

# What are we weighting for?

A mechanistic model for probability weighting

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Mathematisches Institut





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# Main results

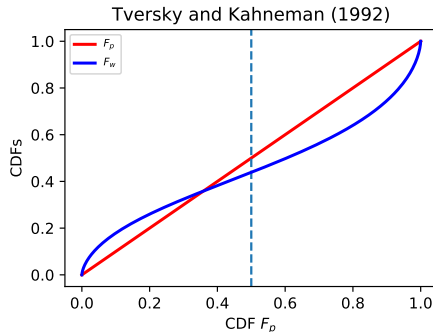
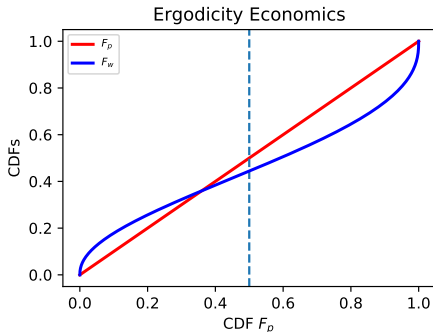
Main Result

Probability  
Weighting

Ergodicity  
Question

Estimation

Conclusion



- 1 inverse-S shape can be explained by difference in uncertainty
- 2 cautious estimation of probabilities generates such differences

► PW K&T 1979



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# Definition of Probability Weighting (PW)

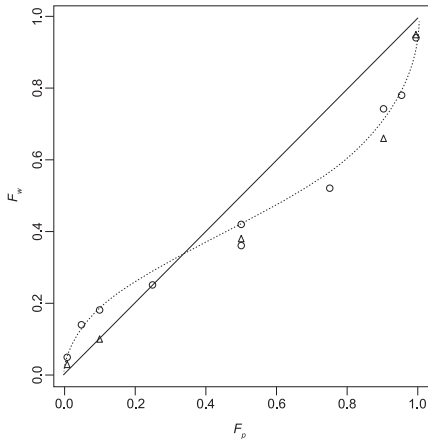
- low probabilities treated as higher; high probabilities treated as lower
- stable empirical pattern: inverse-S shape

Received wisdom:

- PW = maladaptive irrational cognitive bias

In search of a mechanism

- ↪ How does this pattern emerge?
- ↪ Can we derive a functional form (rather than fit a function)?



(Tversky and Kahneman 1992, p. 310, Fig. 1, relabelled axes)

Task: model payout,  $x$ , of a gamble as a random variable.

### Disinterested Observer (DO)



DO assigns  
probabilities  $p(x)$   
CDF  $F_p(x)$

### Decision Maker (DM)



DM assigns different  
probabilities  $w(x)$  (decision weights)  
CDF  $F_w(x)$



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Main Result

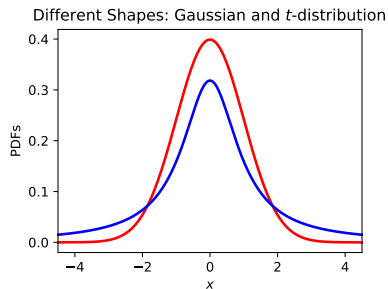
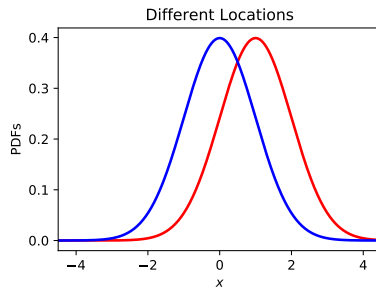
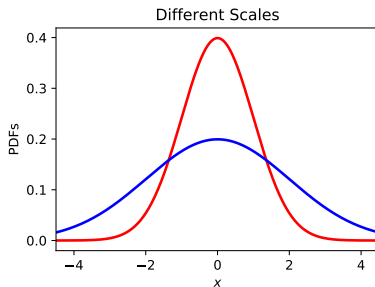
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# Possible model differences





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# Thought experiment: DM assumes greater scale

