

No-Cloning Theorem and Quantum Teleportation

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Bell States

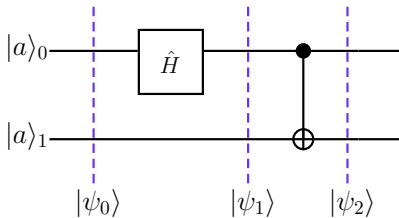


Introduction to Quantum Circuits

We now introduce a quantum circuit to compute **Bell states**



Bell State Circuit (1/4)



Let

$$|a\rangle_0 = |0\rangle \text{ and } |a\rangle_1 = |0\rangle$$

Also, remember that

$$\hat{H} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

$$\hat{C}_{\text{not}} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|$$



Bell State Circuit (2/4)

So,

$$|\psi\rangle_0 = |0\rangle \otimes |0\rangle = |00\rangle$$

$$|\psi\rangle_1 = (\hat{H} \otimes \hat{I})(|0\rangle \otimes |0\rangle) = \hat{H}|0\rangle \otimes \hat{I}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|0\rangle$$

$$|\psi\rangle_2 = \hat{C}_{\text{not}}(\frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|0\rangle)$$

$$= \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

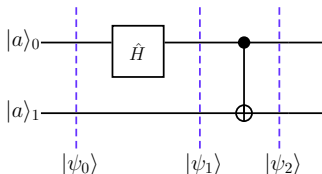
So,

$$|\psi\rangle_2 = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$



Bell State Circuit (3/4)

Exercise



Compute $|\psi\rangle_2$ for

$$|a\rangle_0 = |0\rangle \text{ and } |a\rangle_1 = |1\rangle$$

$$|a\rangle_0 = |1\rangle \text{ and } |a\rangle_1 = |0\rangle$$

$$|a\rangle_0 = |1\rangle \text{ and } |a\rangle_1 = |1\rangle$$



Bell State Circuit (4/4)

Answers

$$\begin{aligned}\hat{C}_{\text{not}}((\hat{H} \otimes \hat{I})|00\rangle) &= \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\ \hat{C}_{\text{not}}((\hat{H} \otimes \hat{I})|01\rangle) &= \frac{|01\rangle + |10\rangle}{\sqrt{2}} \\ \hat{C}_{\text{not}}((\hat{H} \otimes \hat{I})|10\rangle) &= \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\ \hat{C}_{\text{not}}((\hat{H} \otimes \hat{I})|11\rangle) &= \frac{|01\rangle - |10\rangle}{\sqrt{2}}\end{aligned}$$

These states are known as the **Bell states**

$$\begin{aligned}|\Phi^+\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\ |\Phi^-\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\ |\Psi^+\rangle &= \frac{|01\rangle + |10\rangle}{\sqrt{2}} \\ |\Psi^-\rangle &= \frac{|01\rangle - |10\rangle}{\sqrt{2}}\end{aligned}$$



Quantum Entanglement (1/2)

Bell states are examples of entangled states. Bell states are key features of a quantum information transmission protocol known as quantum teleportation.

Quantum entanglement is a unique type of correlation shared between components of a quantum system.

Quantum entanglement and the principle of superposition are two of the main features behind the power of quantum computation and quantum information theory.



Quantum Entanglement (2/2)

Entangled quantum systems are sometimes best used collectively, that is, sometimes an optimal use of entangled quantum systems for information storage and retrieval includes manipulating and measuring those systems as a whole, rather than on an individual basis.



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The No-Cloning Theorem



The No-Cloning Theorem

The No-Cloning Theorem.

It is not possible to create an identical copy of an **unknown** quantum state.

Proof.

Let us suppose that a Unitary operator \hat{U} exists such that, for two independent quantum states $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ (where α, β are unknown complex numbers) and $|e\rangle$ we have:

$$\hat{U}|\psi\rangle \otimes |e\rangle = |\psi\rangle \otimes |\psi\rangle$$



The No-Cloning Theorem

On the one hand, \hat{U} must fulfill the following property:

$$\begin{aligned}\hat{U}|\psi\rangle \otimes |e\rangle &= \hat{U}(\alpha|0\rangle + \beta|1\rangle) \otimes |e\rangle \\ &= (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) \\ &= \alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle \\ &= \alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle\end{aligned}$$



The No-Cloning Theorem

On the other hand, linearity implies the following:

$$\begin{aligned}\hat{U}|\psi\rangle \otimes |e\rangle &= \hat{U}(\alpha|0\rangle + \beta|1\rangle) \otimes |e\rangle \\ &= \hat{U}(\alpha|0\rangle|e\rangle + \beta|1\rangle|e\rangle) \\ &= \alpha\hat{U}|0\rangle|e\rangle + \beta\hat{U}|1\rangle|e\rangle \\ &= \alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle \\ &= \alpha|00\rangle + \beta|11\rangle\end{aligned}$$



