Vector Spaces and Dirac Notation

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Vector Spaces

Def. Vector spaces. Let $\mathbb{V}$ be a set associated to a field $\mathbb{F}$. The elements of $\mathbb{V}$ are called *vectors* and are denoted by bold font variables (like $\mathbf{x}$). The elements of $\mathbb{F}$ are known as *scalars* and are denoted by lowercase letters (like c).

Vector Spaces

We define the notions of vector addition and scalar multiplication in the following lines:

- *Vector addition*. This is a binary operation that takes a pair of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ to produce another vector $\mathbf{x} + \mathbf{y} \in \mathbb{V}$.
- *Scalar multiplication*. This is an operation that takes a vector $\mathbf{x} \in \mathbb{V}$ and a scalar $c \in \mathbb{F}$ to produce another vector $c\mathbf{x} \in \mathbb{V}$.

Vector Spaces

Set $\mathbb{V}$, together with a field $\mathbb{F}$ and the operations known as vector addition and scalar multiplication, is known as a **Vector Space** iff it satisfies the following axioms.

Vector Spaces

1. $\forall\ \mathbf{x}, \mathbf{y} \in \mathbb{V} \Rightarrow \mathbf{x} + \mathbf{y} \in \mathbb{V}$.
2. $\forall\ \mathbf{x}, \mathbf{y} \in \mathbb{V} \Rightarrow \mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
3. $\forall\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{V} \Rightarrow \mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$.
4. $\exists!\ \mathbf{0} \in \mathbb{V}$ such that $\forall\ \mathbf{x} \in \mathbb{V} \Rightarrow \mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$.
5. For each $\mathbf{x} \in \mathbb{V}\ \exists!\ -\mathbf{x} \in \mathbb{V}$ such that $\mathbf{x} + (-\mathbf{x}) = -\mathbf{x} + \mathbf{x} = \mathbf{0}$.

Vector Spaces

6. $\forall\, \mathbf{x} \in \mathbb{V},\ \alpha \in \mathbb{F} \ \Rightarrow\ \alpha\mathbf{x} \in \mathbb{V}$.
7. $\forall\, \mathbf{x} \in \mathbb{V} \Rightarrow 1\mathbf{x} = \mathbf{x}$, where 1 is the multiplicative identity of $\mathbb{F}$.
8. $\forall\, \mathbf{x} \in \mathbb{V} \Rightarrow 0\mathbf{x} = \mathbf{0}$, where 0 is the additive identity of $\mathbb{F}$.
9. $\forall\, \mathbf{x} \in \mathbb{V}, \alpha, \beta \in \mathbb{F}\ (\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$.
10. $\forall\, \mathbf{x} \in \mathbb{V}, \alpha, \beta \in \mathbb{F} \Rightarrow \alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$
11. $\forall\, \mathbf{x}, \mathbf{y} \in \mathbb{V}, \alpha \in \mathbb{F}\ \alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$.

Vector Spaces - Exercises

Let us define the set $\mathbb{C}^2(\mathbb{C})$:

$$\mathbb{C}^2(\mathbb{C}) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| \, a, b \in \mathbb{C} \text{ and scalars } \alpha \in \mathbb{C} \right\}$$

Exercise 1. Prove that $\mathbb{C}^2(\mathbb{C})$ is a vector space.

Exercise 2. Prove that $\mathbb{C}^n(\mathbb{C})$ is a vector space (optional).

Vector Spaces - More exercises

Exercise 3. Prove that $\mathbb{M}_2(\mathbb{C})$ is a vector space.

Exercise 4. Prove that $\mathbb{M}_n(\mathbb{C})$ is a vector space (optional).

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Mathematics for quantum computation

In quantum computation, we use the **Dirac notation** for denoting vectors:

$$\vec{x} = |x\rangle$$

So,

$$\vec{x} = a\hat{\imath} + b\hat{\jmath} \;\Leftrightarrow\; |x\rangle = a|i\rangle + b|j\rangle$$

More on Dirac notation shortly.

