot\pi\cdot t^q)$, shown in Figure 5 and Tables 2 and 3 (it's implicit that the 2d path is defined as $\{t, \sin(k\cdot\pi\cdot t^q)\}$).

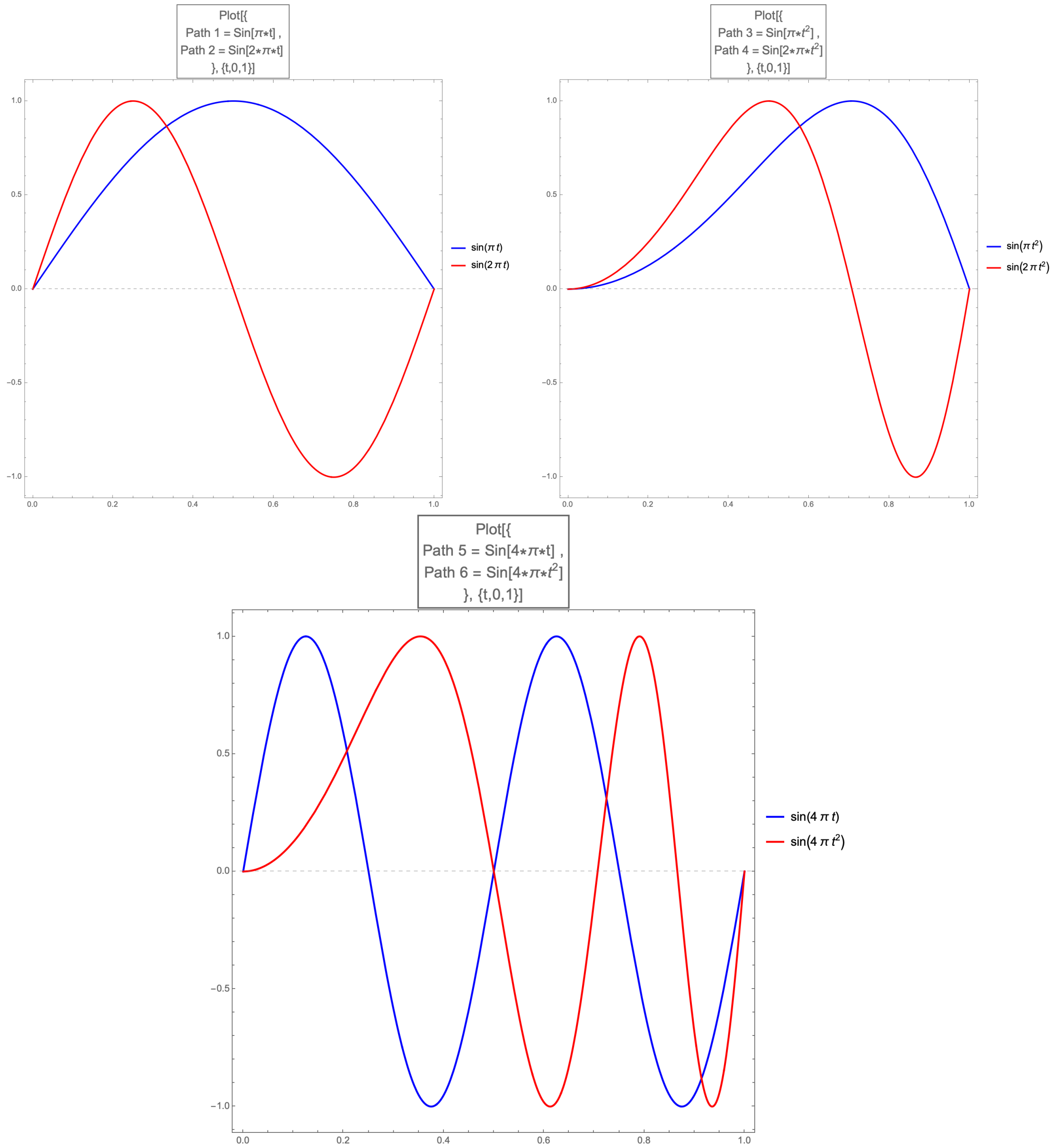

Figure 5: 2d smooth paths with different argument for $\{t, \sin(k \cdot \pi \cdot t^q)\}$

The $\{t, \sin(k \cdot \pi \cdot t)\}$ paths might be seen as adversarial paths, since you need to get more levels on the signature to be able to get the details of the path and they're not piecewise linear. In Tables 2 and 3, $FS(x)$ is the FresnelS integral evaluated at x, and$FC(x)$ is the FresnelC integral evaluated at x.

Level	Path	Signature
1	1	$\begin{matrix} 1 & 0 \end{matrix}$
1	2	$\begin{matrix} 1 & 0 \end{matrix}$
1	3	$\begin{matrix} 1 & 0 \end{matrix}$
1	4	$\begin{matrix} 1 & 0 \end{matrix}$
1	5	$\begin{matrix} 1 & 0 \end{matrix}$
1	6	$\begin{matrix} 1 & 0 \end{matrix}$
2	1	$\begin{matrix} \frac{1}{2} & -\frac{2}{\pi} \\ \frac{2}{\pi} & 0 \end{matrix}$
2	2	$\begin{matrix} \frac{1}{2} & 0 \\ 0 & 0 \end{matrix}$
2	3	$\begin{matrix} \frac{1}{2} & -\frac{FS\left(\sqrt{2}\right)}{\sqrt{2}} \\ \frac{FS\left(\sqrt{2}\right)}{\sqrt{2}} & 0 \end{matrix}$
2	4	$\begin{matrix} \frac{1}{2} & -\frac{FS(2)}{2} \\ \frac{FS(2)}{2} & 0 \end{matrix}$
2	5	$\begin{matrix} \frac{1}{2} & 0 \\ 0 & 0 \end{matrix}$
2	6	$\begin{matrix} \frac{1}{2} & -\frac{FS\left(2\sqrt{2}\right)}{2\sqrt{2}} \\ \frac{FS\left(2\sqrt{2}\right)}{2\sqrt{2}} & 0 \end{matrix}$
3	1	$\begin{matrix} \frac{1}{6} & -\frac{1}{\pi} \\ 0 & \frac{1}{4} \end{matrix} \Bigg\vert \begin{matrix} \frac{1}{\pi} & -\frac{1}{2} \\ \frac{1}{4} & 0 \end{matrix}$
3	2	$\begin{matrix} \frac{1}{6} & \frac{1}{2\pi} \\ -\frac{1}{\pi} & \frac{1}{4} \end{matrix} \Bigg\vert \begin{matrix} \frac{1}{2\pi} & -\frac{1}{2} \\ \frac{1}{4} & 0 \end{matrix}$
3	3	$\begin{matrix} \frac{1}{6} & -\frac{1}{\pi} \\ \frac{2}{\pi} - \frac{FS\left(\sqrt{2}\right)}{\sqrt{2}} & \frac{1}{4} - \frac{FC(2)}{8} \end{matrix} \Bigg\vert \begin{matrix} -\frac{1}{\pi} + \frac{FS\left(\sqrt{2}\right)}{\sqrt{2}} & -\frac{1}{2} + \frac{FC(2)}{4} \\ \frac{1}{4} - \frac{FC(2)}{8} & 0 \end{matrix}$
3	4	$\begin{matrix} \frac{1}{6} & 0 \\ -\frac{FS(2)}{2} & \frac{1}{4} - \frac{FC\left(2\sqrt{2}\right)}{8\sqrt{2}} \end{matrix} \Bigg\vert \begin{matrix} \frac{FS(2)}{2} & -\frac{1}{2} + \frac{FC\left(2\sqrt{2}\right)}{4\sqrt{2}} \\ \frac{1}{4} - \frac{FC\left(2\sqrt{2}\right)}{8\sqrt{2}} & 0 \end{matrix}$
3	5	$\begin{matrix} \frac{1}{6} & \frac{1}{4\pi} \\ -\frac{1}{2\pi} & \frac{1}{4} \end{matrix} \Bigg\vert \begin{matrix} \frac{1}{4\pi} & -\frac{1}{2} \\ \frac{1}{4} & 0 \end{matrix}$
3	6	$\begin{matrix} \frac{1}{6} & 0 \\ -\frac{FS\left(2\sqrt{2}\right)}{2\sqrt{2}} & \frac{1}{4} - \frac{FC(4)}{16} \end{matrix} \Bigg\vert \begin{matrix} \frac{FS\left(2\sqrt{2}\right)}{2\sqrt{2}} & -\frac{1}{2} + \frac{FC(4)}{8} \\ \frac{1}{4} - \frac{FC(4)}{16} & 0 \end{matrix}$

