Postulates of Quantum Mechanics and the Bell State Quantum Circuit

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State space and definition of qubit

Postulate 1. State Space

To each isolated physical system we associate a Hilbert space $\mathcal{H}$, hereinafter known as the **state space** of the system. The physical system is completely described by its **state vector**, which is a unit vector $|\psi\rangle \in \mathcal{H}$. The dimension of $\mathcal{H}$ depends on the specific degrees of freedom of the physical property under consideration.

Postulate 1. State Space

Postulate 1 implies that a linear combination of state vectors is a state vector. This is known as the **superposition principle**. In particular, any vector state $|\psi\rangle$ may be described as a superposition of basis states $\{|e_i\rangle\}$ in $\mathcal{H}$, i.e. $|\psi\rangle = \sum_i c_i|e_i\rangle$, $c_i \in \mathbb{C}$.

Postulate 1. Definition of qubit

In quantum computing, information is stored, manipulated and measured in the form of qubits.

A qubit may be mathematically represented as a unit vector in a two-dimensional Hilbert space $|\psi\rangle \in \mathcal{H}^2$.

A qubit $|\psi\rangle$ may be written in general form as

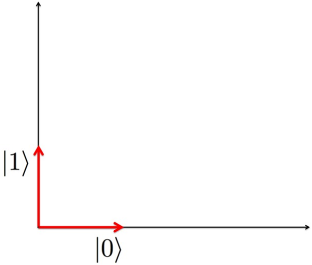
$$|\psi\rangle = \alpha|p\rangle + \beta|q\rangle$$

where $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$ and $\{|p\rangle, |q\rangle\}$ is an arbitrary basis spanning $\mathcal{H}^2$.

Postulate 1. Definition of qubit

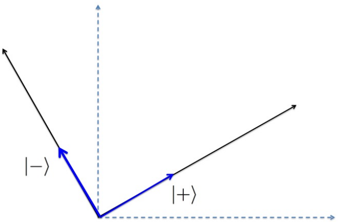
A most important consequence of the vectorial nature of a qubit is the possibility of writing it as a linear combination of elements of any basis.

For example, we may choose $|p\rangle = |0\rangle$ and $|q\rangle = |1\rangle$ to write a qubit. The basis $\{|0\rangle, |1\rangle\}$ is known as **the computational basis**.



Postulate 1. Definition of qubit

We may also choose the **the diagonal basis** $\{|+\rangle, |-\rangle\}$ to write a qubit.



Postulate 1. Definition of qubit

Choosing a concrete vector basis and values for corresponding complex coefficients is known as **preparing a qubit.** For instance,

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$$

In computer science parlance, preparing a qubit (or a set of qubits) is equivalent to variable initialization in a computer program.

Quantum Evolution

Postulate 2. Evolution by Unitary Operator

The evolution of a closed quantum system with state vector $|\Psi\rangle$ is described by a Unitary operator. The state of a system at time t_2 according to its state at time t_1 is given by

$$|\Psi(t_2)\rangle = \hat{U}|\Psi(t_1)\rangle.$$

Postulate 2 only describes the mathematical properties that an evolution operator must have. The specific evolution operator required to describe the behaviour of a particular quantum system depends on the system itself.

Postulate 2. Evolution by Schrödinger equation

Quantum evolution can also be written in terms of differential equations.

The time evolution of a closed quantum system can be described by the Schrödinger equation:

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{\mathbf{H}}|\psi(t)\rangle$$

where $\hbar = \frac{h}{2\pi}$, h is Planck's constant (about $6.62607004 \times 10^{-34} Js$), and $\hat{\mathbf{H}}$ is a Hermitian operator known as the *Hamiltonian* of the system.

Examples of Unitary Operators (1/2)

Pauli operators

$$\hat{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0|;\ \hat{\sigma}_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|;\ \hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

If $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\langle 0| = (1,\ 0)$, and $\langle 1| = (0,\ 1)$

Then we produce the following matrix representations of Pauli operators:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Examples of Unitary Operators (2/2)

The **Hadamard operator**

$$\hat{H} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

Again, if $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\langle 0| = (1,\ 0)$, and $\langle 1| = (0,\ 1)$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(Already known) Example of Evolution by Unitary Operator

Let $\hat{\sigma}_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$ and $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$. Compute $\hat{\sigma}_y|\psi\rangle$.

$$\begin{aligned}
\hat{\sigma}_y|\psi\rangle &= (-i|0\rangle\langle 1| + i|1\rangle\langle 0|)(\tfrac{\sqrt{3}}{2}|0\rangle + \tfrac{i}{2}|1\rangle) \\
&= -\frac{\sqrt{3}i\langle 1|0\rangle}{2}|0\rangle - \frac{i^2\langle 1|1\rangle}{2}|0\rangle + \frac{\sqrt{3}i\langle 0|0\rangle}{2}|1\rangle + \frac{i^2\langle 0|1\rangle}{2}|1\rangle \\
&= \frac{1}{2}|0\rangle + \frac{\sqrt{3}i}{2}|1\rangle
\end{aligned}$$

Example of Evolution by Unitary Operator

In computer science parlance,

- Designing Unitary operators is equivalent to writing a computer program.
- Applying Unitary operators to qubits is equivalent to running a computer program.

