Update to The Hidden Moments of a Probability Distribution

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Abstract

Empirical distributions have their in-sample maxima as natural censoring. We look at the "hidden tail", that is, the part of the distribution in excess of the maximum for a sample size of n. Using extreme value theory, we examine the properties of the hidden tail and calculate its moments of order p. The method is useful in showing how large a bias one can expect, for a given n, between the visible in-sample mean and the true statistical mean (or higher moments), which is considerable for α close to 1. Among other properties, we note that the hidden moment of order 0, that is, the exceedance probability for power law distributions, follows a geometric distribution and has for expectation regardless of the parametrization of the scale and tail index. We show the exact hidden moments of order 1 for some interesting distributions. This paper is an update to "What You See and What You Don't See: The Hidden Moments of a Probability Distribution" ([Taleb (2020)]). Mathematica code is available on GitHub at https://github.com/MarcosCarreira/Lucretius.

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1 Introduction

We should not limit our knowledge what we have observed. Consider the case where we have observed m realizations of some measurable phenomenon. What can we say about the expectation of the maximum value we could observe in the next $n - m$ realizations?

Nassim Taleb has written about the statistical properties of this yet unobserved population in several papers and books, and we're going to focus on "What You See and What You Don't See: The Hidden Moments of a Probability Distribution" ([Taleb (2020)]).

We would like to share some updates on this paper:

In the first result we show how the exponential distribution for the hidden moment of order 0 shown in [Taleb (2020)] is an approximation for a geometrical distribution that is a function only of the number of past observations and the running maximum; the only dependence for this result is independence (ie the realizations are assumed to be independent identically distributed sampled form the same distribution).

The second result uses the first result (ie, the hidden moment of order 0 for any distribution is equal to the hidden moment of the Uniform Distribution) to derive exact formulas for the hidden moments of order 1 for several distributions like the Gaussian and the Pareto.

This work relied heavily on: (i) [Taleb (2020)], the inspiration for the session on Hidden Moments on the Global Uncertainty Reading Group on 10-Jan-2024; (ii) [Embrechts et al (1997)], the main reference for Extreme Value Theory.

Section 2 (Definitions) will describe the problem and provide definitions and examples of distributions.

Section 3 (Records) will describe the results from [Taleb (2020)].

Section 4 (Remark 1) will describe how Remark 1 from [Taleb (2020)] provides the intuition to solve the problem.

Section 5 (Simulations) shows how to validate these new results.

2 Definitions

2.1 Sampling and Running Maximum

We assume that we're observing samples of a distribution $\phi(\ldots)$ with n samples $\{X_1, X_2, \ldots, X_n\}$ observed until now.

We define the Record or Running Maximum as

$$K_n = \max(X_1, X_2, \ldots, X_n) \tag{1}$$

and we want to infer the moments of the distribution and the statistical properties of the expected new Record for the subsequent new samples.

2.2 Moments

It is useful to think of the moments of order p as the sum of two integrals, one up to the current Record and the other representing the hidden moment:

$$\mathbb{E}\left[X^p\right] = \int\limits_{L}^{K_m} x^p \phi\left(x\right) dx + \int\limits_{K_m}^{+\infty} x^p \phi\left(x\right) dx \tag{2}$$

Which can also be written as:

$$\mathbb{E}\left[X^p\right] = \mu_{L,p} + \mu_{K,p} \tag{3}$$

2.3 Distributions

For the InverseCDF, we assume $0 \leq p \leq 1$.

2.3.1 Uniform

- Domain:$[0, 1]$
- PDF: $f\left(x\right) = \begin{cases} 1 & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$
- CDF: $F\left(x\right) = \begin{cases} 0 & x < 0 \\ x & x \in [0,1] \\ 1 & x > 1 \end{cases}$
- Survival Function: $S\left(x\right) = \begin{cases} 1 & x < 0 \\ 1 - x & x \in [0,1] \\ 0 & x > 1 \end{cases}$
- InverseCDF: $I\left(p\right) = p$
- Shadow Moment: $\mu_{K,p} = \begin{cases} \frac{1-K^{p+1}}{p+1} & K \in (0,1) \\ 0 & K \notin (0,1) \end{cases}$

2.3.2 Gaussian

- Domain:$(-\infty, +\infty)$
- PDF: $f\left(\mu, \sigma, x\right) = \frac{e^{-\frac{(x-\mu)^2}{2\cdot\sigma^2}}}{\sqrt{2\pi}\cdot\sigma}$
- CDF: $F\left(\mu, \sigma, x\right) = \frac{1}{2} \cdot \mathrm{Erfc}\left(\frac{\mu - x}{\sqrt{2}\cdot\sigma}\right)$
- Survival Function: $S\left(\mu, \sigma, x\right) = \frac{1}{2}\left(\mathrm{erf}\left(\frac{\mu-x}{\sqrt{2}\cdot\sigma}\right) + 1\right)$
- InverseCDF: $I\left(\mu, \sigma, p\right) = \mu - \sqrt{2} \cdot \sigma \cdot \mathrm{erfc}^{-1}(2 \cdot p)$
- Shadow Moment $(\mu = 0, \sigma = 1)$: $\mu_{K,p} = \frac{2^{\frac{p}{2}-1} \cdot \Gamma\left(\frac{p+1}{2}, \frac{K^2}{2}\right)}{\sqrt{\pi}}$

2.3.3 Cauchy

- Domain:$(-\infty, +\infty)$
- PDF: $f(\mu, \gamma, x) = \frac{1}{\pi \cdot \gamma \cdot \left(\frac{(x-\mu)^2}{\gamma^2}+1\right)}$
- CDF: $F(\mu, \gamma, x) = \frac{1}{2} + \frac{\tan^{-1}\left(\frac{x-\mu}{\gamma}\right)}{\pi}$
- Survival Function: $S(\mu, \gamma, x) = \frac{1}{2} - \frac{\tan^{-1}\left(\frac{x-\mu}{\gamma}\right)}{\pi}$
- InverseCDF: $I(\mu, \sigma, p) = \mu - \gamma \cdot \cot(\pi \cdot p)$
- Shadow Moment $(\mu = 0, p = 0)$: $\mu_{K,0} = \frac{\tan^{-1}\left(\frac{\gamma}{K}\right)}{\pi}$

2.3.4 Student's T

- Domain:$(-\infty, +\infty)$
- PDF: $f(\mu, \sigma, \nu, x) = \frac{\left(\frac{\nu}{\nu+\frac{(x-\mu)^2}{\sigma^2}}\right)^{\frac{\nu+1}{2}}}{\sqrt{\nu} \cdot \sigma \cdot B\left(\frac{\nu}{2}, \frac{1}{2}\right)}$
- CDF: $F(\mu, \sigma, \nu, x) = \begin{cases} \frac{1}{2} I_{\frac{\nu \cdot \sigma^2}{(x-\mu)^2+\nu\sigma^2}}\left(\frac{\nu}{2}, \frac{1}{2}\right) & x \leq \mu \\ 1 - \frac{1}{2} I_{\frac{\nu \cdot \sigma^2}{(x-\mu)^2+\nu\sigma^2}}\left(\frac{\nu}{2}, \frac{1}{2}\right) & x \geq \mu \end{cases}$
- Survival Function: $S(\mu, \sigma, \nu, x) = \begin{cases} 1 - \frac{1}{2} I_{\frac{\nu \cdot \sigma^2}{(x-\mu)^2+\nu\sigma^2}}\left(\frac{\nu}{2}, \frac{1}{2}\right) & x \leq \mu \\ \frac{1}{2} I_{\frac{\nu \cdot \sigma^2}{(x-\mu)^2+\nu\sigma^2}}\left(\frac{\nu}{2}, \frac{1}{2}\right) & x \geq \mu \end{cases}$
- InverseCDF: $I(\mu, \sigma, \nu, p) = \begin{cases} \mu - \sqrt{\nu} \cdot \sigma \cdot \sqrt{\frac{1}{I_{2p}^{-1}\left(\frac{\nu}{2}, \frac{1}{2}\right)} - 1} & p < \frac{1}{2} \\ \mu & p = \frac{1}{2} \\ \mu + \sqrt{\nu} \cdot \sigma \cdot \sqrt{\frac{1}{I_{2-2p}^{-1}\left(\frac{\nu}{2}, \frac{1}{2}\right)} - 1} & p < \frac{1}{2} \end{cases}$
- Shadow Moment $(\mu = 0, \sigma = 1)$: $\mu_{K,p} = \frac{\nu^{\frac{\nu+1}{2}-\frac{1}{2}} \cdot K^{p-\nu} \cdot {}_2F_1\left(\frac{\nu+1}{2}, \frac{\nu-p}{2}; \frac{1}{2} \cdot (-p+\nu+2); -\frac{\nu}{K^2}\right)}{B\left(\frac{\nu}{2}, \frac{1}{2}\right) \cdot (\nu-p)}$

2.3.5 Pareto

- Domain:$(L, +\infty)$
- PDF: $f(L, \alpha, x) = \begin{cases} 0 & x < L \\ \frac{\alpha}{x} \cdot \left(\frac{x}{L}\right)^{-\alpha} & x \geq L \end{cases}$
- CDF: $F(L, \alpha, x) = \begin{cases} 0 & x < L \\ 1 - \left(\frac{x}{L}\right)^{-\alpha} & x \geq L \end{cases}$

- Survival Function: $S(L,\alpha,x) = \begin{cases} 1 & x < L \\ \left(\frac{x}{L}\right)^{-\alpha} & x \geq L \end{cases}$
- InverseCDF: $I(L,\alpha,p) = L \cdot (1-p)^{-1/\alpha}$
- Shadow Moment: $\mu_{K,p} = \frac{\alpha \cdot L^{\alpha} \cdot K^{p-\alpha}}{\alpha - p}$

2.3.6 LogNormal

- Domain:$(-\infty,+\infty)$
- PDF: $f(\mu,\sigma,x) = \begin{cases} 0 & x \leq 0 \\ \frac{e^{-\frac{(\log(x)-\mu)^2}{2\cdot\sigma^2}}}{\sqrt{2\pi}\cdot\sigma\cdot x} & x > 0 \end{cases}$
- CDF: $F(\mu,\sigma,x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{2} \cdot \mathrm{erfc}\left(\frac{\mu - \log(x)}{\sqrt{2}\cdot\sigma}\right) & x > 0 \end{cases}$
- Survival Function: $S(\mu,\sigma,x) = \begin{cases} 1 & x \leq 0 \\ \frac{1}{2} \cdot \mathrm{erfc}\left(\frac{-\mu + \log(x)}{\sqrt{2}\cdot\sigma}\right) & x > 0 \end{cases}$
- InverseCDF: $I(\mu,\sigma,p) = e^{\mu - \sqrt{2}\cdot\sigma\cdot\mathrm{erfc}^{-1}(2\cdot p)}$
- Shadow Moment $(\mu = 0, \sigma = 1)$: $\mu_{K,p} = \frac{1}{2} \cdot e^{\frac{p^2}{2}} \cdot \left(\mathrm{erf}\left(\frac{p - \log(K)}{\sqrt{2}}\right) + 1\right)$

2.4 Inverse CDF charts

Let's start with the Uniform and the Gaussian (Figure 1):

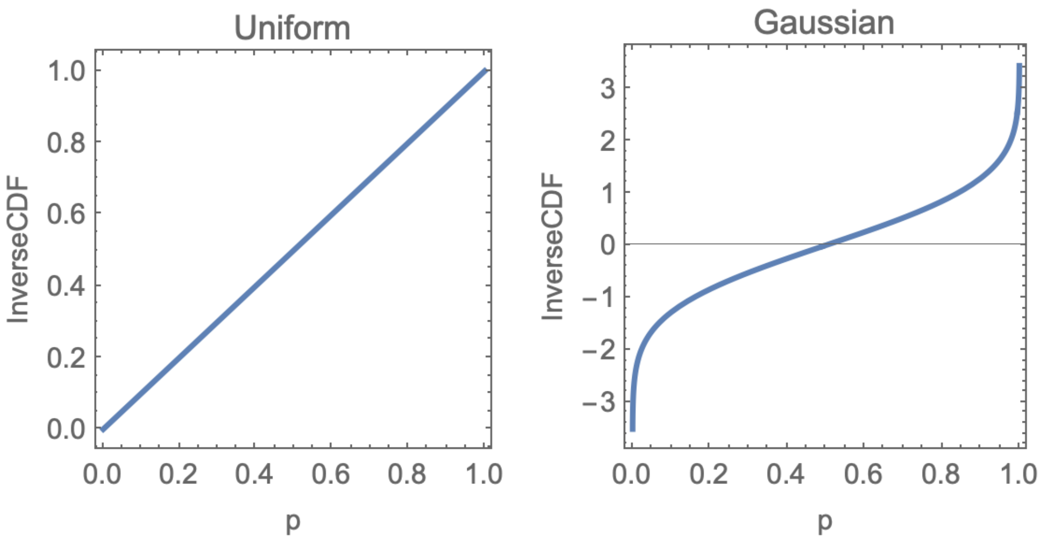

Figure 1: Inverse CDFs: Uniform and Gaussian

Then the Student's T and Pareto (Figure 2):

