# Quantum Teleportation - Detailed Mathematical Description

Salvador E. Venegas Andraca
Facultad de Ciencias, UNAM
svenegas@ciencias.unam.mx

February 24, 2025

## 1 Quantum Teleportation

Text and teleportation circuit diagram taken from [1], equations in detail produced by SEVA.

1. Alice and Bob met long ago but now live far apart.

2. While together they generated an EPR pair (N.B. An EPR pair is one of the four bipartite states produced by the Bell circuit we recently studied in our course), each taking one qubit of the EPR pair when they separated.

3. Many years later, Bob is in hiding, and Alice's mission, should she choose to accept it, is to deliver a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ to Bob.

4. Alice does *not* know the state of the qubit she is meant to send. However, she can certainly manipulate her two qubits, i.e. she can apply quantum unitary operators as well as measurement operators to the qubits on her hands. Moreover, she can only send *classical* information to Bob (for example, she may have a secure telephone line to talk to Bob.)

5. So, the crux of the matter is that Alice has to send a qubit $|\psi\rangle$ (that is, quantum information contained in a quantum physical system) to Bob. Alice has physical domain of the quantum system that contains the qubit $|\psi\rangle$ she is meant to send but she does *not* know the contents of $|\psi\rangle$, i.e. Alice can apply Unitary and measurement operators to her qubits but she *ignores* the actual values of $\alpha$ and $\beta$ in $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

   To fulfill her mission, Alice has the following resources:

   (a) A qubit from an EPR pair. The other qubit of this EPR pair is on Bob's hands.

   (b) The qubit that stores/contains the quantum information we want to send to Bob.
   **Remember**: Alice may think of simply measuring this qubit but she should discard this idea because, as we learned when studying the postulates of quantum mechanics, measuring a quantum system is a probabilistic process *and* the resulting quantum state after the measurement (i.e the post-measurement qubit) is, in general, different from the quantum state held immediately before the measurement (i.e. the pre-measurement qubit.)

   (c) A secure telephone line ( or any other kind of classical communication system) that she can use to speak to Bob. Of course, remember that Alice cannot use this telephone line to tell Bob the information contained in $|\psi\rangle$ because she cannot extract any information from $|\psi\rangle$ without changing the contents of $|\psi\rangle$.

   (d) Access to a laboratory in which an experimental physicist will help her to build any one- and two-qubit gate Alice needs as well as to perform measurements on Alice's qubits.

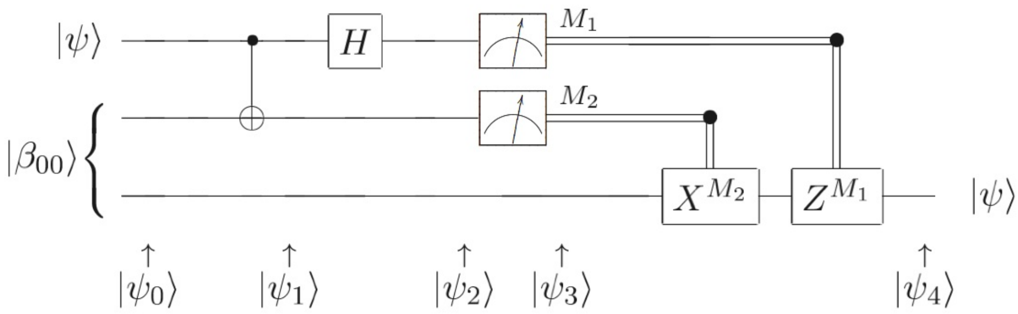


**Figure 1.13.** **Quantum circuit for teleporting a qubit. The two top lines represent Alice's system, while the bottom line is Bob's system. The meters represent measurement, and the double lines coming out of them carry classical bits (recall that single lines denote qubits).**

Figure 1: Quantum Teleportation Circuit (figure taken from [1]).

6. As for Bob, he has access to the following resources:

   (a) A qubit from an EPR pair. The other qubit of this EPR pair is on Alice's hands.

   (b) A secure telephone line ( or any other kind of classical communication system) that he can use to speak to Alice.

   (c) Access to a laboratory in which an experimental physicist will help him to build any one-qubit gate Bob needs.

Alice has chosen to accept this mission. So, in order to send $|\psi\rangle$ to Bob, she will use the quantum circuit presented in Fig. (1).