Table 2: Signatures of the sin paths up to level 3

<table>
<tr><th>Level</th><th>Path</th><th>Signature</th></tr>
<tr><td>4</td><td>1</td><td>$\left.\begin{array}{cc|cc} \frac{1}{24} & \frac{2}{\pi^3}-\frac{1}{2\pi} & \frac{6}{\pi^3}-\frac{1}{2\pi} & \frac{2}{\pi^2}-\frac{1}{4} \\ \frac{1}{2\pi}-\frac{6}{\pi^3} & \frac{1}{8} & \frac{1}{4}-\frac{2}{\pi^2} & -\frac{2}{9\pi} \\ \hline \frac{1}{2\pi}-\frac{2}{\pi^3} & -\frac{2}{\pi^2} & \frac{1}{8} & -\frac{2}{3\pi} \\ \frac{2}{\pi^2}-\frac{1}{4} & \frac{2}{3\pi} & \frac{2}{9\pi} & 0 \end{array}\right.$</td></tr>
<tr><td>4</td><td>2</td><td>$\left.\begin{array}{cc|cc} \frac{1}{24} & \frac{1}{4\pi} & -\frac{1}{4\pi} & -\frac{1}{4} \\ -\frac{1}{4\pi} & \frac{1}{8} & \frac{1}{4} & 0 \\ \hline \frac{1}{4\pi} & 0 & \frac{1}{8} & 0 \\ -\frac{1}{4} & 0 & 0 & 0 \end{array}\right.$</td></tr>
<tr><td>4</td><td>3</td><td>$\left.\begin{array}{cc|cc} \frac{1}{24} & -\frac{FC\left(\sqrt{2}\right)}{4\sqrt{2}\pi}-\frac{1}{4\pi} & -\frac{3FC\left(\sqrt{2}\right)}{4\sqrt{2}\pi}-\frac{FS\left(\sqrt{2}\right)}{2\sqrt{2}}+\frac{5}{4\pi} & \frac{FS\left(\sqrt{2}\right)^2}{4}-\frac{1}{4} \\ \frac{3FC\left(\sqrt{2}\right)}{4\sqrt{2}\pi}-\frac{1}{4\pi} & \frac{1}{8} & -\frac{FC(2)}{8}-\frac{FS\left(\sqrt{2}\right)^2}{4}+\frac{1}{4} & \frac{FS\left(\sqrt{6}\right)}{24\sqrt{6}}-\frac{FS\left(\sqrt{2}\right)}{8\sqrt{2}} \\ \hline \frac{FC\left(\sqrt{2}\right)}{4\sqrt{2}\pi}+\frac{FS\left(\sqrt{2}\right)}{2\sqrt{2}}-\frac{3}{4\pi} & \frac{FS\left(\sqrt{2}\right)^2}{4} & \frac{1}{8}-\frac{FC(2)}{8} & \frac{FS\left(\sqrt{6}\right)}{8\sqrt{6}}-\frac{3FS\left(\sqrt{2}\right)}{8\sqrt{2}} \\ \frac{FC(2)}{4}+\frac{FS\left(\sqrt{2}\right)^2}{4}-\frac{1}{4} & \frac{3FS\left(\sqrt{2}\right)}{8\sqrt{2}}-\frac{FS\left(\sqrt{6}\right)}{8\sqrt{6}} & \frac{FS\left(\sqrt{2}\right)}{8\sqrt{2}}-\frac{FS\left(\sqrt{6}\right)}{24\sqrt{6}} & 0 \end{array}\right.$</td></tr>
<tr><td>4</td><td>4</td><td>$\left.\begin{array}{cc|cc} \frac{1}{24} & \frac{1}{8\pi}-\frac{FC(2)}{16\pi} & -\frac{3FC(2)}{16\pi}-\frac{FS(2)}{4}+\frac{3}{8\pi} & \frac{FS(2)^2}{8}-\frac{1}{4} \\ \frac{3FC(2)}{16\pi}-\frac{3}{8\pi} & \frac{1}{8} & -\frac{FC\left(2\sqrt{2}\right)}{8\sqrt{2}}-\frac{FS(2)^2}{8}+\frac{1}{4} & \frac{FS\left(2\sqrt{3}\right)}{48\sqrt{3}}-\frac{FS(2)}{16} \\ \hline \frac{FC(2)}{16\pi}+\frac{FS(2)}{4}-\frac{1}{8\pi} & -\frac{FS(2)^2}{8} & \frac{1}{8}-\frac{FC\left(2\sqrt{2}\right)}{8\sqrt{2}} & \frac{FS\left(2\sqrt{3}\right)}{16\sqrt{3}}-\frac{3FS(2)}{16} \\ \frac{FC\left(2\sqrt{2}\right)}{4\sqrt{2}}+\frac{FS(2)^2}{8}-\frac{1}{4} & \frac{3FS(2)}{16}-\frac{FS\left(2\sqrt{3}\right)}{16\sqrt{3}} & \frac{FS(2)}{16}-\frac{FS\left(2\sqrt{3}\right)}{48\sqrt{3}} & 0 \end{array}\right.$</td></tr>
<tr><td>4</td><td>5</td><td>$\left.\begin{array}{cc|cc} \frac{1}{24} & \frac{1}{8\pi} & -\frac{1}{8\pi} & -\frac{1}{4} \\ -\frac{1}{8\pi} & \frac{1}{8} & \frac{1}{4} & 0 \\ \hline \frac{1}{8\pi} & 0 & \frac{1}{8} & 0 \\ -\frac{1}{4} & 0 & 0 & 0 \end{array}\right.$</td></tr>
<tr><td>4</td><td>6</td><td>$\left.\begin{array}{cc|cc} \frac{1}{24} & \frac{1}{16\pi}-\frac{FC\left(2\sqrt{2}\right)}{32\sqrt{2}\pi} & -\frac{3FC\left(2\sqrt{2}\right)}{32\sqrt{2}\pi}-\frac{FS\left(2\sqrt{2}\right)}{4\sqrt{2}}+\frac{3}{16\pi} & \frac{FS\left(2\sqrt{2}\right)^2}{16}-\frac{1}{4} \\ \frac{3FC\left(2\sqrt{2}\right)}{32\sqrt{2}\pi}-\frac{3}{16\pi} & \frac{1}{8} & -\frac{FC(4)}{16}-\frac{1}{16}FS\left(2\sqrt{2}\right)^2+\frac{1}{4} & \frac{FS\left(2\sqrt{6}\right)}{48\sqrt{6}}-\frac{FS\left(2\sqrt{2}\right)}{16\sqrt{2}} \\ \hline \frac{FC\left(2\sqrt{2}\right)}{32\sqrt{2}\pi}+\frac{FS\left(2\sqrt{2}\right)}{4\sqrt{2}}-\frac{1}{16\pi} & -\frac{1}{16}FS\left(2\sqrt{2}\right)^2 & \frac{1}{8}-\frac{FC(4)}{16} & \frac{FS\left(2\sqrt{6}\right)}{16\sqrt{6}}-\frac{3FS\left(2\sqrt{2}\right)}{16\sqrt{2}} \\ \frac{FC(4)}{8}+\frac{FS\left(2\sqrt{2}\right)^2}{16}-\frac{1}{4} & \frac{3FS\left(2\sqrt{2}\right)}{16\sqrt{2}}-\frac{FS\left(2\sqrt{6}\right)}{16\sqrt{6}} & \frac{FS\left(2\sqrt{2}\right)}{16\sqrt{2}}-\frac{FS\left(2\sqrt{6}\right)}{48\sqrt{6}} & 0 \end{array}\right.$</td></tr>
</table>