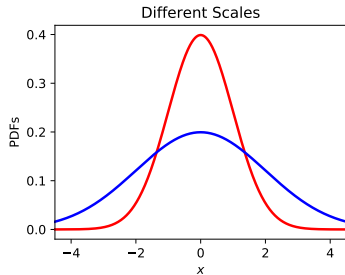
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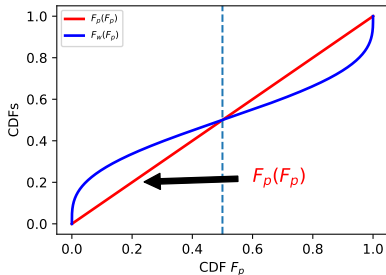
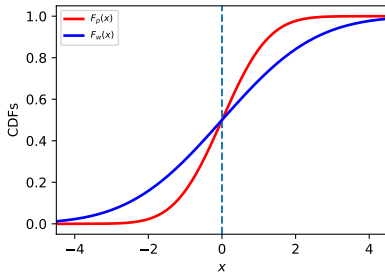
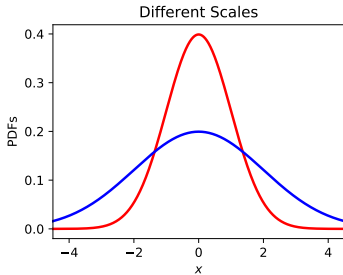
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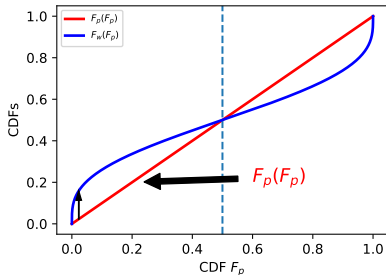
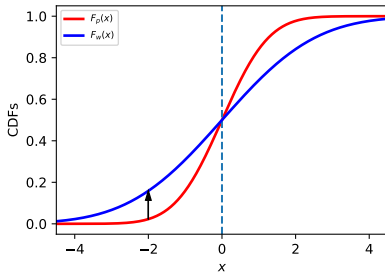
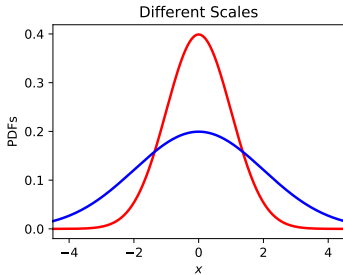
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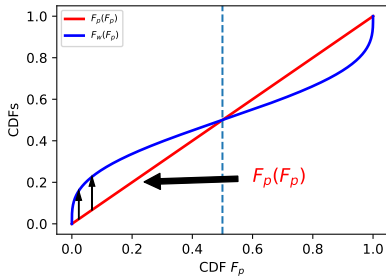
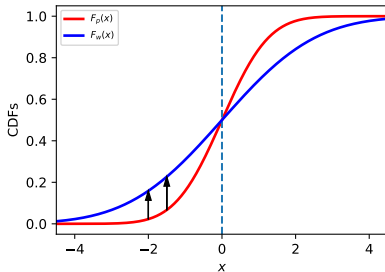
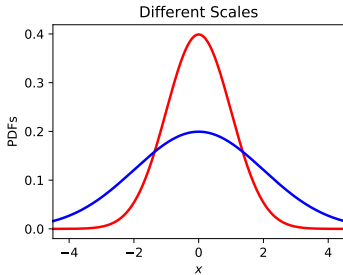
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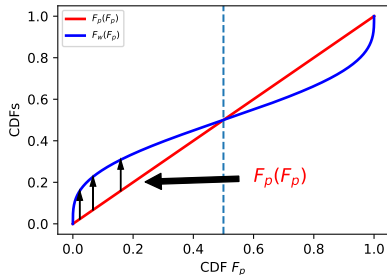
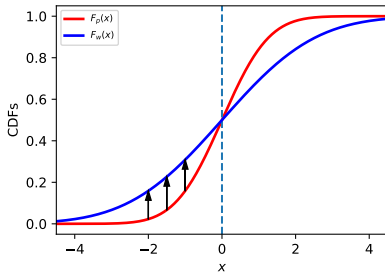
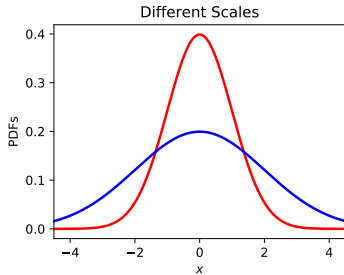
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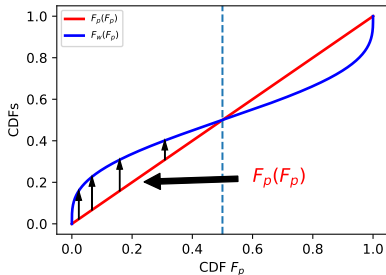
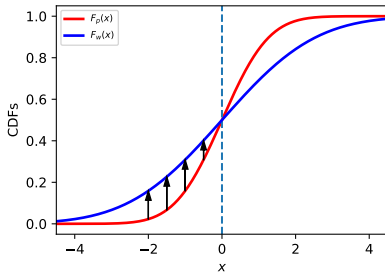
