

# What are we weighting for?

A mechanistic model for probability weighting

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## Abstract

Behavioural economics provides labels for patterns in human economic behaviour. Probability weighting is one such label. It expresses a mismatch between probabilities used in a formal model of a decision problem (*i.e.* model parameters) and probabilities inferred from real people's behaviour faced with the modelled decision problem (the same parameters estimated empirically). The inferred probabilities are called “decision weights.” It is considered a robust observation that decision weights are higher than probabilities for extreme events, and (necessarily, because of normalisation) lower than probabilities for common events. We formalize this observation as an observer and a decision maker using different probability distributions to model the outcome of an experiment. We show how the specific disagreement arises from the decision maker having to estimate probabilities as frequencies in a time series, whereas the observer may know them a-priori.

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# 1 Nomenclature

*Probability weighting* is a concept that originated in prospect theory. It is one way to conceptualise a pattern in human behaviour, of caution with respect to formal models. This is best explained by a thought experiment, in which

- a *disinterested observer* (DO), such as an experimenter, tells
- a *decision maker* (DM)

that an event occurs with some probability. The DO observes DM's behaviour (*e.g.* gambling on the event), and finds it consistent with a behavioural model (*e.g.* expected-utility optimization) in which the DM uses a probability that differs systematically from what the DO has declared. The apparent probabilities, inferred from the DM's decisions, are called "*decision weights*." We will adopt this nomenclature here:

- By "*probabilities*," expressed as probability density functions (PDFs) and denoted  $p(x)$ , we will mean the numbers specified by a DO.
- By "*decision weights*," also expressed as PDFs and denoted  $w(x)$ , we will mean the numbers that best describe the behaviour of a DM in the DO's behavioural model.<sup>1</sup>

Here,  $x$ , the realisation of a random variable,  $X$ , is an observable representing the outcome of the event about which the decision is being made. For example,  $x$  might be the payout of a gamble, in some currency unit, which the DM is invited to accept or decline.

Specifically, it is consistently observed that decision weights,  $w(x) > p(x)$  (used by DMs), are higher than probabilities,  $p$  (declared by DOs), for extreme events (with small  $p$ ). This observation is often summarised visually by comparing

- cumulative density functions (CDFs) for probabilities, denoted

$$F_p(x) = \int_{-\infty}^x p(s)ds \quad (1)$$

- and CDFs for decision weights, denoted

$$F_w(x) = \int_{-\infty}^x w(s)ds . \quad (2)$$

In Fig. 1 we reproduce the first such visual summary from [Tversky and Kahneman \(1992, p. 310\)](#).

As a final piece of nomenclature, we will use the terms *location*, *scale*, and *shape* when discussing probability distributions. Consider a standard normal distribution  $\mathcal{N}(0, 1)$  – here, the parameters indicate location 0 and squared scale 1 (for a Gaussian the location is the mean and scale is the standard deviation). For a general random variable  $X$ , with arbitrary parameters for location  $\mu_X$  and scale  $\sigma_X$ , the transformation in (Eq. 3) obtains

<sup>1</sup>In the literature, decision weights are not always normalised, but for simplicity we will work with normalised decision weights. Mathematically speaking, they are therefore proper probabilities even though we don't call them that. Our results are unaffected because normalising just means dividing by a constant (the sum or integral of the non-normalised decision weights).



Figure 1: **Empirical phenomenon of probability weighting.** Cumulative decision weights  $F_w$  (used by decision makers) versus cumulative probabilities  $F_p$  (used by disinterested observers), as reported by [Tversky and Kahneman \(1992, p. 310, Fig. 1, relabelled axes\)](#). The figure is to be read as follows: pick a point along the horizontal axis (the cumulative probability  $F_p$  used by a DO) and look up the corresponding value on the vertical axis of the dotted inverse-S curve (the cumulative decision weight  $F_w$  used by a DM). Low cumulative probabilities (left) are exceeded by their corresponding cumulative decision weights, and for high cumulative probabilities it's the other way around. It's the inverse-S shape of the curve that indicates this qualitative relationship.

the identically-shaped location-0 and scale-1 distribution for the so standardised random variable

$$Z = \frac{X - \overbrace{\mu_X}^{\text{location}}}{\underbrace{\sigma_X}_{\text{scale}}} . \quad (3)$$

Thus the PDF of  $Z$ ,  $p(z)$  is a density with location  $\mu_Z = 0$  and scale  $\sigma_Z = 1$ . In a graph of a distribution, a change of location shifts the curve to the left or right, and a change in scale shrinks or blows up the width of its features. Neither operation changes the *shape* of the distribution.

## 2 Probability weighting as a difference between models

Behavioural economics interprets Fig. 1 as evidence for a cognitive bias of the DM, an error of judgment. We will keep a neutral stance. We don't assume the DO to know "the truth" – he has a model of the world. Nor do we assume the DM to know "the truth" – he has another model of the world. From our perspective Fig. 1 merely shows that the two models differ. It says nothing about who is right or wrong.