Table 3: Signatures of the sin paths - level 4

Of special interest is the signature of a line segment described by:

$$\{\alpha_1 + \beta_1 \cdot t, \alpha_2 + \beta_2 \cdot t, \ldots, \alpha_d + \beta_d \cdot t\}$$

Over the domain $t \in [a, b]$. If some paths exhibit high-frequency components that force us to dig deeper to find the information that describes the path, a line segment reveals itself on the first level:

$$\{b_1 \cdot (b-a), b_2 \cdot (b-a), \ldots, b_d \cdot (b-a)\}$$

All levels are defined by:

$$S(X)_{a,b}^{i_1,\ldots,i_k} = \frac{\beta_{i_1} \cdot \beta_{i_2} \cdot \ldots \cdot \beta_{i_k} \cdot (b-a)^k}{k!} \tag{8}$$

2.2 Properties

2.2.1 Translation invariance

From the definition of the signature of a line segment, we can see that we lost information about the initial level; the signature encoded the shape of the path but not the initial location. This is helpful for the composition of the paths, but it also means that we will need to keep the initial value stored outside the signature.

2.2.2 Shuffle product

Let's start with the formal definitions (and some informal definitions as well):

A (a, b)-shuffle of a set with size $a + b$ is a permutation that keeps the relative order of the two blocks with sizes a and b of the set. We can also talk about one of the shuffles of the blocks A and B to describe the same thing.

So given $\{1, 2\}$ and $\{3, 4\}$ as the two pieces of the set, $\{1, 3, 2, 4\}$ is a valid shuffle; $\{3, 1, 2, 4\}$ is a valid shuffle as well; but $\{2, 1, 3, 4\}$ is not a valid shuffle. This example uses different values to make it easier to see the relative positions, but it's all about the relative positions.

Consider two multi-indexes I and J with sizes k_i and k_j, respectively:

$$I = \{i_1, i_2, \ldots, i_{k_i}\}$$

$$J = \{j_1, j_2, \ldots, j_{k_j}\}$$

For $d = 2$ an example is:

$$I = \{1, 2\}$$

$$J = \{2, 1\}$$

We define the shuffle product of I and J ($I \sqcup\!\sqcup J$) as the set of all multi-indexes of length $k_i + k_j$ that are valid shuffles of I and J.

In our example:

$I \sqcup\!\sqcup \{3\} = \{\{1, 2, 3\}, \{1, 3, 2\}, \{3, 1, 2\}\}$

$I \sqcup\!\sqcup \{3, 4\} = \{\{1, 2, 3, 4\}, \{1, 3, 2, 4\}, \{1, 3, 4, 2\}, \{3, 1, 2, 4\}, \{3, 1, 4, 2\}, \{3, 4, 1, 2\}\}$

$I \sqcup\!\sqcup J = \{\{1, 2, 2, 1\}, \{1, 2, 2, 1\}, \{1, 2, 1, 2\}, \{2, 1, 2, 1\}, \{2, 1, 1, 2\}, \{2, 1, 1, 2\}\}$

There are many ways to generate these shuffles (Shuffle product of two lists (Mathematica Stack Exchange)). We will use the top answer, which uses the recursive properties below (ε is the empty list, a and b are single elements, u and v are arbitrary lists), but inverted (the single element comes first in the function):

$u \sqcup\!\sqcup \varepsilon = u \sqcup\!\sqcup \varepsilon = u$

$ua \sqcup\!\sqcup vb = (u \sqcup\!\sqcup vb)\, a + (ua \sqcup\!\sqcup v)\, b$

Algorithm 2 The shuffle product function

```
shp[u:{a_,x___}, v:{b_, y___}, c___]:=
  Join[shp[{x}, v, c, a], shp[u, {y}, c, b]];
shp[{x___}, {y___}, c___] := {{c, x, y}} ;
```

The Shuffle product identity for a path $X : [a, b] \to \mathbb{R}^d$ and two multi-indexes I and J is defined as:

$$S(X)^I_{a,b} \cdot S(X)^J_{a,b} = \sum_{K=I \sqcup\!\sqcup J} S(X)^K_{a,b} \tag{9}$$

As an example, take Path 1 on Table 1:

$$S(X)^2_{0,5} \cdot S(X)^{1,2}_{0,5} = S(X)^{2,1,2}_{0,5} + S(X)^{1,2,2}_{0,5} + S(X)^{1,2,2}_{0,5}$$

Because:

$$\{2\} \sqcup\!\sqcup \{1, 2\} = \{\{2, 1, 2\}, \{1, 2, 2\}, \{1, 2, 2\}\}$$

Substituting:

$$55 \cdot \frac{475}{3} = \frac{7250}{3} + \frac{18875}{6} + \frac{18875}{6}$$

$$\frac{26125}{3} = \frac{26125}{3}$$

As expected.