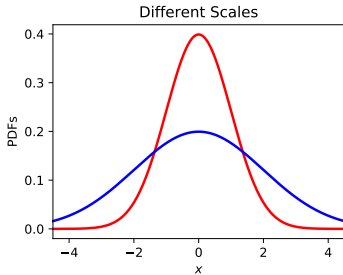
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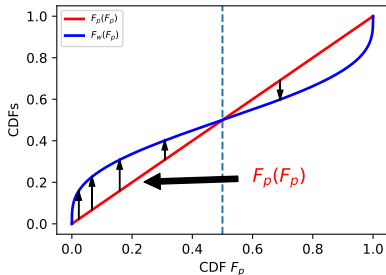
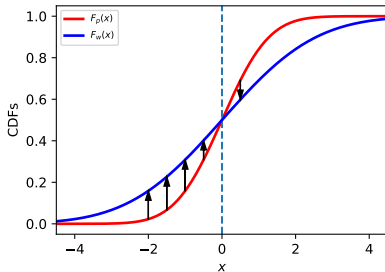
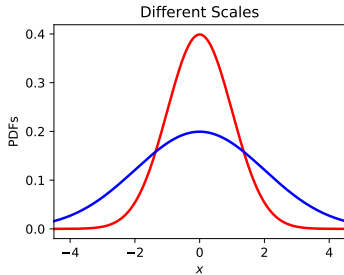
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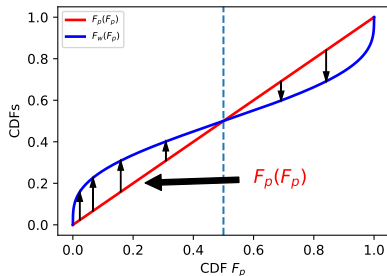
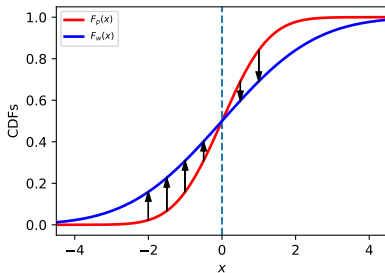
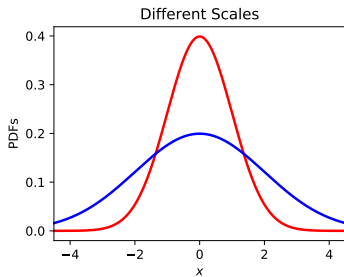
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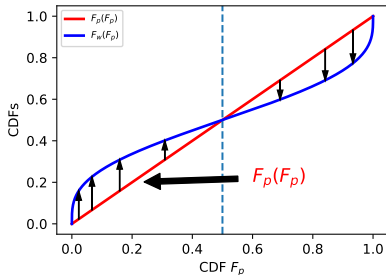
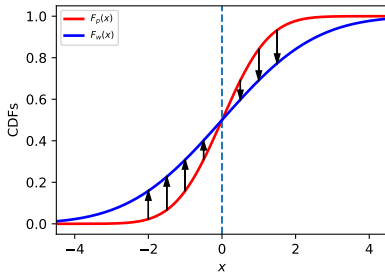
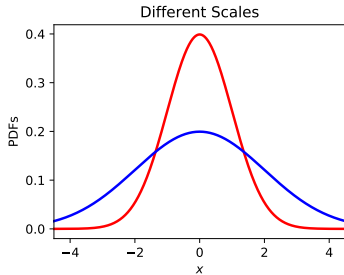
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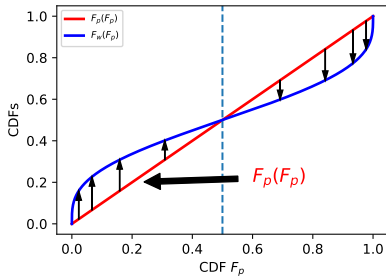
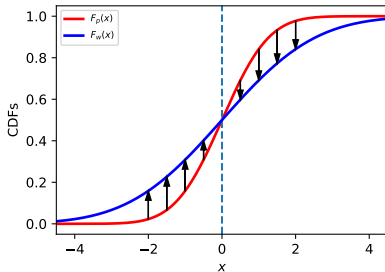
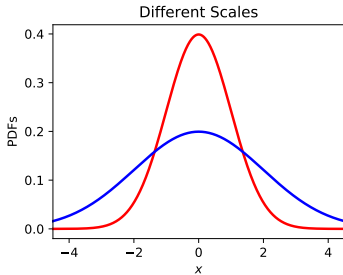
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Numerically easy for any pair of distributions (models):