## 2.1 The inverse-S curve

### 2.1.1 Tversky and Kahneman

Tversky and Kahneman (1992) chose to fit the empirical data in Fig. 1 with the following function

$$\tilde{F}_w^{TK}(F_p; \gamma) = (F_p)^\gamma \frac{1}{[(F_p)^\gamma + (1 - F_p)^\gamma]^{1/\gamma}}. \quad (4)$$

We note that no mechanistic motivation was given for fitting this specific family of CDFs (parameterised by  $\gamma$ ). The motivation is purely phenomenological: with  $\gamma < 1$ , this family “looks a bit like the data.” The function  $\tilde{F}_w^{TK}(F_p; \gamma)$  has only one free parameter,  $\gamma$ . For  $\gamma = 1$  it is the identity, and the CDFs coincide,  $\tilde{F}_w^{TK}(F_p) = F_p$ . Further,  $\tilde{F}_w^{TK}$  has the following property: any curvature moves the intersection with the diagonal away from the mid-point  $1/2$ . This means if the function is used to fit an inverse S (where  $\gamma < 1$ ), the fitting procedure itself introduces a shift of the intersection to the left. Because of this, we consider the key observation to be the inverse-S shape, whereas the shift to the left may be an artefact of the function chosen for the fit.

### 2.1.2 Scale, location, and the inverse S

We now make explicit how the robust qualitative observation of the inverse-S shape in Fig. 1 emerges when the DM uses a larger scale in his model of the world than the DO.

We illustrate this with a Gaussian distribution. Let’s assume that a DO models an observable  $x$  – which will often be a future change in wealth – as a Gaussian with location  $\mu$  and variance  $\sigma^2$ . And let’s further assume that a DM models the same observable as a Gaussian with the same location,  $\mu$ , but with a greater scale, so that the variance is  $(\alpha\sigma)^2$ . The DM simply assumes a broader range –  $\alpha$  times greater – of plausible values, left panel of Fig. 2.

Generically, if the DM uses a greater scale in his model, then decision weights are higher than probabilities for low-probability events, and (because of normalisation), lower than probabilities for high-probability events. We can express this by plotting, for any value of  $x$ , the decision weight *vs.* the probability observed at  $x$ , right panel of Fig. 2.

In the Gaussian case we can write the distributions explicitly

$$w(x) = \frac{1}{\sqrt{2\pi\alpha^2\sigma^2}} \exp \left[ \frac{-(x - \mu)^2}{2(\alpha^2\sigma^2)} \right] \quad (5)$$

and

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ \frac{-(x - \mu)^2}{2\sigma^2} \right], \quad (6)$$

and solve (Eq. 6) for  $(x - \mu)^2$ , substitute that in in (Eq. 5), and obtain the following expression for decision weights directly as a function of probabilities

$$w(p) = p^{\frac{1}{\alpha^2}} \frac{(2\pi\sigma_1^2)^{\frac{1-\alpha^2}{2\alpha^2}}}{\alpha}, \quad (7)$$

which is precisely what’s plotted in the right panel of Fig. 2. As a sanity check, consider the shape of the  $w(p)$  (blue curve, right panel Fig. 2): for a given value of  $\alpha$ , it is just a power law in  $p$  with some pre-factor that ensures normalization. If  $\alpha > 1$  it means that the DM uses a greater standard deviation than the DO. In this case, the exponent of  $p$

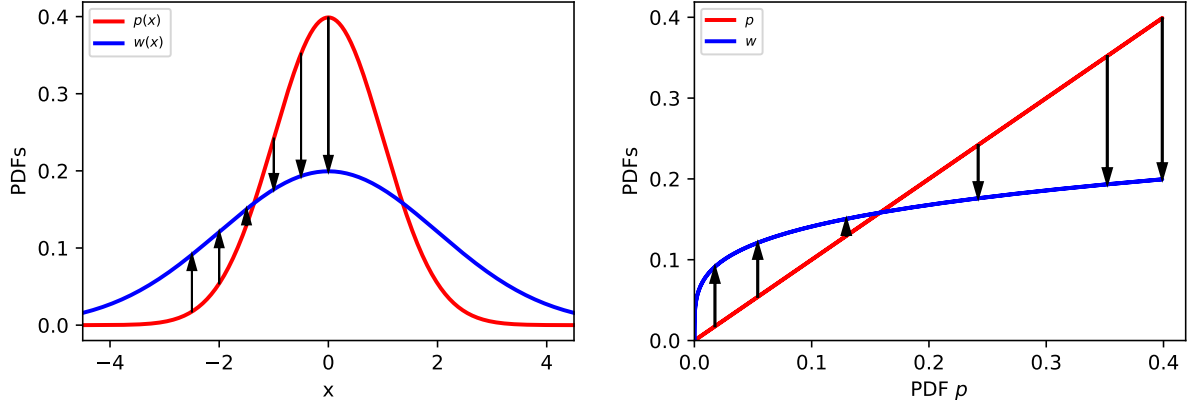


Figure 2: **Mapping PDFs.** Left: probability PDF (red), estimated by a DO; and decision-weight PDF (blue), estimated by a DM. The DO models  $x$  with a best estimate for the scale (standard deviation) and assumes the true frequency distribution is the red line. The DM models  $x$  with a greater scale (here 2 times greater,  $\alpha = 2$ ), and assumes the true frequency distribution is the blue line. Comparing the two curves, the DM appears to the DO as someone who over-estimates probabilities of low-probability events and underestimates probabilities of high-probability events, indicated by vertical arrows. Right: the difference between decision weights and probabilities can also be expressed by directly plotting, for any value of  $x$ , the decision weight  $vs.$  the probability observed at  $x$ . This corresponds to a non-linear distortion of the horizontal axis. The arrows on the left correspond to the same  $x$ -values as on the right. They therefore start and end at identical vertical positions as on the left. Because of the non-linear distortion of the horizontal axis, they are shifted to different locations horizontally.

satisfies  $\frac{1}{\alpha^2} < 1$ , and the blue curve is above the diagonal for small arguments and below it for large arguments.