Measurement of quantum states

Quantum Measurement

In quantum mechanics, measurement is a non-trivial and highly counter-intuitive process because:

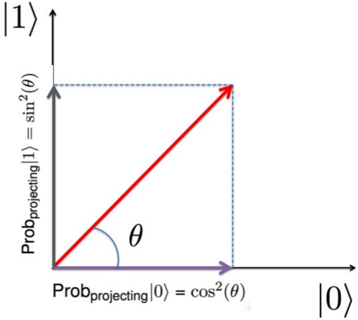
a) Measurement outcomes are inherently probabilistic. Regardless of the carefulness in the preparation of a measurement procedure, the possible outcomes of such measurement are produced/generated according to a certain probability distribution.

Quantum Measurement

b) Pre- and post-measurement quantum states are different. Once a measurement has been performed, a quantum system is unavoidably altered due to the interaction with the measurement apparatus. Consequently, for an arbitrary quantum system, pre-measurement and post-measurement quantum states are different in general.

Quantum Measurement

c) Measuring $\Leftrightarrow$ Projecting. Measuring a quantum system is equivalent to projecting (as in analytic geometry) a vector onto a vector basis.



Quantum Measurement

The **KEY** points to remember are:

1. Measuring a quantum state is equivalent to information retrieval.
2. Measuring a quantum system is a probabilistic process.
3. Measuring a quantum system makes irreversible changes in the information contained in that quantum system, i.e. in general, pre-measurement and post-measurement states of a quantum system are different.
4. Measuring a quantum system is equivalent to projecting its corresponding quantum state onto one of the vectors of the chosen measurement basis.

Quantum Measurement

Suppose that we have a quantum system $|\psi\rangle$ living in an n-dimensional Hilbert space $\mathcal{H}^n$.

The dimension of the Hilbert space is equal to the number of degrees of freedom of the quantum system and, consequently, it is also equal to the number of different possible outcomes of a measurement performed on such quantum system.

Quantum Measurement

For example, let us focus on the spin of an electron. Experimental results show that performing a measurement on the spin of an electron always produces one of two possible outcomes: spin up or spin down.

So, the spin of an electron is an example of a quantum property that has two degrees of freedom. Therefore, a quantum state representing the spin of an electron must live in a two-dimensional Hilbert space.

Quantum Measurement

We may write a quantum state $|\psi\rangle$ that represents the spin of an electron as follows:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $|0\rangle$ represents the quantum state spin up, $|1\rangle$ represents the quantum state spin down, and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$.

Formal description of Quantum Measurement

The mathematical description of quantum measurement requires the following components:

- A quantum state $|\psi\rangle$ and the dimension n of the Hilbert space $\mathcal{H}$ in which $|\psi\rangle$ lives.
- A set of measurement outcomes $\{a_i | i \in \{0, 1, \ldots, n-1\}\}$.
- An orthonormal basis $B = \{|i\rangle | i \in \{0, 1, \ldots, n-1\}\}$ of $\mathcal{H}$. The elements of B will be the vectors on which we will project $|\psi\rangle$.
- A set of measurement operators $\{\hat{M}_{a_i} = |i\rangle\langle i|\}$ which will be built using the elements of basis B. Index i labels the different measurement outcomes, which act on the state space of the system being measured.
- Mathematical expressions for computing probability distributions and post-measurement quantum states.

Formal description of Quantum Measurement

Let $|\psi\rangle \in \mathcal{H}^n$ be the state of a quantum system immediately before the measurement. Also, let $\{a_i\}$ be the set of measurement outcomes and $\{\hat{M}_{a_i} = |i\rangle\langle i|\}$ be the set of measurement operators built using basis $B = \{|i\rangle\}$, where $i \in \{0, 1, \ldots, n-1\}$.

Then, the probability that outcome a_i occurs is given by

$$p(a_i) = \langle\psi|\hat{M}^\dagger_{a_i}\hat{M}_{a_i}|\psi\rangle$$

where $\hat{M}^\dagger_{a_i}$ is the result of applying the dagger operator $\dagger$ to the projection operator $\hat{M}_{a_i}$.

Finally, the post-measurement quantum state that corresponds to measurement outcome a_i is given by

$$|\psi\rangle^{a_i}_{pm} = \frac{\hat{M}_{a_i}|\psi\rangle}{\sqrt{p(a_i)}}$$

Quantum Measurement - Example 01

Let $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ be a qubit (i.e., a mathematical representation of a two-dimensional quantum system) defined over the **computational** basis $\{|0\rangle, |1\rangle\}$.

