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Subjectively Weighted Utility: A Descriptive Extension of the Expected Utility Model

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A parsimonious extension to the Expected Utility (EU) model is proposed involving one additional parameter. The model is an example of a Subjectively Weighted Utility (SWU) model, which differs from the EU model only in the way probabilities are incorporated. The probability transformation is of a form that has appeared in the literature on human information processing and the additional parameter α has been interpreted as an index of information processing performance.

The SWU model is shown to have considerable descriptive ability. It is discussed in the context of the substitutability axiom, distinct utility curves, insurance purchasing and gambling behavior, and probability variance preferences.

Utility theory and the Expected Utility (EU) model are said to have originated with Bernoulli's analysis (1738) of the so-called St. Petersburg Paradox. Important subsequent developments in the field were the axiomatization of utility theory by Von Neumann and Morgenstern (1947) and the incorporation of subjective probability by Savage (1954) who also provided an operational form of the axioms for the theory. The notion of a Subjectively Expected Utility (SEU) model was also advanced by Edwards (1955).

Although utility theory originated as a descriptive model for human behavior, its current status is that of a normative model regarded in the behavioral literature as a poor descriptor of empirically observed decision making behavior. Attempts have been made to extend the model in various ways to improve its descriptive ability. One such class of extensions are the Subjectively Weighted Utility (SWU) models where the criterion for decision making is a weighted sum of utilities, and the weights are some transformations of probabilities. Another class replaces utilities with some function of the moments of the probability distribution over outcomes (Coombs & Huang, 1970). All these methods are similar in spirit in that they propose a single criterion defined on the attributes of the decision problem, and describe decision behavior in terms of maximization of the criterion. The major competitors to criterion models are *process* models (also termed *protocol* or *contingency* models) where the sequen-

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tial information processing nature of decision making is emphasized. Payne (1973) has an excellent discussion of the relative merits of these approaches.

Since the model presented in this paper is a criterion model, it seems appropriate to comment briefly on the motivation for presenting another such model. Criterion models are general in the sense that the same form of model applies to all decision makers. This enables a comparison of behavior across individuals in terms of the parameters of the model. The major strength of criterion models lies in their analytical tractability and explicit structure. The implications of a given model can be explored and used to suggest experimentally verifiable hypotheses.

The strongest objections to criterion models are based on limitations in their descriptive ability. These are best made apparent by counter examples such as the Allais (1953) paradox. Other objections arise from the results of empirical investigations which cannot be explained by EU–SEU models. Some of these issues will be addressed in this paper. It appears that some of the descriptive shortcomings of most criterion models arise from the rigid way in which probabilities are incorporated in the criterion. A major exception is the Value Theory model proposed by Kahneman and Tversky (Note 1). The model studied in this paper is related to Value Theory in the way probabilities are treated. However, unlike Value Theory, no special restrictions are placed on the utility function.

Thus, the model described in this paper is descriptive in intent. At the same time, the structure of the model is in the formal tradition of utility theory. In this context it is useful to recall the descriptive beginnings of utility theory, and the impact of concepts such as risk aversion, risk premiums, and certainty equivalents, despite the descriptive shortcomings of the theory. In the same spirit, the model presented here does not describe each and every aspect of observed decision making behavior. However, it does explain certain important phenomena which are difficult or impossible to model with an EU theory. Furthermore, this extended descriptive ability is captured by a single additional parameter which is capable of intuitive interpretation.

THE MODEL

The model is an example of an SWU (Subjectively Weighted Utility) model. Prizes are mapped into utilities in the usual manner: $x \to U(x)$. In addition, the model maps probabilities into subjective weights: $p_i \to w_i$. This weight function is defined implicitly by the relation

$$ln\left(\frac{w_i}{1 - w_i}\right) = \alpha \, ln\left(\frac{p_i}{1 - p_i}\right)$$

where $0 < \alpha < \infty$. Or

$$\frac{w_i}{1 - w_i} = \left(\frac{p_i}{1 - p_i}\right)^\alpha$$

and hence

$$w_i = \frac{(\text{Odds}_i)^\alpha}{1 + (\text{Odds}_i)^\alpha}; \ \text{Odds}_i = \left(\frac{p_i}{1 - p_i}\right)$$