In the following pages, I provide the readership with a detailed derivation of each stage of the quantum teleportation circuit, each labelled $|\psi\rangle_i$, with $i \in \{0, 1, 2, 3, 4\}$. We denote our EPR pair by $|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

- $|\psi_0\rangle$

$$\begin{aligned}
|\psi\rangle_0 =& |\psi\rangle \otimes |\beta_{00}\rangle \\
=& (\alpha|0\rangle + \beta|1\rangle) \otimes (\frac{|00\rangle + |11\rangle}{\sqrt{2}}) \\
=& \frac{1}{\sqrt{2}}(\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)) \\
=& \frac{1}{\sqrt{2}}(\alpha|0\rangle|00\rangle + \alpha|0\rangle|11\rangle + \beta|1\rangle|00\rangle + \beta|1\rangle|11\rangle) \\
=& \frac{1}{\sqrt{2}}(\alpha|00\rangle|0\rangle + \alpha|01\rangle|1\rangle + \beta|10\rangle|0\rangle + \beta|11\rangle|1\rangle)
\end{aligned}$$

So,

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(\alpha|00\rangle|0\rangle + \alpha|01\rangle|1\rangle + \beta|10\rangle|0\rangle + \beta|11\rangle|1\rangle) \tag{1}$$

- $|\psi_1\rangle$

$$
\begin{aligned}
|\psi_1\rangle =&\hat{C}_{\text{NOT}} \otimes \hat{\mathbb{I}}\ |\psi_0\rangle \\
=&(\hat{C}_{\text{NOT}} \otimes \hat{\mathbb{I}})[\frac{1}{\sqrt{2}}(\alpha|00\rangle|0\rangle + \alpha|01\rangle|1\rangle + \beta|10\rangle|0\rangle + \beta|11\rangle|1\rangle)] \\
=&\frac{1}{\sqrt{2}}[\hat{C}_{\text{NOT}} \otimes \hat{\mathbb{I}}\ (\alpha|00\rangle|0\rangle) + \hat{C}_{\text{NOT}} \otimes \hat{\mathbb{I}}\ (\alpha|01\rangle|1\rangle) + \hat{C}_{\text{NOT}} \otimes \hat{\mathbb{I}}\ (\beta|10\rangle|0\rangle) + \hat{C}_{\text{NOT}} \otimes \hat{\mathbb{I}}\ (\beta|11\rangle|1\rangle)] \\
=&\frac{1}{\sqrt{2}}[\alpha(\hat{C}_{\text{NOT}}\ |00\rangle \otimes \hat{\mathbb{I}}\ |0\rangle) + \alpha(\hat{C}_{\text{NOT}}\ |01\rangle \otimes \hat{\mathbb{I}}|1\rangle) + \beta(\hat{C}_{\text{NOT}}\ |10\rangle \otimes \hat{\mathbb{I}}|0\rangle) + \beta(\hat{C}_{\text{NOT}}\ |11\rangle \otimes \hat{\mathbb{I}}|1\rangle)] \\
=&\frac{1}{\sqrt{2}}(\alpha|00\rangle|0\rangle + \alpha|01\rangle|1\rangle + \beta|11\rangle|0\rangle + \beta|10\rangle|1\rangle) \\
=&\frac{1}{\sqrt{2}}(\alpha|0\rangle|00\rangle + \alpha|0\rangle|11\rangle + \beta|1\rangle|10\rangle + \beta|1\rangle|01\rangle)
\end{aligned}
$$

So,

$$
|\psi_1\rangle = \frac{1}{\sqrt{2}}(\alpha|0\rangle|00\rangle + \alpha|0\rangle|11\rangle + \beta|1\rangle|10\rangle + \beta|1\rangle|01\rangle) \tag{2}
$$

- $|\psi_2\rangle$

$$
\begin{aligned}
|\psi_2\rangle =&(\hat{H} \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}})\ |\psi_1\rangle \\
=&(\hat{H} \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}})\ (\frac{1}{\sqrt{2}}[\alpha|0\rangle|00\rangle + \alpha|0\rangle|11\rangle + \beta|1\rangle|10\rangle + \beta|1\rangle|01\rangle]) \\
=&(\hat{H} \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}})\ (\frac{1}{\sqrt{2}}[\alpha|0\rangle|0\rangle|0\rangle + \alpha|0\rangle|1\rangle|1\rangle + \beta|1\rangle|1\rangle|0\rangle + \beta|1\rangle|0\rangle|1\rangle]) \\
=&\frac{1}{\sqrt{2}}[\alpha\hat{H}|0\rangle \otimes \hat{\mathbb{I}}|0\rangle \otimes \hat{\mathbb{I}}|0\rangle + \alpha\hat{H}|0\rangle \otimes \hat{\mathbb{I}}|1\rangle \otimes \hat{\mathbb{I}}|1\rangle + \beta\hat{H}|1\rangle \otimes \hat{\mathbb{I}}|1\rangle \otimes \hat{\mathbb{I}}|0\rangle + \beta\hat{H}|1\rangle \otimes \hat{\mathbb{I}}|0\rangle \otimes \hat{\mathbb{I}}|1\rangle] \\
=&\frac{1}{\sqrt{2}}[\alpha(\frac{|0\rangle + |1\rangle}{\sqrt{2}})|00\rangle + \alpha(\frac{|0\rangle + |1\rangle}{\sqrt{2}})|11\rangle + \beta(\frac{|0\rangle - |1\rangle}{\sqrt{2}})|10\rangle + \beta(\frac{|0\rangle - |1\rangle}{\sqrt{2}})|01\rangle] \\
=&\frac{1}{\sqrt{2}^2}[\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle] \\
=&\frac{1}{2}[|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)]
\end{aligned}
$$