Measure $|\psi\rangle$ with the basis $\{|0\rangle, |1\rangle\}$, i.e compute

$$p(a_0), p(a_1), |\psi\rangle_{pm}^{a_0}, |\psi\rangle_{pm}^{a_1}$$

where measurement outcomes are labelled as a_0, a_1.

Quantum Measurement - Example 01

The first step is to compute measurement operators $\hat{M}_{a_0}, \hat{M}_{a_1}$ and corresponding operators $\hat{M}^\dagger_{a_0}, \hat{M}^\dagger_{a_1}$.

- Since we will use the computational basis $\{|0\rangle, |1\rangle\}$ to measure, then measurement operators $\hat{M}_{a_0}$ and $\hat{M}_{a_1}$ are defined as:

$$\hat{M}_{a_0} = |0\rangle\langle 0| \text{ and } \hat{M}_{a_1} = |1\rangle\langle 1|$$

- Note that $\hat{M}^\dagger_{a_0} = (|0\rangle\langle 0|)^\dagger = |0\rangle\langle 0|$ and $\hat{M}^\dagger_{a_1} = (|1\rangle\langle 1|)^\dagger = |1\rangle\langle 1|$
- Furthermore,

$\hat{M}^\dagger_{a_0}\hat{M}_{a_0} = (|0\rangle\langle 0|)(|0\rangle\langle 0|) = |0\rangle\langle 0| = \hat{M}_{a_0}$ (Exercise 04, Outer Product Section)

$\hat{M}^\dagger_{a_1}\hat{M}_{a_1} = (|1\rangle\langle 1|)(|1\rangle\langle 1|) = |1\rangle\langle 1| = \hat{M}_{a_1}$ (Exercise 04, Outer Product Section)

Quantum Measurement - Example 01

Let us now calculate the probability of outcome a_0

$$
\begin{aligned}
p(a_0) &= \langle\psi|\hat{M}^{\dagger}_{a_0}\hat{M}_{a_0}|\psi\rangle = \langle\psi|\hat{M}_{a_0}|\psi\rangle \\
&= \left(\tfrac{1}{\sqrt{2}}\langle 0| + \tfrac{1}{\sqrt{2}}\langle 1|\right)\left[(|0\rangle\langle 0|)\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)\right] \\
&= \left(\tfrac{1}{\sqrt{2}}\langle 0| + \tfrac{1}{\sqrt{2}}\langle 1|\right)\left(\tfrac{1}{\sqrt{2}}\langle 0|0\rangle|0\rangle + \tfrac{1}{\sqrt{2}}\langle 0|1\rangle|0\rangle\right) \\
&= \left(\tfrac{1}{\sqrt{2}}\langle 0| + \tfrac{1}{\sqrt{2}}\langle 1|\right)\left(\tfrac{1}{\sqrt{2}}|0\rangle\right) \\
&= \left(\tfrac{1}{\sqrt{2}}\right)^2\langle 0|0\rangle + \left(\tfrac{1}{\sqrt{2}}\right)^2\langle 1|0\rangle \\
&= \tfrac{1}{2}
\end{aligned}
$$

So, $\mathbf{p}(a_0) = \frac{1}{2}$

Quantum Measurement - Example 01

We now compute the post-measurement quantum state $|\psi\rangle_{pm}^{a_0}$

$$
\begin{aligned}
|\psi\rangle_{pm}^{a_0} &= \frac{\hat{M}_{a_0}|\psi\rangle}{\sqrt{p(a_0)}} \\
&= \frac{\frac{1}{\sqrt{2}}|0\rangle}{\frac{1}{\sqrt{2}}} \\
&= |0\rangle
\end{aligned}
$$

So, $|\psi\rangle_{pm}^{a_0} = |0\rangle$

Quantum Measurement - Example 01

As for outcome a_1:

$$
\begin{aligned}
p(a_1) &= \langle\psi|\hat{M}^{\dagger}_{a_1}\hat{M}_{a_1}|\psi\rangle = \langle\psi|\hat{M}_{a_1}|\psi\rangle \\
&= (\tfrac{1}{\sqrt{2}}\langle 0| + \tfrac{1}{\sqrt{2}}\langle 1|)[(|1\rangle\langle 1|)(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle)] \\
&= (\tfrac{1}{\sqrt{2}}\langle 0| + \tfrac{1}{\sqrt{2}}\langle 1|)(\tfrac{1}{\sqrt{2}}\langle 1|0\rangle|1\rangle + \tfrac{1}{\sqrt{2}}\langle 1|1\rangle|1\rangle) \\
&= (\tfrac{1}{\sqrt{2}}\langle 0| + \tfrac{1}{\sqrt{2}}\langle 1|)(\tfrac{1}{\sqrt{2}}|1\rangle) \\
&= (\tfrac{1}{\sqrt{2}})^2\langle 0|1\rangle + (\tfrac{1}{\sqrt{2}})^2\langle 1|1\rangle \\
&= \tfrac{1}{2}
\end{aligned}
$$