- 1 list values of DO's CDF,  $F_p(x)$ , at set  $x_i$
- 2 list values of DM's CDF,  $F_w(x)$ , at same  $x_i$
- 3 plot  $F_w(x)$  vs.  $F_p(x)$





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Main Result

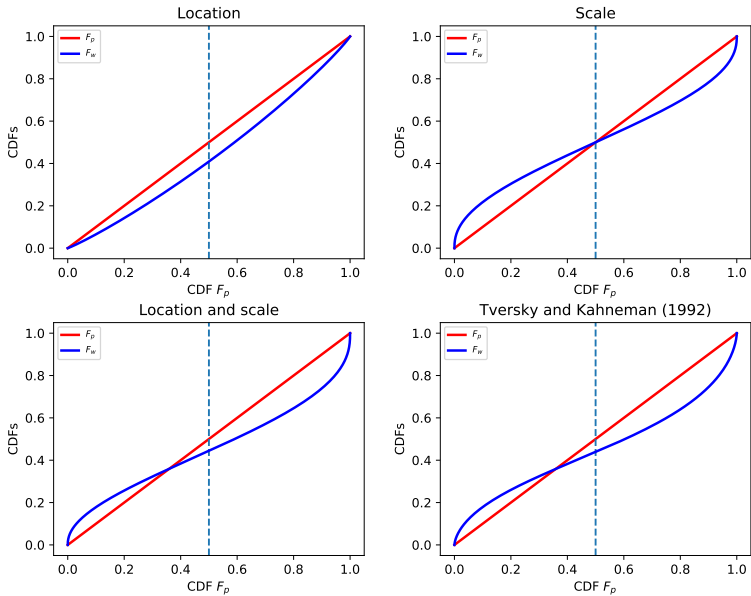
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# Asymmetry from different locations





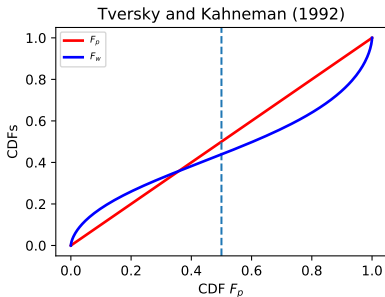
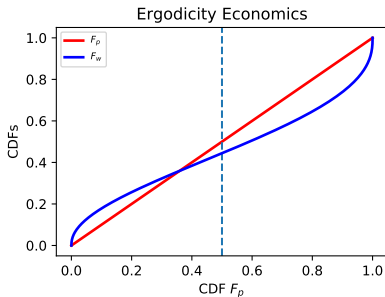
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# Interim conclusion

- DM's greater scale gives inverse-S shape (unimodal distributions)
- difference in locations gives asymmetry
- reproduces observations of probability weighting

*Job done. Thank you for your attention ;)*

► Functional Forms





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# The Ergodicity Question

Main Result

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## Typical DO concern

What happens on average to  
the **ensemble** of subjects?