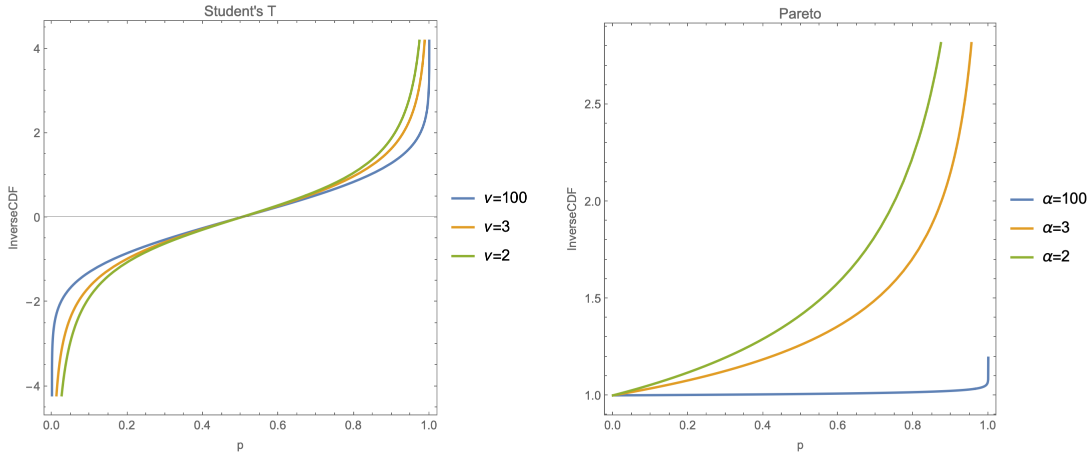

Figure 2: Inverse CDFs: Student's T and Pareto

3 Records

3.1 Fisher-Tippett-Gnedenko

Following [Embrechts et al (1997)], the Fisher-Tippett-Gnedenko theorem answers two questions: (i) What are the possible (non-degenerate) limit laws for the maxima K_m when properly normalized and centered? and (ii) Which distributions satisfy for all n $>$ 2 the identity in law $\max(X_1, X_2, \ldots, X_m) \stackrel{d}{=} c_m X + d_m$ for appropriate constants $c_m > 0$ and $d_m \in \mathbb{R}$ (norming constants)?

The answer lies in the Max-stable distributions $\left(z = \frac{x-\mu}{\beta}\right)$:

- Fréchet: $PDF\left[\Phi_\alpha(\alpha, \beta, \mu, x)\right] = \frac{\alpha e^{-z^{-\alpha}} z^{-\alpha-1}}{\beta}$, $x > \mu$ (Cauchy, Pareto, Student's T)
- Gumbel: $PDF\left[\Lambda(\beta, \mu, x)\right] = \frac{e^{-\left(z+e^{-z}\right)}}{\beta}$ (Gaussian, LogNormal)
- Weibull: $PDF\left[\psi_\alpha(\alpha, \beta, \mu, x)\right] = \frac{\alpha e^{-(-z)^\alpha}(-z)^{\alpha-1}}{\beta}$, $x < \mu$ (Uniform, Beta)

3.2 Original paper deductions

We're at the tail, after a point K, where the survival function can be approximated by:

$$P\left[X > x\right] \approx \left(\frac{x}{L}\right)^{-\alpha}$$

With a PDF given by:

$$\frac{\partial\left[1 - \left(\frac{x}{L}\right)^{-\alpha}\right]}{\partial x} = \alpha \cdot L^\alpha \cdot x^{-\alpha-1}$$

As we saw for the Pareto. The Student's T has this behavior (Figure 3):

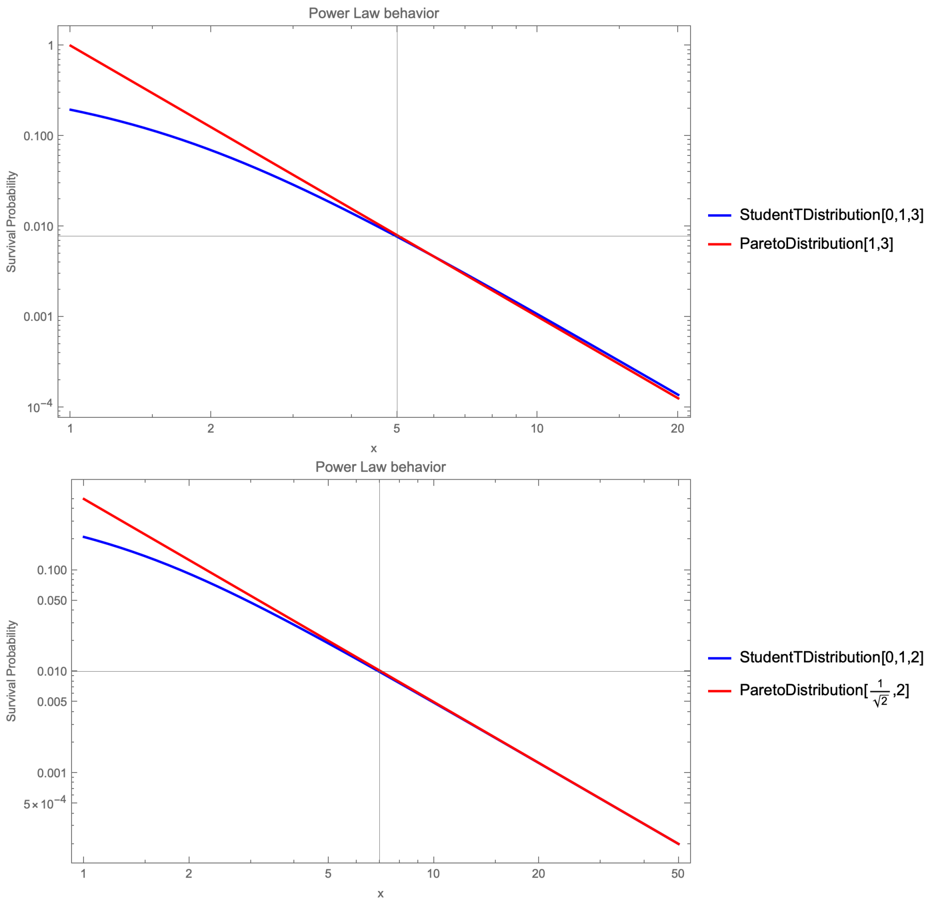

Figure 3: Inverse CDFs: Student's T and Pareto

The hidden moments of order p (for $p < \alpha$) are:

$$\mu_{K,p} = \int_K^\infty \left[x^p \cdot \alpha \cdot L^\alpha \cdot x^{-\alpha-1}\right] dx$$
$$\mu_{K,p} = \int_K^\infty \left[\alpha \cdot L^\alpha \cdot x^{p-\alpha-1}\right] dx$$

$$\mu_{K,p} = \frac{\alpha \cdot L^\alpha \cdot K^{p-\alpha}}{\alpha - p} \tag{4}$$

Now we will substitute K for its distribution and use the Transformed Distributions method described in A.1.

The original paper [Taleb (2020)] compares the PDF of the Record of a Pareto after n observations:

$$\psi(n, x) = n \cdot \left(1 - \left(\frac{L}{x}\right)^\alpha\right)^{n-1} \cdot \frac{\alpha}{x} \cdot \left(\frac{L}{x}\right)^\alpha$$

With the PDF of the Frechét distribution as the appropriate Max-stable distribution corresponding to the Pareto:

$$\varphi(x) = \frac{\alpha}{x} \cdot \left(\frac{\beta}{x}\right)^\alpha \cdot \exp\left(-\left(\frac{x}{\beta}\right)^{-\alpha}\right)$$

We could match the means of both distributions:

$$-\frac{L \cdot n \cdot \Gamma\left(n\right) \cdot \Gamma\left(-\frac{1}{\alpha}\right)}{\alpha\left(n-\frac{1}{\alpha}\right) \cdot \Gamma\left(n-\frac{1}{\alpha}\right)} = \beta \cdot \left(-\frac{1}{\alpha}\right) \cdot \Gamma\left(-\frac{1}{\alpha}\right)$$

Leading to:

$$\beta = \frac{L \cdot n \cdot \Gamma(n)}{\left(n-\frac{1}{\alpha}\right) \cdot \Gamma\left(n-\frac{1}{\alpha}\right)} \tag{5}$$

Or follow [Taleb 2020 SCOFT] and look at the limit:

$$\lim_{x\to\infty}\left(\frac{\psi\left(n,x\right)}{\varphi\left(x\right)}\right) = \lim_{x\to\infty}\left(n \cdot \left(\frac{L}{\beta}\right)^{\alpha} \cdot e^{\left(\frac{x}{\beta}\right)^{-\alpha}} \cdot \left(1-L^{\alpha}x^{-\alpha}\right)^{n-1}\right) = n\left(\frac{L}{\beta}\right)^{\alpha}$$

And make it equal to 1; this leads to (isolating β):

$$\beta = L \cdot n^{\frac{1}{\alpha}} \tag{6}$$

The two formulas for β produce similar results, so we will use the simpler formula (Equation 6).

Substituting:

$$\varphi\left(x\right) = \frac{\alpha}{x} \cdot \left(\frac{L \cdot n^{\frac{1}{\alpha}}}{x}\right)^{\alpha} \cdot \exp\left(-\left(\frac{x}{L \cdot n^{\frac{1}{\alpha}}}\right)^{-\alpha}\right)$$

$$\varphi\left(x\right) = \frac{\alpha}{x} \cdot \left(\frac{L}{x}\right)^{\alpha} \cdot n \cdot \exp\left(-\left(\frac{L}{x}\right)^{\alpha} \cdot n\right)$$

Now applying the formula for the Transformed Distribution:

$$f_Y\left(y\right) = f_X\left(v\left(y\right)\right) \cdot \left|\frac{\partial v\left(y\right)}{\partial y}\right| \tag{7}$$

With:

$$y = u\left(K\right) = \frac{\alpha \cdot L^{\alpha} \cdot K^{p-\alpha}}{\alpha - p}$$

$$v\left(y\right) = \left(\frac{y \cdot \left(\alpha - p\right)}{\alpha \cdot L^{\alpha}}\right)^{\frac{1}{p-\alpha}}$$

$$\left|\frac{\partial v\left(y\right)}{\partial y}\right| = \left|\left(-\frac{1}{\alpha - p}\right) \cdot \left(\frac{y \cdot \left(\alpha - p\right)}{\alpha \cdot L^{\alpha}}\right)^{\frac{1}{p-\alpha}-1} \cdot \left(\frac{\alpha - p}{\alpha \cdot L^{\alpha}}\right)\right|$$

$$\left|\frac{\partial v\left(y\right)}{\partial y}\right| = \left|\left(\frac{y \cdot \left(\alpha - p\right)}{\alpha \cdot L^{\alpha}}\right)^{\frac{1}{p-\alpha}} \cdot \left(\frac{\alpha \cdot L^{\alpha}}{y \cdot \left(\alpha - p\right)}\right) \cdot \left(\frac{1}{\alpha \cdot L^{\alpha}}\right)\right|$$

$$\left|\frac{\partial v\left(y\right)}{\partial y}\right| = \left|\left(\frac{y \cdot \left(\alpha - p\right)}{\alpha \cdot L^{\alpha}}\right)^{\frac{1}{p-\alpha}} \cdot \left(\frac{1}{y \cdot \left(\alpha - p\right)}\right)\right|$$

And:

$$f_X\left(v\left(y\right)\right) = \alpha \cdot L^{\alpha} \cdot \left(v\left(y\right)\right)^{-\alpha-1} \cdot n \cdot \exp\left(-\left(\frac{L}{v\left(y\right)}\right)^{\alpha} \cdot n\right)$$

$$f_X\left(v\left(y\right)\right) = \alpha \cdot L^{\alpha} \cdot \left(\frac{y \cdot (\alpha - p)}{\alpha \cdot L^{\alpha}}\right)^{-\frac{\alpha+1}{p-\alpha}} \cdot n \cdot \exp\left(-L^{\alpha} \cdot \left(\frac{y \cdot (\alpha - p)}{\alpha \cdot L^{\alpha}}\right)^{-\frac{\alpha}{p-\alpha}} \cdot n\right)$$