So, $\mathbf{p}(a_1) = \frac{1}{2}$

Quantum Measurement - Example 01

Finally, we compute the post-measurement quantum state $|\psi\rangle_{pm}^{a_1}$:

$$\begin{aligned} |\psi\rangle_{pm}^{a_1} &= \frac{\hat{M}_{a_1}|\psi\rangle}{\sqrt{p(a_1)}} \\ &= \frac{\frac{1}{\sqrt{2}}|1\rangle}{\frac{1}{\sqrt{2}}} \\ &= |1\rangle \end{aligned}$$

So, $|\psi\rangle_{pm}^{a_1} = |1\rangle$

Quantum Measurement - Exercise 02

Let $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ be the same qubit of Exercise 01.

Now, let us measure $|\psi\rangle$ with the **diagonal** basis $\{|+\rangle, |-\rangle\}$, i.e. compute

$$p(a_+), p(a_-), |\psi\rangle_{pm}^{a_+}, |\psi\rangle_{pm}^{a_-}$$

where measurement outcomes are labelled as a_+, a_-.

Quantum Measurement - Example 01

As in Exercise 01, the first step is to compute measurement operators $\hat{M}_{a_+}, \hat{M}_{a_-}$ and corresponding operators $\hat{M}^\dagger_{a_+}, \hat{M}^\dagger_{a_-}$.

- Since we will use the diagonal basis $\{|+\rangle, |-\rangle\}$ to measure, then measurement operators $\hat{M}_{a_+}$ and $\hat{M}_{a_-}$ are defined as:

$$\hat{M}_{a_+} = |+\rangle\langle+| \text{ and } \hat{M}_{a_-} = |-\rangle\langle-|$$

- Note that $\hat{M}^\dagger_{a_+} = (|+\rangle\langle+|)^\dagger = |+\rangle\langle+|$ and $\hat{M}^\dagger_{a_-} = (|-\rangle\langle-|)^\dagger = |-\rangle\langle-|$

- It can be easily proved that

$$\hat{M}^\dagger_{a_+}\hat{M}_{a_+} = (|+\rangle\langle+|)(|+\rangle\langle+|) = |+\rangle\langle+| = \hat{M}_{a_+}$$

$$\hat{M}^\dagger_{a_-}\hat{M}_{a_-} = (|-\rangle\langle-|)(|-\rangle\langle-|) = |-\rangle\langle-| = \hat{M}_{a_-}$$

Quantum Measurement - Exercise 02

Now, let us calculate the probability of getting outcome a_+:

$$\begin{aligned}
p(a_+) &= \langle\psi|\hat{M}^\dagger_{a_+}\hat{M}_{a_+}|\psi\rangle = \langle\psi|\hat{M}_{a_+}|\psi\rangle \\
&= (\tfrac{1}{\sqrt{2}}\langle 0| + \tfrac{1}{\sqrt{2}}\langle 1|)[(|+\rangle\langle +|)(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle)] \\
&= (\langle +|)[(|+\rangle\langle +|)(|+\rangle)] \\
&= (\langle +|)[(\langle +|+\rangle)(|+\rangle)] \\
&= \langle +|+\rangle \\
&= 1
\end{aligned}$$

So, $\mathbf{p}(a_+) = 1$

Quantum Measurement - Exercise 02

Corresponding post-measurement quantum state $|\psi\rangle_{pm}^{a_+}$ is:

$$
\begin{aligned}
|\psi\rangle_{pm}^{a_+} &= \frac{\hat{M}_{a_+}|\psi\rangle}{\sqrt{p(a_+)}} \\
&= \frac{|+\rangle}{1} \\
&= |+\rangle
\end{aligned}
$$

Quantum Measurement - Exercise 02

As for outcome a_-:

$$
\begin{aligned}
p(a_-) &= \langle\psi|\hat{M}^\dagger_{a_-}\hat{M}_{a_-}|\psi\rangle = \langle\psi|\hat{M}_{a_-}|\psi\rangle \\
&= (\tfrac{1}{\sqrt{2}}\langle 0| + \tfrac{1}{\sqrt{2}}\langle 1|)[(|-\rangle\langle -|)(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle)] \\
&= (\langle +|)[(|-\rangle\langle -|)(|+\rangle)] \\
&= (\langle +|)[(\langle -|+\rangle)(|+\rangle)] \\
&= (\langle +|)(0|+\rangle) \\
&= (\langle +|)(|\vec{0}\rangle) \\
&= 0
\end{aligned}
$$

So, $\mathbf{p}(a_-) = 0$

Quantum Measurement - Exercise 02

Finally, we compute the post-measurement quantum state $|\psi\rangle_{pm}^{a_-}$:

$$
\begin{aligned}
|\psi\rangle_{pm}^{a_-} &= \frac{\hat{M}_{a_-}|\psi\rangle}{\sqrt{p(a_-)}} \\
&= \frac{|-\rangle}{0} \text{ which, of course, is undefined.}
\end{aligned}
$$

Why did we get $\frac{|-\rangle}{0}$? The answer is:

Remember that $p(a_-) = 0$.
We cannot compute a quantum state that will never exist!