$\neq$

## Typical DM concern

What happens to me  
**on average over time**?



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# Why DM's greater scale?

- DM has no control over experiment
- experiment may be unclear to DM
- DM may not trust DO
- ...



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# Experiencing probabilities

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- probabilities are not observable
- probabilities encountered as
  - known frequency in ensemble of experiments (DO)
  - frequencies estimated over time (DM)

↪ **estimates have uncertainties – cautious DM accounts for these**



## Rare Event

- $p(x) = 0.001$
- 100 observations
- $\sim 99.5\%$  of such time series will contain 0 or 1 events
- Naïve estimation:  $\hat{p}(x) = 0$  or  $\hat{p}(x) = 0.01$

↪ 1000% error in  $\hat{p}(x)$

## Common Event

- $p(x) = 0.5$
- 100 observations
- $\sim 99.5\%$  of time series would contain between 35 and 65 events,
- Naïve estimation:  $0.35 < \hat{p}(x) < 0.65$

↪  $< 50\%$  error in  $\hat{p}(x)$

↪ small  $p(x)$ , small count

↪ small count, big uncertainty



# Relative estimation error is large for rare events

Asymptotic probability	Most likely count	Standard error in count	Standard error in probability	Relative error in probability
0.1	1000	32	0.003	3%
0.01	100	10	0.001	10%
0.001	10	3	0.0003	30%
0.0001	1	1	0.0001	100%

**Table:**  $T = 10\,000$ , assuming Poisson statistics, relative estimation errors  $\sim 1/\sqrt{\text{count}}$

$\hookrightarrow$  small  $p(x)$ , small count  
 $\hookrightarrow$  small count, big uncertainty



To avoid surprises, let's say DMs **add estimation uncertainty**  $\varepsilon [p(x)]$  to every estimated probability, then normalize, s.t.

$$w(x) = \frac{p(x) + \varepsilon [p(x)]}{\int (p(s) + \varepsilon [p(s)]) ds}$$

This allows us to derive a functional form, e.g. for the Gaussian case

$$w(p) = p^{\frac{1}{\alpha^2}} \frac{(2\pi\sigma^2)^{\frac{1-\alpha^2}{2\alpha^2}}}{\alpha},$$

where DM's scale is  $\alpha\sigma$

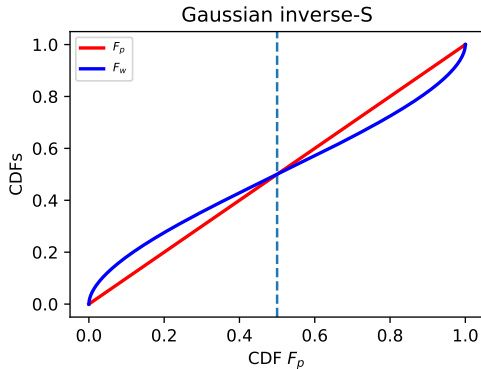
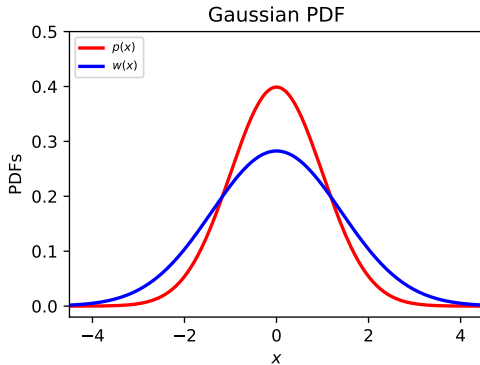
- $\alpha < 1 \rightarrow$  S shape
- $\alpha > 1 \rightarrow$  inverse-S shape





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# Estimating probabilities for two Gaussians



ToDo: Make inverse-S more pronounced

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## Ergodicity Economics and probability weighting

- inverse-S shape appears as neutral indicator of a difference in opinion
  - reported observations consistent with DM's extra uncertainty
  - may arise from DM estimating probabilities over time
- ↪ Probability weighting is rational cautious behaviour under uncertainty over time
- 
- Manuscript at <https://www.researchers.one/article/2020-04-14>
  - Interactive code at <https://bit.ly/lml-pw-count-b>