Leading to:

$$f_Y\left(y\right) = f_X\left(v\left(y\right)\right) \cdot \left|\frac{\partial v\left(y\right)}{\partial y}\right|$$

$$f_Y\left(y\right) = \frac{\alpha \cdot L^{\alpha}}{y \cdot (\alpha - p)} \cdot n \cdot \left(\frac{y \cdot (\alpha - p)}{\alpha \cdot L^{\alpha}}\right)^{-\frac{\alpha}{p-\alpha}} \cdot \exp\left(-L^{\alpha} \cdot \left(\frac{y \cdot (\alpha - p)}{\alpha \cdot L^{\alpha}}\right)^{-\frac{\alpha}{p-\alpha}} \cdot n\right)$$

$$f_Y\left(y\right) = n \cdot \left(\frac{y \cdot (\alpha - p)}{\alpha \cdot L^{\alpha}}\right)^{-\frac{\alpha}{p-\alpha}-1} \cdot \exp\left(-L^{\alpha} \cdot \left(\frac{y \cdot (\alpha - p)}{\alpha \cdot L^{\alpha}}\right)^{-\frac{\alpha}{p-\alpha}} \cdot n\right)$$

$$f_Y\left(y\right) = n \cdot \left(\frac{y \cdot (\alpha - p)}{\alpha \cdot L^{\alpha}}\right)^{\frac{p}{\alpha-p}} \cdot \exp\left(-n \cdot L^{\alpha} \cdot \left(\frac{y \cdot (\alpha - p)}{\alpha \cdot L^{\alpha}}\right)^{\frac{\alpha}{\alpha-p}}\right)$$

$$f_Y\left(y\right) = n \cdot L^{-\frac{\alpha \cdot p}{\alpha-p}} \cdot \left(y \cdot \left(1 - \frac{p}{\alpha}\right)\right)^{\frac{p}{\alpha-p}} \cdot \exp\left(-n \cdot L^{-\frac{\alpha \cdot p}{\alpha-p}} \cdot \left(y \cdot \left(1 - \frac{p}{\alpha}\right)\right)^{\frac{\alpha}{\alpha-p}}\right) \tag{8}$$

Which corresponds to Proposition 1 in [Taleb (2020)].
Let's simplify the formula:

$$n \cdot L^{-\frac{\alpha \cdot p}{\alpha-p}} = \lambda$$

$$y \cdot \left(1 - \frac{p}{\alpha}\right) = z$$

$$f_Z\left(z\right) = \lambda \cdot \left(z\right)^{\frac{p}{\alpha-p}} \cdot \exp\left(-\lambda \cdot \left(z\right)^{\frac{\alpha}{\alpha-p}}\right) \tag{9}$$

Integrating it:

$$\int_0^{+\infty} \left[\frac{f_Z\left(z\right)}{\left(1 - \frac{p}{\alpha}\right)}\right] dz = 1$$

The mean is:

$$\int_0^{+\infty} \left[z \cdot \frac{f_Z\left(z\right)}{\left(1 - \frac{p}{\alpha}\right)}\right] dz = \lambda^{-\frac{\alpha-p}{\alpha}} \cdot \Gamma\left(2 - \frac{p}{\alpha}\right)$$

Substituting and going from z to y:

$$\mathbb{E}\left[y\right]=\frac{\left(n\cdot L^{-\frac{\alpha\cdot p}{\alpha-p}}\right)^{-\frac{\alpha-p}{\alpha}}\cdot\left(1-\frac{p}{\alpha}\right)\cdot\Gamma\left(1-\frac{p}{\alpha}\right)}{\left(1-\frac{p}{\alpha}\right)}$$

$$\mathbb{E}\left[y\right]=\left(n\right)^{-\frac{\alpha-p}{\alpha}}\cdot\left(L\right)^{p}\cdot\Gamma\left(1-\frac{p}{\alpha}\right) \tag{10}$$

Which is also in [Taleb (2020)].

4 Remark 1

4.1 Frechét

Following Equation 8 and substituting $p=0$, we have Remark 1 from [Taleb (2020)]:

$$f_{Y,p=0}\left(y\right)=n\cdot L^{-\frac{\alpha\cdot 0}{\alpha-0}}\cdot\left(y\cdot\left(1-\frac{0}{\alpha}\right)\right)^{\frac{0}{\alpha-0}}\cdot\exp\left(-n\cdot L^{-\frac{\alpha\cdot 0}{\alpha-0}}\cdot\left(y\cdot\left(1-\frac{0}{\alpha}\right)\right)^{\frac{\alpha}{\alpha-0}}\right)$$

$$f_{Y,p=0}\left(y\right)=n\cdot\exp\left(-n\cdot y\right) \tag{11}$$

Which shows that exceedance probability is independent of the thickness of the tails (α).

But (as Fergal asked during our discussion) what is the support of y? Shouldn't it be $[0,1]$ since it's a probability itself?

4.2 Uniform

Let's do the same calculations for the Uniform distribution. The CDF and PDF for the Record after n observations are:

$$H_U\left(n,x\right)=\left(x\right)^n$$

$$h_U\left(n,x\right)=n\cdot\left(x\right)^{n-1}$$

$$\mathbb{E}\left[h_U\left(n,x\right)\right]=\frac{n}{n+1}$$

The PDF and the mean are shown in Figure 4:

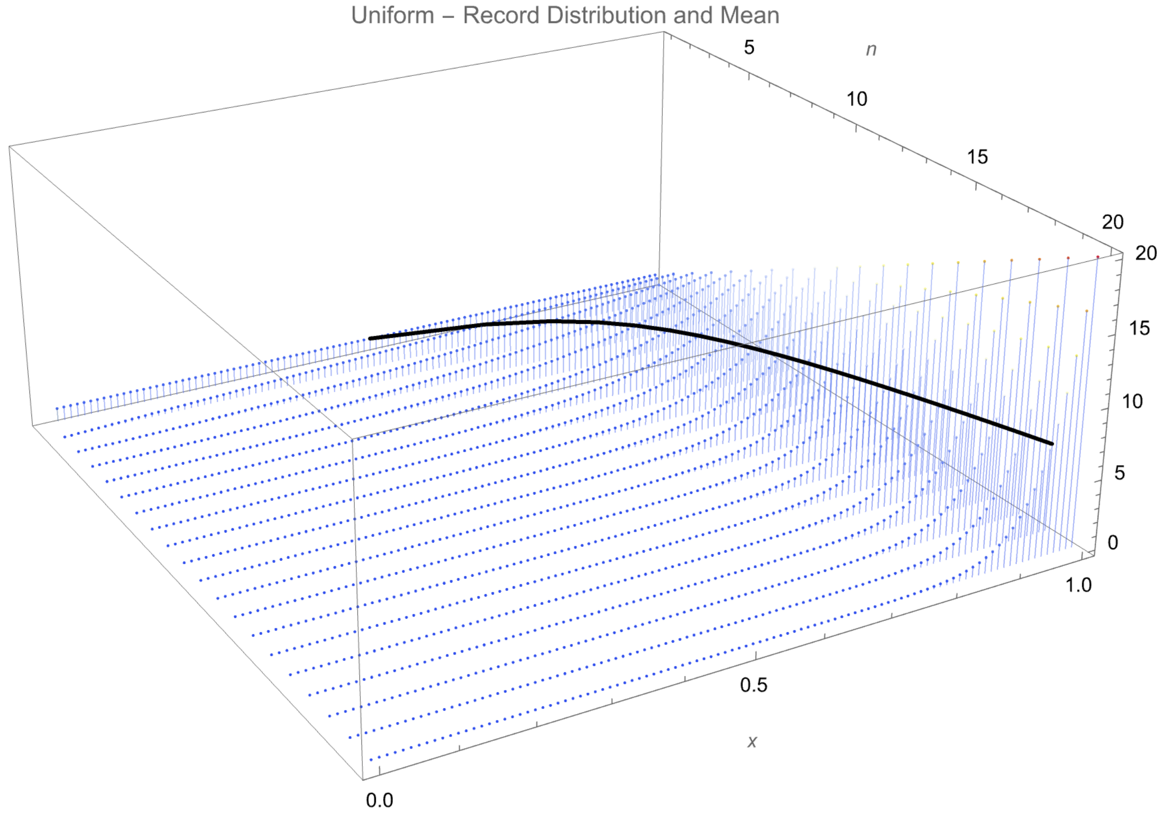

Figure 4: Uniform - Record Distribution and Mean

4.3 Inverse CDF trick

Integrate the product of the PDF of the Record for the Uniform using a probability p as the variable and the InverseCDF for the distribution.

Why? For IID random variables (any distribution!), the higher the record the more I'll wait for the next record (see [Embrechts et al (1997)], 6.2: "Exploratory Data Analysis for Extremes").

For the Uniform the average waiting time is:

$$\delta t\left(K\right)=\sum_{j=1}^{\infty}\left\{\left(1-K\right)\cdot K^{j-1}\cdot j\right\}=\frac{1}{1-K}$$

And the expected number of records up to n points is:

$$\mathbb{E}\left[N_K\right]=\log\left(n\right)+\gamma$$

Let's check these results. For 100K runs of $U\left[0,1\right]$ (10K points each) we mark the new record and for how many draws it persisted as the new record. This produces for each path a series of points $\left(K_j,\delta t_j\right)$ where $\delta t_j=t_{j+1}-t_j$ and t_j marks the draw where the jth new record appeared. We then plot the histogram of the pairs $\left(K_j,1-\frac{1}{\delta t_j}\right)$ as $\left(\log\left(1-K_j\right),\log\left(\frac{1}{\delta t_j}\right)\right)$ in Figure 5:

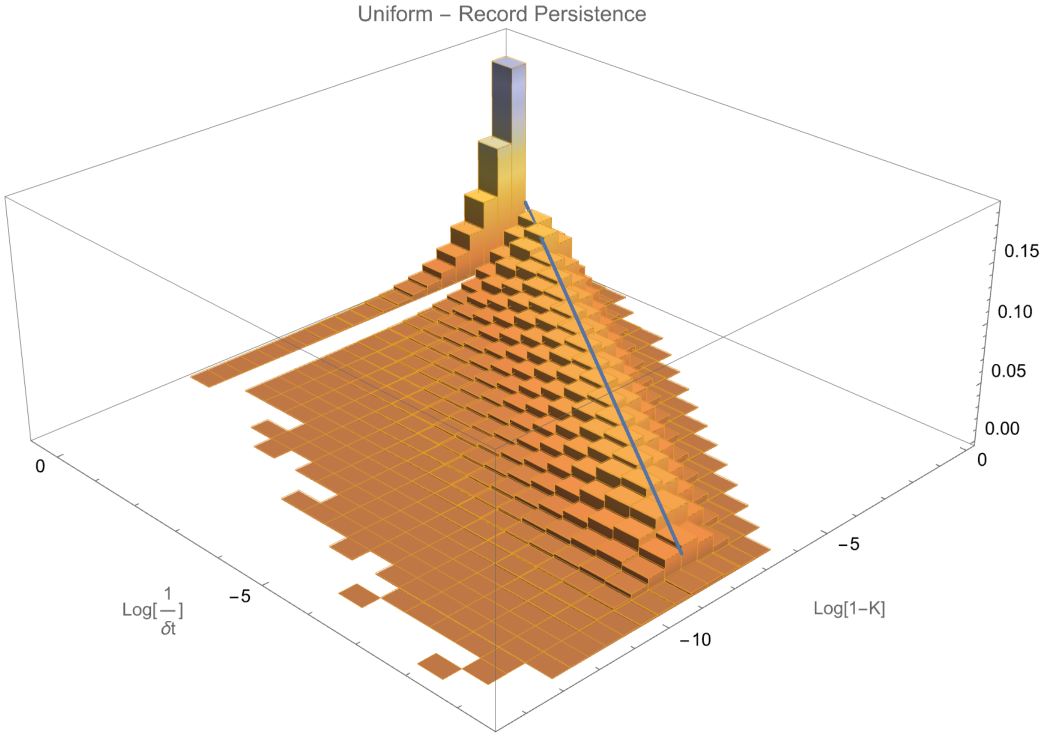

Figure 5: Uniform - Record Persistence

We can see how it follows the diagonal, as expected, since:

$$\delta t\,(K)\cdot(1-K)=1$$

$$K=1-\frac{1}{\delta t\,(K)}$$

$$1-K=\frac{1}{\delta t\,(K)}$$

For the same 100K runs of $U\,[0,1]$ (10K points each) we count the number of new records in each run and plot the histogram of the counts, centered around the expected mean of $\log(10000)+\gamma=9.79$ in Figure 6:

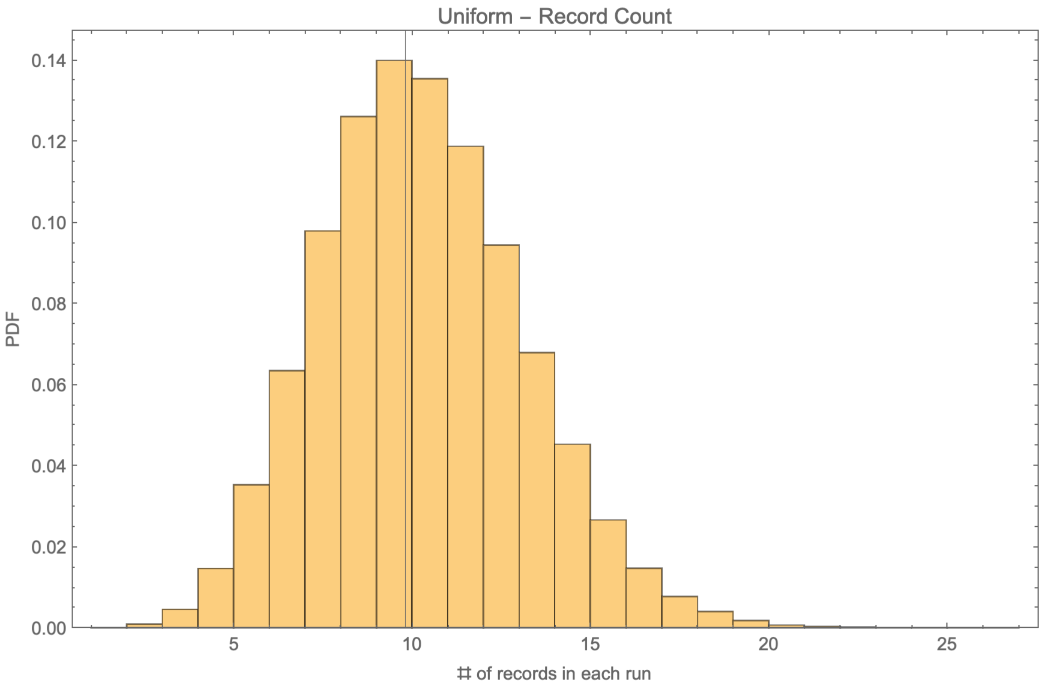

Figure 6: Uniform - Record Count

And for any distribution we can think about the representation of the records on the InverseCDF plot as: Every new record is a record on the Uniform, and the probability of exceedance is independent of the distribution (Remark 1), and the InverseCDF gives us the expected value of the exceedance (higher for fat-tailed distributions, etc.).

