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(5 pts)

Problema 1: Formulación de tensión plana

Escribir las siguientes relaciones para la formulación de tensión plana:

- Deformación-desplazamiento
- Ecuaciones de equilibrio
- Ecuaciones de Hooke
- Ecuaciones de Navier
- Ecuaciones de Michell-Beltrami

El tensor de esfuerzos de Cauchy para esta formulación está dado como:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (1.1)$$

Para escribir la relación de deformación-desplazamiento debemos escribir el tensor de deformación en términos de los desplazamientos, i.e.,

$$\varepsilon_{ij} = \frac{1}{2}[u_{i,j} + u_{j,i}]. \quad (1.2)$$

Así,

$$\varepsilon_{ij} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{bmatrix}$$

Recordando que para esta formulación $w = 0$ y que u, v no dependen de z , tal que

$$\frac{\partial u_i}{\partial z} = \frac{\partial w}{\partial x^i} = 0. \quad (1.3)$$

Entonces, la relación de deformación-desplazamiento se reduce a

$$\varepsilon_{ij} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & 0 \\ \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (1.4)$$

Mientras las ecuaciones de equilibrios están dadas mediante

$$\sigma_{ji,j} + B_i = 0.$$

Estas ecuaciones son

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + B_x &= 0, \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + B_y &= 0, \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + B_z &= 0. \end{aligned}$$

Y por (1.1) se reducen a

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + B_x &= 0, \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + B_y &= 0. \end{aligned} \quad (1.5)$$

La relación inversa de la ley de Hooke está dada por

$$\varepsilon_{ij} = \frac{1-\nu}{E} \sigma_{ij} - \frac{\nu}{E} \delta_{ij} \sigma_{kk}, \quad (1.6)$$

tal que para esta formulación se ve como:

$$\varepsilon_{xx} = \frac{1-\nu}{E}\sigma_{xx} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}),$$

$$\varepsilon_{yy} = \frac{1-\nu}{E}\sigma_{yy} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}),$$

$$\varepsilon_{zz} = \frac{1-\nu}{E}\sigma_{zz} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}),$$

$$\varepsilon_{xy} = \frac{1-\nu}{E}\sigma_{xy},$$

$$\varepsilon_{yz} = \frac{1-\nu}{E}\sigma_{yz},$$

$$\varepsilon_{xz} = \frac{1-\nu}{E}\sigma_{xz}.$$

Que por (1.1) se reducen a

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy}, \quad (1.7)$$

$$\varepsilon_{yy} = \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{xx}, \quad (1.8)$$

$$\varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}), \quad (1.9)$$

$$\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}, \quad (1.10)$$

$$\varepsilon_{xz} = \varepsilon_{yz} = 0.$$

Invertimos (1.7) a (1.10); primero sumamos y restamos (1.7) y (1.8),

$$\varepsilon_{xx} + \varepsilon_{yy} = \frac{1-\nu}{E}(\sigma_{xx} + \sigma_{yy}) \implies \sigma_{xx} + \sigma_{yy} = \frac{E}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy}), \quad (1.11)$$

$$\varepsilon_{xx} - \varepsilon_{yy} = \frac{1+\nu}{E}(\sigma_{xx} - \sigma_{yy}) \implies \sigma_{xx} - \sigma_{yy} = \frac{E}{1+\nu}(\varepsilon_{xx} - \varepsilon_{yy}). \quad (1.12)$$

Para obtener σ_{xx} sumamos (1.11) y (1.12),

$$\begin{aligned}
 2\sigma_{xx} &= \frac{E}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy}) + \frac{E}{1+\nu}(\varepsilon_{xx} - \varepsilon_{yy}) \\
 &= E \left[\frac{(1+\nu)(\varepsilon_{xx} + \varepsilon_{yy}) + (1-\nu)(\varepsilon_{xx} - \varepsilon_{yy})}{1-\nu^2} \right], \\
 &= \frac{E}{1-\nu^2} [2\varepsilon_{xx} + 2\nu\varepsilon_{yy}], \\
 \sigma_{xx} &= \frac{E}{1-\nu^2} [\varepsilon_{xx} + \nu\varepsilon_{yy}].
 \end{aligned} \tag{1.13}$$