Alternatively, we can express the difference between models by plotting the CDFs  $F_w$  and  $F_p$ . We do this in Fig. 3, where the inverted S emerges purely from the DM's greater assumed scale,  $\alpha\sigma$ .



Figure 3: **Mapping CDFs.** Left: The DO assumes the observable  $X$  follows Gaussian distribution  $X \sim \mathcal{N}(0, 1)$ , which results in the red CDF of the standard normal,  $F_p(x) = \Phi_{0,1}(x)$ . The DM is more cautious, in his model the same observable  $X$  follows a wider Gaussian distribution,  $X \sim \mathcal{N}(0, 3)$  depicted by  $F_w$  (blue). Following the vertical arrows (left to right), we see that for low values of the event probability  $x$  the DM's CDF is larger than the DO's CDF,  $F_p(x) < F_w(x)$ ; the curves coincide at 0.5 because no difference in location is assumed; necessarily for large values of the event probability  $x$  the DM's CDF must be lower than the DO's. Right: the same CDFs as on the left but now plotted not against  $x$  but against the CDF  $F_p$ . Trivially, the CDF  $F_p$  plotted against itself is the diagonal; the CDF  $F_w$  now displays the generic inverse-S shape known from prospect theory. The arrows start and end at the same vertical values as on the left. Because the horizontal axis is non-linearly stretched (as the argument changed from  $x$  to  $F_p$ ), their horizontal locations are shifted.

## 2.2 Different scales and locations

In Fig. 4 we explore what happens if both the scales and the locations of the DO's and DM's models differ. Visually, this produces an excellent fit to empirical data, to which we will return in Sec. 4. A difference in assumed scales and locations, for simple Gaussian distributions, is sufficient to reproduce the observations. This suggests a different nomenclature and a conceptual clarification. The inverse-S curve does not mean that "probabilities are re-weighted" – it just means that experimenters and their subjects have different views about what might be an appropriate response to a situation.

