Dinámica de Medios Deformables

Grupo 8253 – Sem. 2024-2

Tarea 5

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(5 pts)

Problema 1: Formulación de tensión plana

Escribir las siguientes relaciones para la formulación de tensión plana:

- Deformación-desplazamiento
- Ecuaciones de equilibrio
- Ecuaciones de Hooke
- Ecuaciones de Navier
- Ecuaciones de Michell-Beltrami

El tensor de esfuerzos de Cauchy para esta formulación está dado como:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0. \end{bmatrix} \tag{1.1}$$

Para escribir la relación de deformación-desplazamiento debemos escribir el tensor de deformación en términos de los desplazamientos, *i.e.*,

$$\varepsilon_{ij} = \tfrac{1}{2}[u_{i,j} + u_{j,i}]. \tag{1.2}$$

Así,

$$\varepsilon_{ij} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{bmatrix}$$

Recordando que para esta formulación $w = 0$ y que u, v no dependen de z, tal que

$$\frac{\partial u_i}{\partial z} = \frac{\partial w}{\partial x^i} = 0. \tag{1.3}$$

Entonces, la relación de deformación-desplazamiento se reduce a

$$\boxed{\varepsilon_{ij} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 0 \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix}.} \tag{1.4}$$

Mientras las ecuaciones de equilibrios están dadas mediante

$$\sigma_{ji,j} + B_i = 0.$$

Estas ecuaciones son

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{xx}}{\partial z} + B_x = 0, \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + B_y = 0, \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + B_z = 0. \end{aligned}$$

Y por (1.1) se reducen a

$$\boxed{\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + B_x = 0, \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + B_y = 0. \end{aligned}} \tag{1.5}$$

La relación inversa de la ley de Hooke está dada por

$$\varepsilon_{ij} = \frac{1-\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk}, \tag{1.6}$$

tal que para esta formulación se ve como:

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{1-\nu}{E}\sigma_{xx} - \frac{\nu}{E}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}),\\
\varepsilon_{yy} &= \frac{1-\nu}{E}\sigma_{yy} - \frac{\nu}{E}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}),\\
\varepsilon_{zz} &= \frac{1-\nu}{E}\sigma_{zz} - \frac{\nu}{E}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}),\\
\varepsilon_{xy} &= \frac{1-\nu}{E}\sigma_{xy},\\
\varepsilon_{yz} &= \frac{1-\nu}{E}\sigma_{yz},\\
\varepsilon_{xz} &= \frac{1-\nu}{E}\sigma_{xz}.
\end{aligned}
$$

Que por (1.1) se reducen a

$$
\boxed{
\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy}, && (1.7)\\
\varepsilon_{yy} &= \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{xx}, && (1.8)\\
\varepsilon_{zz} &= -\frac{\nu}{E}(\sigma_{xx}+\sigma_{yy}), && (1.9)\\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy}, && (1.10)\\
\varepsilon_{xz} &= \varepsilon_{yz} = 0.
\end{aligned}
}
$$

Invertimos (1.7) a (1.10); primero sumamos y restamos (1.7) y (1.8),

$$
\varepsilon_{xx}+\varepsilon_{yy} = \frac{1-\nu}{E}(\sigma_{xx}+\sigma_{yy}) \quad\Longrightarrow\quad \sigma_{xx}+\sigma_{yy} = \; = \frac{E}{1-\nu}(\varepsilon_{xx}+\varepsilon_{yy}), \tag{1.11}
$$

$$
\varepsilon_{xx}-\varepsilon_{yy} = \frac{1+\nu}{E}(\sigma_{xx}-\sigma_{yy}) \quad\Longrightarrow\quad \sigma_{xx}-\sigma_{yy} = \; = \frac{E}{1+\nu}(\varepsilon_{xx}-\varepsilon_{yy}). \tag{1.12}
$$

Para obtener σ_{xx} sumamos (1.11) y (1.12),

$$\begin{aligned}
2\sigma_{xx} &= \frac{E}{1-\nu}(\varepsilon_{xx}+\varepsilon_{yy}) + \frac{E}{1+\nu}(\varepsilon_{xx}-\varepsilon_{yy}) \\
&= E\left[\frac{(1+\nu)(\varepsilon_{xx}+\varepsilon_{yy}) + (1-\nu)(\varepsilon_{xx}-\varepsilon_{yy})}{1-\nu^2}\right], \\
&= \frac{E}{1-\nu^2}[2\varepsilon_{xx}+2\nu\varepsilon_{yy}], \\
\sigma_{xx} &= \frac{E}{1-\nu^2}[\varepsilon_{xx}+\nu\varepsilon_{yy}]. \qquad (1.13)
\end{aligned}$$

Y para obtener σ_{yy} restamos (1.11) y (1.12),

$$\begin{aligned}
2\sigma_{yy} &= E\left[\frac{(1+\nu)(\varepsilon_{xx}+\varepsilon_{yy}) - (1-\nu)(\varepsilon_{xx}-\varepsilon_{yy})}{1-\nu^2}\right], \\
\sigma_{yy} &= \frac{E}{1-\nu^2}[\nu\varepsilon_{xx}+\varepsilon_{yy}]. \qquad (1.14)
\end{aligned}$$