So,

$$
|\psi_2\rangle = \frac{1}{2}[|00\rangle_A(\alpha|0\rangle + \beta|1\rangle) + |01\rangle_A(\alpha|1\rangle + \beta|0\rangle) + |10\rangle_A(\alpha|0\rangle - \beta|1\rangle) + |11\rangle_A(\alpha|1\rangle - \beta|0\rangle)] \tag{3}
$$

where I use the notation $|ij\rangle_A$, $i, j \in \{0, 1\}$ (as in $|00\rangle_A$) to stress the fact that those two qubits are (and have always been) under the domain of Alice, i.e. Alice can manipulate them and measure them at will.

- $|\psi_3\rangle$

  Now, $|\psi_3\rangle$ is the quantum state produced after measuring Alice's qubits.

  In this step, we shall measure Alice's two qubits, $|\psi\rangle$ and her EPR qubit, *simultaneously*. Let us start by defining the following one-qubit measurement operators:

$$\hat{P}_{a_0}^{|\psi\rangle} = |0\rangle\langle 0| \tag{4a}$$

$$\hat{P}_{a_1}^{|\psi\rangle} = |1\rangle\langle 1| \tag{4b}$$

$$\hat{P}_{b_0}^{|\beta_{00}\rangle} = |0\rangle\langle 0| \tag{4c}$$

$$\hat{P}_{b_1}^{|\beta_{00}\rangle} = |1\rangle\langle 1| \tag{4d}$$

  Operators from Eqs. (4a,4b) would be used to measure $|\psi\rangle$, while operators from Eqs. (4c, 4d) would be employed to measure Alice's qubit from the EPR pair she shares with Bob.

  Notes:
  1. We use the labels $\{a_0, a_1\}$ and $\{b_0, b_1\}$ to refer to the *possible* measurement outcomes for $|\psi\rangle$ and Alice's EPR qubit, respectively.
  2. Hence, only four outcomes are possible: $\{a_0, b_0\}$, $\{a_0, b_1\}$, $\{a_1, b_0\}$ or $\{a_1, b_1\}$.

  Based on Eqs. (4a-4d) we now define the following two-qubit measurement operators:

$$\hat{P}_{\{a_0,b_0\}} = \hat{P}_{a_0}^{|\psi\rangle} \otimes \hat{P}_{b_0}^{|\beta_{00}\rangle} = |0_{a_0}\rangle\langle 0_{a_0}| \otimes |0_{b_0}\rangle\langle 0_{b_0}| = |0_{a_0}0_{b_0}\rangle\langle 0_{a_0}0_{b_0}| = |00\rangle\langle 00| \tag{5a}$$

$$\hat{P}_{\{a_0,b_1\}} = \hat{P}_{a_0}^{|\psi\rangle} \otimes \hat{P}_{b_1}^{|\beta_{00}\rangle} = |0_{a_0}\rangle\langle 0_{a_0}| \otimes |1_{b_1}\rangle\langle 1_{b_1}| = |0_{a_0}1_{b_1}\rangle\langle 0_{a_0}1_{b_1}| = |01\rangle\langle 01| \tag{5b}$$

$$\hat{P}_{\{a_1,b_0\}} = \hat{P}_{a_1}^{|\psi\rangle} \otimes \hat{P}_{b_0}^{|\beta_{00}\rangle} = |1_{a_1}\rangle\langle 1_{a_1}| \otimes |0_{b_0}\rangle\langle 0_{b_0}| = |1_{a_1}0_{b_0}\rangle\langle 1_{a_1}0_{b_0}| = |10\rangle\langle 10| \tag{5c}$$

$$\hat{P}_{\{a_1,b_1\}} = \hat{P}_{a_1}^{|\psi\rangle} \otimes \hat{P}_{b_1}^{|\beta_{00}\rangle} = |1_{a_1}\rangle\langle 1_{a_1}| \otimes |1_{b_1}\rangle\langle 1_{b_1}| = |1_{a_1}1_{b_1}\rangle\langle 1_{a_1}1_{b_1}| = |11\rangle\langle 11| \tag{5d}$$

  Let us now calculate the probability distribution and post-measurement states for outcomes $\{a_0, b_0\}$, $\{a_0, b_1\}$, $\{a_1, b_0\}$, $\{a_1, b_1\}$.