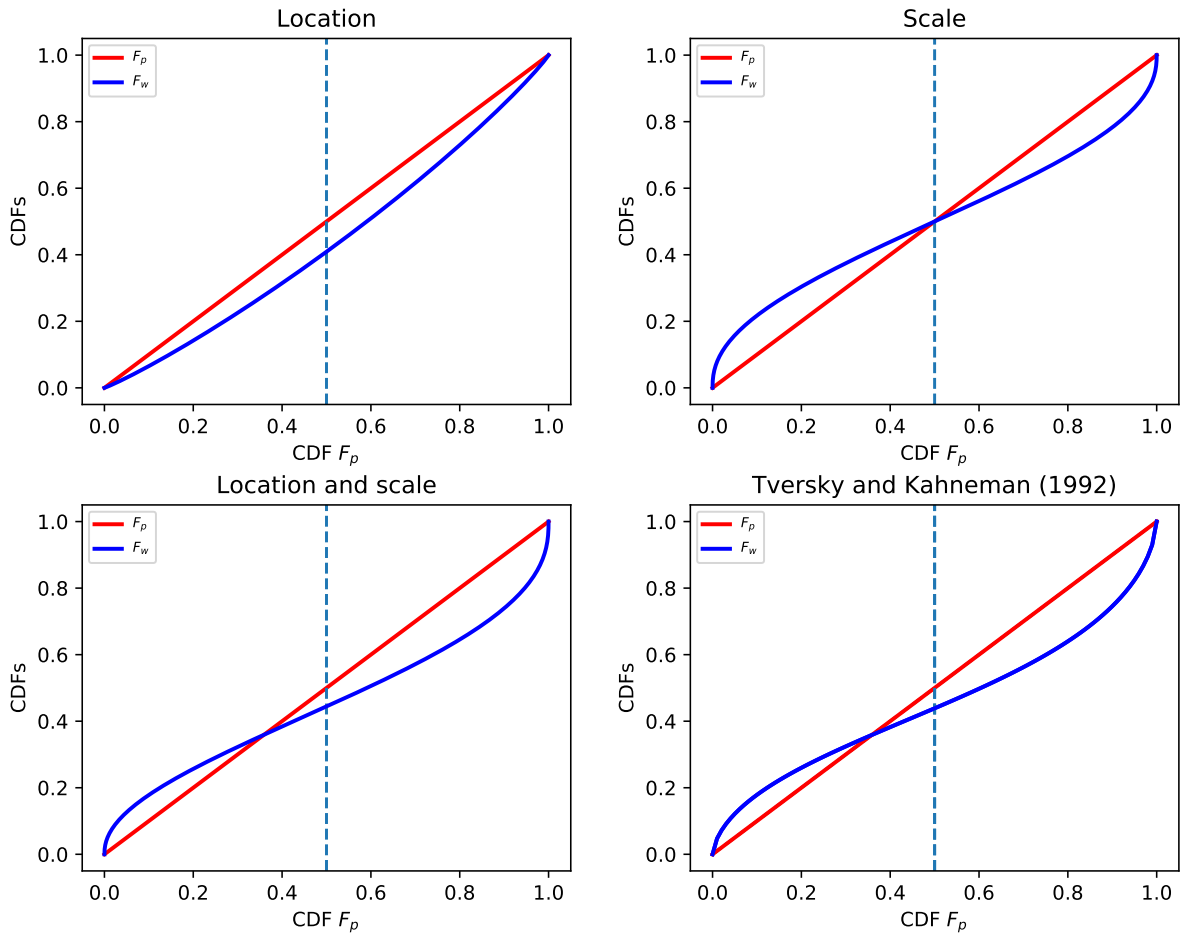


Figure 4: **CDF maps, Gaussian distribution.**

Top left: Difference in scale. DO assumes location 0, scale 1; DM assumes location 0, scale 1.64 (broader than DO).

Top right: Difference in location. DO assumes location 0, scale 1; DM assumes location 0.18 (bigger than DO), scale 1.

Bottom left: Differences in scale and location. DO assumes location 0, scale 1; DM assumes location 0.18 (bigger than DO), scale 1.64 (broader than DO).

Bottom right: Fit to observations reported by [Tversky and Kahneman \(1992\)](#). This is (Eq. 4) with  $\gamma = 0.65$ . Note the similarity to bottom left.

## 2.3 Different shapes and locations

Numerically, our procedure can be applied to arbitrary distributions:

1. construct a list of values for the CDF assumed by the DO,  $F_p(x)$ .
2. construct a list of values for the CDF assumed by the DM,  $F_w(x)$ .
3. plot  $F_w(x)$  vs  $F_p(x)$ .

Of course, the DM could even assume a distribution whose shape differs from that of the DO's distribution. The inverse S arises whenever a DM assumes a greater scale for a unimodal distribution. To illustrate the generality of the procedure, in Fig. 5 we carry it out for Student's (power-law tailed)  $t$ -distributions (which we refer to as  $t$ -distributions), where DO and DM use different shape parameters and different locations.<sup>2</sup> The result is qualitatively similar to the bottom right panel of Fig. 3, corresponding to (Eq. 4).

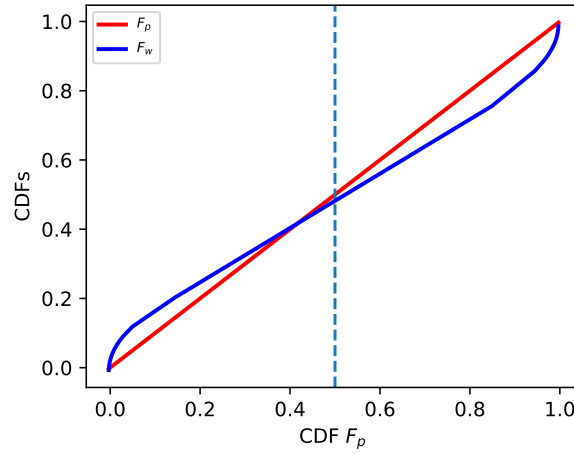


Figure 5: Probability weighting for  $t$ -distributions, where the DM uses a different shape parameter (1) and a different location parameter (0) from those of the DO (2 and 0.2, respectively).

<sup>2</sup>The PDF of the  $t$ -distribution is

$$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}\sigma} \left(1 + \frac{1}{\nu} \left(\frac{x-\mu}{\sigma}\right)^2\right)^{-\frac{\nu+1}{2}}, \quad (8)$$

where  $\nu$  is the shape parameter,  $\sigma$  is the scale parameter, and  $\mu$  is the location parameter. The corresponding CDF is

$$F(x) = \begin{cases} 1 - \frac{1}{2} I_{\frac{\nu}{(\frac{x-\mu}{\sigma})^2 + \nu}} \left(\frac{\nu}{2}, \frac{1}{2}\right) & \text{if } x - \mu \geq 0; \\ \frac{1}{2} I_{\frac{\nu}{(\frac{x-\mu}{\sigma})^2 + \nu}} \left(\frac{\nu}{2}, \frac{1}{2}\right) & \text{if } x - \mu < 0, \end{cases} \quad (9)$$

where  $I_x(a, b)$  is the incomplete beta function.

In the limit  $\nu \rightarrow \infty$ , the  $t$ -distribution converges to a Gaussian with location  $\mu$  and scale  $\sigma$ . We assume by default that  $\sigma = 1$ , so the  $t$ -distribution is effectively characterised by two parameters – shape ( $\nu$ ) and location ( $\mu$ ).



### 3 Reasons for different models

“Probability weighting” is a term that suggests a detrimental cognitive bias, and the empirical findings are often interpreted that way. We caution against this interpretation. At least we should keep in mind that it is unclear who suffers from the bias: experimenter or test subject?

Whatever the answer, two observations are robust and interesting: first, disagreement is common; and, second, the disagreement tends to go in the same direction: DMs tend to assume a greater range of plausible outcomes than DOs.

An explanation for the first observation is that probability is a slippery concept, and the word is used to mean different things. The second observation may be explained as follows. A DO often has control over, and essentially perfect knowledge of, the decision problem he poses. A DM does not have such knowledge, and this ignorance will often translate into additional assumed uncertainty. For example, the DO may know the true probabilities of some gamble in an experiment; the DM may in addition have doubts about the DO’s sincerity, or whether he (the DM himself) fully understands the rules of the game. We will return to this in Sec. 3.2.