4.4 Pareto (again)

4.4.1 Record

As derived above:

$$h_P\left(n,x\right) = n\cdot\left(1-\left(\frac{L}{x}\right)^{\alpha}\right)^{n-1}\cdot\frac{\alpha}{x}\cdot\left(\frac{L}{x}\right)^{\alpha}$$

The expected value integral will be helped by a change of variables.

$$\mathbb{E}\left[h_P\left(n,x\right)\right] = \int_0^{+\infty}\left[x\cdot h_P\left(n,x\right)\right]dx = -\frac{L\cdot n\cdot B\left(n,-\frac{1}{\alpha}\right)}{\alpha\cdot\left(n-\frac{1}{\alpha}\right)}$$

But we can also integrate the product of the PDF of the Record for the Uniform using the probability p as the variable and the InverseCDF for the distribution:

$$\int_0^1\left[L\left(1-p\right)^{-\frac{1}{\alpha}}\right]\left[n\cdot p^{n-1}\right]dp = -\frac{L\cdot n\cdot B\left(n,-\frac{1}{\alpha}\right)}{\alpha\cdot\left(n-\frac{1}{\alpha}\right)}$$

This integral is easier and can be calculated only at the tail (very small relative error for large values of n):

$$\int_{0}^{1}\left[L\,(1-p)^{-\frac{1}{\alpha}}\right]\left[n\cdot p^{n-1}\right]dp \approx \int_{1-\varepsilon}^{1}\left[L\,(1-p)^{-\frac{1}{\alpha}}\right]\left[n\cdot p^{n-1}\right]dp$$

The PDF and the mean are shown in Figure 7:

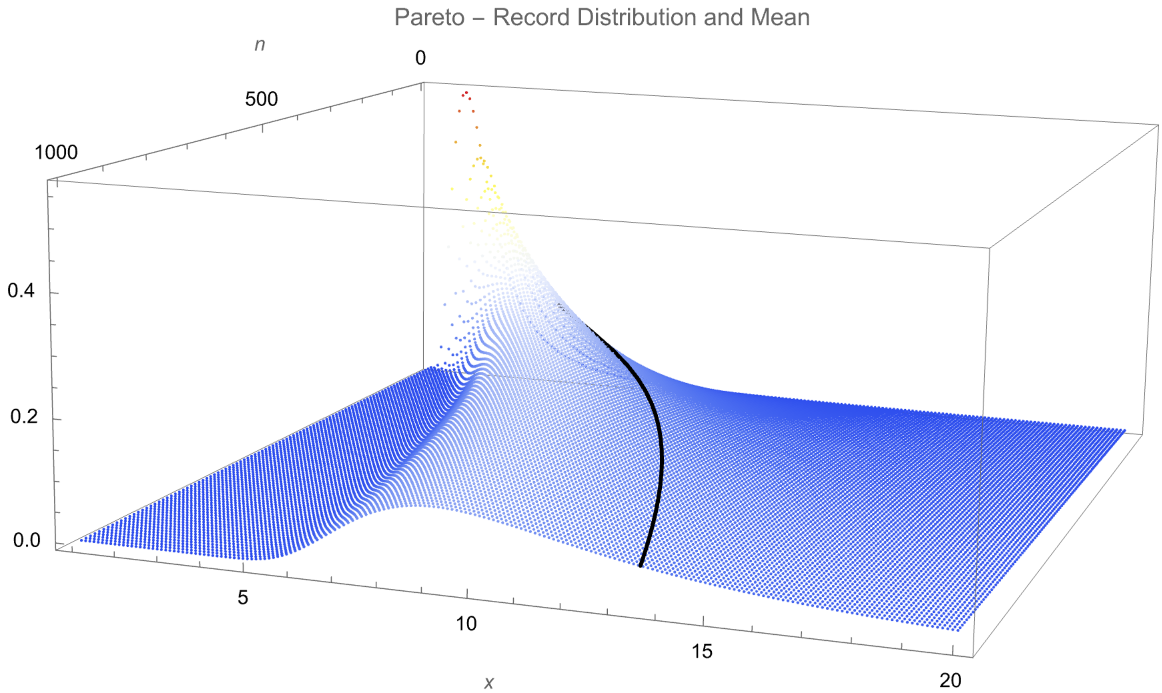

Figure 7: Pareto - Record Distribution and Mean

4.4.2 A new trick

And now we can do another trick:
Let's solve for X in the formula of the CDF of the Record:

$$Y = \left(1 - L^{\alpha}\cdot X^{-\alpha}\right)^{n} \Rightarrow X = L\cdot\left(1 - Y^{\frac{1}{n}}\right)^{-\frac{1}{\alpha}}$$

And replace K in the Shadow Moment formula (Equation 4):

$$\mu_{K,p} = \frac{\alpha\cdot L^{\alpha}\cdot K^{p-\alpha}}{(\alpha-p)} = \frac{\alpha\cdot L^{\alpha}\cdot\left(L\cdot\left(1-Y^{\frac{1}{n}}\right)^{-\frac{1}{\alpha}}\right)^{p-\alpha}}{(\alpha-p)}$$

$$\mu_{K,p} = \frac{\alpha\cdot L^{\alpha+p-\alpha}\cdot\left(1-Y^{\frac{1}{n}}\right)^{\frac{\alpha-p}{\alpha}}}{(\alpha-p)}$$

And now solve for Y:

$$\left(1-Y^{\frac{1}{n}}\right)^{\frac{\alpha-p}{\alpha}} = \frac{\mu_{K,p}\cdot(\alpha-p)}{\alpha\cdot L^{p}}$$

$$\left(1-Y^{\frac{1}{n}}\right)=\left(\mu_{K,p}\cdot L^{-p}\cdot\left(1-\frac{p}{\alpha}\right)\right)^{\frac{\alpha}{\alpha-p}}$$

$$Y=\left(1-\left(\mu_{K,p}\cdot L^{-p}\cdot\left(1-\frac{p}{\alpha}\right)\right)^{\frac{\alpha}{\alpha-p}}\right)^{n} \tag{12}$$

We have just found the Survival Function for the Shadow Moment of order p. So the CDF and PDF are, respectively (substituting $\mu_{K,p}\to x$):

$$\Omega\left(x\right)=1-\left(1-\left(x\cdot L^{-p}\cdot\left(1-\frac{p}{\alpha}\right)\right)^{\frac{\alpha}{\alpha-p}}\right)^{n} \tag{13}$$

$$\omega\left(x\right)=-n\cdot\left(1-\left(x\cdot L^{-p}\cdot\left(1-\frac{p}{\alpha}\right)\right)^{\frac{\alpha}{\alpha-p}}\right)^{n-1}\cdot f_1$$

$$f_1=\left(-\left(\frac{\alpha}{\alpha-p}\right)\cdot\left(x\cdot L^{-p}\cdot\left(1-\frac{p}{\alpha}\right)\right)^{\frac{\alpha}{\alpha-p}-1}\right)\cdot L^{-p}\cdot\left(1-\frac{p}{\alpha}\right)$$

$$f_1=-L^{-p}\cdot\left(x\cdot L^{-p}\cdot\left(1-\frac{p}{\alpha}\right)\right)^{\frac{p}{\alpha-p}}$$

$$f_1=-L^{-p\cdot\left(1+\frac{p}{\alpha-p}\right)}\cdot\left(x\cdot\left(1-\frac{p}{\alpha}\right)\right)^{\frac{p}{\alpha-p}}$$

$$f_1=-L^{-\left(\frac{\alpha\cdot p}{\alpha-p}\right)}\cdot\left(x\cdot\left(1-\frac{p}{\alpha}\right)\right)^{\frac{p}{\alpha-p}}$$

$$f_1=-\left(x\cdot L^{-\alpha}\cdot\left(1-\frac{p}{\alpha}\right)\right)^{\frac{p}{\alpha-p}}$$

$$\omega\left(x\right)=n\cdot\left(x\cdot L^{-\alpha}\cdot\left(1-\frac{p}{\alpha}\right)\right)^{\frac{p}{\alpha-p}}\cdot\left(1-\left(x\cdot L^{-p}\cdot\left(1-\frac{p}{\alpha}\right)\right)^{\frac{\alpha}{\alpha-p}}\right)^{n-1} \tag{14}$$

For $p=0$:

$$\omega\left(x\right)=n\cdot\left(1-x\right)^{n-1} \tag{15}$$

With mean:

$$\mathbb{E}\left[\omega\left(x\right)\right]=\frac{1}{n+1} \tag{16}$$

A Geometric Distribution equivalent to the Exponential Distribution of Remark 1. The PDF and the mean are shown in Figure 8:

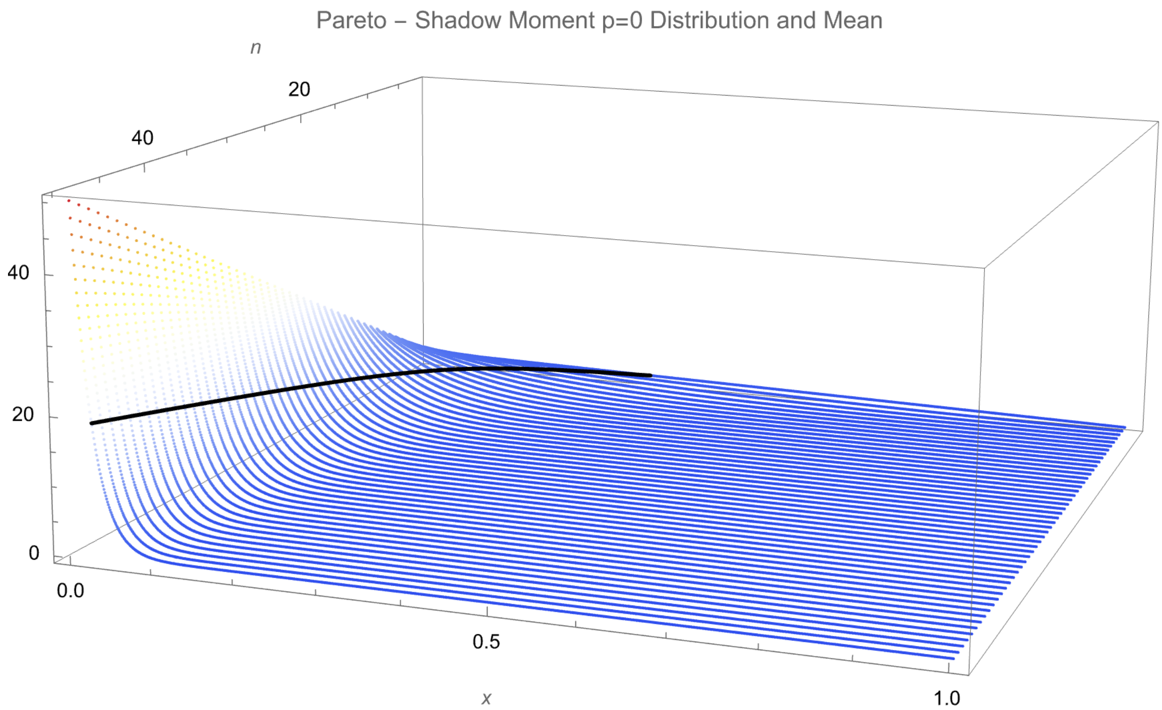

Figure 8: Pareto - Shadow Moment p=0 Distribution and Mean

For $p = 1$:

$$\omega(x) = n \cdot \left(x \cdot L^{-\alpha} \cdot \left(1 - \frac{1}{\alpha}\right)\right)^{\frac{1}{\alpha-1}} \cdot \left(1 - \left(x \cdot L^{-1} \cdot \left(1 - \frac{1}{\alpha}\right)\right)^{\frac{\alpha}{\alpha-1}}\right)^{n-1} \tag{17}$$