We always have $0 \leqslant w_i \leqslant 1$, and for binary gambles where $p_1 + p_2 = 1$, we have $w_1 + w_2 = 1$. However, if the number of outcomes is greater than two, the weights need not (and usually will not) sum to 1. For $\alpha \neq 1$ the mapping has three fixed points: 0, ½, and 1. Thus, equiprobability, certainty, and impossibility are not affected by the mapping. The mapping is sketched in Fig. 1. As α tends to zero, every outcome is seen as equally likely; at $\alpha = 1$, the weights correspond to the true probabilities. If values of $\alpha > 1$ are considered, the implication as α goes to infinity is that an event is thought impossible if its probability is less than ½ and certain if the probability is greater than ½. (This can give rise to certain paradoxes so we require $\alpha < \infty$.)

We can regard α as a measure of information processing performance. Low values of α can be thought of as "underprocessing," and values of $\alpha > 1$, as "overprocessing" of the available information. In another sense, low values of $\alpha(<1)$ can be interpreted as excessive uncertainty and high values (>1) as excessive certainty. In the following sections we will restrict our attention to cases where $\alpha < 1$.

There is considerable motivation in the literature for considering the general shape of weight function used here. For example, Dale (1959) has reported that subjects commonly tend to overestimate low probabilities

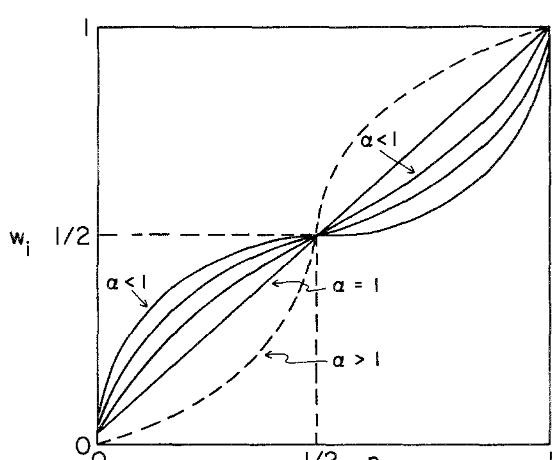

FIG. 1. Transformation of probabilities.

and underestimate high ones. A similar shape is also suggested by Kahneman and Tversky (Note 1) in their Value Theory model, although no functional form is specified, and the theory requires discontinuity of the function at the end points.

As for the particular functional form that has been chosen, its desirable properties include its qualitative shape, simplicity, and continuity. In addition, work in the literature on modeling human probability processing has used an information processing model which is very similar (Phillips & Edwards, 1966; Phillips, Hays, & Edwards, 1966). The model is based on a transformation to "log-odds" and involves only one parameter, α.

THE SWU CRITERION AND UTILITY THEORY AXIOMS

The criterion for comparing lotteries in this model is the SWU measure which is defined for a lottery $l = \{(p_i, x_i), i = 1, \ldots, n\}$ as:

$$\mathrm{SWU} = \sum_{i=1}^{n} w_i \, U(x_i) / \sum_{i=1}^{n} w_i \ .$$

This model is a parsimonious extension of the Expected Utility (EU) model in the sense that only one additional parameter is introduced. Note that the criterion differs from the usual SEU model, in that the effective weights are normalized, and will sum to one. Thus the weight on each outcome depends on the probabilities for other outcomes through the normalization factor $\sum_{i=1}^{n} w_i$. We begin by discussing the relationship between the SWU and EU models.

The SWU model differs from the EU model in the way probabilities are incorporated in the criterion. All the axioms underlying the EU model are satisfied (necessarily) by the SWU model except for the axiom of substitutability. This axiom requires that in any gamble, substitution of a prize with an equivalent prize or gamble does not change the evaluation of the gamble. Due to the bias introduced by the weight function, this property is not possessed by the SWU model.