Summary of Quantum Measurement Exercises

- $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ is the quantum state we want to measure.
- Computational basis $\{|0\rangle, |1\rangle\}$
- Diagonal basis $\{|+\rangle, |-\rangle\}$
- Remember that $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Measurement basis	Outcomes	Outcome Probabilities and Post-measurement quantum states
$\{|0\rangle, |1\rangle\}$	a_0, a_1	$p(a_0) = 0.5 \; |\psi\rangle_{pm}^{a_0} = |0\rangle$ $p(a_1) = 0.5 \; |\psi\rangle_{pm}^{a_1} = |1\rangle$
$\{|+\rangle, |-\rangle\}$	a_+, a_-	$p(a_+) = 1 \; |\psi\rangle_{pm}^{a_+} = |+\rangle$ $p(a_-) = 0 \; \nexists |\psi\rangle_{pm}^{a_-}$

More results (1/3)

Similarly, it is possible to prove the following

- $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ is the quantum state we want to measure.
- Computational basis $\{|0\rangle, |1\rangle\}$
- Diagonal basis $\{|+\rangle, |-\rangle\}$
- Remember that $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Measurement basis	Outcomes	Outcome Probabilities and Post-measurement quantum states
$\{|0\rangle, |1\rangle\}$	a_0, a_1	$p(a_0) = 0.5 \ |\psi\rangle_{pm}^{a_0} = |0\rangle$ $p(a_1) = 0.5 \ |\psi\rangle_{pm}^{a_1} = -|1\rangle = |1\rangle$ get rid of global phase
$\{|+\rangle, |-\rangle\}$	a_+, a_-	$p(a_+) = 0 \quad \nexists |\psi\rangle_{pm}^{a_+}$ $p(a_-) = 1 \ |\psi\rangle_{pm}^{a_-} = |-\rangle$

More results (2/3)

- $|\psi\rangle = |0\rangle$ is the quantum state we want to measure.
- Computational basis $\{|0\rangle, |1\rangle\}$
- Diagonal basis $\{|+\rangle, |-\rangle\}$
- Remember that $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Measurement basis	Outcomes	Outcome Probabilities and Post-measurement quantum states
$\{|0\rangle, |1\rangle\}$	a_0, a_1	$p(a_0) = 1 \; |\psi\rangle_{pm}^{a_0} = |0\rangle$ $p(a_1) = 0 \; \nexists |\psi\rangle_{pm}^{a_1}$
$\{|+\rangle, |-\rangle\}$	a_+, a_-	$p(a_+) = 0.5 \; |\psi\rangle_{pm}^{a_+} = |+\rangle$ $p(a_-) = 0.5 \; |\psi\rangle_{pm}^{a_-} = |-\rangle$

More results (3/3)

- $|\psi\rangle = |1\rangle$ is the quantum state we want to measure.
- Computational basis $\{|0\rangle, |1\rangle\}$
- Diagonal basis $\{|+\rangle, |-\rangle\}$
- Remember that $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Measurement basis	Outcomes	Outcome Probabilities and Post-measurement quantum states
$\{|0\rangle, |1\rangle\}$	a_0, a_1	$p(a_0) = 0 \ \nexists|\psi\rangle^{a_0}_{pm}$ $p(a_1) = 1 \ \ |\psi\rangle^{a_1}_{pm} = |1\rangle$
$\{|+\rangle, |-\rangle\}$	a_+, a_-	$p(a_+) = 0.5 \ \ |\psi\rangle^{a_+}_{pm} = |+\rangle$ $p(a_-) = 0.5 \ |\psi\rangle^{a_-}_{pm} = |-\rangle$

Composite Quantum Systems

Postulate 4. Composite quantum systems

Quantum registers

Let $|\psi\rangle \in \mathcal{H}^2$ be a qubit. We know that the most general state of a qubit written in terms of the computational basis is

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $\alpha, \beta \in \mathbb{C}$ and $||\alpha||^2 + ||\beta||^2 = 1$.

Moreover, we know that measuring $|\psi\rangle$ will produce one of two mutually exclusive outcomes that we could labeled as 0 and 1.

Postulate 4. Composite quantum systems

Quantum registers

Now, what about a pair of qubits?
What is the most general state of a pair of qubits?

- Let $|\psi\rangle_1 \in \mathcal{H}_1$ where $|\psi\rangle_1 = a|0\rangle + b|1\rangle$ and $|\psi\rangle_2 \in \mathcal{H}_2$ where $|\psi\rangle_2 = c|0\rangle + d|1\rangle$.
- Note that , if we measure **both** qubits **at once**, we will get one out of four possible outcomes: 0 and 0, 0 and 1, 1 and 0, 1 and 1.