Assuming $L = 1$ and using the properties described in A.2 the mean is easier to calculate:

$$\mathbb{E}[\omega(x)] = \frac{n \cdot B\left(n, 2 - \frac{1}{\alpha}\right)}{\left(1 - \frac{1}{\alpha}\right)} \tag{18}$$

The PDF and the mean for $\alpha = 3$ are shown in Figure 9:

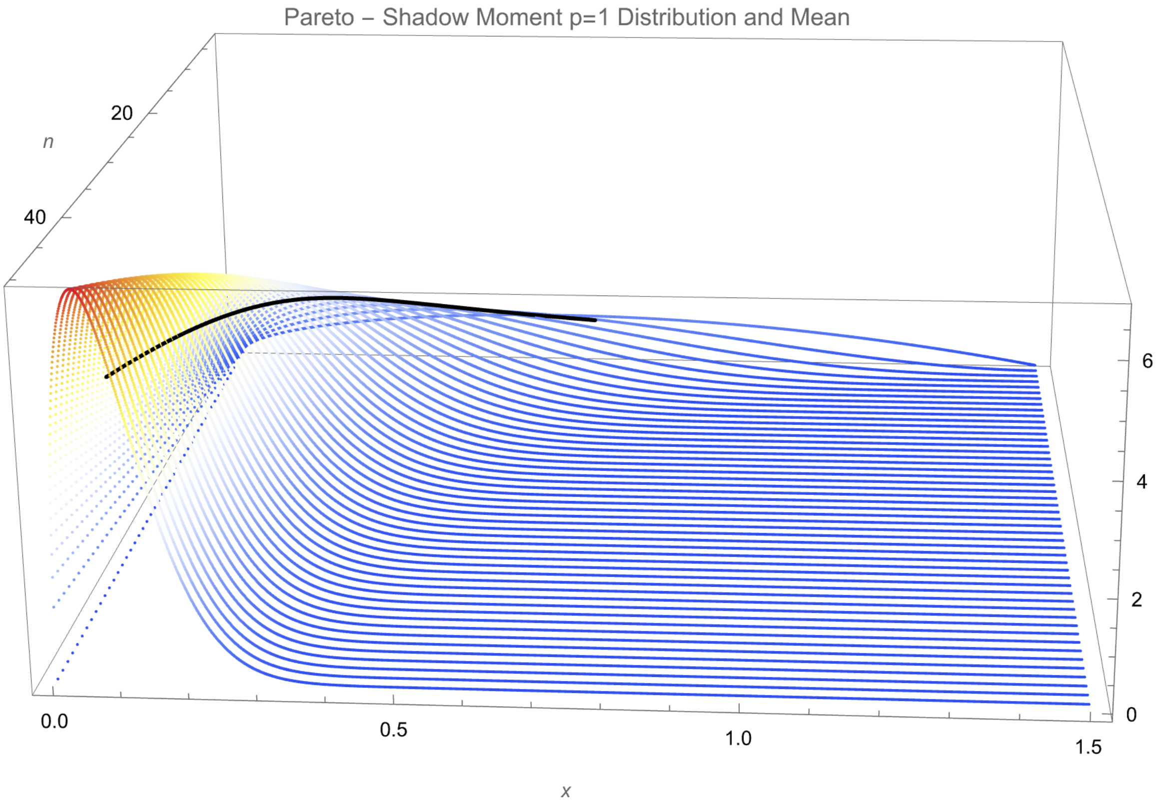

Figure 9: Pareto - Shadow Moment p=1 Distribution and Mean

4.4.3 Conditional Means

We can compare the means for the moments for each n and divide the means (equivalent to $\frac{n\cdot(n+1)\cdot B\left(n,2-\frac{1}{\alpha}\right)}{\left(1-\frac{1}{\alpha}\right)}$) as shown in Figure 10:

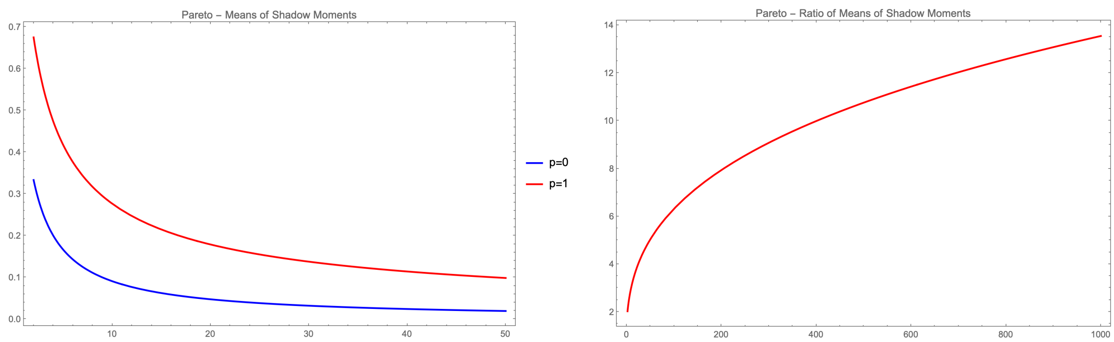

Figure 10: Pareto - Shadow Moment p=0 and p=1 - Means and Ratio (p=1 / p=0)

The equivalent LogLog plots are shown in Figure 11:

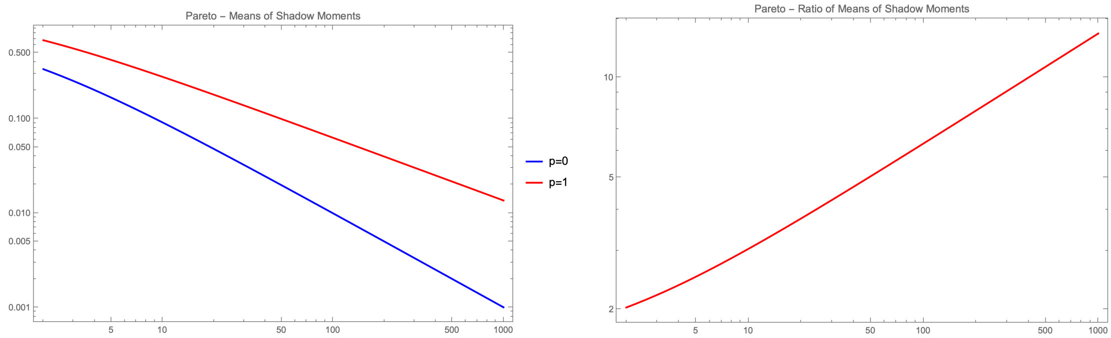

Figure 11: Pareto - Shadow Moment p=0 and p=1 - Means and Ratio (p=1 / p=0) LogLog

We can also look at the formula:

$$\frac{\mu_{K,1}}{\mu_{K,0}} = \frac{\alpha \cdot L^{\alpha} \cdot K^{1-\alpha}}{(\alpha-1)} \cdot \frac{(\alpha-0)}{\alpha \cdot L^{\alpha} \cdot K^{0-\alpha}} = \frac{\alpha \cdot K}{(\alpha-1)}$$

And substitute K by:

$$h_P\left(n,x\right) = n \cdot \left(1-\left(\frac{L}{x}\right)^{\alpha}\right)^{n-1} \cdot \frac{\alpha}{x} \cdot \left(\frac{L}{x}\right)^{\alpha}$$

To reach:

$$n \cdot \left(1-\left(\frac{L}{x}\right)^{\alpha}\right)^{n-1} \cdot \frac{\alpha}{x} \cdot \frac{\alpha}{(\alpha-1)} \cdot \left(\frac{L}{x}\right)^{\alpha}$$

Which we plot against the ratio of the means of the shadow moments in Figure 12:

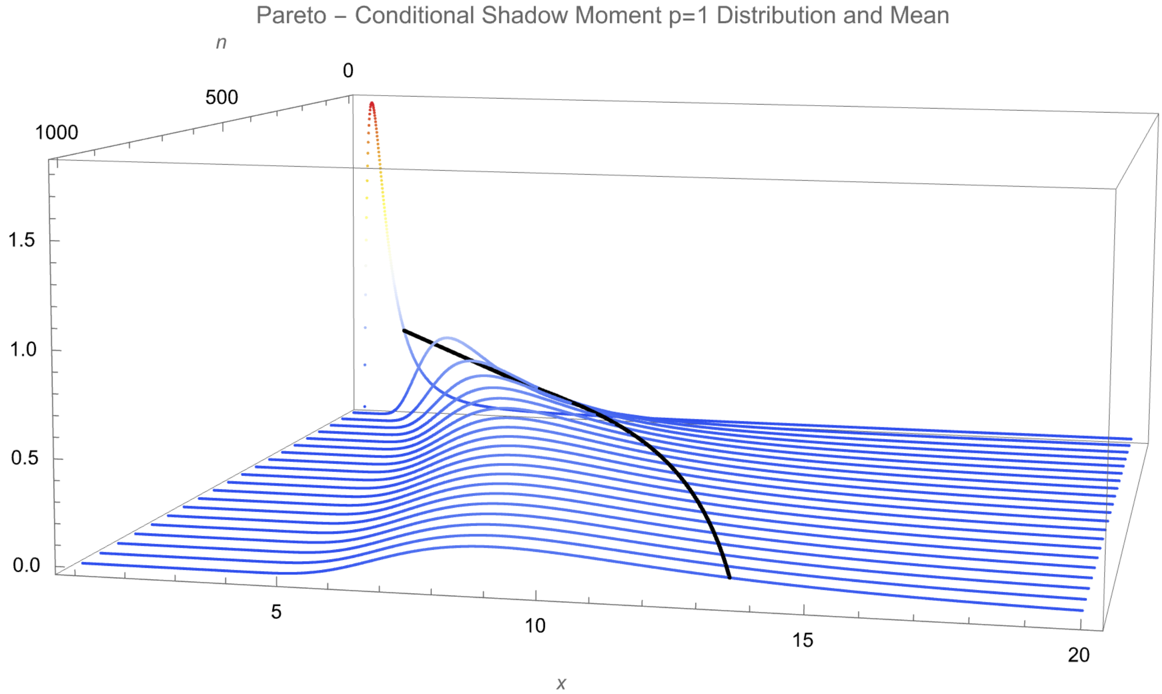

Figure 12: Pareto - Conditional Shadow Moment p=1 Distribution and Mean

4.4.4 What has changed?

Let's compare Equation 8 with Equation 14:

$$f_Y(y) = n \cdot L^{-\frac{\alpha \cdot p}{\alpha - p}} \cdot \left(y \cdot \left(1 - \frac{p}{\alpha}\right)\right)^{\frac{p}{\alpha - p}} \cdot \exp\left(-n \cdot L^{-\frac{\alpha \cdot p}{\alpha - p}} \cdot \left(y \cdot \left(1 - \frac{p}{\alpha}\right)\right)^{\frac{\alpha}{\alpha - p}}\right)$$

$$\omega(x) = n \cdot L^{-\frac{\alpha \cdot p}{\alpha - p}} \cdot \left(x \cdot \left(1 - \frac{p}{\alpha}\right)\right)^{\frac{p}{\alpha - p}} \cdot \left(1 - L^{-\frac{\alpha \cdot p}{\alpha - p}} \cdot \left(x \cdot \left(1 - \frac{p}{\alpha}\right)\right)^{\frac{\alpha}{\alpha - p}}\right)^{n-1}$$

The only change is the substitution of the exponential distribution by a geometric distribution.

4.4.5 Limits

Looking for the x that solves:

$$1 - L^{-\frac{\alpha \cdot p}{\alpha - p}} \cdot \left(x_{max} \cdot \left(1 - \frac{p}{\alpha}\right)\right)^{\frac{\alpha}{\alpha - p}} = 0$$

We find:

$$x_{max} = \left(\frac{\alpha}{\alpha - p}\right) \cdot \left(L^{\frac{\alpha \cdot p}{\alpha - p}}\right)^{1 - \frac{p}{\alpha}}$$

$$x_{max} = \left(\frac{\alpha}{\alpha - p}\right) \cdot L^p$$

For $p = 0$:

$$x_{max} = 1$$

For $p = 1$:

$$x_{max} = \left(\frac{\alpha}{\alpha - 1}\right) \cdot L$$

What about other distributions?

4.5 Gaussian

4.5.1 Record

Integrating the PDF of the Record to estimate the mean:

$$\int_{-\infty}^{+\infty} \left[\frac{2^{\frac{1}{2} - n} n e^{-\frac{x^2}{2}} \operatorname{erfc}\left(-\frac{x}{\sqrt{2}}\right)^{n-1}}{\sqrt{\pi}}\right] [x] \, dx$$

Integrating the product of the PDF of the Record for the Uniform using a probability p as the variable and the InverseCDF for the distribution:

$$\int_0^1 \left[0 - \sqrt{2} \cdot 1 \cdot \text{erfc}^{-1}(2 \cdot p)\right] \left[n \cdot p^{n-1}\right] dp$$

Both integrals give us the same result when evaluated numerically, and have the same symmetry (ie, $\int_0^{+\infty}$ corresponds to $\int_{\frac{1}{2}}^{1}$).