Hilbert space

A **Hilbert space** $\mathcal{H}$ is a (complete) complex inner-product vector space. An example of a Hilbert space is $\mathbb{C}^2(\mathbb{C})$, the complex bidimensional vector space defined over the field of complex numbers :

$$\mathbb{C}^2(\mathbb{C}) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \;\middle|\; a, b \in \mathbb{C} \text{ and scalars } \alpha \in \mathbb{C} \right\}$$

Kets and Bras

The Dirac Notation, also known as the Bra-Ket notation, is a standard representation to describe quantum states.

The Dirac notation is widely used in quantum mechanics and quantum computation.

Let us now formally define the notions of Ket and Bra.

Kets

Let $\mathcal{H}$ be a Hilbert space. A vector $\psi \in \mathcal{H}$ is denoted by $|\psi\rangle$ and it is referred to as a **ket**.

We can represent elements $|\psi\rangle$ of $\mathcal{H}$ as column vectors by choosing a basis for $\mathcal{H}$. For example, let $\mathcal{H} = \mathbb{C}^2$ and let us choose the vector basis $\{|0\rangle, |1\rangle\}$, where

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then, every element $|\psi\rangle \in \mathcal{H}$ can be written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \alpha, \beta \in \mathbb{C}$$

Example of kets

$|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \in \mathbb{C}^2$ may be written as

$$|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$$

Exercise - Kets

Let $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$. Write $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in terms of $|+\rangle, |-\rangle$.

Answer to exercise - Kets

Let $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$.
Write $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in terms of $|+\rangle, |-\rangle$.

Note that
$|+\rangle + |-\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} + \frac{|0\rangle-|1\rangle}{\sqrt{2}} = \frac{2|0\rangle}{\sqrt{2}} \Rightarrow |0\rangle = \frac{|+\rangle+|-\rangle}{\sqrt{2}}$

Similarly,
$|+\rangle - |-\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} - (\frac{|0\rangle-|1\rangle}{\sqrt{2}}) = \frac{2|1\rangle}{\sqrt{2}} \Rightarrow |1\rangle = \frac{|+\rangle-|-\rangle}{\sqrt{2}}$

Hence,
$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \alpha\frac{|+\rangle+|-\rangle}{\sqrt{2}} + \beta\frac{|+\rangle-|-\rangle}{\sqrt{2}} = \frac{\alpha+\beta}{\sqrt{2}}|+\rangle + \frac{\alpha-\beta}{\sqrt{2}}|-\rangle$

Therefore,

$$|\psi\rangle = \frac{\alpha+\beta}{\sqrt{2}}|+\rangle + \frac{\alpha-\beta}{\sqrt{2}}|-\rangle$$

Bras

Bras. Formally speaking, bras are functionals (i.e. functions of vector spaces into corresponding fields) and in practice, they can be thought of as **row** vectors:

$$|\psi\rangle = \alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle \text{ if and only if } \langle\psi| = \alpha^*\langle\mathbf{0}| + \beta^*\langle\mathbf{1}|$$

where

$$\alpha, \beta, \alpha^*, \beta^* \in \mathbb{C}$$

$$\alpha = a + bi, \beta = c + di$$

$$\alpha^* = a - bi, \beta^* = c - di$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\langle 0| = (1, 0) \text{ and } \langle 1| = (0, 1)$$

Bras

For example, let us define $|\psi\rangle$ as follows:

$$|\psi\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \frac{i}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\frac{i}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}$$

The corresponding bra $\langle\psi|$ is

$$\langle\psi| = \frac{-i}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1| = \frac{-i}{\sqrt{2}}(1,\ 0) + \frac{1}{\sqrt{2}}(0,\ 1) = (\frac{-i}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

Exercises - Bras

1. Compute $\langle +|$ and $\langle -|$

2. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $||\alpha||^2 + ||\beta||^2 = 1$. Does it follow that $\langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1|$ *where* $||\alpha^*||^2 + ||\beta^*||^2 = 1$?

Answers to exercises - Bras

1. Compute $\langle +|$ and $\langle -|$

Answer: $\langle +| = \frac{\langle 0|+\langle 1|}{\sqrt{2}}$ and $\langle -| = \frac{\langle 0|-\langle 1|}{\sqrt{2}}$

Answers to exercises - Bras

2. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $||\alpha||^2 + ||\beta||^2 = 1$. Does it follow that $\langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1|$ *where* $||\alpha^*||^2 + ||\beta^*||^2 = 1$?