2.2.3 Chen's Identity

Given two paths $X : [a, b] \to \mathbb{R}^d$ and $Y : [b, c] \to \mathbb{R}^d$ with the same dimension d and consecutive intervals, we can define the concatenated path Z:

$$Z = X * Y : [a, c] \to \mathbb{R}^d$$

With:

$$\begin{cases} Z_t = X_t & t \in [a, b] \\ Z_t = X_b + (Y_t - Y_b) & t \in [b, c] \end{cases}$$

Chen's Identity defines the signature of the concatenated path as the product $\otimes$ of the two signatures:

$$S(Z)_{a,c} = S(X * Y)_{a,c} = S(X)_{a,b} \otimes S(Y)_{b,c} \tag{10}$$

Let's define the shortcuts:

$$S(X)^{i_1,\ldots,i_k}_{a,b} = x_{i_1,\ldots,i_k} \tag{11}$$

$$S\left(Y\right)_{b,c}^{i_1,\ldots,i_k} = y_{i_1,\ldots,i_k} \tag{12}$$

$$S\left(Z\right)_{a,c}^{i_1,\ldots,i_k} = z_{i_1,\ldots,i_k} \tag{13}$$

This product $\otimes$ of the two signatures can be better seen in Table 4 (for $d = 2$):

	1	$y_1 \quad y_2$	$\begin{matrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{matrix}$	$S\left(Y\right)_{a,b}^{i_1,i_2,i_3}$
1	1	$y_1 \quad y_2$	$\begin{matrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{matrix}$	$S\left(Y\right)_{a,b}^{i_1,i_2,i_3}$
$x_1 \quad x_2$	$x_1 \quad x_2$	$x_1 \quad x_2 \otimes y_1 \quad y_2$	$x_1 \quad x_2 \otimes \begin{matrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{matrix}$	$x_1 \quad x_2 \otimes S\left(Y\right)_{a,b}^{i_1,i_2,i_3}$
$\begin{matrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{matrix}$	$\begin{matrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{matrix}$	$\begin{matrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{matrix} \otimes y_1 \quad y_2$	$\begin{matrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{matrix} \otimes \begin{matrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{matrix}$	$\begin{matrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{matrix} \otimes S\left(Y\right)_{a,b}^{i_1,i_2,i_3}$
$S\left(X\right)_{a,b}^{i_1,i_2,i_3}$	$S\left(X\right)_{a,b}^{i_1,i_2,i_3}$	$S\left(X\right)_{a,b}^{i_1,i_2,i_3} \otimes y_1 \quad y_2$	$S\left(X\right)_{a,b}^{i_1,i_2,i_3} \otimes \begin{matrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{matrix}$	$S\left(X\right)_{a,b}^{i_1,i_2,i_3} \otimes S\left(Y\right)_{a,b}^{i_1,i_2,i_3}$

Table 4: The product $\otimes$ of the two signatures (sums on the top-right to bottom-left diagonals)

This develops into Table 5 (for $d = 2$):

	1	$y_1 \quad y_2$	$\begin{matrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{matrix}$
1	1	$y_1 \quad y_2$	$\begin{matrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{matrix}$
$x_1 \quad x_2$	$x_1 \quad x_2$	$\begin{matrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{matrix}$	$\begin{matrix} x_1 y_{1,1} & x_1 y_{1,2} \\ x_1 y_{2,1} & x_1 y_{2,2} \end{matrix} \Big\vert \begin{matrix} x_2 y_{1,1} & x_2 y_{1,2} \\ x_2 y_{2,1} & x_2 y_{2,2} \end{matrix}$
$\begin{matrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{matrix}$	$\begin{matrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{matrix}$	$\begin{matrix} x_{1,1} y_1 & x_{1,1} y_2 \\ x_{1,2} y_1 & x_{1,2} y_2 \end{matrix} \Big\vert \begin{matrix} x_{2,1} y_1 & x_{2,1} y_2 \\ x_{2,2} y_1 & x_{2,2} y_2 \end{matrix}$	$\begin{matrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{matrix} \otimes \begin{matrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{matrix}$
$S\left(X\right)_{a,b}^{i_1,i_2,i_3}$	$S\left(X\right)_{a,b}^{i_1,i_2,i_3}$	$S\left(X\right)_{a,b}^{i_1,i_2,i_3} \otimes y_1 \quad y_2$	$S\left(X\right)_{a,b}^{i_1,i_2,i_3} \otimes \begin{matrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{matrix}$

Table 5: The product $\otimes$ of the two signatures developed (sums on the top-right to bottom-left diagonals)

So for each level of Z we have:

$$S\left(Z\right)_{a,c}^{i_1} = S\left(X\right)_{a,b}^{i_1} + S\left(Y\right)_{b,c}^{i_1} \tag{14}$$

$$S\left(Z\right)_{a,c}^{i_1,i_2} = S\left(X\right)_{a,b}^{i_1,i_2} + S\left(X\right)_{a,b}^{i_1} \cdot S\left(Y\right)_{b,c}^{i_2} + S\left(Y\right)_{b,c}^{i_1,i_2} \tag{15}$$

$$S\left(Z\right)_{a,c}^{i_1,i_2,i_3} = S\left(X\right)_{a,b}^{i_1,i_2,i_3} + S\left(X\right)_{a,b}^{i_1,i_2} \cdot S\left(Y\right)_{b,c}^{i_3} + S\left(X\right)_{a,b}^{i_1} \cdot S\left(Y\right)_{b,c}^{i_2,i_3} + S\left(Y\right)_{b,c}^{i_1,i_2,i_3} \tag{16}$$

$$S(Z)_{a,c}^{i_1,\ldots,i_k} = \sum_{j=0}^{k} S(X)_{a,b}^{i_1,\ldots,i_j} \cdot S(Y)_{b,c}^{i_{j+1},\ldots,i_k} \tag{17}$$

Where:

$$S(X)_{a,b}^{i_1,\ldots,i_0} = S(Y)_{b,c}^{i_{k+1},\ldots,i_k} = 1 \tag{18}$$

Using Formula 17 frees us from computing the whole signature of the concatenated path if we just want some particular terms; it is valid for any kind of path, as we can see in the example below (Figure 6):