#### 3.1 Some meanings of “probability”

Many thousands of pages have been written about the meaning of probability. We will not attempt a summary of the philosophical debate and instead focus on a few relevant points.

**Frequency-in-an-ensemble interpretation of probability** Consider the simple probabilistic statement: “the probability of rain here tomorrow is 70%.” Tomorrow only happens once, so one might ask: in 70% of what will it rain? The technical answer to this question is often: rain happens in 70% of the members of an ensemble of computer simulations, run by a weather service, of what may happen tomorrow. So one interpretation of “probability” is “relative frequency in a hypothetical ensemble of simulated possible futures.”

It is thus a statement about a model. How exactly it is linked to physical reality is not completely clear.

**Frequency-over-time interpretation of probability** In some situations, the statement “70% chance of rain here tomorrow” refers to the relative frequency over time. Before the advent of computer models in weather forecasting, people used to compare recent measurements (of wind and pressure today, say) to measurements further in the past – weeks, months, years earlier, that were similar and where one had reason to believe that what had happened 1 day later back then would be similar to what will happen tomorrow.

Rather than a statement about outcomes of an in-silico model, the statement may thus be a summary of real-world observations over a long time.

**Degree-of-belief interpretation of probability** No matter how “probability” relates to a frequentist physical statement (whether with respect to an ensemble of simultaneously possible futures or to a sequence of actual past futures), it also corresponds to a

mental state of believing something with a degree of conviction: “I’m 90% sure I left my wallet in that taxi.”

For our purpose it suffices to say that there’s no guarantee that a probabilistic statement will be interpreted by the receiver (the DM) as it was intended by whoever made the statement (the DO).

## 3.2 Consistent differences between DO and DM

### Estimation errors for probabilities

Let’s assume that both the DO and the DM mean by “probability” the relative frequency of an event in an infinitely long time series of observations. Of course, real time series have finite length, so probabilities defined this way are model parameters. But they can be estimated from a real time series, by counting. If an event occurs in  $n$  out of  $T$  observations, then our best estimate for its probability is  $n/T$ .

As the probability of an event gets smaller, so does the number of times we see it in a finite time series. Moreover, the uncertainty in our estimate of the probability becomes relatively larger as an event becomes rarer. Take an extreme example of an event which occurs in 0.1% observations asymptotically, of which we have a time series of 100 observations. Almost all such series (around 99.5% of them) will contain 0 or 1 events, meaning that our best estimate of the probability will be either 0 or 1%. In other words, we estimate the event as either impossible or occurring ten times more frequently than it really would in a long series. If, however, the event occurs 50% of the time asymptotically, then a probability estimate from 100 observations would likely (in around 95% of series) be in the range 40–60%, leading to a much smaller relative error.

If the relative uncertainty in a probability estimate becomes larger as an event becomes rarer, then a DM who must estimate probabilities from observations is well advised to account for this in his decision-making. Specifically, the DM should acknowledge that, due to his lack of information, *prima facie* rare events may be rather more common than his data imply, while common events, being revealed more often, are more easily characterised. In such circumstances, caution may dictate that the DM assign to rare events higher probabilities than his estimates, commensurate with his uncertainty in those estimates. This would look like probability weighting to a DO and, indeed, would constitute a mechanistic reason for it.

Let’s build a simple model to see if it matches the stylised facts of probability weighting, including the inverse-S curve. We model the arrival of events as a Poisson process, whose rate parameter is the asymptotic probability,  $p$ , of the event. Put simply, this means that the count per unit time converges to  $p$  over long time. The count,  $n(T)$ , over the finite observation time,  $T$ , is a Poisson-distributed random variable with intensity,  $pT$ . It has mean value,  $pT$ , and standard deviation,  $\sqrt{pT}$ . If the DM makes one observation per time unit, then the measured relative frequency,  $\hat{p} = n(T)/T$ , is an unbiased estimator of the asymptotic probability,  $p$ , with standard error,  $\sqrt{\hat{p}/T}$ .

The standard error in an estimated probability shrinks as the probability decreases. However, the relative error in the estimate is  $1/\sqrt{\hat{p}T}$ , which grows as the event becomes rarer. This is consistent with our claim, that low probabilities come with larger relative errors, further exemplified in Tab. 1.

The most important message is that errors in probability estimates behave differently for low probabilities than for high probabilities: absolute errors are smaller for lower probabilities, but relative errors are larger.

Asymptotic probability	Most likely count	Standard error in count	Standard error in probability	Relative error in probability
0.1	1000	32	0.003	3%
0.01	100	10	0.001	10%
0.001	10	3	0.0003	30%
0.0001	1	1	0.0001	100%

Table 1: This table assumes  $T = 10000$  observed time intervals. To be read as follows (first line): for an event of asymptotic probability 0.1, the most likely count in 10000 trials is 1000. Assuming Poisson statistics, this comes with an estimation error of  $\sqrt{1000} = 32$  in the count and  $32/10000 = 0.003$  in the probability, which is  $0.003/0.1 = 3\%$  of the asymptotic probability.