Postulate 4. Composite quantum systems

Quantum registers

So, we may think inductively and propose an ansatz for mathematical representation of two qubits:

$$|\phi\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$$

where $|\phi\rangle$ is the result of 'mixing' $|\psi\rangle_1$ and $|\psi\rangle_2$, $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$, and $\sum_i ||\alpha_i||^2 = 1$.

Postulate 4. Composite quantum systems

Quantum registers

In fact, the expression

$$|\phi\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$$

is an example of a well established method employed to describe *multipartite* quantum systems: the tensor product.

Postulate 4. Composite quantum systems

We now focus on the mathematical description of a composite quantum system, i.e. a system made up of several different physical systems.

The state space of a composite quantum system is the tensor product of the component system state spaces.

If we have n quantum systems expressed as *state vectors*, labeled $|\psi\rangle_1, |\psi\rangle_2, \ldots, |\psi\rangle_n$ then the joint state of the total system is given by $|\psi\rangle_T = |\psi\rangle_1 \otimes |\psi\rangle_2 \otimes \ldots \otimes |\psi\rangle_n$.

Please note this crucial property: $\otimes_{i=1}^{n} \mathcal{H}_i^2 = \mathcal{H}^{2^n}$.

Postulate 4. Composite quantum systems

Let $\hat{A} : \mathcal{H}_1 \to \mathcal{H}_1$ and $\hat{B} : \mathcal{H}_2 \to \mathcal{H}_2$ be linear operators. Then, the following equalities hold $\forall\ |a\rangle_1, |a\rangle_2 \in \mathcal{H}_1$, $|b\rangle_1, |b\rangle_2 \in \mathcal{H}_2$, $\alpha \in \mathbb{C}$:

1) $\alpha(|a\rangle_1 \otimes |b\rangle_1) = (\alpha|a\rangle_1) \otimes |b\rangle_1 = |a\rangle_1 \otimes (\alpha|b\rangle_1)$
2) $(|a\rangle_1 + |a\rangle_2) \otimes |b\rangle_1 = |a\rangle_1 \otimes |b\rangle_1 + |a\rangle_2 \otimes |b\rangle_1$
3) $|a\rangle_1 \otimes (|b\rangle_1 + |b\rangle_2) = |a\rangle_1 \otimes |b\rangle_1 + |a\rangle_1 \otimes |b\rangle_2$
4) $\hat{A} \otimes \hat{B}(|a\rangle_1 \otimes |b\rangle_1) = \hat{A}|a\rangle_1 \otimes \hat{B}|b\rangle_1$
5) Let $|a\rangle_i \in \mathcal{H}_1$, $|b\rangle_i \in \mathcal{H}_2$ and $\alpha_i \in \mathbb{C} \Rightarrow \hat{A} \otimes \hat{B}(\sum_i \alpha_i |a\rangle_i \otimes |b\rangle_i) = \sum_i \alpha_i \hat{A}|a\rangle_i \otimes \hat{B}|b\rangle_i$

Postulate 4. Composite quantum systems

- We may write $|a\rangle \otimes |b\rangle$ as $|ab\rangle$ or $|a, b\rangle$.

- Moreover, the tensor product of $|a\rangle$ with itself n times may be written as $|a\rangle \otimes |a\rangle \otimes \ldots \otimes |a\rangle = |a\rangle^{\otimes n}$.

Postulate 4. Composite quantum systems

Example 1. Let $|\psi\rangle_1 \in \mathcal{H}_1$, $|\psi\rangle_2 \in \mathcal{H}_2$ where $|\psi\rangle_1 = a|0\rangle + b|1\rangle$ and $|\psi\rangle_2 = c|0\rangle + d|1\rangle$. Then

$$\begin{aligned}
|\psi\rangle_1 \otimes |\psi\rangle_2 &= (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) \\
&= (a|0\rangle + b|1\rangle) \otimes c|0\rangle + (a|0\rangle + b|1\rangle) \otimes d|1\rangle \\
&= ac|0\rangle|0\rangle + bc|1\rangle|0\rangle + ad|0\rangle|1\rangle + bd|1\rangle|1\rangle \\
&= ac|00\rangle + bc|10\rangle + ad|01\rangle + bd|11\rangle \\
&= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle \\
&= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle
\end{aligned}$$

Postulate 4. Composite quantum systems

Example 2. Let $|\psi\rangle_1 \in \mathcal{H}_1$, $|\psi\rangle_2 \in \mathcal{H}_2$, $|\psi\rangle_3 \in \mathcal{H}_3$ where

$$|\psi\rangle_1 = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$|\psi\rangle_2 = \frac{1}{2}|0\rangle + i\frac{\sqrt{3}}{2}|1\rangle$$

$$|\psi\rangle_3 = \cos\frac{3\pi}{4}|0\rangle + i\sin\frac{3\pi}{4}|1\rangle$$

Compute

$$|\psi\rangle_1 \otimes |\psi\rangle_2$$

$$|\psi\rangle_1 \otimes |\psi\rangle_3$$

$$|\psi\rangle_1 \otimes |\psi\rangle_2 \otimes |\psi\rangle_3$$

Postulate 4. Composite quantum systems

Example 3.