1. $p(a_0,b_0) = \langle\psi_2|\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle$

$$\begin{aligned}\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle =& |00\rangle\langle 00|\big[\frac{1}{2}\big(|00\rangle(\alpha|0\rangle+\beta|1\rangle)+|01\rangle(\alpha|1\rangle+\beta|0\rangle)+|10\rangle(\alpha|0\rangle-\beta|1\rangle)+|11\rangle(\alpha|1\rangle-\beta|0\rangle)\big)\big]\\ =& \frac{1}{2}\big[\,\langle 00|00\rangle\,|00\rangle(\alpha|0\rangle+\beta|1\rangle)+\langle 00|01\rangle\,|00\rangle(\alpha|1\rangle+\beta|0\rangle)\\ &+\langle 00|10\rangle\,|00\rangle(\alpha|0\rangle-\beta|1\rangle)+\langle 00|11\rangle\,|00\rangle(\alpha|1\rangle-\beta|0\rangle)\big]\\ =& \frac{1}{2}|00\rangle(\alpha|0\rangle+\beta|1\rangle)\end{aligned}$$

That is,

$$\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle = \frac{1}{2}|00\rangle(\alpha|0\rangle+\beta|1\rangle) \tag{6}$$

Now,

$$\begin{aligned}\langle\psi_2|\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle =& \frac{1}{2}\big[\langle 00|(\alpha^*\langle 0|+\beta^*\langle 1|)+\langle 01|(\alpha^*\langle 1|+\beta^*\langle 0|)+\langle 10|(\alpha^*\langle 0|-\beta^*\langle 1|)+\langle 11|(\alpha^*\langle 1|-\beta^*\langle 0|)\big]\\ &\big[\frac{1}{2}|00\rangle(\alpha|0\rangle+\beta|1\rangle)\big]\\ =& \frac{1}{4}\big[\,\langle 00|00\rangle\,(\alpha^*\langle 0|+\beta^*\langle 1|)(\alpha|0\rangle+\beta|1\rangle)+\langle 01|00\rangle\,(\alpha^*\langle 1|+\beta^*\langle 0|)(\alpha|0\rangle+\beta|1\rangle)\\ &+\langle 10|00\rangle\,(\alpha^*\langle 0|-\beta^*\langle 1|)(\alpha|0\rangle+\beta|1\rangle)+\langle 11|00\rangle\,(\alpha^*\langle 1|-\beta^*\langle 0|)(\alpha|0\rangle+\beta|1\rangle)\big]\\ =& \frac{1}{4}\big[(\alpha^*\langle 0|+\beta^*\langle 1|)(\alpha|0\rangle+\beta|1\rangle)\big]\\ =& \frac{1}{4}\big[\alpha^*\alpha\,\langle 0|0\rangle+\alpha^*\beta\,\langle 0|1\rangle+\beta^*\alpha\,\langle 1|0\rangle+\beta^*\beta\,\langle 1|1\rangle\,\big]\\ =& \frac{1}{4}\big[||\alpha||^2+||\beta||^2\big]\\ =& \frac{1}{4}\end{aligned}$$

That is,

$$p(a_0,b_0) = \langle\psi_2|\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle = \frac{1}{4} \tag{7}$$

The corresponding post-measurement state $|\psi\rangle^{\text{pm}}_{\{a_0,b_0\}}$ is given by

$$|\psi\rangle^{\mathbf{pm}}_{\{a_0,b_0\}} = \frac{\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle}{\sqrt{\langle\psi_2|\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle}} = \frac{\frac{1}{2}|00\rangle(\alpha|0\rangle+\beta|1\rangle)}{\sqrt{1/4}} = |00\rangle(\alpha|0\rangle+\beta|1\rangle)$$

Therefore,

$$p(a_0,b_0) = \langle\psi_2|\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle = \frac{1}{4} \tag{8}$$

$$|\psi\rangle^{\text{pm}}_{\{a_0,b_0\}} = |00\rangle_A(\alpha|0\rangle_B+\beta|1\rangle_B) \tag{9}$$

where subindex $A$ is used to explicitly state that qubits $|00\rangle_A$ are on Alice's hands, while subindex $B$ is used to explicitly state that qubit $\alpha|0\rangle_B+\beta|1\rangle_B$ is on Bob's hands.

2. $p(a_0, b_1) = \langle\psi_2|\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle$

$$\begin{aligned}\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle =&|01\rangle\langle 01|\big[\frac{1}{2}\big(|00\rangle(\alpha|0\rangle+\beta|1\rangle)+|01\rangle(\alpha|1\rangle+\beta|0\rangle)+|10\rangle(\alpha|0\rangle-\beta|1\rangle)+|11\rangle(\alpha|1\rangle-\beta|0\rangle)\big)\big]\\ =&\frac{1}{2}\big[\ \langle 01|00\rangle\ |01\rangle(\alpha|0\rangle+\beta|1\rangle)+\langle 01|01\rangle\ |01\rangle(\alpha|1\rangle+\beta|0\rangle)\\ &+\langle 01|10\rangle\ |01\rangle(\alpha|0\rangle-\beta|1\rangle)+\langle 01|11\rangle\ |01\rangle(\alpha|1\rangle-\beta|0\rangle)\big]\\ =&\frac{1}{2}|01\rangle(\alpha|1\rangle+\beta|0\rangle)\end{aligned}$$