Let's make one more assumption: DMs don't like surprises. To avoid surprises, a DM will incorporate the uncertainty in probability estimates by assuming that the actual long-time relative frequency of an event is his best estimate plus two standard errors. In the notation of density functions, suppose he estimates the probability of an event,  $x \leq X \leq x + \delta x$ , as  $\hat{p}(x)\delta x$  from a time series of length  $T$ . To this he adds two standard errors to get his decision weight for the event,

$$w(x)\delta x = \hat{p}(x)\delta x + 2\sqrt{\frac{\hat{p}(x)}{T}}\delta x = \left( \hat{p}(x) + 2\sqrt{\frac{\hat{p}(x)}{T}} \right) \delta x \quad (10)$$

\*\*\*OP: The uncertainty is only in the count  $n$ , whereas the interval width  $\delta x$  is known and can be treated as a constant factor. If the uncertainty in  $n$  is  $\sqrt{n}$ , then the uncertainty in  $\delta x n$  is  $\delta x \sqrt{n}$ . OP\*\*\*

\*\*\*OP: We don't need to assume 2 standard errors, one is also ok – I think I did it to make the figure look clearer, but we can do that by tuning  $T$  now. OP\*\*\*

\*\*\*OP: We're overloading notation, with  $\hat{p}$  being both a probability and a density... OP\*\*\*  
for small bin width,  $\delta x \ll 1$ . Repeating this procedure across the domain of the density function, *i.e.* for all possible events, gives a decision weight density of

$$w(x) = \frac{\hat{p}(x) + 2\sqrt{\hat{p}(x)/T}}{\int_{-\infty}^{\infty} [\hat{p}(s) + 2\sqrt{\hat{p}(s)/T}] ds}, \quad (11)$$

where the denominator ensures normalization.

\*\*\*OP: check code for factors of 2 (there was one missing in the numerator). OP\*\*\*

Note the role of  $T$  here: as  $T$  grows large, *i.e.* as the DM collects more information, the cautionary correction term vanishes, and both  $w(x)$  and  $\hat{p}(x)$  converge to the asymptotic density,  $p(x)$ . With perfect information, a DM does not need to adjust decisions to account for uncertainty.

\*\*\*OP: We might want to do this numerically to make sure we understand the role of  $T$ , *i.e.* simulate time series, estimate  $p$  and its error etc., compare resulting  $w$  to  $p$ . OP\*\*\*

Figure 6 shows the resulting PDFs and the CDF mappings, for a Gaussian distribution and for a fat-tailed  $t$ -distribution. As one might expect, the effect is more pronounced for the fat-tailed distribution.