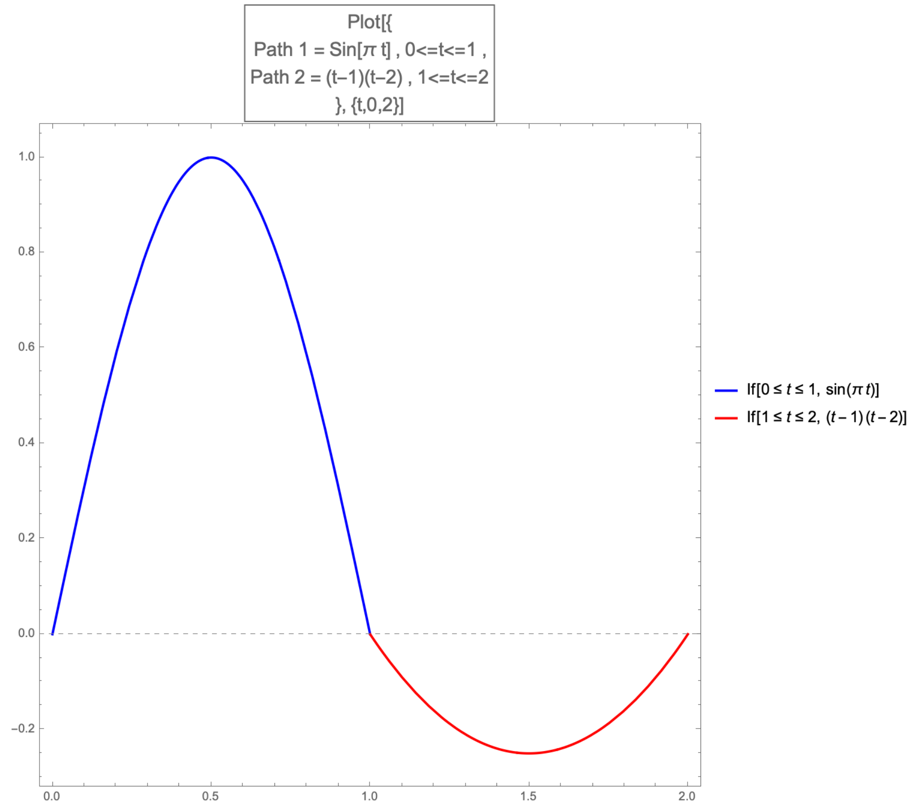

Figure 6: Concatenation of paths

The signature of the first path is full of πs, because of the Sin function; the signature of the second path is full of fractions of the form $\pm\frac{1}{m}$; but Formula 6 holds:

$$S(Z)_{0,2}^{1,1,2} = \frac{1}{4} - \frac{1}{\pi}$$

$$S(Z)_{0,2}^{2,1,1,2} = -\frac{1}{72} + \frac{1}{3\pi} - \frac{2}{\pi^2}$$

We get the same results from the signature of the concatenated path (applying the SIGNAT function) and from Formula 6. The Mathematica notebook has this example and other examples can be checked easily.

3 Core Signatures

3.1 Log signatures

An useful transform of the signatures are the log signatures. Without going into the details of formal power series and Lie brackets (please refer to [Reizenstein (2018)], [Reizenstein and Graham (2020)], [Reizenstein (2016)] and [Reizenstein (2015)], and the iisignature Python package in general for all the details that go into an efficient computation of log signatures), we're interested in the dimensionality reduction shown. We reproduce Table 2 from [Reizenstein (2015)] (Table 6) and adding an approximation for $d = 2$:

$$core2app = 2 + \sum_{n=2}^{m} \sum_{k=1}^{n-1} \left\lfloor \frac{1}{n} \binom{n}{k} \right\rfloor \tag{19}$$

level m	d=2			d=3		d=4		d=5	
	sig	core2app	log	sig	log	sig	log	sig	log
1	2	2	2	3	3	4	4	5	5
2	6	3	3	12	6	20	10	30	15
3	14	5	5	39	14	84	30	155	55
4	30	8	8	120	32	340	90	780	205
5	62	14	14	363	80	1364	294	3905	829
6	126	23	23	1092	196	5460	964	19530	3409
7	254	41	41	3279	508	21844	3304	97655	14569
8	510	71	71	9840	1318	87380	11464	488280	63319
9	1022	127	127	29523	3502	349524	40584	2441405	280319
10	2046	228	226	88572	9382	1398100	145338	12207030	1256567
11	4094	414	412	265719	25486	5592404	526638	61035155	5695487
12	8190	753	747	797160	69706	22369620	1924378	305175780	26039187

Table 6: Sizes of signatures and log signatures from [Reizenstein (2015)], and the approximation core2app

Remember that Formula 7 gives us the size of the signature.

How do we get a reduction of approximately $1/n$ of dimensionality? Do we really need the log to get this reduction? [Améndola, Friz and Sturmfels (2019)] hints that no, we might not need logs.

3.2 Shuffle products and core signatures

Formula 9 helps us to define higher level terms of the signature in terms of the lower level terms, but there are not enough equations to map one level to another; every new level is expected to have new information.

Let's start with $d = 2$:

3.2.1 d=2, level 1

Level 1 starts with 2 new variables, where V_m is the set of the values of the signature at level m:

$$V_1 = \{s_1, s_2\}$$

V_1 is new information (the chord); let's define C_m the set of core variables up to level m:

$$C_1 = \{s_1, s_2\}$$

3.2.2 d=2, level 2

Level 2 starts with 4 new variables:

$$V_2 = \{s_{1,1}, s_{1,2}, s_{2,1}, s_{2,2}\}$$

And we have 4 equations coming from the shuffle product:

$2s_{1,1} = s_1^2$
$s_{1,2} + s_{2,1} = s_1 s_2$
$s_{1,2} + s_{2,1} = s_1 s_2$
$2s_{2,2} = s_2^2$

But there's symmetry involved; the 2nd and 3rd equations are the same. Solving the system (more on this later) yields the solution set P_2, where P_m is the set of solutions found on level m and Q_m is the complete set of substitution rules up to level m:

$s_{1,1} \to \frac{s_1^2}{2}$
$s_{2,1} \to s_1 s_2 - s_{1,2}$
$s_{2,2} \to \frac{s_2^2}{2}$

Yes, we could have expressed it in terms of $s_{2,1}$ instead of $s_{1,2}$, but we will discuss the structure of the solutions later. We can scan the right side of the substitution rules to find C_2 :

s_1, s_2
$s_{1,2}$

3.2.3 d=2, level 3

Level 3 starts with 8 new variables, and we have 8 equations coming from the shuffle product (levels 1 vs 2).

Solving the system yields the solution set P_3, with 6 elements (we'll list Q_6 in the Appendix).

Scanning $Q_3 = Q_2 \cup P_3$, we find C_3:

s_1, s_2
$s_{1,2}$
$s_{1,1,2}, s_{1,2,2}$

3.2.4 d=2, level 4

Level 4 starts with 16 new variables and we have 32 equations (not unique) coming from the shuffle product (levels 1 vs 3 and 2 vs 2).

Solving the system yields the solution set P_4, with 13 elements.