That is,

$$\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle = \frac{1}{2}|01\rangle(\alpha|1\rangle+\beta|0\rangle) \tag{10}$$

Now,

$$\begin{aligned}\langle\psi_2|\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle =&\frac{1}{2}\big[\langle 00|(\alpha^*\langle 0|+\beta^*\langle 1|)+\langle 01|(\alpha^*\langle 1|+\beta^*\langle 0|)+\langle 10|(\alpha^*\langle 0|-\beta^*\langle 1|)+\langle 11|(\alpha^*\langle 1|-\beta^*\langle 0|)\big]\\ &\big[\frac{1}{2}|01\rangle(\alpha|1\rangle+\beta|0\rangle)\big]\\ =&\frac{1}{4}\big[\ \langle 00|01\rangle\ (\alpha^*\langle 0|+\beta^*\langle 1|)(\alpha|1\rangle+\beta|0\rangle)+\langle 01|01\rangle\ (\alpha^*\langle 1|+\beta^*\langle 0|)(\alpha|1\rangle+\beta|0\rangle)\\ &+\langle 10|01\rangle\ (\alpha^*\langle 0|-\beta^*\langle 1|)(\alpha|1\rangle+\beta|0\rangle)+\langle 11|01\rangle\ (\alpha^*\langle 1|-\beta^*\langle 0|)(\alpha|1\rangle+\beta|0\rangle)\big]\\ =&\frac{1}{4}\big[(\alpha^*\langle 1|+\beta^*\langle 0|)(\alpha|1\rangle+\beta|0\rangle)\big]\\ =&\frac{1}{4}\big[\alpha^*\alpha\ \langle 1|1\rangle+\alpha^*\beta\ \langle 1|0\rangle+\beta^*\alpha\ \langle 0|1\rangle+\beta^*\beta\ \langle 0|0\rangle\ \big]\\ =&\frac{1}{4}\big[||\alpha||^2+||\beta||^2\big]\\ =&\frac{1}{4}\end{aligned}$$

That is,

$$p(a_0, b_1) = \langle\psi_2|\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle = \frac{1}{4} \tag{11}$$

The corresponding post-measurement state $|\psi\rangle^{\text{pm}}_{\{a_0,b_1\}}$ is given by

$$|\psi\rangle^{\mathbf{pm}}_{\{a_0,b_1\}} = \frac{\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle}{\sqrt{\langle\psi_2|\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle}} = \frac{\frac{1}{2}|01\rangle(\alpha|1\rangle+\beta|0\rangle)}{\sqrt{1/4}} = |01\rangle(\alpha|1\rangle+\beta|0\rangle)$$

Therefore,

$$p(a_0, b_1) = \langle\psi_2|\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle = \frac{1}{4} \tag{12}$$

$$|\psi\rangle^{\text{pm}}_{\{a_0,b_1\}} = |01\rangle_A(\alpha|1\rangle_B+\beta|0\rangle_B) \tag{13}$$

where subindex $A$ is used to explicitly state that qubits $|01\rangle_A$ are on Alice's hands, while subindex $B$ is used to explicitly state that qubit $\alpha|1\rangle_B+\beta|0\rangle_B$ is on Bob's hands.

3. $p(a_1, b_0) = \langle\psi_2|\hat{P}_{\{a_1,b_0\}}|\psi_2\rangle$

$$\begin{aligned}
\hat{P}_{\{a_1,b_0\}}|\psi_2\rangle =& |10\rangle\langle 10|\big[\frac{1}{2}\big(|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)\big)\big] \\
=& \frac{1}{2}\big[\, \langle 10|00\rangle\, |10\rangle(\alpha|0\rangle + \beta|1\rangle) + \langle 10|01\rangle\, |10\rangle(\alpha|1\rangle + \beta|0\rangle) \\
& + \langle 10|10\rangle\, |10\rangle(\alpha|0\rangle - \beta|1\rangle) + \langle 10|11\rangle\, |10\rangle(\alpha|1\rangle - \beta|0\rangle)\big] \\
=& \frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle)
\end{aligned}$$

That is,

$$\hat{P}_{\{a_1,b_0\}}|\psi_2\rangle = \frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle) \tag{14}$$