Answer:
$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \Rightarrow \langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1|$.
Now, since $\alpha, \beta \in \mathbb{C}$ then let us write $\alpha = a + bi$ and $\beta = c + di$.
Also, note that $\alpha^* = a - bi$ and $\beta^* = c - di$
Furthermore, $||\alpha||^2 = a^2 + b^2$ and $||\beta||^2 = c^2 + d^2 \Rightarrow$
$||\alpha||^2 + ||\beta||^2 = a^2 + b^2 + c^2 + d^2 = 1$
Finally, please note that $||\alpha^*||^2 = a^2 + b^2$ and $||\beta^*||^2 = c^2 + d^2 \Rightarrow$
$||\alpha^*||^2 + ||\beta^*||^2 = a^2 + b^2 + c^2 + d^2 = 1$

So, the answer is **Yes, it does**.

Summary of Kets and Bras

Thus, if $\mathcal{H}$ is an n-dimensional Hilbert space then

- A ket $|\psi\rangle \in \mathcal{H}$ can be represented as an n-dimensional column vector.
- Its corresponding bra $\langle\psi| \in \mathcal{H}^*$ can be seen as an n-dimensional row vector

$|\psi\rangle \leftrightarrow \langle\psi|$ **corresponds to transposition and conjugation**.

Inner product on *Complex* Vector Spaces

Definition. Let $\mathbb{V}(\mathbb{C})$ denote a vector space $\mathbb{V}$ defined over the set of complex numbers $\mathbb{C}$. Also, let $|a\rangle, |b\rangle \in \mathbb{V}(\mathbb{C})$. We define the inner product function $(,)$ as follows

$$(,) : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$$

with the following properties:

1. $\forall |a\rangle \in \mathbb{V} \Rightarrow (|a\rangle, |a\rangle) \geq 0$ and $(|a\rangle, |a\rangle) = 0 \Leftrightarrow |a\rangle = 0$.
2. $\forall |a\rangle, |b\rangle \in \mathbb{V} \Rightarrow (|a\rangle, |b\rangle) = (|b\rangle, |a\rangle)^*$
3. $\forall |a\rangle, |b\rangle_i \in \mathbb{V}, \alpha_i \in \mathbb{C}, i \in \mathbb{N} \Rightarrow (|a\rangle, \sum_i \alpha_i |b\rangle_i) = \sum_i \alpha_i (|a\rangle, |b\rangle_i)$

Inner product

We define the inner product in $\mathbb{C}^n$, which is the usual row-column matrix multiplication.

Let $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{C}^n \Rightarrow$

$$\left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right) = (a_1^*, \ldots, a_n^*) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \sum_{i=1}^{n} a_i^* b_i$$

where a_i^* is the conjugate of complex number a_i, $\forall i \in \{1, \ldots n\}$

Inner product

We can use the Dirac notation to make calculations.

Let $|\phi\rangle, |\psi\rangle \in \mathbb{C}^2$. We denote the inner product in $\mathbb{C}^2$ as follows:

$$(|\phi\rangle, |\psi\rangle) = \langle\phi||\psi\rangle = \langle\phi|\psi\rangle$$

So, if $|\phi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ and $|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ then

$$\langle\phi|\psi\rangle = (\phi_1^*, \phi_2^*) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \phi_1^*\psi_1 + \phi_2^*\psi_2$$

Inner product

For example, let us take the representations of $|0\rangle$ and $|1\rangle$ given in previous slides

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Note that $|0\rangle \perp |1\rangle$ as well as the fact that both vectors have unitary norm. Consequently, the inner product of $|0\rangle$ and $|1\rangle$ must be zero and the inner product of each vector with itself must be equal to one:

Inner product

$$\langle \mathbf{0}|\mathbf{1}\rangle = (1,\ 0)\begin{pmatrix}0\\1\end{pmatrix} = (1\times 0 + 0\times 1) = \mathbf{0} = (0,\ 1)\begin{pmatrix}1\\0\end{pmatrix} = \langle \mathbf{1}|\mathbf{0}\rangle$$