We know that

$$\hat{A} \otimes \hat{B}(|a\rangle_1 \otimes |b\rangle_1) = \hat{A}|a\rangle_1 \otimes \hat{B}|b\rangle_1$$

Since

$$\hat{H}^{\otimes 2}|0\rangle^{\otimes 2} = \hat{H}^{\otimes 2}|00\rangle = \hat{H} \otimes \hat{H}(|0\rangle \otimes |0\rangle)$$

verify that

$$\hat{H} \otimes \hat{H}(|0\rangle \otimes |0\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

Postulate 4. Composite quantum systems

Example 4. We know that $\hat{A}\otimes\hat{B}(|a\rangle_1\otimes|b\rangle_1)=\hat{A}|a\rangle_1\otimes\hat{B}|b\rangle_1$.
Let

$$\hat{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0|$$
$$\hat{\sigma}_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$$
$$\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$
$$\hat{H} = \tfrac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$
$$|\psi\rangle_1 = \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle$$
$$|\psi\rangle_2 = \tfrac{1}{2}|0\rangle + i\tfrac{\sqrt{3}}{2}|1\rangle$$

$$|\psi\rangle_3 = \cos\frac{3\pi}{4}|0\rangle + i\sin\frac{3\pi}{4}|1\rangle$$

Compute as many combinations as you may wish, for instance

$$\hat{\sigma}_x\otimes\hat{\sigma}_y\otimes\hat{H}(|\psi\rangle_1\otimes|\psi\rangle_2\otimes|\psi\rangle_3)$$

Postulate 4. Composite quantum systems

Example 5. Let $\hat{H}^{\otimes 2}$ be the tensor product of the Hadamard operator with itself. Prove that

$$\begin{aligned}\hat{H}^{\otimes 2} \quad = \quad & \tfrac{1}{2}(|00\rangle\langle 00| + |01\rangle\langle 00| + |10\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 01| \\ & -|01\rangle\langle 01| + |10\rangle\langle 01| - |11\rangle\langle 01| + |00\rangle\langle 10| + |01\rangle\langle 10| \\ & -|10\rangle\langle 10| - |11\rangle\langle 10| + |00\rangle\langle 11| - |01\rangle\langle 11| - |10\rangle\langle 11| \\ & +|11\rangle\langle 11|)\end{aligned}$$

Key mathematical rule:

$$|a\rangle\langle b| \otimes |c\rangle\langle d| = |ac\rangle\langle bd|$$

For instance,

$$|0\rangle\langle 1| \otimes |0\rangle\langle 0| = |0\rangle|0\rangle\langle 1|\langle 0| = |00\rangle\langle 10|$$

Postulate 4. Composite quantum systems

Example 6. Let $\hat{H}^{\otimes 2}$ be the tensor product of the Hadamard operator with itself and let $|\psi\rangle = |0\rangle \otimes |0\rangle = |00\rangle$. Prove that

$$\begin{aligned}
\hat{H}^{\otimes 2}\,|00\rangle &= \tfrac{1}{2}(|00\rangle\langle 00| + |01\rangle\langle 00| + |10\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 01| \\
&\quad -|01\rangle\langle 01| + |10\rangle\langle 01| - |11\rangle\langle 01| + |00\rangle\langle 10| + |01\rangle\langle 10| \\
&\quad -|10\rangle\langle 10| - |11\rangle\langle 10| + |00\rangle\langle 11| - |01\rangle\langle 11| - |10\rangle\langle 11| \\
&\quad +|11\rangle\langle 11|)|00\rangle \\
&= \tfrac{1}{2}(\langle 00|00\rangle|00\rangle + \langle 00|00\rangle|01\rangle + \langle 00|00\rangle|10\rangle + \langle 00|00\rangle|11\rangle \\
&= \tfrac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)
\end{aligned}$$

That is, the notion of inner product on orthonormal bases easily extends to $\mathcal{H}^{2^n}$.

Postulate 4. Composite quantum systems

Example 7. Introducing $\hat{C}_{\text{not}}$.
Let us introduce $\hat{C}_{\text{not}}$, a two-qubit gate that behaves as follows:

$$\hat{C}_{\text{not}} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|$$

Then

$$\begin{aligned} \hat{C}_{\text{not}}|00\rangle &= |00\rangle \\ \hat{C}_{\text{not}}|01\rangle &= |01\rangle \\ \hat{C}_{\text{not}}|10\rangle &= |11\rangle \\ \hat{C}_{\text{not}}|11\rangle &= |10\rangle \end{aligned}$$

The operator $\hat{C}_{\text{not}}$ acts as a *Controlled Not*: $\hat{C}_{\text{not}}$ flips the second qubit iff the first qubit is $|1\rangle$, otherwise $\hat{C}_{\text{not}}$ behaves like a buffer.