Now,

$$\begin{aligned}
\langle\psi_2|\hat{P}_{\{a_1,b_0\}}|\psi_2\rangle =& \frac{1}{2}\big[\langle 00|(\alpha^*\langle 0| + \beta^*\langle 1|) + \langle 01|(\alpha^*\langle 1| + \beta^*\langle 0|) + \langle 10|(\alpha^*\langle 0| - \beta^*\langle 1|) + \langle 11|(\alpha^*\langle 1| - \beta^*\langle 0|)\big] \\
& \big[\frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle)\big] \\
=& \frac{1}{4}\big[\, \langle 00|10\rangle\, (\alpha^*\langle 0| + \beta^*\langle 1|)(\alpha|0\rangle - \beta|1\rangle) + \langle 01|10\rangle\, (\alpha^*\langle 1| + \beta^*\langle 0|)(\alpha|0\rangle - \beta|1\rangle) \\
& + \langle 10|10\rangle\, (\alpha^*\langle 0| - \beta^*\langle 1|)(\alpha|0\rangle - \beta|1\rangle) + \langle 11|10\rangle\, (\alpha^*\langle 1| - \beta^*\langle 0|)(\alpha|0\rangle - \beta|1\rangle)\big] \\
=& \frac{1}{4}\big[(\alpha^*\langle 0| - \beta^*\langle 1|)(\alpha|0\rangle - \beta|1\rangle)\big] \\
=& \frac{1}{4}\big[\alpha^*\alpha\,\langle 0|0\rangle - \alpha^*\beta\,\langle 0|1\rangle - \beta^*\alpha\,\langle 1|0\rangle + \beta^*\beta\,\langle 1|1\rangle\,\big] \\
=& \frac{1}{4}\big[||\alpha||^2 + ||\beta||^2\big] \\
=& \frac{1}{4}
\end{aligned}$$

That is,

$$p(a_1, b_0) = \langle\psi_2|\hat{P}_{\{a_1,b_0\}}|\psi_2\rangle = \frac{1}{4} \tag{15}$$

The corresponding post-measurement state $|\psi\rangle^{\text{pm}}_{\{a_1,b_0\}}$ is given by

$$|\psi\rangle^{\mathbf{pm}}_{\{a_1,b_0\}} = \frac{\hat{P}_{\{a_1,b_0\}}|\psi_2\rangle}{\sqrt{\langle\psi_2|\hat{P}_{\{a_1,b_0\}}|\psi_2\rangle}} = \frac{\frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle)}{\sqrt{1/4}} = |10\rangle(\alpha|0\rangle - \beta|1\rangle)$$

Therefore,

$$p(a_1, b_0) = \langle\psi_2|\hat{P}_{\{a_1,b_0\}}|\psi_2\rangle = \frac{1}{4} \tag{16}$$

$$|\psi\rangle^{\text{pm}}_{\{a_1,b_0\}} = |10\rangle_A(\alpha|0\rangle_B - \beta|1\rangle_B) \tag{17}$$

where subindex $A$ is used to explicitly state that qubits $|10\rangle_A$ are on Alice's hands, while subindex $B$ is used to explicitly state that qubit $\alpha|0\rangle_B - \beta|1\rangle_B$ is on Bob's hands.

4. $p(a_1, b_1) = \langle\psi_2|\hat{P}_{\{a_1,b_1\}}|\psi_2\rangle$

$$
\begin{aligned}
\hat{P}_{\{a_1,b_1\}}|\psi_2\rangle =&|11\rangle\langle 11|\Big[\frac{1}{2}\Big(|00\rangle(\alpha|0\rangle+\beta|1\rangle)+|01\rangle(\alpha|1\rangle+\beta|0\rangle)+|10\rangle(\alpha|0\rangle-\beta|1\rangle)+|11\rangle(\alpha|1\rangle-\beta|0\rangle)\Big)\Big]\\
=&\frac{1}{2}\Big[\,\langle 11|00\rangle\,|11\rangle(\alpha|0\rangle+\beta|1\rangle)+\langle 11|01\rangle\,|11\rangle(\alpha|1\rangle+\beta|0\rangle)\\
&+\langle 11|10\rangle\,|11\rangle(\alpha|0\rangle-\beta|1\rangle)+\langle 11|11\rangle\,|11\rangle(\alpha|1\rangle-\beta|0\rangle)\Big]\\
=&\frac{1}{2}|11\rangle(\alpha|1\rangle-\beta|0\rangle)
\end{aligned}
$$

That is,

$$\hat{P}_{\{a_1,b_1\}}|\psi_2\rangle = \frac{1}{2}|11\rangle(\alpha|1\rangle-\beta|0\rangle) \tag{18}$$