To illustrate this, consider the following example. Suppose that a decision maker is indifferent between a gamble offering even chances at prizes of \$0 and \$100, and a prize of \$C for certain.

$$l_1 \sim \begin{array}{ll} \tfrac{1}{2} & \$100 \\ \tfrac{1}{2} & \$0 \end{array} \qquad \sim \qquad \$C \ .$$

Now consider the choice between the lotteries l_2 and l_3.

$$l_2 \sim \begin{array}{ll} \tfrac{1}{2} & \$100 \\ \tfrac{1}{2} & \$C \end{array} \qquad\qquad l_3 \sim \begin{array}{ll} \tfrac{3}{4} & \$100 \\ \tfrac{1}{4} & \$0 \end{array}$$

By substitutability, these two lotteries are equivalent under the EU criterion. However, since the weight function underestimates probabilities greater than ½, the SWU criterion values the gamble l_3 lower than the EU model. Hence, the SWU model predicts that l_2 would be preferred by the decision maker over l_3. This example also suggests a simple experiment to support the descriptive ability of the SWU model.

DETERMINATION OF UTILITY CURVES

This lack of substitutability has been found in preliminary testing by Karmarkar (Note 2) in the context of a more extensive experimental framework. The purpose of that paper was to demonstrate the existence of specific probability effects outside the expected utility model even where all probabilities were canonical. Briefly, the experiment consisted of determining utility curves first using even gambles (in the usual way) and then using other (fixed) probabilities. Under the expected utility model the same utility curve should have been obtained regardless of the probabilities of the gambles. In fact, it was found that distinct utility curves were obtained for different probabilities.

The SWU model displays this phenomenon of distinct utility curves. This type of behavior can be demonstrated by simulating the deter-

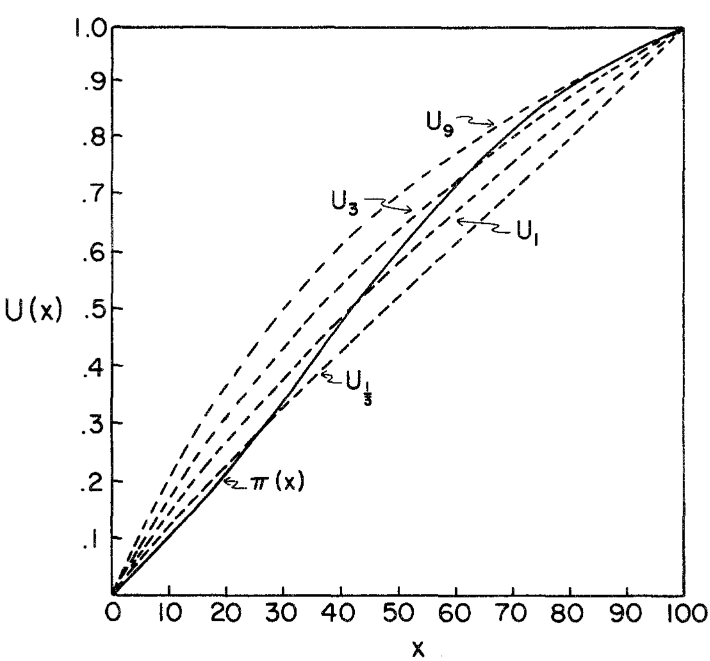

FIG. 2. Utility curves generated by the SWU model for different values of the odds. ($U_1(x) = \log(x + 100)$, $\alpha = 0.8$.)

mination of utility curves for such a model. The utility function $U(x) = \log(x + 100)$ was used in an SWU model with $\alpha = 0.8$. Separate utility curves were determined based on certainty equivalents to binary gambles parametrized by different values of odds. The odds are stated in terms of higher payoff and the values used were ⅓, 1, 3, and 9. The results are plotted in Fig. 2 where the subscript on U indicates the odds used. The apparent increase in risk aversion with increasing odds that was reported by Karmarkar (Note 2) can be seen here. This is in accord with the empirically observed phenomenon of a "preference for variance." That is to say, if two lotteries have equal expected utility in the usual sense, the SWU model predicts that the model with higher odds favoring will be *less* preferred. This may seem counterintuitive, but becomes plausible when examples of such gambles are examined (such an example was presented above). The apparent preference for variance displayed by the SWU model is examined more carefully in a later section.