Y para obtener σ_{yy} restamos (1.11) y (1.12),

$$\begin{aligned}
 2\sigma_{yy} &= E \left[\frac{(1+\nu)(\varepsilon_{xx} + \varepsilon_{yy}) - (1-\nu)(\varepsilon_{xx} - \varepsilon_{yy})}{1-\nu^2} \right], \\
 \sigma_{yy} &= \frac{E}{1-\nu^2} [\nu\varepsilon_{xx} + \varepsilon_{yy}].
 \end{aligned} \tag{1.14}$$

De (1.1) sabemos que $\sigma_{zz} = 0$. Finalmente, resolvemos (1.10) para σ_{xy} ,

$$\sigma_{xy} = \frac{E}{1+\nu} \varepsilon_{xy}. \tag{1.15}$$

Para las ecuaciones de Navier, primero sustituimos (1.4) en (1.13) a (1.15),

$$\begin{aligned}
 \sigma_{xx} &= \frac{E}{1-\nu^2} \left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} \right), \\
 \sigma_{yy} &= \frac{E}{1-\nu^2} \left(\nu \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \\
 \sigma_{xy} &= \frac{E}{2(1+\nu)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \\
 \sigma_{zz} &= \sigma_{yz} = \sigma_{xz} = 0.
 \end{aligned}$$

Introduciendo las relaciones anteriores en las ecuaciones de equilibrio (1.5) obtenemos las ecuaciones de Navier

$$\begin{aligned}
 \frac{\partial}{\partial x} \left[\frac{E}{1-\nu^2} \left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[\frac{E}{2(1+\nu)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + B_x &= 0, \\
 \frac{\partial}{\partial x} \left[\frac{E}{2(1+\nu)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[\frac{E}{1-\nu^2} \left(\nu \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] + B_y &= 0.
 \end{aligned}$$

Finalmente, para las ecuaciones de Michell-Beltrami, partimos de las ecuaciones de compatibilidad, de las que “sobrevive” únicamente

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}. \quad (1.16)$$

Introduciendo las relaciones (1.7) a (1.10),

$$\begin{aligned} \frac{1}{E} \frac{\partial^2 \sigma_{xx}}{\partial y^2} - \frac{\nu}{E} \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{1}{E} \frac{\partial^2 \sigma_{yy}}{\partial x^2} - \frac{\nu}{E} \frac{\partial^2 \sigma_{xx}}{\partial x^2} &= 2(1 + \nu) \frac{\partial^2 \sigma_{xy}}{\partial x \partial y}, \\ \left(\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{xx}}{\partial y^2} \right) + \left(\frac{\partial^2 \sigma_{yy}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} \right) - \nu \left(\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} \right) &= (1 + \nu) \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2}, \\ \nabla^2 (\sigma_{xx} + \sigma_{yy}) &= (1 + \nu) \left(2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} \right). \end{aligned} \quad (1.17)$$

Derivando las relaciones de equilibrio (1.5) respecto a x y y , respectivamente,

$$\begin{aligned} \frac{\partial^2 \sigma_{xx}}{\partial x \partial x} + \frac{\partial^2 \sigma_{yx}}{\partial y \partial x} + \frac{\partial B_x}{\partial x} &= 0, \\ \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_{yy}}{\partial y \partial y} + \frac{\partial B_y}{\partial y} &= 0. \end{aligned}$$

Sumando y resolviendo para las fuerzas de volumen,

$$\begin{aligned} 2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} &= 0, \\ 2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} &= - \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} \right). \end{aligned} \quad (1.18)$$

Sustituyendo este resultado en (1.17), tenemos que las ecuaciones de Michell-Beltrami para la formulación de tensión plana son

$$\nabla^2 (\sigma_{xx} + \sigma_{yy}) = -(1 + \nu) \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} \right).$$

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Problema 2: Formulación de deformación plana

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