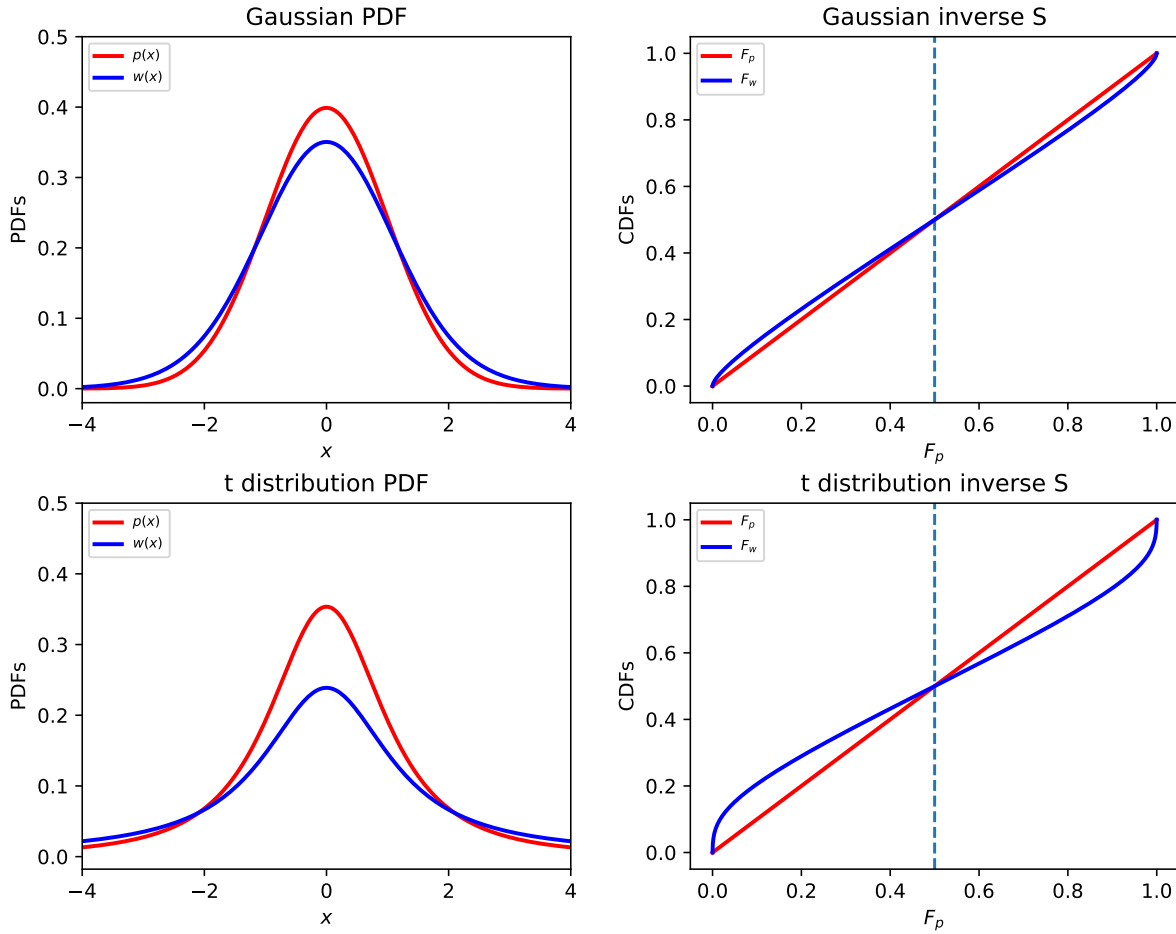


Figure 6: PDFs and inverse-S curves arising when the DO assumes a Gaussian (scale 1, location 0, top line) or a *t*-distribution (shape 2, location 0, bottom line), and the DM uses decision weights according to (Eq. 11) with  $T = 10$  observations. For the fat-tailed *t*-distribution (in the bottom line) the difference between  $p$  and  $w$  is more pronounced.

### 3.2.1 Typical situations of DO and DM: ergodicity

To recap: behavioural economics observes that DOs tend to assign lower weights to low-probability events than DMs. While behavioural economics commonly assumes that the DM is wrong, we make no such judgement. In any decision problem, the aim of the decision must be taken into account, and that crucially depends on the situation of the individual.

The two types of modellers (DO and DM) pursue different goals. Broadly, the DO tends to be a behavioural scientist without personal exposure to the success or failure of the DM (who tends to be a test subject or someone whose behaviour is being observed in the wild). The DM, of course, has such exposure. Throughout the history of economics, it has been a common mistake, by DOs, to assume that DMs optimise what happens to them on average in an ensemble. To the DM what happens to the ensemble is usually not a primary concern – instead, the concern of the DM is what happens to him over time. Not distinguishing between these two perspectives is only permissible if they lead to identical predictions, meaning only if the relevant observables are ergodic (Peters 2019).

It is now well known that this is usually not the case in the following sense: DMs are usually observed making choices that affect their wealth, and wealth is usually modelled

as a stochastic process that is not ergodic. The ensemble average of wealth does not behave like the time average of wealth.

The most striking example is the universally important case of noisy multiplicative growth – universal because it is the fundamental process that drives evolution: noise generates the diversity of phenotypes necessary for evolution and multiplicative growth (self-reproduction) is how successful phenotypes spread their traits in a population. This process operates on amoeba, as it does on forms of institutions, and on investment strategies.

The simplest model of noisy multiplicative growth is geometric Brownian motion,  $dx = x(\mu dt + \sigma dW)$ . The average over the full statistical ensemble (often studied by the DO) of geometric Brownian motion grows as  $\exp(\mu t)$ . The individual trajectory of geometric Brownian motion, on the other hand, grows in the long run as  $\exp[(\mu - \frac{\sigma^2}{2})t]$ .

If the DO takes the ensemble perspective, he will deem the fluctuations irrelevant. But from the DM's time perspective, they reduce growth. While a DO interested in the ensemble may get away with disregarding low-probability events, for the DM's success hedging against rare events is of crucial importance (not only but ultimately also for survival).

The difference between how these two perspectives evaluate the effects of probabilistic events is qualitatively in line with the observed phenomena we set out to explain. The DM typically has large uncertainties, especially for low-probability events, and has an evolutionary incentive to err on the side of caution, *i.e.* to behave as though low-probability (extreme) events had a higher probability than in the DO's model.

## 4 Fitting the model to experimental results

Visually, looking at the figures and the level of noise in the data in Fig. 1, one would conclude that Tversky and Kahneman's fitting exercise of the inverse-S curve by way of the physically unmotivated function  $\tilde{F}_w^{TK}$ , (Eq. 4), resembles the data no better than our mechanistically constrained model. This is particularly evident in the bottom two panels of Fig. 4, which show that a Gaussian  $w(x)$  whose scale and location differ from those of  $p(x)$  reproduces the fitted functional shape of  $F_w$ .

For completeness and scientific hygiene, in the present section we fit location and scale parameters in the Gaussian and  $t$  models for  $F_w$  to experimental data from Tversky and Kahneman (1992) (depicted in circles in Fig. 1) and from Tversky and Fox (1995). Specifically, in the Gaussian model we fit the location and scale parameters  $\mu$  and  $\sigma$  in the CDF

$$F_w(x) = \Phi\left(\frac{\Phi^{-1}(F_p(x)) - \mu}{\sigma}\right), \quad (12)$$

where  $\Phi$  is the CDF of the standard normal distribution. In the  $t$ -model we fit the location parameter  $\mu$  and the shape parameter  $\nu$  in the CDF  $F_w(x)$  of a  $t$ -distributed random variable (see Sec. 2.3), assuming  $p$  follows a standard normal distribution.

In addition to (Eq. 4) used by Tversky and Kahneman, we fit the function

$$\tilde{F}_L(F_p; \delta, \gamma) = \frac{\delta F_p^\gamma}{\delta F_p^\gamma + (1 - F_p)^\gamma}, \quad (13)$$

suggested by Lattimore et al. (1992) to parametrically describe probability weighting (also used by Tversky and Wakker (1995) and Prelec (1998)). The reason for fitting

(Eq. 13) is to ensure a fair comparison: the Gaussian and  $t$  models are characterised by two parameters, whereas (Eq. 4) only has one free parameter. Equation (13) has two parameters.