Note that the underlying utility curve can be recovered by a sequence of even (50:50) lotteries. On the other hand, the definition of a utility function in the Savage axiomatization depends on being able to state probabilities $\pi(c)$ for any prize c, such that

$$c \sim \begin{cases} \pi(c) & c^* \\ 1-\pi(c) & c_* \end{cases} \qquad ; c_* \leqslant c \leqslant c^*$$

where c^*, c_* are the "most preferred" and "least preferred" outcomes in the range being considered. This approach is inconvenient to use in practice because of subjective difficulties in stating $\pi(c)$. In fact, in the SWU model, such a procedure would not lead to the underlying utility curve at all. For example, in the situation described earlier, the curve that would be obtained has been indicated by the solid curve $\pi(x)$ in Fig. 2.

RISK, INSURANCE, AND GAMBLING

In most of the following we will assume that the underlying utility function is linear; $U(x) = x$. This serves to emphasize that many of the behavioral implications of the SWU model are essentially due to the bias introduced by the weighting function $w(\cdot)$. Furthermore, we will commonly be using binary gambles or lotteries as examples and will denote such a lottery in the usual way as $l = (p, x, y)$ to indicate

$$l \sim \begin{cases} p & x \\ (1-p) & y \end{cases}$$

The following notation will be used

$E(l)$ = Expected Value of the lottery $= px + (1 - p)y$

$$
\begin{aligned}
V(l) &= \text{Variance of the lottery} = p(1-p)(x-y)^2 \\
EU(l) &= \text{Expected Utility} = pU(x) + (1-p)\,U(x) \\
&\qquad\qquad\qquad\qquad\quad = E(l) \text{ when } U(x) = x \\
SWU(l) &= \text{weighted utility} = wU(x) + (1-w)\,U(y) \\
CE(l) &= \text{Certainty Equivalent} = U^{-1}\,(EU(l)) \\
&= E(l) \text{ when } U(x) = x \\
SWE(l) &= \text{Subjective Equivalent} = U^{-1}\,(SWU(l)) \\
&= SWU(l) \text{ when } U(x) = x \\
R(l) &= \text{Risk Premium} = E(l) - CE(l) \\
&= 0 \text{ when } U(x) = x \\
P(l) &= \text{Probability Bias} = CE(l) - SWE(l) \\
B(l) &= \text{Total Bias} = E(l) - SWE(l) \\
&= E(l) - SWU(l) \text{ when } U(x) = x \\
&= (p-w)(x-y)
\end{aligned}
$$

The dependence on the lottery will usually be suppressed for notational convenience.

In the SWU model, the probability structure of gambles has an effect on the apparent risk aversion of the decision maker. We define the evaluation of a lottery as being risk averse when the total bias $B(l)$ is positive and risk prone when $B(l)$ is negative. The SWU model can exhibit both risk aversion and risk proneness even when the underlying utility function is linear (or risk neutral). This is easily seen by examining the Total Bias of a lottery $l = (p, x, y)$ which is given by $B = (p - w)(x - y)$. Assuming $x > y$, we can see that $B > 0$ indicating risk aversion for $\frac{1}{2} < p < 1$ and $B < 0$ (risk prone) for $0 < p < \frac{1}{2}$.

If we fix x and y and allow p to vary, the bias due to the probability structure will be maximized when $(p - w)$ is maximized. This maximum is achieved for some $\frac{1}{2} < p < 1$. For example, for $\alpha = 0.75$, the maximum is achieved at $p = 0.855$. This has implications for the phenomenon of distinct utility curves described earlier. If $\alpha = 0.75$, for example, the apparent risk aversion of the separate curves will increase as p is increased up to 0.855 and will decrease thereafter. The bias will disappear as p tends to 1. By symmetry, the apparent increase in risk proneness will be highest when p is reduced to 0.145. This behavior relative to the underlying utility curve depends only on α and not on the utility curve itself.

This is an interesting phenomenon from a behavioral point of view. It suggests that although a person may not appear to be very risk averse in terms of his cardinal utility function, he may in fact behave in a much more risk averse (or risk prone) manner in particular circumstances. For example, suppose that the lottery being evaluated is exactly that—a public lottery. The gamble here involves a small chance p at a large prize x, and a complementary chance at a small loss y. Even if the lottery is not favorable on an expected value basis, the probability bias can cause the lottery to appear attractive. Specifically, the small probability p is overvalued, and the bias is magnified by the large prize x.