Scanning $Q_4 = Q_3 \cup P_4$, we find C_4:

s_1, s_2
$s_{1,2}$
$s_{1,1,2}, s_{1,2,2}$
$s_{1,1,1,2}, s_{1,1,2,2}, s_{1,2,2,2}$

3.2.5 d=2, level 5

Level 5 starts with 32 new variables and we have 64 equations (not unique) coming from the shuffle product (levels 1 vs 4 and 2 vs 3).

Solving the system yields the solution set P_5, with 26 elements.

Scanning $Q_5 = Q_4 \cup P_5$, we find C_5:

s_1, s_2
$s_{1,2}$
$s_{1,1,2}, s_{1,2,2}$
$s_{1,1,1,2}, s_{1,1,2,2}, s_{1,2,2,2}$
$s_{1,1,1,1,2}, s_{1,1,1,2,2}, s_{1,1,2,1,2}, s_{1,1,2,2,2}, s_{1,2,1,2,2}, s_{1,2,2,2,2}$

3.2.6 d=2, level 6

Level 6 starts with 64 new variables and we have 192 equations (not unique) coming from the shuffle product (levels 1 vs 5, 2 vs 4 and 3 vs 3).

Solving the system (we solve the partial system on the 1 vs 5 equations first, then 2 vs 4 and then 3 vs 3) yields the solution set P_6, with 55 elements.

Scanning $Q_6 = Q_5 \cup P_6$, we find C_6:

s_1, s_2
$s_{1,2}$
$s_{1,1,2}, s_{1,2,2}$
$s_{1,1,1,2}, s_{1,1,2,2}, s_{1,2,2,2}$
$s_{1,1,1,1,2}, s_{1,1,1,2,2}, s_{1,1,2,1,2}, s_{1,1,2,2,2}, s_{1,2,1,2,2}, s_{1,2,2,2,2}$
$s_{1,1,1,1,1,2}, s_{1,1,1,1,2,2}, s_{1,1,1,2,1,2}, s_{1,1,1,2,2,2}, s_{1,1,2,1,2,2}, s_{1,1,2,2,1,2}, s_{1,1,2,2,2,2}, s_{1,2,1,2,2,2}, s_{1,2,2,2,2,2}$

So our core sets have sizes:

$$2, 3, 5, 8, 14, 23$$

Matching the log signatures.

3.2.7 d=3

We can make similar calculations for d=3 to find core sizes:

$$3, 6, 14, 32, 80, 196$$

Also matching the log signatures.

3.3 Gröbner basis and the structure of the solutions

Let's look in depth at the structure of the solutions for $d = 2$.

3.3.1 Level 1

We have two core variables, s_1 and s_2.

3.3.2 Level 2

Our 4 equations are in fact 3:

$2s_{1,1} = s_1^2$
$s_{1,2} + s_{2,1} = s_1 s_2$
$2s_{2,2} = s_2^2$

We can find the Gröbner basis for this polynomial system to group the variables (we use Mathematica's GroebnerBasis with $Method \to "Buchberger"$, which for this problem is much faster than the default $"GroebnerWalk"$):

$2s_{2,2} - s_2^2$
$s_{1,2} + s_{2,1} - s_1 s_2$
$2s_{1,1} - s_1^2$

We see that components like $s_{j,j,j,...}$ don't bring new information - they all are functions of s_j. We can start to group the number of core variables by the number of 1s on the multi-index). So s_1 and $s_{1,2}$ have one 1, and s_2 has none.

	Core variables		
	Number of 1s		
Level	2	1	0
1		1	1
2		1	

3.3.3 Level 3

The Gröbner basis is:

$3s_{2,2,2} - s_2 s_{2,2}$
$-s_2 s_{2,1} + s_{2,1,2} + 2s_{2,2,1}$
$-s_2 s_{1,2} + s_2 s_{2,1} + 2s_{1,2,2} - 2s_{2,2,1}$
$-s_1 s_{2,1} + s_{1,2,1} + 2s_{2,1,1}$
$-s_2 s_{1,1} + s_1 s_{2,1} + s_{1,1,2} - s_{2,1,1}$
$3s_{1,1,1} - s_1 s_{1,1}$

We see now that components where the multi-indexes are permutations of each other get grouped together: $s_{2,1,2}$ and $s_{2,2,1}$ are alone in one equation, and the same happens with $s_{1,2,2}$ and $s_{2,2,1}$. So for each group of 3 variables there's only one term with new information, and, as expected, there's no new information in $\{s_{1,1,1}, s_{2,2,2}\}$.

	Core variables		
	Number of 1s		
Level	2	1	0
1		1	1
2		1	
3	1	1	

3.3.4 Level 4

We see now 3 permutation components and, surprisingly, there's no new information in $\{s_{1,2,2,1}, s_{2,1,1,2}\}$. So 4 of the 16 points are isolated, and we have 3 star-like isolated graphs (Figure 7):

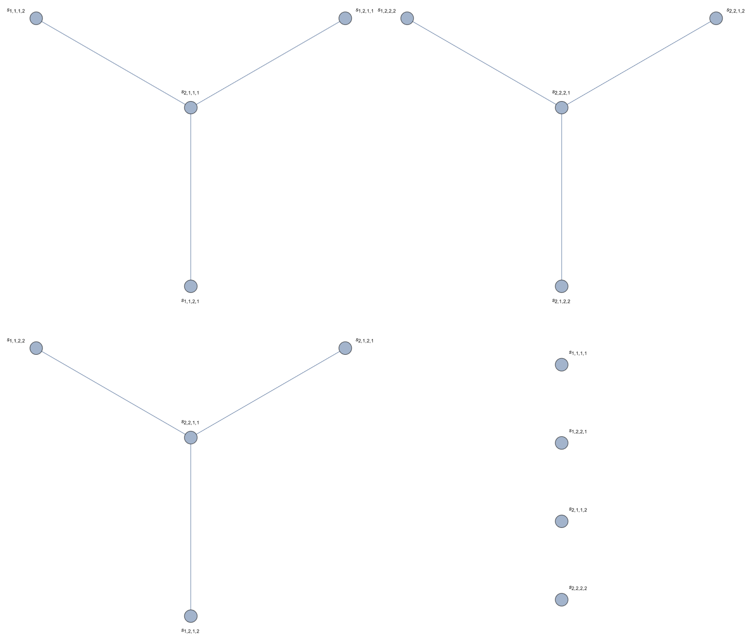

Figure 7: Graph of the Groebner basis, d=2, m=4

And updating our table:

	Core variables			
	Number of 1s			
Level	3	2	1	0
1			1	1
2			1	
3		1	1	
4	1	1	1	

3.3.5 Level 5

We see now 4 permutation components, with 2 of them exhibiting loops. So 2 of the 32 points are isolated, we have 2 star-like isolated graphs with 5 points each (Figure 8):