Now,

$$
\begin{aligned}
\langle\psi_2|\hat{P}_{\{a_1,b_1\}}|\psi_2\rangle =&\frac{1}{2}\Big[\langle 00|(\alpha^*\langle 0|+\beta^*\langle 1|)+\langle 01|(\alpha^*\langle 1|+\beta^*\langle 0|)+\langle 10|(\alpha^*\langle 0|-\beta^*\langle 1|)+\langle 11|(\alpha^*\langle 1|-\beta^*\langle 0|)\Big]\\
&\Big[\frac{1}{2}|11\rangle(\alpha|1\rangle-\beta|0\rangle)\Big]\\
=&\frac{1}{4}\Big[\,\langle 00|11\rangle\,(\alpha^*\langle 0|+\beta^*\langle 1|)(\alpha|1\rangle-\beta|0\rangle)+\langle 01|11\rangle\,(\alpha^*\langle 1|+\beta^*\langle 0|)(\alpha|1\rangle-\beta|0\rangle)\\
&+\langle 10|11\rangle\,(\alpha^*\langle 0|-\beta^*\langle 1|)(\alpha|1\rangle-\beta|0\rangle)+\langle 11|11\rangle\,(\alpha^*\langle 1|-\beta^*\langle 0|)(\alpha|1\rangle-\beta|0\rangle)\Big]\\
=&\frac{1}{4}\Big[(\alpha^*\langle 1|-\beta^*\langle 0|)(\alpha|1\rangle-\beta|0\rangle)\Big]\\
=&\frac{1}{4}\Big[\alpha^*\alpha\,\langle 1|1\rangle-\alpha^*\beta\,\langle 1|0\rangle-\beta^*\alpha\,\langle 0|1\rangle+\beta^*\beta\,\langle 0|0\rangle\,\Big]\\
=&\frac{1}{4}\Big[||\alpha||^2+||\beta||^2\Big]\\
=&\frac{1}{4}
\end{aligned}
$$

That is,

$$p(a_1, b_1) = \langle\psi_2|\hat{P}_{\{a_1,b_1\}}|\psi_2\rangle = \frac{1}{4} \tag{19}$$

The corresponding post-measurement state $|\psi\rangle^{\text{pm}}_{\{a_1,b_1\}}$ is given by

$$|\psi\rangle^{\mathbf{pm}}_{\{a_1,b_1\}} = \frac{\hat{P}_{\{a_1,b_1\}}|\psi_2\rangle}{\sqrt{\langle\psi_2|\hat{P}_{\{a_1,b_1\}}|\psi_2\rangle}} = \frac{\frac{1}{2}|11\rangle(\alpha|1\rangle-\beta|0\rangle)}{\sqrt{1/4}} = |11\rangle(\alpha|1\rangle-\beta|0\rangle)$$

Therefore,

$$p(a_1, b_1) = \langle\psi_2|\hat{P}_{\{a_1,b_1\}}|\psi_2\rangle = \frac{1}{4} \tag{20}$$

$$|\psi\rangle^{\text{pm}}_{\{a_1,b_1\}} = |11\rangle_A(\alpha|1\rangle_B-\beta|0\rangle_B) \tag{21}$$

where subindex $A$ is used to explicitly state that qubits $|11\rangle_A$ are on Alice's hands, while subindex $B$ is used to explicitly state that qubit $\alpha|1\rangle_B-\beta|0\rangle_B$ is on Bob's hands.

- $|\psi_4\rangle$

In summary, we have four cases described by Eqs.(8,9,12,13,16,17,20,21):

**Case 1**. Outcome $\{a_0, b_0\}$.

The probability of getting outcome $\{a_0, b_0\}$ is

$$p(a_0, b_0) = \langle\psi_2|\hat{P}_{\{a_0,b_0\}}|\psi_2\rangle = \frac{1}{4}$$

Furthermore, post-measurement quantum state for outcome $\{a_0, b_0\}$ is given by

$$|\psi\rangle^{\rm pm}_{\{a_0,b_0\}} = |00\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B)$$

In this case, Alice *knows* that her qubits are in state $|00\rangle_A$. Moreover, she *knows* that Bob's qubit is in the state $\alpha|0\rangle_B + \beta|1\rangle_B$, that is, the qubit Alice was expected to send! Therefore, she calls Bob *via a classical channel* (a telephone line, for instance) to tell him 'your qubit is ready!', i.e.

$$|\psi_4\rangle = \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_B[|00\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B)] \tag{22}$$

**Case 2**. Outcome $\{a_0, b_1\}$.

The probability of getting outcome $\{a_0, b_1\}$ is

$$p(a_0, b_1) = \langle\psi_2|\hat{P}_{\{a_0,b_1\}}|\psi_2\rangle = \frac{1}{4}$$

Furthermore, post-measurement quantum state for outcome $\{a_0, b_1\}$ is given by

$$|\psi\rangle^{\rm pm}_{\{a_0,b_1\}} = |01\rangle_A(\alpha|1\rangle_B + \beta|0\rangle_B)$$

In this case, Alice *knows* that her qubits are in state $|01\rangle_A$. Moreover, she *knows* that Bob's qubit is in the state $\alpha|1\rangle_B + \beta|0\rangle_B$. Now, since

$$\begin{aligned}\hat{\sigma}_x(\alpha|1\rangle_B + \beta|0\rangle_B) =&(|0\rangle\langle 1| + |1\rangle\langle 0|)(\alpha|1\rangle_B + \beta|0\rangle_B)\\ =&\alpha\,\langle 1|1\rangle\,|0\rangle + \beta\,\langle 1|0\rangle\,|0\rangle + \alpha\,\langle 0|1\rangle\,|1\rangle + \beta\,\langle 0|0\rangle\,|1\rangle\\ =&\alpha|0\rangle + \beta|1\rangle\end{aligned}$$

Then Alice calls Bob *via a classical channel* (a telephone line, for instance) to tell him that

$$|\psi_4\rangle = \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_A \otimes (\hat{\sigma}_x)_B[|01\rangle_A(\alpha|1\rangle_B + \beta|0\rangle_B)] \tag{23}$$

That is, Bob only needs to apply $\hat{\sigma}_x$ operator to his qubit in order to have it ready!