Figure 7 presents the fit results. We obtain very good fits to data for both Gaussian and  $t$ -distributions, as well as for (Eq. 4) and (Eq. 13), in the two experiments. It is practically impossible to distinguish between the fitted functions within standard errors. We conclude that our model fits the data well, and unlike (Eq. 4) or (Eq. 13), the fitted functions are directly derived from a physically plausible mechanism, and are not simply phenomenological.

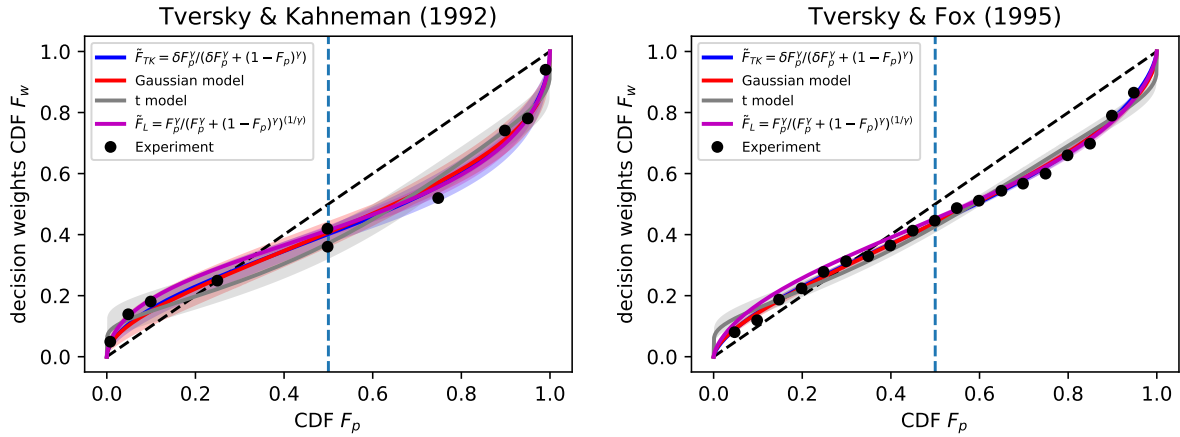


Figure 7: Model fitting to experimental data from [Tversky and Kahneman \(1992\)](#) (left) and [Tversky and Fox \(1995\)](#) (right). Left) [Lattimore et al. \(1992\)](#):  $\delta = 0.67$  ( $SE = 0.04$ ),  $\gamma = 0.58$  ( $\pm 0.03$ ); Gaussian model:  $\mu = 0.38$  ( $\pm 0.06$ ),  $\sigma = 1.60$  ( $\pm 0.10$ );  $t$  model:  $\nu = 1.27$  ( $\pm 0.28$ ),  $\mu = 0.40$  ( $\pm 0.07$ ); [Tversky and Kahneman \(1992\)](#) (Eq. 4):  $\gamma = 0.60$  ( $\pm 0.02$ ). Right) [Lattimore et al. \(1992\)](#):  $\delta = 0.77$  ( $\pm 0.01$ ),  $\gamma = 0.69$  ( $\pm 0.01$ ); Gaussian model:  $\mu = 0.22$  ( $\pm 0.01$ ),  $\sigma = 1.41$  ( $\pm 0.03$ );  $t$  model:  $\nu = 1.41$  ( $\pm 0.21$ ),  $\mu = 0.22$  ( $\pm 0.03$ ); [Tversky and Kahneman \(1992\)](#) (Eq. 4):  $\gamma = 0.68$  ( $\pm 0.01$ ). Shaded areas indicate two standard errors in the fitted parameter values. The fit was done by implementing the Levenberg-Marquardt algorithm [Levenberg \(1944\)](#) for non-linear least squares curve fitting.



## 5 Discussion

On 28 February 2020, [Sunstein \(2020\)](#), a behavioural economist, legal scholar, and former United States Administrator of the Office of Information and Regulatory Affairs, diagnosed that people’s concern about a potential coronavirus outbreak in the US was attributable to an extreme case of probability weighting – they neglected the fact, supposedly, that such an event had a low probability. When the piece was published, many commented that it seemed quite reasonable to them to take precautions, and Sunstein himself may have underestimated the severity of what lay ahead. One month later the US became the epicentre of the global coronavirus pandemic.

The episode illustrates that an inverted S-curve is a neutral indicator of a difference in opinion. It says nothing about who is right and who is wrong.

The term “probability weighting” suggests an obscure mental process, where a DM carries out operations on probabilities. It seems more natural to us to consider a DM modelling events he is unsure about. From this latter point of view, it is easy to think of reasons for a DM’s model to differ from a DO’s. Specifically DMs will often include additional uncertainty, leading to the frequently observed inverse-S shape.

The model of estimating probabilities from real time series, which we discuss in [Sec. 3](#) has qualitative features that display a degree of universality. Relative errors in the DM’s probability estimates are always greater for rarer events. A dislike of the unexpected, which explains the systematic overestimation of low probabilities, is similarly common. “Probability weighting” is purely descriptive and comes with the ill-conceived connotation of DMs suffering from a cognitive error. The phenomenon is better thought of as DMs making wise decisions given the information available to them, which is constrained because they must collect such information as time passes.

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