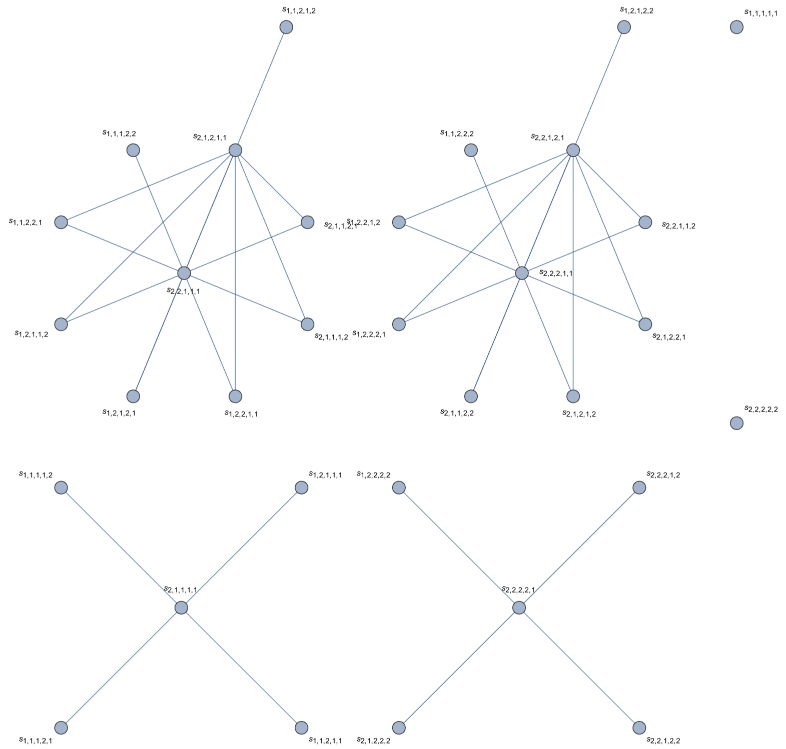

Figure 8: Graph of the Groebner basis, d=2, m=5

. So the structure of the groups seems to follow quite well Pascal's Triangle:

	Core variables				
	Number of 1s				
Level	4	3	2	1	0
1				1	1
2				1	
3			1	1	
4		1	1	1	
5	1	2	2	1	

3.3.6 Level 6

We run the GroebnerBasis function for the equations arising from the 1 vs 5 shuffle, as it is the most sparse of the resulting systems (Figure 9). We have similar results running the GroebnerBasis function for the other shuffles.

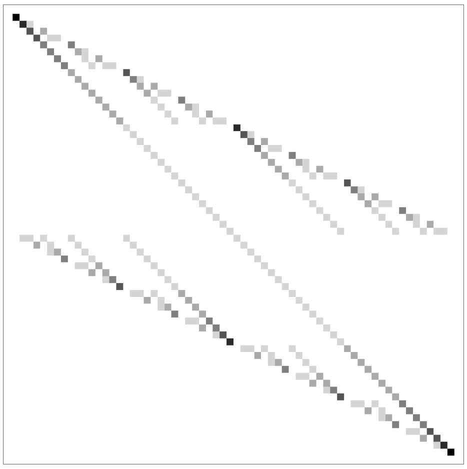

Figure 9: ArrayPlot of the system of equations arising from the 1 vs 5 shuffle

We see now 5 permutation components, with 3 of them exhibiting loops. So 2 of the 64 points are isolated, we have 2 star-like isolated graphs with 6 points each (Figure 10):

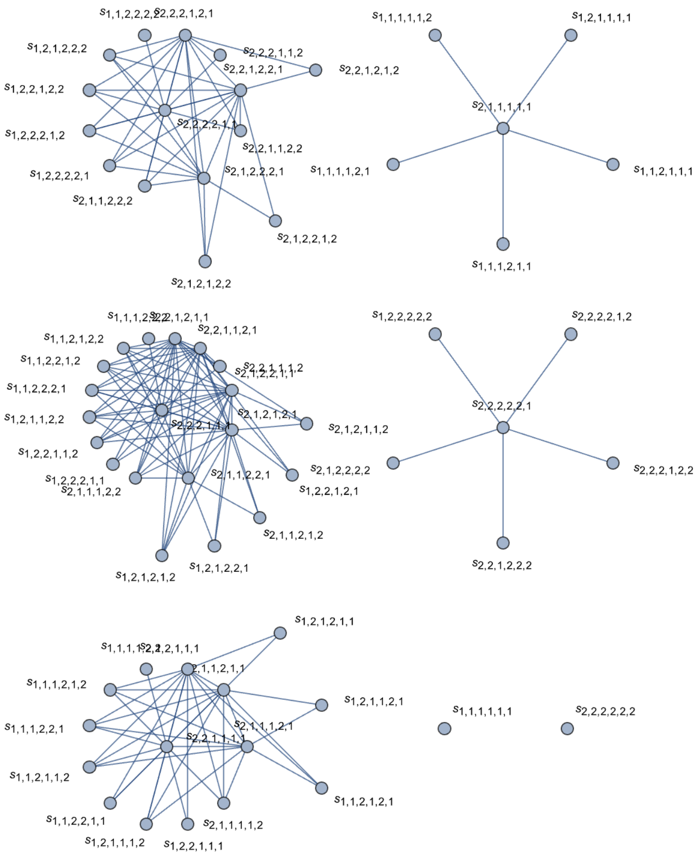

Figure 10: Graph of the Groebner basis of the 1 vs 5 shuffle, d=2, m=6

Now we can look at the results and suggest a rule for the number of solutions in each permutation group. We can group the solutions we found before:

$s_{1,1,1,1,1,2}$
$s_{1,1,1,1,2,2}$, $s_{1,1,1,2,1,2}$
$s_{1,1,1,2,2,2}$, $s_{1,1,2,1,2,2}$, $s_{1,1,2,2,1,2}$
$s_{1,1,2,2,2,2}$, $s_{1,2,1,2,2,2}$
$s_{1,2,2,2,2,2}$

Defining Formula 20, which we saw inside the sums on Formula 19:

$$core2app\left(m,k\right)=\left\lfloor\frac{1}{m}\binom{m}{k}\right\rfloor\quad,\ m\geq 2 \tag{20}$$

And applying it to Pascal's Triangle:

level m	6	5	4	3	2	1	0
1						1	1
2						1	
3					1	1	
4				1	1	1	
5			1	2	2	1	
6		1	2	3	2	1	

level m	6	5	4	3	2	1	0
1						1	1
2						1	
3					1	1	
4				1	1	1	
5			1	2	2	1	
6		1	2	3	2	1	

Table 7: Distribution of core variables (d=2) by the number of 1s in the index vs the approximation 20

This helps to explain why we get a reduction of approximately $1/n$ of dimensionality.

This approximation starts to overestimate the number of core variables for $m=10$, but it is very helpful to understand the structure of the solutions.

4 Inversion of Core Signatures

4.1 Time-ordered paths

4.1.1 Chen's Identity for two pieces

Let's start with a simple (but useful) case: a 2d-path $Z:[0,T]\to\mathbb{R}^2$ where $Z\left(t\right)=\{t,z_t\}$, so $Z_1\left(t\right)$ goes from 0 to T (no loops or retrograde movements): a term structure defined as a function of t.