**Case 3**. Outcome $\{a_1, b_0\}$.

The probability of getting outcome $\{a_1, b_0\}$ is

$$p(a_1, b_0) = \langle\psi_2|\hat{P}_{\{a_1,b_0\}}|\psi_2\rangle = \frac{1}{4}$$

Furthermore, post-measurement quantum state for outcome $\{a_1, b_0\}$ is given by

$$|\psi\rangle^{\text{pm}}_{\{a_1,b_0\}} = |10\rangle_A(\alpha|0\rangle_B - \beta|1\rangle_B)$$

In this case, Alice *knows* that her qubits are in state $|10\rangle_A$. Moreover, she *knows* that Bob's qubit is in the state $\alpha|0\rangle_B - \beta|1\rangle_B$. Now, since

$$\begin{aligned}
\hat{\sigma}_z(\alpha|0\rangle_B - \beta|1\rangle_B) =&(|0\rangle\langle 0| - |1\rangle\langle 1|)(\alpha|0\rangle_B - \beta|1\rangle_B) \\
=&\alpha\,\langle 0|0\rangle\,|0\rangle - \beta\,\langle 0|1\rangle\,|0\rangle - \alpha\,\langle 1|0\rangle\,|1\rangle + \beta\,\langle 1|1\rangle\,|1\rangle \\
=&\alpha|0\rangle + \beta|1\rangle
\end{aligned}$$

Then Alice calls Bob *via a classical channel* (a telephone line, for instance) to tell him that

$$|\psi_4\rangle = \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_A \otimes (\hat{\sigma}_z)_B[|01\rangle_A(\alpha|0\rangle_B - \beta|1\rangle_B)] \tag{24}$$

That is, Bob only needs to apply $\hat{\sigma}_z$ operator to his qubit in order to have it ready!

**Case 4**. Outcome $\{a_1, b_1\}$.

The probability of getting outcome $\{a_1, b_1\}$ is

$$p(a_1, b_1) = \langle\psi_2|\hat{P}_{\{a_1,b_1\}}|\psi_2\rangle = \frac{1}{4}$$

Furthermore, post-measurement quantum state for outcome $\{a_1, b_1\}$ is given by

$$|\psi\rangle^{\text{pm}}_{\{a_1,b_1\}} = |11\rangle_A(\alpha|1\rangle_B - \beta|0\rangle_B)$$

In this case, Alice *knows* that her qubits are in state $|11\rangle_A$. Moreover, she *knows* that Bob's qubit is in the state $\alpha|1\rangle_B - \beta|0\rangle_B$. Now, since

$$\begin{aligned}
\hat{\sigma}_z(\hat{\sigma}_x(\alpha|1\rangle_B - \beta|0\rangle_B)) =&\hat{\sigma}_z(|0\rangle\langle 1| + |1\rangle\langle 0|)(\alpha|1\rangle_B - \beta|0\rangle_B) \\
=&\hat{\sigma}_z(\alpha\,\langle 1|1\rangle\,|0\rangle - \beta\,\langle 1|0\rangle\,|0\rangle + \alpha\,\langle 0|1\rangle\,|1\rangle - \beta\,\langle 0|0\rangle\,|1\rangle) \\
=&\hat{\sigma}_z(\alpha|0\rangle - \beta|1\rangle) \\
=&(|0\rangle\langle 0| - |1\rangle\langle 1|)(\alpha|0\rangle - \beta|1\rangle) \\
=&\alpha\,\langle 0|0\rangle\,|0\rangle - \beta\,\langle 0|1\rangle\,|0\rangle - \alpha\,\langle 1|0\rangle\,|1\rangle + \beta\,\langle 1|1\rangle\,|1\rangle \\
=&\alpha|0\rangle + \beta|1\rangle
\end{aligned}$$

Then Alice calls Bob *via a classical channel* (a telephone line, for instance) to tell him that

$$|\psi_4\rangle = \hat{\mathbb{I}}_A \otimes \hat{\mathbb{I}}_A \otimes (\hat{\sigma}_z\hat{\sigma}_x)_B[|01\rangle_A(\alpha|1\rangle_B - \beta|0\rangle_B)] \tag{25}$$

That is, Bob only needs to apply $\hat{\sigma}_z\hat{\sigma}_x$ operators (in that precise order) to his qubit in order to have it ready!

# References

[1] M.A. Nielsen and I.L. Chuang. Quantum Computation and Quantum Information. Cambridge University Press (2000)