De (1.1) sabemos que $\sigma_{zz}=0$. Finalmente, resolvemos (1.10) para σ_{xy},

$$\sigma_{xy} = \frac{E}{1+\nu}\varepsilon_{xy}. \qquad (1.15)$$

Para las ecuaciones de Navier, primero sustituimos (1.4) en (1.13) a (1.15),

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}\right), \\
\sigma_{yy} &= \frac{E}{1-\nu^2}\left(\nu\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right), \\
\sigma_{xy} &= \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \\
\sigma_{zz} &= \sigma_{yz} = \sigma_{xz} = 0.
\end{aligned}$$

Introduciendo las relaciones anteriores en las ecuaciones de equilibrio (1.5) obtenemos las ecuaciones de Navier

$$\boxed{\begin{aligned}
\frac{\partial}{\partial x}\left[\frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] + B_x = 0, \\
\frac{\partial}{\partial x}\left[\frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\frac{E}{1-\nu^2}\left(\nu\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right] + B_y = 0.
\end{aligned}}$$

Finalmente, para las ecuaciones de Michell-Beltrami, partimos de las ecuaciones de compatibilidad, de las que "sobrevive" únicamente

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x\,\partial y}. \tag{1.16}$$

Introduciendo las relaciones (1.7) a (1.10),

$$\begin{aligned}
\frac{1}{E}\frac{\partial^2 \sigma_{xx}}{\partial y^2} - \frac{\nu}{E}\frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{1}{E}\frac{\partial^2 \sigma_{yy}}{\partial x^2} - \frac{\nu}{E}\frac{\partial^2 \sigma_{xx}}{\partial x^2} &= 2(1+\nu)\frac{\partial^2 \sigma_{xy}}{\partial x\,\partial y},\\
\left(\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{xx}}{\partial y^2}\right) + \left(\frac{\partial^2 \sigma_{yy}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2}\right) - \nu\left(\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2}\right) &= (1+\nu)\frac{\partial^2 \sigma_{xy}}{\partial x\,\partial y} + \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2},\\
\nabla^2(\sigma_{xx}+\sigma_{yy}) &= (1+\nu)\left(2\frac{\partial^2 \sigma_{xy}}{\partial x\,\partial y} + \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2}\right).
\end{aligned} \tag{1.17}$$

Derivando las relaciones de equilibrio (1.5) respecto a x y y, respectivamente,

$$\begin{aligned}
\frac{\partial^2 \sigma_{xx}}{\partial x\,\partial x} + \frac{\partial^2 \sigma_{yx}}{\partial y\,\partial x} + \frac{\partial B_x}{\partial x} &= 0,\\
\frac{\partial^2 \sigma_{xy}}{\partial x\,\partial y} + \frac{\partial^2 \sigma_{yy}}{\partial y\,\partial y} + \frac{\partial B_y}{\partial y} &= 0.
\end{aligned}$$

Sumando y resolviendo para las fuerzas de volumen,

$$\begin{aligned}
2\frac{\partial^2 \sigma_{xy}}{\partial x\,\partial y} + \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} &= 0,\\
2\frac{\partial^2 \sigma_{xy}}{\partial x\,\partial y} + \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} &= -\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right).
\end{aligned} \tag{1.18}$$

Sustituyendo este resultado en (1.17), tenemos que las ecuaciones de Michell-Beltrami para la formulación de tensión plana son

$$\boxed{\nabla^2(\sigma_{xx}+\sigma_{yy}) = -(1+\nu)\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right).}$$

(5 pts)

Problema 2: Formulación de deformación plana

Escribir las siguientes relaciones para la formulación de deformación plana:

- Deformación-desplazamiento
- Ecuaciones de equilibrio
- Ecuaciones de Hooke
- Ecuaciones de Navier
- Ecuaciones de Michell-Beltrami

Para esta formulación tenemos que el tensor de deformaciones es

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0. \end{bmatrix} \tag{2.1}$$

Y que la relación de deformación-desplazamiento es igual a la de tensión plana (1.4), *i.e.*,

$$\boxed{\varepsilon_{ij} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 0 \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix}.} \tag{2.2}$$

Mientras que las ecuaciones de equilibrio son iguales a las de tensión plana (1.5),

$$\boxed{\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + B_x = 0, \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + B_y = 0. \end{aligned}} \tag{2.3}$$

La ley de Hooke para este caso está dada por

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}, \tag{2.4}$$

tal que,

$$
\begin{aligned}
\sigma_{xx} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{xx}, \\
\sigma_{yy} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{yy}, \\
\sigma_{zz} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{zz}, \\
\sigma_{xy} &= 2\mu\varepsilon_{xy} = \sigma_{yx}, \\
\sigma_{yz} &= 2\mu\varepsilon_{yz} = \sigma_{zy}, \\
\sigma_{xz} &= 2\mu\varepsilon_{xz} = \sigma_{zx}.
\end{aligned}
$$