In an analogous manner, a small chance at a large loss will lead to behavior that is more risk averse than the underlying utility curve would warrant. In effect, the probability bias is able to account for simultaneous gambling and insurance purchasing behavior, even when the underlying utility curve is risk neutral. This makes unnecessary the device of postulating kinks in the utility curve (Friedman & Savage, 1948). If we consider utility functions that are other than linear, we can obtain richer descriptions of behavior. If the utility curve is linear or risk averse, then increasing levels of loss will make insurance purchase more attractive and this effect will be reinforced by the bias towards risk aversion induced by the probability structure. However, suppose the utility curve is *risk prone* for large negative changes in asset position. (This is implied by the boundedness requirements of the Savage axiomatization, and is also suggested in the Value Theory model of Kahneman & Tversky, Note 1). Then for moderate levels of loss, the risk aversion induced by the probability bias can make insurance purchase seem attractive, although at large levels of loss, the risk proneness of the utility curve will dominate. It is also interesting to note that the probability bias is greatest (in relative terms) for probabilities that are fairly high in an insurance context.

APPARENT VARIANCE PREFERENCES

Certain empirical studies of decision making behavior have purported to show "preferences for variance" (Coombs & Pruitt, 1960). That is to say, under specified conditions, subjects exhibit preferences for high or low variance gambles. Typical patterns of preferences can be described in the context of a set of gambles such as those displayed in Table 1. Here, all the gambles displayed have approximately equal expected values. Gambles in the same row have the same variance and those in the same column have the same probability p of winning.

In a common pattern of "variance preference" subjects were found to prefer high variance gambles in sets 1 and 2, and low variance in sets 4 and 5. In this section we show that the SWU model displays this type of variance preference due to its probability bias alone. We will assume that $U(x) = x$.

If we fix the expected values of a set of gambles, we can write x in terms of other parameters as

$$x = \mathrm{E}/p + (1 - 1/p)y \ .$$

Substituting this expression into the definitions of SWU and V, we get:

$$\mathrm{SWU} = (1 - \frac{w}{p})y + (\frac{w}{p})\mathrm{E}$$

TABLE 1*
BETS (p, x, y) USED TO DETERMINE PROBABILITY AND VARIANCE PREFERENCES

		Set				
		1	2	3	4	5
Set	Variance	p = 1/10	p = 3/10	p = 5/10	p = 7/10	p = 9/10
A	.09	+$ .90 − .10	+$ .45 − .20	+$ .30 − .30	+$ .20 − .45	+$ .10 − .90
B	1.00	+ 2.97 − .33	+ 1.50 − .65	+ 1.00 − 1.00	+ .65 − 1.50	+ .33 − 2.97
C	5.00	+ 6.75 − .75	+ 3.40 − 1.45	+ 2.25 − 2.25	+ 1.45 − 3.40	+ .75 − 6.75
D	25.00	+ 14.85 − 1.65	+ 7.60 − 3.25	+ 5.00 − 5.00	+ 3.25 − 7.60	+ 1.65 − 14.85
E	90.00	+ 28.40 − 3.15	+ 14.45 − 6.20	+ 9.50 − 9.50	+ 6.20 − 14.45	+ 3.15 − 28.40

Note. The upper entry in each cell is x, the lower is y.
* Taken from Slovic & Lichtenstein (1968).

$$V = \frac{(1-p)}{p}(E-y)^2 .$$

Suppose now that we fix $\frac{1}{2} < p < 1$ so that $w/p < 1$. We can plot SWU and V as functions of the remaining free parameter y (Fig. 3).