The PDF and the mean are shown in Figure 13:

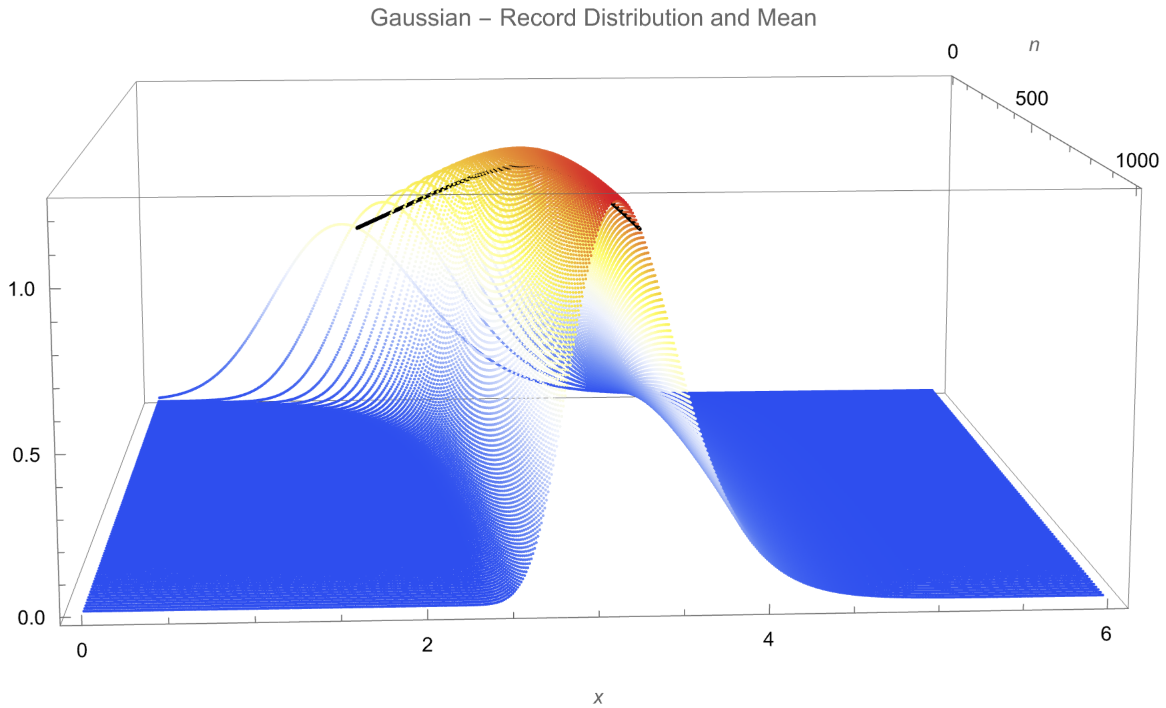

Figure 13: Gaussian - Record Distribution and Mean

4.5.2 The new trick

Let's solve for X in the formula of the CDF of the Record:

$$Y = \left(\frac{1}{2} \cdot \text{Erfc}\left(\frac{\mu - X}{\sqrt{2} \cdot \sigma}\right)\right)^n \Rightarrow X = \mu - \sqrt{2} \cdot \sigma \cdot \text{erfc}^{-1}\left(2 \cdot (Y)^{1/n}\right)$$

For the Standard Gaussian:

$$Y = \left(\frac{1}{2} \cdot \text{Erfc}\left(\frac{X}{\sqrt{2}}\right)\right)^n \Rightarrow X = -\sqrt{2} \cdot \text{erfc}^{-1}\left(2 \cdot (Y)^{1/n}\right)$$

And replace K in the Shadow Moment formula:

$$\mu_{K,p} = \frac{2^{\frac{p}{2}-1} \cdot \Gamma\left(\frac{p+1}{2}, \frac{K^2}{2}\right)}{\sqrt{\pi}}$$

$$\mu_{K,p} = \frac{2^{\frac{p}{2}-1} \cdot \Gamma\left(\frac{p+1}{2}, \left(\text{erfc}^{-1}\left(2 \cdot (Y)^{1/n}\right)\right)^2\right)}{\sqrt{\pi}}$$

And now solve for Y:
For $p=0$:

$$\mu_{K,0}=\frac{2^{-1}\cdot\Gamma\left(\frac{1}{2},\left(\operatorname{erfc}^{-1}\left(2\cdot(Y)^{1/n}\right)\right)^2\right)}{\sqrt{\pi}}$$

But

$$\Gamma\left(\frac{1}{2},z\right)=\sqrt{\pi}\cdot\operatorname{Erfc}\left(\sqrt{z}\right)$$

Which means that we have solutions for two values of Y such that:

$$\operatorname{erfc}^{-1}\left(2\cdot(Y_1)^{1/n}\right)=-\operatorname{erfc}^{-1}\left(2\cdot(Y_2)^{1/n}\right)$$

So:

$$2\cdot(Y_2)^{1/n}=2-2\cdot(Y_1)^{1/n}$$

$$(Y_2)^{1/n}=1-(Y_1)^{1/n}$$

Solving for Y_1:

$$\mu_{K,0}=\frac{2^{-1}\cdot\sqrt{\pi}\cdot\left(2\cdot(Y_1)^{1/n}\right)}{\sqrt{\pi}}$$

$$\mu_{K,0}=Y_1^{1/n}$$

$$Y_1=\mu_{K,0}^n$$

And therefore:

$$Y_2=(1-\mu_{K,0})^n$$

We have just found two Survival Functions for the Shadow Moment of order $p=0$. We will use the second solution $Y_2=(1-\mu_{K,0})^n$.

So the CDF and PDF for $p=0$ are, respectively (substituting $\mu_{K,p}\to x$):

$$\Omega(x)=1-(1-x)^n \tag{19}$$

$$\omega(x)=n\cdot(1-x)^{n-1} \tag{20}$$

The same as the solution for the Pareto.
Now for $p=1$:

$$\mu_{K,1}=\frac{2^{-\frac{1}{2}}\cdot\Gamma\left(1,\left(\operatorname{erfc}^{-1}\left(2\cdot(Y)^{1/n}\right)\right)^2\right)}{\sqrt{\pi}}$$

$$\mu_{K,1} = \frac{e^{-\text{erfc}^{-1}\left(2\cdot Y^{1/n}\right)^2}}{\sqrt{2\pi}}$$

$$\log\left(\sqrt{2\pi}\cdot\mu_{K,1}\right) = -\text{erfc}^{-1}\left(2\cdot Y^{1/n}\right)^2$$

Which means that we have solutions for two values of Y such that:

$$(Y_2)^{1/n} = 1 - (Y_1)^{1/n}$$

Solving for Y_1:

$$\sqrt{-\log\left(\sqrt{2\pi}\cdot\mu_{K,1}\right)} = \text{erfc}^{-1}\left(2\cdot Y_1^{1/n}\right)$$

$$Y_1^{1/n} = \frac{1}{2}\cdot\text{Erfc}\left(\sqrt{-\log\left(\sqrt{2\pi}\cdot\mu_{K,1}\right)}\right)$$

$$Y_1 = \left(\frac{1}{2}\cdot\text{Erfc}\left(\sqrt{-\log\left(\sqrt{2\pi}\cdot\mu_{K,1}\right)}\right)\right)^n$$

And therefore:

$$Y_2 = \left(1-\frac{1}{2}\cdot\text{Erfc}\left(\sqrt{-\log\left(\sqrt{2\pi}\cdot\mu_{K,1}\right)}\right)\right)^n$$

We have just found two Survival Functions for the Shadow Moment of order $p = 1$. We will use the second solution $Y_2 = \left(1-\frac{1}{2}\cdot\text{Erfc}\left(\sqrt{-\log\left(\sqrt{2\pi}\cdot\mu_{K,1}\right)}\right)\right)^n$.

So the CDF and PDF for $p = 1$ are, respectively (substituting $\mu_{K,p} \to x$):

$$\Omega(x) = 1 - \left(1-\frac{1}{2}\cdot\text{Erfc}\left(\sqrt{-\log\left(\sqrt{2\pi}\cdot\mu_{K,1}\right)}\right)\right)^n \tag{21}$$

$$\omega(x) = n\cdot\left(1-\frac{1}{2}\cdot\text{Erfc}\left(\sqrt{-\log\left(\sqrt{2\pi}\cdot x\right)}\right)\right)^{n-1}\cdot\frac{1}{2}\cdot\frac{\sqrt{2}}{\sqrt{-\log\left(\sqrt{2\pi}x\right)}}$$

$$\omega(x) = \frac{\sqrt{2}\cdot n}{\sqrt{-2\cdot\log\left(\sqrt{2\pi}x\right)}}\cdot\left(1-\frac{1}{2}\cdot\text{Erfc}\left(\sqrt{-\log\left(\sqrt{2\pi}\cdot x\right)}\right)\right)^{n-1} \tag{22}$$

And with $-\log\left(\sqrt{2\pi}\cdot x\right) > 0$ we find that the support of x is:

$$x \in \left[0, \frac{1}{\sqrt{2\pi}}\right]$$

Other distributions like the Student's T should show clearly the transition between the Gaussian/Uniform behavior and the Pareto/Cauchy behavior.

The analysis of LogNormal distribution in this framework is an interesting goal left to the reader.

4.6 Truncated Pareto

4.6.1 Motivation

Following [Cirillo Taleb 2016], we look at the Truncated Pareto distribution where losses can be large but not too large.

4.6.2 Definition

- Domain:(L, H) , $H > L$
- PDF: $f(L, H, \alpha, x) = \begin{cases} 0 & x < L \\ \left(\frac{1}{1-\left(\frac{L}{H}\right)^{\alpha}}\right) \cdot \frac{\alpha}{x} \cdot \left(\frac{x}{L}\right)^{-\alpha} & L \leq x \leq H \\ 0 & x > H \end{cases}$
- CDF: $F(L, \alpha, x) = \begin{cases} 0 & x < L \\ \left(\frac{1}{1-\left(\frac{L}{H}\right)^{\alpha}}\right) \cdot \left(1 - \left(\frac{x}{L}\right)^{-\alpha}\right) & L \leq x \leq H \\ 1 & x > H \end{cases}$
- Survival Function: $S(L, \alpha, x) = \begin{cases} 1 & x < L \\ 1 - \left(\frac{1}{1-\left(\frac{L}{H}\right)^{\alpha}}\right) \cdot \left(1 - \left(\frac{x}{L}\right)^{-\alpha}\right) & L \leq x \leq H \\ 0 & x > H \end{cases}$
- InverseCDF: $I(L, H, \alpha, p) = L \cdot \left(1 - p \cdot \left(1 - \left(\frac{L}{H}\right)^{\alpha}\right)\right)^{-1/\alpha}$
- Shadow Moment: $\mu_{K,p} = \left(\frac{1}{1-\left(\frac{L}{H}\right)^{\alpha}}\right) \cdot \frac{\alpha \cdot L^{\alpha} \cdot \left(K^{p-\alpha} - H^{p-\alpha}\right)}{(\alpha - p)}$

4.6.3 Shadow Moment Distribution

Let's define η:

$$\eta = \left(1 - \left(\frac{L}{H}\right)^{\alpha}\right) \tag{23}$$

Let's solve for X in the formula of the CDF of the Record:

$$Y = \left(\frac{1 - L^{\alpha} \cdot X^{-\alpha}}{1 - L^{\alpha} \cdot H^{-\alpha}}\right)^{n} \Rightarrow X = L \cdot \left(1 - \eta \cdot Y^{\frac{1}{n}}\right)^{-\frac{1}{\alpha}}$$

And replace K in the Shadow Moment formula:

$$\mu_{K,p} = \left(\frac{1}{\eta}\right) \cdot \frac{\alpha \cdot L^{\alpha} \cdot \left(K^{p-\alpha} - H^{p-\alpha}\right)}{(\alpha - p)}$$

$$\mu_{K,p} = \frac{\alpha \cdot L^{\alpha} \cdot \left(L^{p-\alpha} \cdot \left(1 - \eta \cdot Y^{\frac{1}{n}}\right)^{\frac{\alpha - p}{\alpha}} - H^{p-\alpha}\right)}{\eta \cdot (\alpha - p)}$$

$$\mu_{K,p} = \frac{\alpha \cdot \left(L^p \cdot \left(1 - \eta \cdot Y^{\frac{1}{n}}\right)^{\frac{\alpha-p}{\alpha}} - H^p \cdot \left(\frac{L}{H}\right)^{\alpha}\right)}{\eta \cdot (\alpha - p)}$$

$$\mu_{K,p} = \frac{\alpha \cdot L^p \cdot \left(\left(1 - \eta \cdot Y^{\frac{1}{n}}\right)^{\frac{\alpha-p}{\alpha}} - \left(\frac{L}{H}\right)^{\alpha-p}\right)}{\eta \cdot (\alpha - p)}$$