Que por (2.1) se reducen a

$$
\boxed{
\begin{aligned}
\sigma_{xx} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy}) + 2\mu\varepsilon_{xx}, \\
\sigma_{yy} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy}) + 2\mu\varepsilon_{yy}, \\
\sigma_{zz} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy}), \\
\sigma_{xy} &= 2\mu\varepsilon_{xy} = \sigma_{yx}, \\
\sigma_{xz} &= \sigma_{yz} = 0.
\end{aligned}
}
\tag{2.5}
$$

Ahora, para las ecuaciones de Navier primero sustituimos (2.2) en (2.5),

$$
\begin{aligned}
\sigma_{xx} &= (\lambda + 2\mu)\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y}, \\
\sigma_{yy} &= (\lambda + 2\mu)\frac{\partial v}{\partial y} + \lambda\frac{\partial u}{\partial x}, \\
\sigma_{zz} &= \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \\
\sigma_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).
\end{aligned}
$$

Introduciendo las relaciones anteriores en las ecuaciones de equilibrio (2.3) tenemos que las ecuaciones de Navier son:

$$
\begin{aligned}
\implies \frac{\partial}{\partial x}\left[(\lambda+2\mu)\frac{\partial u}{\partial x}+\lambda\frac{\partial v}{\partial y}\right]+\frac{\partial}{\partial y}\left[\mu\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big)\right]+B_x=0,\\
\lambda\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial x^2}+\lambda\frac{\partial^2 v}{\partial x\,\partial y}+\mu\frac{\partial^2 u}{\partial y^2}+\mu\frac{\partial^2 v}{\partial x\,\partial y}+B_x=0,\\
\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+B_x=0.
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
\implies \frac{\partial}{\partial x}\left[\mu\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big)\right]+\frac{\partial}{\partial y}\left[(\lambda+2\mu)\frac{\partial v}{\partial y}+\lambda\frac{\partial u}{\partial x}\right]+B_y=0,\\
\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+B_y=0.
\end{aligned}
\tag{2.7}
$$

Reescribiendo (2.6) y (2.7) en notación vectorial, las ecuaciones de Navier para la deformación plana son

$$
\boxed{
\begin{aligned}
\mu\,\nabla^2 u+(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+B_x=0,\\
\mu\,\nabla^2 v+(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+B_y=0.
\end{aligned}
}
$$

Para obtener las ecuaciones de Michell-Beltrami, invertimos las relaciones (2.5), *i.e.*, es análogo al caso de tensión plana,

$$
\begin{aligned}
\varepsilon_{xx}+\varepsilon_{yy}=\frac{1}{2(\lambda+\mu)}(\sigma_{xx}+\sigma_{yy}),\\
\varepsilon_{xx}-\varepsilon_{yy}=\frac{1}{2\mu}(\sigma_{xx}-\sigma_{yy}).
\end{aligned}
$$

Para ε_{xx},

$$
\varepsilon_{xx}=\frac{(1+\nu)}{E}((1-\nu)\sigma_{xx}-\nu\sigma_{yy}).
\tag{2.8}
$$

Para ε_{yy},

$$
\varepsilon_{yy}=\frac{(1+\nu)}{E}((1-\nu)\sigma_{yy}-\nu\sigma_{xx}).
\tag{2.9}
$$

Y, además, por (2.1) $\varepsilon_{zz} = 0$. Para ε_{xy},

$$\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}. \tag{2.10}$$

Introduciendo (2.8) a (2.10) en las ecuaciones de compatibilidad (1.16),

$$\frac{\partial^2}{\partial y^2}\left[\left(\frac{1+\nu}{E}\right)((1-\nu)\sigma_{xx} - \nu\sigma_{yy})\right] + \frac{\partial^2}{\partial x^2}\left[\left(\frac{1+\nu}{E}\right)((1-\nu)\sigma_{yy} - \nu\sigma_{xx})\right] = 2\left(\frac{1+\nu}{E}\right)\frac{\partial^2\sigma_{xy}}{\partial x\,\partial y},$$

$$\left(\frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{xx}}{\partial y^2}\right) + \left(\frac{\partial^2\sigma_{yy}}{\partial x^2} + \frac{\partial^2\sigma_{yy}}{\partial y^2}\right) - \nu\left(\frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{xx}}{\partial y^2}\right) - \nu\left(\frac{\partial^2\sigma_{yy}}{\partial x^2} + \frac{\partial^2\sigma_{yy}}{\partial y^2}\right) = 2\frac{\partial^2\sigma_{xy}}{\partial x\,\partial y} + \frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{yy}}{\partial y^2},$$

$$(1-\nu)\,\nabla^2(\sigma_{xx} + \sigma_{yy}) = 2\frac{\partial^2\sigma_{xy}}{\partial x\,\partial y} + \frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{yy}}{\partial y^2}.$$

Sustituyendo (1.18) en la expresión anterior, tenemos que la ecuación de Michell-Beltrami para la formulación de deformación plana es

$$\boxed{\nabla^2(\sigma_{xx} + \sigma_{yy}) = -\frac{1}{1-\nu}\left(\frac{\partial B_x}{\partial x} + \frac{\partial By}{\partial y}\right).}$$