We have the truncated signature $S\left(Z\right)_{0,T}$ up to a level m. Can we find a piecewise linear approximation for Z? What is our criteria for what constitutes a good approximation?

Let's assume we do not have an adversarial case like $\{t,\sin\left(2\cdot\pi\cdot t\right)\}$ or $\{t,\sin\left(4\cdot\pi\cdot t\right)\}$.

We want to find an approximation that:

1. Follows the chord (ie, starts and ends at the same points of the original path) - this means the signature of the concatenated path has the same first level as the original signature
2. No backtracking: the midpoint is located at $\{t_1,z_{t_1}\}$ with $0<t_1<T$

3. Has the same Levy area (the signature of the concatenated path has the same second level as the original signature); this is our "global" curvature

4. Has the same value for one of the core elements of the third level (more details below)

This means we're looking for the concatenation of two line segments $X(t)_{0,t_1} = \{t, \beta_1 \cdot t\}$ and $Y(t)_{t_1,T} = \{t, \beta_1 \cdot T + \beta_2 \cdot t\}$; the signature of Y will not depend on the $\beta_1 \cdot T$ term, so let's think about finding the solutions for $\{T, t_1, \beta_1, \beta_2\}$ given the core elements of the signature $S(Z)_{0,T}$up to level 3: $\{z_1, z_2, z_{1,2}, z_{1,1,2}, z_{1,2,2}\}$.

Applying Chen's Identity for the concatenation of X and Y, we have the following equations (Table 8):

Level	Eq #	Equation
1	1	$T = z_1$
1	2	$\beta_1 t_1 + \beta_2 (T - t_1) = z_2$
2	3	$\frac{1}{2}\beta_1 t_1^2 + \frac{1}{2}\beta_2 (T - t_1)^2 + \beta_2 t_1 (T - t_1) = z_{1,2}$
3	4	$\frac{1}{6}\beta_1 t_1^3 + \frac{1}{6}\beta_2 (T - t_1)^3 + \frac{1}{2}\beta_2 t_1 (T - t_1)^2 + \frac{1}{2}\beta_2 t_1^2 (T - t_1) = z_{1,1,2}$
3	5	$\frac{1}{6}\beta_1^2 t_1^3 + \frac{1}{6}\beta_2^2 (T - t_1)^3 + \frac{1}{2}\beta_2^2 t_1 (T - t_1)^2 + \frac{1}{2}\beta_1 \beta_2 t_1^2 (T - t_1) = z_{1,2,2}$

Table 8: System of equations for two pieces of a term structure

So, 4 variables and 5 equations. Let's solve the system without equation #5 first and then solve it again but without equation #4 (Table 9):

Variable	Solution 1 (without Eq #5)	Solution 2 (without Eq #4)
T	z_1	z_1
t_1	$\frac{2(z_1 z_{1,2} - 3 z_{1,1,2})}{z_1 z_2 - 2 z_{1,2}}$	$\frac{2\left(3 z_1 z_{1,2,2} - 2 z_{1,2}^2\right)}{z_2 (z_1 z_2 - 2 z_{1,2})}$
β_1	$\frac{-2 z_{1,2}(z_1 z_2 - 2 z_{1,2}) - 6 z_2 z_{1,1,2} + z_1^2 z_2^2}{2 z_1 (z_1 z_{1,2} - 3 z_{1,1,2})}$	$\frac{z_2^2 (z_1 z_2 - 2 z_{1,2}) + 6 z_2 z_{1,2,2} - 2 z_2^2 z_{1,2}}{2\left(3 z_1 z_{1,2,2} - 2 z_{1,2}^2\right)}$
β_2	$\frac{2\left(3 z_2 z_{1,1,2} - 2 z_{1,2}^2\right)}{z_1^2 (z_1 z_2 - 2 z_{1,2}) + 6 z_1 z_{1,1,2} - 2 z_1^2 z_{1,2}}$	$\frac{2 z_2 (z_2 z_{1,2} - 3 z_{1,2,2})}{-2 z_{1,2}(z_1 z_2 - 2 z_{1,2}) - 6 z_1 z_{1,2,2} + z_1^2 z_2^2}$

Table 9: Solutions for two pieces of a term structure

Nice symmetries all around. Substituting the solutions on the respective dropped equations (ie, the solutions on the first column applied on Equation #5, the solutions on the second column applied on Equation #4), we find the same consistency condition:

$$2 z_{1,2}^2 + z_1 z_2 z_{1,2} = 3 z_1 z_{1,2,2} + 3 z_2 z_{1,1,2} \tag{21}$$

Because a concatenation of two line segments doesn't have a lot of freedom.

Let's try it on the core signature of $\{t, \sin(\pi \cdot t)\}$:

$$\{z_1, z_2, z_{1,2}, z_{1,1,2}, z_{1,2,2}\} = \left\{1, 0, -\frac{2}{\pi}, -\frac{1}{\pi}, \frac{1}{4}\right\}$$

.

Solution 1 is:

$$\{T, t_1, \beta_1, \beta_2\} = \left\{1, \frac{1}{2}, \frac{8}{\pi}, -\frac{8}{\pi}\right\}$$

Solution 2 has a problem with $z_2 = 0$.
Now we check $\{t, \sin(\pi \cdot t^2)\}$:

$$\{z_1, z_2, z_{1,2}, z_{1,1,2}, z_{1,2,2}\} = \left\{1, 0, -\frac{FS\left(\sqrt{2}\right)}{\sqrt{2}}, -\frac{1}{\pi}, \frac{2 - FC(2)}{8}\right\}$$

.

Solution 1 is:

$$\{T, t_1, \beta_1, \beta_2\} = \left\{1, \frac{\frac{6}{\pi} - \sqrt{2} \cdot FS\left(\sqrt{2}\right)}{\sqrt{2} \cdot FS\left(\sqrt{2}\right)}, \frac{2 \cdot FS\left(\sqrt{2}\right)^2}{\frac{6}{\pi} - \sqrt{2} \cdot FS\left(\sqrt{2}\right)}, -\frac{2 \cdot FS\left(\sqrt{2}\right)^2}{2\sqrt{2} \cdot FS\left(\sqrt{2}\right) - \frac{6}{\pi}}\right\}$$

Solution 2 has the same problem with $z_2 = 0$.

4.1.2 Adversarial Cases

4.1.3 Chen's Identity for three pieces

4.1.4 More than three pieces

4.2 General 2d paths

5 Term structures in finance

5.1 Interest rates

5.2 Volatility

6 Conclusions

We would like to thank Sam Cohen (Oxford) for his help.

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Appendix

A Appendix

The charts and tables for the daily estimates are below. The only real worry is the instability of the parameters of the $T \to T$ kernels, although the intensities $\frac{\alpha}{\beta}$ are stable. The upper bound for $\beta\,(T \to T)$ was hit on day 16.