The No-Cloning Theorem

So, we have that

$$\hat{U}|\psi\rangle \otimes |e\rangle = \alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle$$

as well as

$$\hat{U}|\psi\rangle \otimes |e\rangle = \alpha|00\rangle + \beta|11\rangle$$

Thus,

$$\alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle = \alpha|00\rangle + \beta|11\rangle$$



The No-Cloning Theorem

$$\alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle = \alpha|00\rangle + \beta|11\rangle$$

is true only for:

$$[\alpha = 0 \text{ and } \beta = 1] \Leftrightarrow [\hat{U}|1\rangle|e\rangle = |1\rangle|1\rangle] \quad (1)$$

$$[\alpha = 1 \text{ and } \beta = 0] \Leftrightarrow [\hat{U}|0\rangle|e\rangle = |0\rangle|0\rangle] \quad (2)$$

$$[\alpha = 0 \text{ and } \beta = 0] \quad (3)$$

Eqs. (1,2) correspond to the case of **copying classical information**. Eq. (3) is a most uninteresting case. *Q.E.D.*



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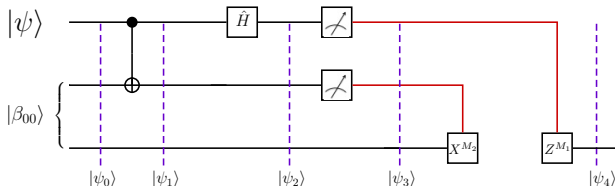
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The Quantum Teleportation Protocol



Quantum Teleportation Circuit (1/3)



$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(\alpha|00\rangle|0\rangle + \alpha|01\rangle|1\rangle + \beta|10\rangle|0\rangle + \beta|11\rangle|1\rangle)$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(\alpha|0\rangle|00\rangle + \alpha|0\rangle|11\rangle + \beta|1\rangle|10\rangle + \beta|1\rangle|01\rangle)$$

$$|\psi_2\rangle = \frac{1}{2} [|00\rangle_A (\alpha|0\rangle + \beta|1\rangle) + |01\rangle_A (\alpha|1\rangle + \beta|0\rangle) + \\ |10\rangle_A (\alpha|0\rangle - \beta|1\rangle) + |11\rangle_A (\alpha|1\rangle - \beta|0\rangle)]$$



Quantum Teleportation Circuit (2/3)

$$|\psi_3\rangle$$

$$p(a_0, b_0) = \langle \psi_2 | \hat{P}_{\{a_0, b_0\}} | \psi_2 \rangle = \frac{1}{4} \text{ and } |\psi\rangle_{\{a_0, b_0\}}^{\text{pm}} = |00\rangle_A (\alpha|0\rangle_B + \beta|1\rangle_B)$$

$$p(a_0, b_1) = \langle \psi_2 | \hat{P}_{\{a_0, b_1\}} | \psi_2 \rangle = \frac{1}{4} \text{ and } |\psi\rangle_{\{a_0, b_1\}}^{\text{pm}} = |01\rangle_A (\alpha|1\rangle_B + \beta|0\rangle_B)$$

$$p(a_1, b_0) = \langle \psi_2 | \hat{P}_{\{a_1, b_0\}} | \psi_2 \rangle = \frac{1}{4} \text{ and } |\psi\rangle_{\{a_1, b_0\}}^{\text{pm}} = |10\rangle_A (\alpha|0\rangle_B - \beta|1\rangle_B)$$

$$p(a_1, b_1) = \langle \psi_2 | \hat{P}_{\{a_1, b_1\}} | \psi_2 \rangle = \frac{1}{4} \text{ and } |\psi\rangle_{\{a_1, b_1\}}^{\text{pm}} = |11\rangle_A (\alpha|1\rangle_B - \beta|0\rangle_B)$$



Quantum Teleportation Circuit (3/3)

$$|\psi_4\rangle$$

If outcome $\{a_0, b_0\} \Rightarrow |\psi_4\rangle = \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_B[|00\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B)]$

If outcome $\{a_0, b_1\} \Rightarrow |\psi_4\rangle = \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_A \otimes (\hat{\sigma}_x)_B[|01\rangle_A(\alpha|1\rangle_B + \beta|0\rangle_B)]$

If outcome $\{a_1, b_0\} \Rightarrow |\psi_4\rangle = \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_A \otimes (\hat{\sigma}_z)_B[|10\rangle_A(\alpha|0\rangle_B - \beta|1\rangle_B)]$

If outcome $\{a_1, b_1\} \Rightarrow |\psi_4\rangle = \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_A \otimes (\hat{\sigma}_z \hat{\sigma}_x)_B[|11\rangle_A(\alpha|1\rangle_B - \beta|0\rangle_B)]$

where $\hat{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0|$ and $\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$