Postulate 4. Composite quantum systems

The Kronecker product is a matrix representation of the tensor product and it is defined as follows.

Let $A = (a_{ij}), B = (b_{ij})$ be two matrices of order $m \times n$ and $p \times q$ respectively. Then $A \otimes B$ is given by

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \dots & A_{mn}B \end{pmatrix}.$$

$A \otimes B$ is of order $mp \times nq$.

Postulate 4. Composite quantum systems

Example 8. Let

$$\hat{H} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

be the Hadamard operator.

Write

$$\hat{H} \otimes \hat{H}$$

in matrix notation.

Postulate 4. Composite quantum systems

Example 8.

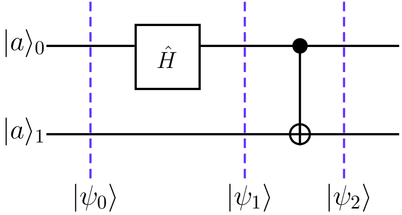
$$
\begin{aligned}
H \otimes H &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & 1\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ 1\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & -1\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}
\end{aligned}
$$

Table of Contents

Introduction to Quantum Circuits

We now introduce a quantum circuit to compute Bell states.

Bell State Circuit (1/5)



Let

$$|a\rangle_0 = |0\rangle \text{ and } |a\rangle_1 = |0\rangle$$

Also, remember that

$$\hat{H} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

$$\hat{C}_{\text{not}} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|$$

Bell State Circuit (2/5)

So,

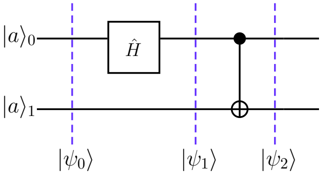
$$
\begin{aligned}
|\psi\rangle_0 &= |0\rangle \otimes |0\rangle = |00\rangle \\
|\psi\rangle_1 &= (\hat{H} \otimes \hat{\mathbb{I}})(|0\rangle \otimes |0\rangle) = \hat{H}|0\rangle \otimes \hat{\mathbb{I}}|0\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \\
&= \tfrac{1}{\sqrt{2}}|0\rangle|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle|0\rangle \\
|\psi\rangle_2 &= \hat{C}_{\text{not}}(\tfrac{1}{\sqrt{2}}|0\rangle|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle|0\rangle) \\
&= \tfrac{1}{\sqrt{2}}|0\rangle|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle|1\rangle = \tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle
\end{aligned}
$$

So,

$$
|\psi\rangle_2 = \frac{|00\rangle + |11\rangle}{\sqrt{2}}
$$

Bell State Circuit (3/5)

Exercise



Compute $|\psi\rangle_2$ for

$$|a\rangle_0 = |0\rangle \text{ and } |a\rangle_1 = |1\rangle$$
$$|a\rangle_0 = |1\rangle \text{ and } |a\rangle_1 = |0\rangle$$
$$|a\rangle_0 = |1\rangle \text{ and } |a\rangle_1 = |1\rangle$$

Bell State Circuit (4/5)

Answers

$$\hat{C}_{\text{not}}((\hat{H} \otimes \hat{\mathbb{I}})|00\rangle) = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$
$$\hat{C}_{\text{not}}((\hat{H} \otimes \hat{\mathbb{I}})|01\rangle) = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$
$$\hat{C}_{\text{not}}((\hat{H} \otimes \hat{\mathbb{I}})|10\rangle) = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$
$$\hat{C}_{\text{not}}((\hat{H} \otimes \hat{\mathbb{I}})|11\rangle) = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

These states are known as the **Bell states**

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$
$$|\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$
$$|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$
$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Bell State Circuit (5/5)

An interesting property: try writing the Bell state $|\Phi^+\rangle$ as the tensor product of two arbitrary qubits, i.e. find the values of α, β, γ, and δ such that

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

Try the same procedure with the other three Bell states

$$|\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$
$$|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$
$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Quantum Entanglement (1/2)

Bell states are examples of entangled states. Bell states are key features of a quantum information transmission protocol known as quantum teleportation.

Quantum entanglement is a unique type of correlation shared between components of a quantum system.

Quantum entanglement and the principle of superposition are two of the main features behind the power of quantum computation and quantum information theory.

Quantum Entanglement (2/2)

Entangled quantum systems are sometimes best used collectively, that is, sometimes an optimal use of entangled quantum systems for information storage and retrieval includes manipulating and measuring those systems as a whole, rather than on an individual basis.