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**Thank you for your attention!**



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Back Up

References

# BACK UP



# Probability Weighting as an Estimation Issue

“It is important to distinguish [overweighting](#), which refers to a property of decision weights, from the [overestimation](#) that is commonly found in the assessment of the probability of rare events. [...] In many real-life situations, overestimation and overweighting may both operate to increase the impact of rare events.” (Kahneman and Tversky [1979](#), p. 281)

↪ distinguish between

- uncertainty estimation and
- “weighting”

we analyse the former and find very good agreement with the empirical inverse-S pattern

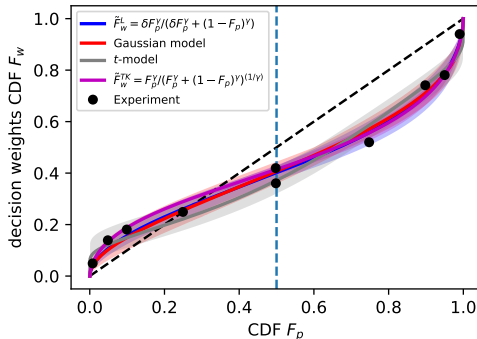
↪ How big is the residual “probability weighting” after accounting for uncertainty estimation?



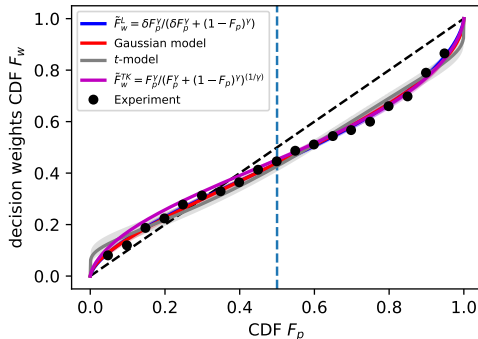
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# Estimation Error Explains 99% of Probability Weighting

Tversky & Kahneman (1992)



Tversky & Fox (1995)



- similar fits of Gaussian &  $t$ -distributed model

→ How big is the residual “probability weighting” after accounting for estimation errors?



Tversky and Kahneman ([1992](#),  $\gamma = 0.68$ )

$$\tilde{F}_w^{TK}(F_p; \gamma) = (F_p)^\gamma \frac{1}{\left[(F_p)^\gamma + (1 - F_p)^\gamma\right]^{1/\gamma}} \quad (1)$$

Lattimore, Baker, and Witte ([1992](#))

$$\tilde{F}_w^L(F_p; \delta, \gamma) = \frac{\delta F_p^\gamma}{\delta F_p^\gamma + (1 - F_p)^\gamma} \quad (2)$$

We derive decision weight as a function of probability with  $(\alpha\sigma)^2$  as the DM's scale

$$w(p) = p^{\frac{1}{\alpha^2}} \frac{(2\pi\sigma^2)^{\frac{1-\alpha^2}{2\alpha^2}}}{\alpha}, \quad (3)$$

which is a power law in  $p$  with a pre-factor to ensure normalisation



Kahneman, Daniel and Amos Tversky (1979). "Prospect Theory: An Analysis of Decision under Risk". *Econometrica* 47 (2), pp. 263–291. DOI:[10.2307/1914185](https://doi.org/10.2307/1914185) (cit. on p. 29).



Lattimore, Pamela K., Joanna R. Baker, and A. Dryden Witte (1992). "Influence of Probability on Risky Choice: A Parametric Examination". *Journal of Economic Behavior and Organization* 17 (3), pp. 377–400. DOI:[10.1016/S0167-2681\(95\)90015-2](https://doi.org/10.1016/S0167-2681(95)90015-2) (cit. on p. 31).



Tversky, Amos and Daniel Kahneman (1992). "Advances in Prospect Theory: Cumulative Representation of Uncertainty". *Journal of Risk and Uncertainty* 5 (4), pp. 297–323. DOI:[10.1007/BF00122574](https://doi.org/10.1007/BF00122574) (cit. on pp. 3, 31).