And now solve for Y:

$$\left(1 - \eta \cdot Y^{\frac{1}{n}}\right)^{\frac{\alpha-p}{\alpha}} - \left(\frac{L}{H}\right)^{\alpha-p} = \eta \cdot \frac{\mu_{K,p} \cdot (\alpha - p)}{\alpha \cdot L^p}$$

$$\left(1 - \eta \cdot Y^{\frac{1}{n}}\right)^{\frac{\alpha-p}{\alpha}} = \eta \cdot \frac{\mu_{K,p} \cdot (\alpha - p)}{\alpha \cdot L^p} + \left(\frac{L}{H}\right)^{\alpha-p}$$

$$1 - \eta \cdot Y^{\frac{1}{n}} = \left(\eta \cdot \mu_{K,p} \cdot \left(1 - \frac{p}{\alpha}\right) \cdot L^{-p} + \left(\frac{L}{H}\right)^{\alpha-p}\right)^{\frac{\alpha}{\alpha-p}}$$

$$Y = \left(\left(\frac{1}{\eta}\right) \cdot \left(1 - \left(\eta \cdot \mu_{K,p} \cdot \left(1 - \frac{p}{\alpha}\right) \cdot L^{-p} + \left(\frac{L}{H}\right)^{\alpha-p}\right)^{\frac{\alpha}{\alpha-p}}\right)\right)^{n} \tag{24}$$

We have just found the Survival Function for the Shadow Moment of order p. So the CDF and PDF are, respectively (substituting $\mu_{K,p} \to x$):

$$\Omega\left(x\right) = 1 - \left(\left(\frac{1}{\eta}\right) \cdot \left(1 - \left(\eta \cdot x \cdot \left(1 - \frac{p}{\alpha}\right) \cdot L^{-p} + \left(\frac{L}{H}\right)^{\alpha-p}\right)^{\frac{\alpha}{\alpha-p}}\right)\right)^{n} \tag{25}$$

And defining the auxiliary function $g\left(x\right)$:

$$g\left(x\right) = \eta \cdot x \cdot \left(1 - \frac{p}{\alpha}\right) \cdot L^{-p} + \left(\frac{L}{H}\right)^{\alpha-p} \tag{26}$$

We now have:

$$\Omega\left(x\right) = 1 - \left(\frac{1 - \left(g\left(x\right)\right)^{\frac{\alpha}{\alpha-p}}}{\eta}\right)^{n}$$

$$\omega\left(x\right) = -n \cdot \left(\frac{1 - \left(g\left(x\right)\right)^{\frac{\alpha}{\alpha-p}}}{\eta}\right)^{n-1} \cdot \left(\frac{1}{\eta}\right) \cdot \left(-\frac{\alpha}{\alpha - p}\right) \cdot \left(g\left(x\right)\right)^{\frac{\alpha}{\alpha-p}-1} \cdot g^{'}\left(x\right)$$

$$g^{'}\left(x\right) = \eta \cdot \left(1 - \frac{p}{\alpha}\right) \cdot L^{-p} \tag{27}$$

$$\omega\left(x\right) = n \cdot \left(\frac{\alpha}{\alpha - p}\right) \cdot \left(\frac{1}{\eta}\right) \cdot \eta \cdot \left(1 - \frac{p}{\alpha}\right) \cdot L^{-p} \cdot \left(g\left(x\right)\right)^{\frac{p}{\alpha-p}} \cdot \left(\frac{1 - \left(g\left(x\right)\right)^{\frac{\alpha}{\alpha-p}}}{\eta}\right)^{n-1}$$

$$\omega(x) = n \cdot L^{-p} \cdot (g(x))^{\frac{p}{\alpha-p}} \cdot \left(\frac{1-(g(x))^{\frac{\alpha}{\alpha-p}}}{\eta}\right)^{n-1} \tag{28}$$

We now look at the limits:
Looking for the x that solves:

$$1-(g(x))^{\frac{\alpha}{\alpha-p}} = 0$$

We find:

$$x_{max} = \left(\frac{\alpha}{\alpha-p}\right) \cdot L^p \cdot \left(\frac{1-\left(\frac{L}{H}\right)^{\alpha-p}}{1-\left(\frac{L}{H}\right)^{\alpha}}\right)$$

For $p=0$:

$$x_{max} = 1$$

For $p=1$:

$$x_{max} = \left(\frac{\alpha}{\alpha-1}\right) \cdot L \cdot \left(\frac{1-\left(\frac{L}{H}\right)^{\alpha-1}}{1-\left(\frac{L}{H}\right)^{\alpha}}\right)$$

Back to the calculation of the particular values of the shadow moments.
For $p=0$:

$$\omega(x) = n \cdot \left(\left(\frac{1}{\eta}\right) \cdot (1-g(x))\right)^{n-1}$$

$$\omega(x) = n \cdot \left(\left(\frac{1}{\eta}\right) \cdot \left(1-\eta \cdot x - \left(\frac{L}{H}\right)^{\alpha}\right)\right)^{n-1}$$

$$\omega(x) = n \cdot \left(\left(\frac{1}{\eta}\right) \cdot (\eta - \eta \cdot x)\right)^{n-1}$$

$$\omega(x) = n \cdot (1-x)^{n-1} \tag{29}$$

With mean:

$$\mathbb{E}[\omega(x)] = \frac{1}{n+1} \tag{30}$$

For $p=1$:

$$\omega(x) = n \cdot L^{-1} \cdot (g(x))^{\frac{1}{\alpha-1}} \cdot \left(\left(\frac{1-(g(x))^{\frac{\alpha}{\alpha-1}}}{\eta}\right)\right)^{n-1}$$

$$g(x) = \eta \cdot x \cdot \left(1-\frac{1}{\alpha}\right) \cdot L^{-1} + \left(\frac{L}{H}\right)^{\alpha-1}$$

Assuming $L=1$ and scaling $H \to h$:

$$\eta = \left(1 - \left(\frac{1}{h}\right)^{\alpha}\right)$$

$$g(x) = \eta \cdot x \cdot \left(1 - \frac{1}{\alpha}\right) + \left(\frac{1}{h}\right)^{\alpha - 1}$$

$$\omega(x) = n \cdot \left(\eta \cdot x \cdot \left(1 - \frac{1}{\alpha}\right) + \left(\frac{1}{h}\right)^{\alpha-1}\right)^{\frac{1}{\alpha-1}} \cdot \left(\left(\frac{1 - \left(\eta \cdot x \cdot \left(1 - \frac{1}{\alpha}\right) + \left(\frac{1}{h}\right)^{\alpha-1}\right)^{\frac{\alpha}{\alpha-1}}}{\eta}\right)\right)^{n-1}$$

With mean:

$$\mathbb{E}\left[\omega(x)\right] = \left(n \cdot B\left(n, 2 - \frac{1}{\alpha}\right) - h^{1-\alpha} \cdot \left(\left(1 - h^{-\alpha}\right)^{n} - 1\right)\right) \cdot \left(\frac{\alpha}{\alpha - 1}\right) \cdot \left(1 - h^{-\alpha}\right)^{-n-1} \tag{31}$$

For $H \to +\infty$ we have the previous formula for the Pareto:

$$\lim_{h \to +\infty} \left(\mathbb{E}\left[\omega(x)\right]\right) = \frac{n \cdot B\left(n, 2 - \frac{1}{\alpha}\right)}{\left(1 - \frac{1}{\alpha}\right)} \tag{32}$$

5 Simulations

5.1 Setup

We generate a fixed sample of 100K runs of $U[0,1]$ (10K points each) we mark the new record and in which draw it happened. We then use the InverseCDF of the Gaussian and the InverseCDF of the Pareto to generate the corresponding samples. This enables us to produce beautiful LogLog charts like Figure 14:

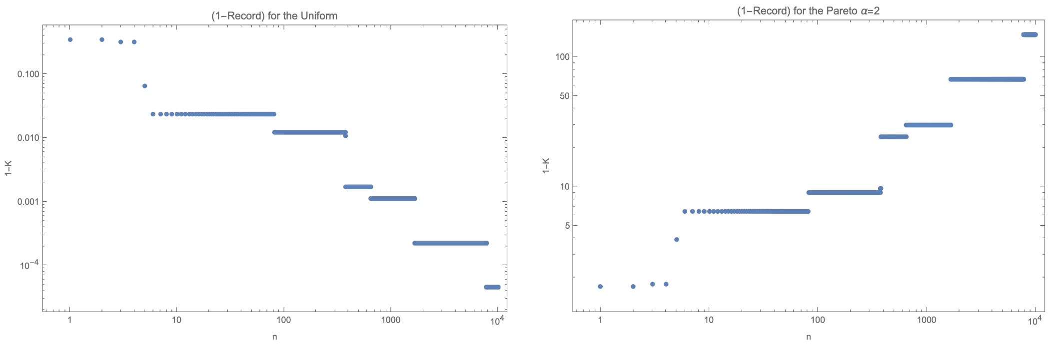

Figure 14: Running maximum for Uniform and Pareto

We then calculate the shadow mean over each path by:

Algorithm 1 Shadow mean over a population

1. Determine the running maximum path (*samplemax*) over the path *sample* (length n)
2. Create the list of records *records* "paired" with the location list *locnewmax* by splitting *samplemax* (Yes, it's Mathematica)
   (a) *records* will include either the first element of *sample* or 0
   (b) *locnewmax* doesn't start with 1
3. Create the list *shadow*:
   (a) For each location *loc* in *locnewmax* (except the last)
      i. Create *nextpts* by selecting the rest of the path *sample* (ie drop the first *loc* points of sample, and you're left with $n - loc$ points)
      ii. Sum all the points of *nextpts* greater than the record corresponding to *loc*
      iii. Divide the sum by $n - loc$
      iv. Add this mean to *shadow*
4. Output:
   (a) A transposed list where each tuple has elements of (i) *locnewmax* (ii) *records* without its last element for a shifted pairing with *locnewmax* (iii)*shadow*

5.2 Shadow Mean as a function of n

Uniform: Figure 15:

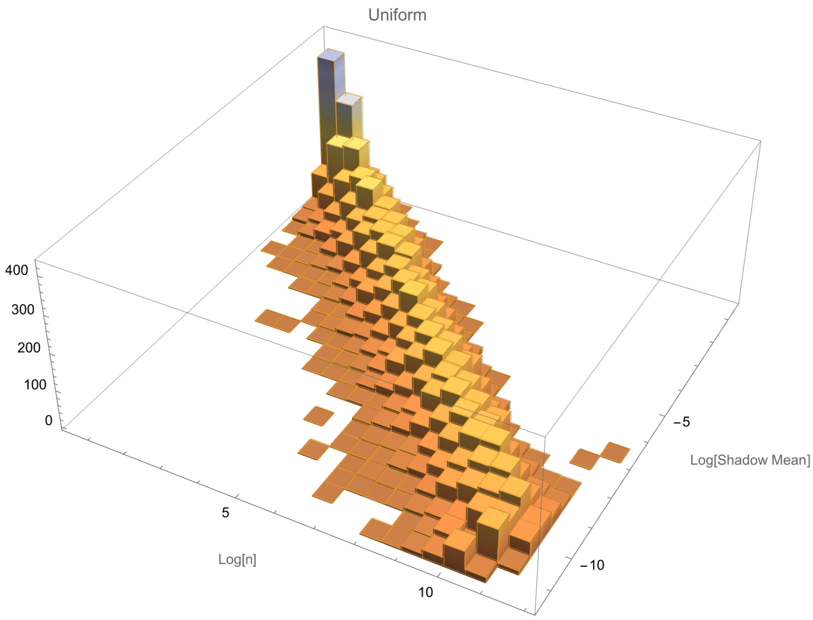

Figure 15: Uniform - Shadow Mean as a function of n

Gaussian: Figure 16:

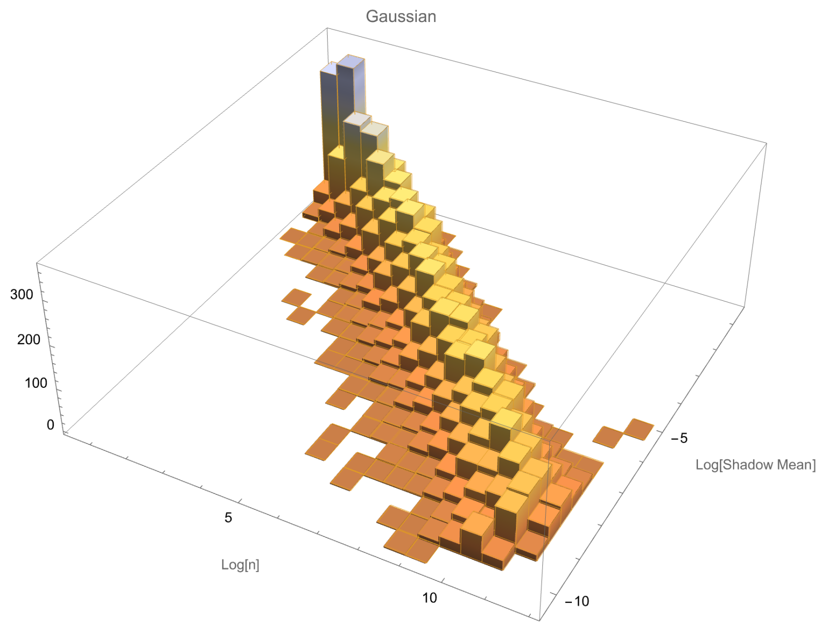

Figure 16: Gaussian - Shadow Mean as a function of n

The Shadow Mean for the Gaussian falls as quickly as the Uniform. In contrast, the Pareto fall more slowly, as shown in Figures 17 (Log[n] but no Log on the Shadow Mean) and 18:

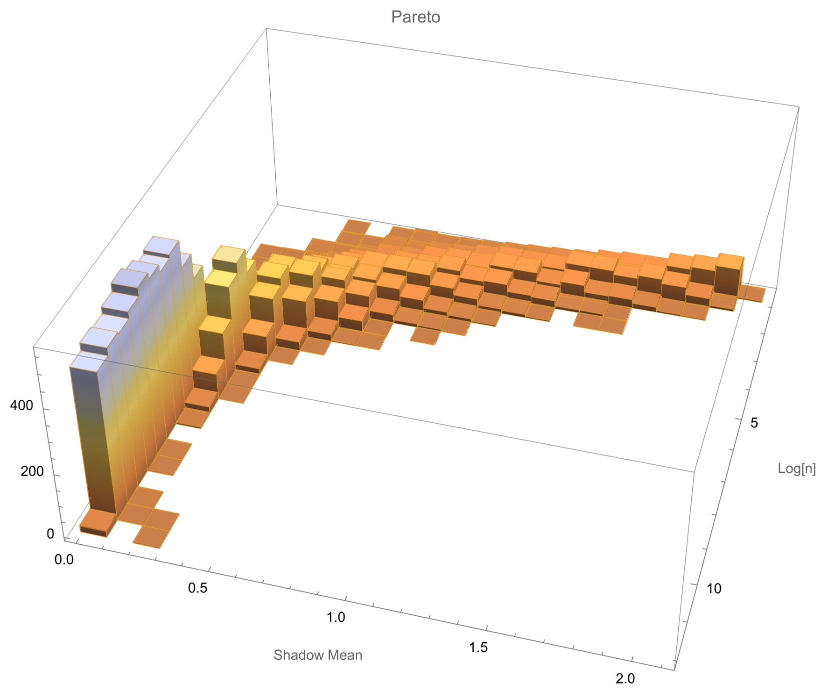

Figure 17: Pareto - Shadow Mean as a function of n, not LogLog

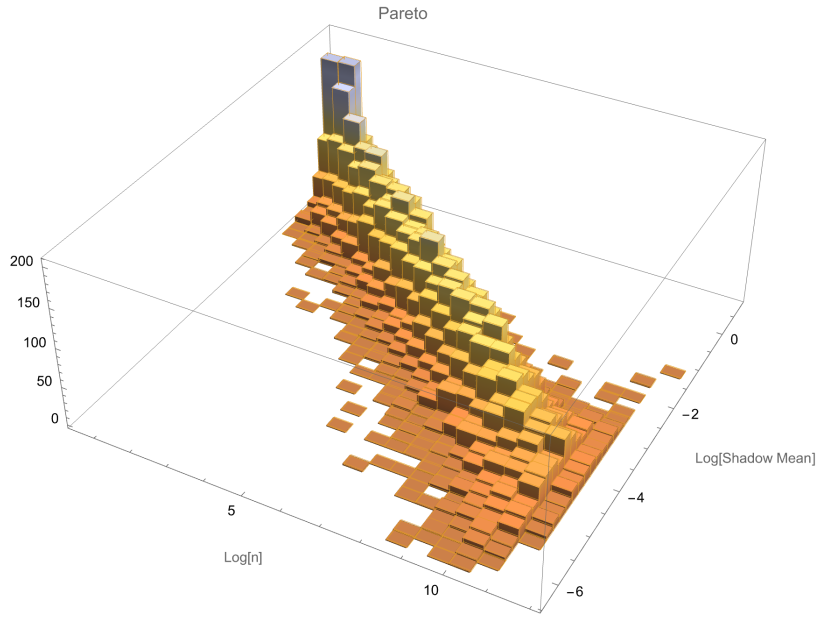

Figure 18: Pareto - Shadow Mean as a function of n, LogLog

Please note the similarities between Figure 9 and Figure 17. On Figure 18 we see that the slope in Log of the Pareto is about half of the slope in Log of the Uniform and Gaussian. We can see this more clearly in the scatterplots below (Figures 19, 20 and 21):

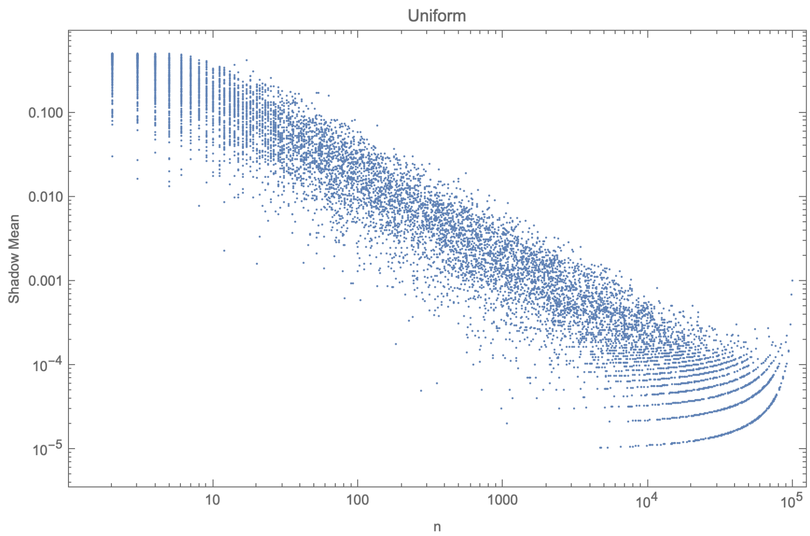

Figure 19: Uniform - Shadow Mean as a function of n - Scatterplot

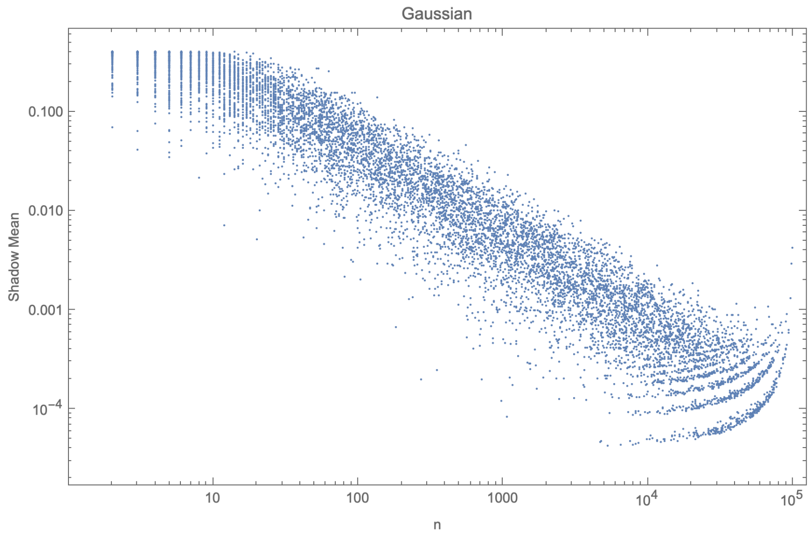

Figure 20: Gaussian - Shadow Mean as a function of n - Scatterplot

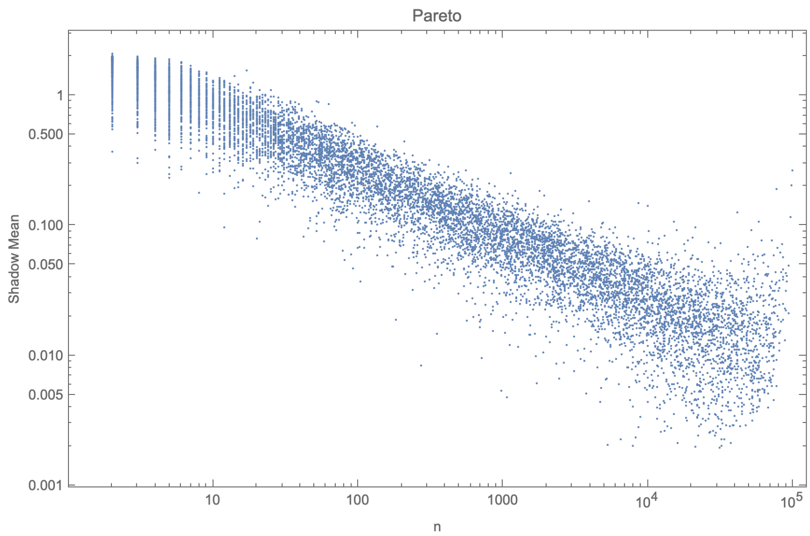

Figure 21: Pareto - Shadow Mean as a function of n - Scatterplot

6 Conclusions

In this paper we update the results from [Taleb (2020)], showing that they were qualitatively correct, and add some exact results, which are easier to calculate focusing on the properties of IID random variables and applying results from the Uniform Distribution to the Inverse CDFs of particular distributions. The observation of new records in rapid succession would suggest the presence of a trend rather than samples from the same distribution.

And again we show the difference between trying to predict the occurrence of an extreme event and trying to predict the magnitude of the excess associated with this event - the former does not inform the latter.

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Appendix

A Appendix

A.1 Transformed Distributions

Good and accessible explanations of the change of variable technique for Transformed Distributions are available at [Siegrist] and [STAT_414_Penn_State]. We will use the following results:

If X is a continuous random variable with domain $[x_1, x_2]$ and PDF $f_X(x)$, and we want the transformed distribution given by $Y = u(X)$ where u is a differentiable and invertible function (with an inverse function $v = u^{-1}$ such that $X = v(Y)$. Then the PDF $f_Y(y)$ is given by:

$$f_Y(y) = f_X(v(y)) \cdot \left| \frac{\partial v(y)}{\partial y} \right|$$

Equation 4 is our function $u(K)$.

We want the distribution of K when K is the record after n observations. The PDF and CDF of K_n are:

$$h(n, x) = n \cdot (F_X(x))^{n-1} \cdot f_X(x)$$

$$H(n, x) = (F_X(x))^n$$

For the Pareto (used in the original paper):

$$\psi(n, x) = n \cdot \left(1 - \left(\frac{L}{x}\right)^{\alpha}\right)^{n-1} \cdot \frac{\alpha}{x} \cdot \left(\frac{L}{x}\right)^{\alpha}$$

A.2 Pareto Integrals

Let's use a change of variables to solve $\int_0^{+\infty} [x \cdot h_P(n, x)]\, dx$ and define $u = \left(\frac{L}{x}\right)^{\alpha}$. Mathematica returns (using IntegrateChangeVariables):

$$\int_0^{+\infty} [x \cdot h_P(n, x)]\, dx = -\frac{\pi \cdot \csc(\pi \cdot n) \cdot L \cdot \Gamma\left(\frac{\alpha-1}{\alpha}\right)}{\Gamma(-n) \cdot \Gamma\left(n - \frac{1}{\alpha} + 1\right)}$$

Using the identities:

$$\Gamma(n) \cdot \Gamma(1-n) = \pi \cdot \csc(\pi \cdot n)$$

$$\Gamma(1 + n - \frac{1}{\alpha}) = (n - \frac{1}{\alpha}) \cdot \Gamma(n - \frac{1}{\alpha})$$

$$\Gamma(1-n) = (-n) \cdot \Gamma(-n)$$

$$\Gamma(1-\frac{1}{\alpha}) = (-\frac{1}{\alpha}) \cdot \Gamma(-\frac{1}{\alpha})$$

We arrive at:

$$\int_0^{+\infty} \left[x \cdot h_P\left(n,x\right)\right] dx = -\frac{L \cdot n \cdot \Gamma\left(n\right) \cdot \Gamma\left(-\frac{1}{\alpha}\right)}{\alpha \cdot \left(n-\frac{1}{\alpha}\right) \cdot \Gamma\left(n-\frac{1}{\alpha}\right)} = -\frac{L \cdot n \cdot B\left(n,-\frac{1}{\alpha}\right)}{\alpha \cdot \left(n-\frac{1}{\alpha}\right)}$$

The integral:

$$\int\limits_0^1 \left[L\left(1-p\right)^{-\frac{1}{\alpha}}\right] \left[n \cdot p^{n-1}\right] dp = -\frac{L \cdot n \cdot \Gamma\left(n\right) \cdot \Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(1+n-\frac{1}{\alpha}\right)}$$

Simplifies to:

$$\int\limits_0^1 \left[L\left(1-p\right)^{-\frac{1}{\alpha}}\right] \left[n \cdot p^{n-1}\right] dp = -\frac{L \cdot n \cdot \Gamma\left(n\right) \cdot \Gamma\left(-\frac{1}{\alpha}\right)}{\alpha \cdot \left(n-\frac{1}{\alpha}\right) \cdot \Gamma\left(n-\frac{1}{\alpha}\right)} = -\frac{L \cdot n \cdot B\left(n,-\frac{1}{\alpha}\right)}{\alpha \cdot \left(n-\frac{1}{\alpha}\right)}$$