Consider the two regions $y < E$ and $y > E$. In the former case $y < x$ and the odds favor the higher prize (corresponding to sets 4 and 5). It can be seen that in this region, increasing SWU corresponds to

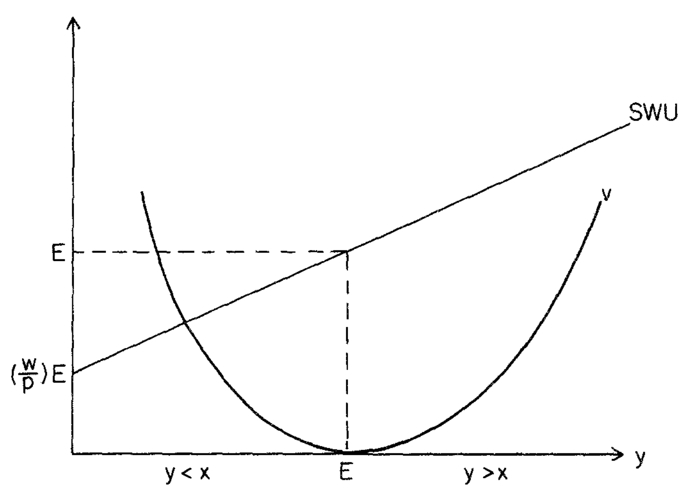

FIG. 3. Illustration of variance preference.

decreasing variance so that lower variance bets are preferred. In the latter case, $y > x$, the odds favor the lower prize, and higher variance bets are preferred (sets 1 and 2). These effects are qualitatively in accord with the results of Payne (1975) and Payne and Braunstein (1971) who describe shifts in variance preference depending on whether probability-to-win is greater or less than probability-to-lose.

APPARENT PROBABILITY PREFERENCES

Investigations such as that by Coombs and Pruitt (1960) have also studied "probability preferences." For the type of variance preference described in the previous section, Slovic and Lichtenstein (1968) stated that there is a related pattern of probability preference: For any fixed value of variance, subjects preferred the gamble with the lowest probability of winning.

To study this situation, suppose we fix the variance V of a set of gambles, in addition to E. We can solve for y in terms of E, V, and p as

$$y = \mathrm{E} - (p\mathrm{V} / (1 - p))^{1/2} .$$

Using this expression we can write SWU as

$$\mathrm{SWU} = \mathrm{E} + (w - p)\,(\mathrm{V} / p(1 - p))^{1/2} .$$

The remaining free parameter in this expression is p. For such a set of gambles with fixed E and V, the most preferred gamble will be that which maximizes SWU; i.e., that which maximizes

$$(w - p) / (p(1 - p)^{1/2} .$$

It can be demonstrated that for $\alpha < 1/2$, this expression tends to infinity as p tends to zero. Hence, it follows that lower probability gambles are preferred. For $\alpha > 1/2$, the expression is maximized for small values of p. For example, for $\alpha = 0.75$ the expression is maximized at $p = .035$.

Another version of "probability preference" can be constructed by looking at a different set of gambles. Consider the case where the expected value E and y are kept fixed with $y < \mathrm{E}$. Then $\mathrm{SWU} = y + w/p$ $(\mathrm{E} - y)$. The preferred probability in this case is that which maximizes (w/p). Again it can be seen that (w/p) goes to infinity as p tends to zero. We would expect that for the subjects with the type of behavior described above (with approximately risk neutral utility functions), that low probability gambles would be preferred in this set.

A theoretical discussion of probability, variance and skewness preferences can be found in the paper by Pollatsek (1971). It is shown there that certain patterns of preferences can be constructed which are inconsistent with *any* SEU model. However, it is not known how prevalent such patterns are.

SUMMARY

This paper has presented an extension of the expected utility model that, in effect, sacrifices the substitutability property for a gain in descriptive ability. The SWU model transforms probabilities into weights. The shape of the weighting function tallies with some of the literature on human probability perception. Utilities for prizes are incorporated in the same way as in the EU model—they can be defined in this case by a sequence of 50–50 lotteries.

It is shown that this model overcomes some of the descriptive limitations of EU models. For example, the model exhibits simultaneous gambling and insurance behavior. The model also exhibits apparent variance and probability preferences of a type described in empirical studies.

The model suggests possible experimental hypotheses. Amongst these are verification of the violation of the substitutability axiom in the manner described earlier, the distinct utility curve phenomenon and specific variance and probability preference. It would also be interesting to test the predictive ability of the model. Finally, from a theoretical point of view, the implications of the model for specific choices of utility function have not been explored.

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