Moreover

$$\langle \mathbf{0}|\mathbf{0}\rangle = (1,\ 0)\begin{pmatrix}1\\0\end{pmatrix} = (1\times 1 + 0\times 0) = \mathbf{1} = (0,\ 1)\begin{pmatrix}0\\1\end{pmatrix} = \langle \mathbf{1}|\mathbf{1}\rangle$$

Exercises inner product

1. Let $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|\phi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$. Compute (1.a) $\langle\psi|\phi\rangle$ and (1.b) $\langle\phi|\psi\rangle$
2. Let $|\psi\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$ and $|\phi\rangle = \frac{3}{4}|0\rangle + \frac{\sqrt{7}i}{4}|1\rangle$. Compute (2.a) $\langle\psi|\phi\rangle$ and (2.b) $\langle\phi|\psi\rangle$.

Answers to exercises inner product

1.a) $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|\phi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$. Compute $\langle\psi|\phi\rangle$.

Since $\langle\psi| = \frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|$ then

$$\begin{aligned}
\langle\psi|\phi\rangle &= \left(\tfrac{1}{\sqrt{2}}\langle 0| + \tfrac{1}{\sqrt{2}}\langle 1|\right)\left(\tfrac{\sqrt{3}}{2}|0\rangle + \tfrac{1}{2}|1\rangle\right) \\
&= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\langle 0|0\rangle + \frac{1}{\sqrt{2}} \times \frac{1}{2}\langle 0|1\rangle + \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\langle 1|0\rangle + \frac{1}{\sqrt{2}} \times \frac{1}{2}\langle 1|1\rangle \\
&= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} \times 1 + \frac{1}{\sqrt{2}} \times \frac{1}{2} \times 0 + \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} \times 0 + \frac{1}{\sqrt{2}} \times \frac{1}{2} \times 1 \\
&= \boldsymbol{\frac{\sqrt{3}+1}{2\sqrt{2}}}
\end{aligned}$$

Answers to exercises inner product

1.b) $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|\phi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$. Compute $\langle\phi|\psi\rangle$.

Since $\langle\phi| = \frac{\sqrt{3}}{2}\langle 0| + \frac{1}{2}\langle 1|$ then

$$
\begin{aligned}
\langle\phi|\psi\rangle &= \left(\frac{\sqrt{3}}{2}\langle 0| + \frac{1}{2}\langle 1|\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \\
&= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\langle 0|0\rangle + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\langle 0|1\rangle + \frac{1}{2} \times \frac{1}{\sqrt{2}}\langle 1|0\rangle + \frac{1}{2} \times \frac{1}{\sqrt{2}}\langle 1|1\rangle \\
&= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} \times 1 + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} \times 0 + \frac{1}{2} \times \frac{1}{\sqrt{2}} \times 0 + \frac{1}{2} \times \frac{1}{\sqrt{2}} \times 1 \\
&= \boldsymbol{\frac{\sqrt{3}+1}{2\sqrt{2}}}
\end{aligned}
$$

Answers to exercises inner product

2.a) $|\psi\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$ and $|\phi\rangle = \frac{3}{4}|0\rangle + \frac{\sqrt{7}i}{4}|1\rangle$. Compute $\langle\psi|\phi\rangle$.

Since $\langle\psi| = \frac{-i}{\sqrt{2}}\langle 0| + \frac{-i}{\sqrt{2}}\langle 1|$ then

$$
\begin{aligned}
\langle\psi|\phi\rangle &= \left(\tfrac{-i}{\sqrt{2}}\langle 0| + \tfrac{-i}{\sqrt{2}}\langle 1|\right)\left(\tfrac{3}{4}|0\rangle + \tfrac{\sqrt{7}i}{4}|1\rangle\right) \\
&= \frac{-i}{\sqrt{2}} \times \frac{3}{4}\langle 0|0\rangle + \frac{-i}{\sqrt{2}} \times \frac{\sqrt{7}i}{4}\langle 0|1\rangle + \frac{-i}{\sqrt{2}} \times \frac{3}{4}\langle 1|0\rangle + \frac{-i}{\sqrt{2}} \times \frac{\sqrt{7}i}{4}\langle 1|1\rangle \\
&= \frac{-i}{\sqrt{2}} \times \frac{3}{4} \times 1 + \frac{-i}{\sqrt{2}} \times \frac{\sqrt{7}i}{4} \times 0 + \frac{-i}{\sqrt{2}} \times \frac{3}{4} \times 0 + \frac{-i}{\sqrt{2}} \times \frac{\sqrt{7}i}{4} \times 1 \\
&= \mathbf{\frac{\sqrt{7}}{4\sqrt{2}} - \frac{3}{4\sqrt{2}}i}
\end{aligned}
$$

Answers to exercises inner product

2.b) $|\psi\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$ and $|\phi\rangle = \frac{3}{4}|0\rangle + \frac{\sqrt{7}i}{4}|1\rangle$. Compute $\langle\phi|\psi\rangle$.

Since $\langle\phi| = \frac{3}{4}\langle 0| + \frac{-\sqrt{7}i}{\sqrt{4}}\langle 1|$ then

$$
\begin{aligned}
\langle\phi|\psi\rangle &= \left(\tfrac{3}{4}\langle 0| + \tfrac{-\sqrt{7}i}{4}\langle 1|\right)\left(\tfrac{i}{\sqrt{2}}|0\rangle + \tfrac{i}{\sqrt{2}}|1\rangle\right) \\
&= \tfrac{3}{4} \times \tfrac{i}{\sqrt{2}}\langle 0|0\rangle + \tfrac{3}{4} \times \tfrac{i}{\sqrt{2}}\langle 0|1\rangle + \tfrac{-\sqrt{7}i}{4} \times \tfrac{i}{\sqrt{2}}\langle 1|0\rangle + \tfrac{-\sqrt{7}i}{4} \times \tfrac{i}{\sqrt{2}}\langle 1|1\rangle \\
&= \tfrac{3}{4} \times \tfrac{i}{\sqrt{2}} \times 1 + \tfrac{3}{4} \times \tfrac{i}{\sqrt{2}} \times 0 + \tfrac{-\sqrt{7}i}{4} \times \tfrac{i}{\sqrt{2}} \times 0 + \tfrac{-\sqrt{7}i}{4} \times \tfrac{i}{\sqrt{2}} \times 1 \\
&= \mathbf{\frac{\sqrt{7}}{4\sqrt{2}} + \frac{3}{4\sqrt{2}}i}
\end{aligned}
$$

Linear operator

We need to define one more operation, the outer product. To do so, let us define a key notion in Linear Algebra: **Linear Operators**.

Linear operator

Def. Linear operator. A linear operator between vector spaces $\mathbb{V}$ and $\mathbb{W}$ is defined as any function $\hat{A} : \mathbb{V} \to \mathbb{W}$ which is linear in its inputs,

$$\hat{A}\left(\sum_i \alpha_i |\psi_i\rangle\right) = \sum_i \alpha_i \hat{A} |\psi_i\rangle$$

Adjoint/Hermitian Conjugate Operator (1/2)

Let $\hat{A} : \mathcal{H} \to \mathcal{H}$ be a linear operator that induces the map $|\psi\rangle \to |\psi'\rangle$.

The operator $\hat{A}^\dagger$, known as $\hat{A}$ dagger, the adjoint of $\hat{A}$ or the Hermitian Conjugate of $\hat{A}$, induces the map $\langle\psi| \to \langle\psi'|$ on the corresponding bras.

In other words,

$$\hat{A}|\psi\rangle = |\psi'\rangle$$

$$\langle\psi|\hat{A}^\dagger = \langle\psi'|$$

Adjoint/Hermitian Conjugate Operator (2/2)

In matrix notation, $\hat{A}^\dagger$ is $(A^t)^*$ where t denotes transposition and $*$ denotes complex conjugation. For example, let A be the following 3×3 matrix:

$$A = \begin{pmatrix} 3+i & -i & 4 \\ 5+\pi i & 0 & 1-2i \\ -3 & \sqrt{7}i & 7 \end{pmatrix}$$

Then, $(A^t)^*$ the Hermitian Conjugate of A, is given by

$$(A^t)^* = \begin{pmatrix} 3-i & 5-\pi i & -3 \\ i & 0 & -\sqrt{7}i \\ 4 & 1+2i & 7 \end{pmatrix}$$

Unitary Operators (1/2)

Unitary operator. Let $\mathcal{H}$ be a Hilbert space and $\hat{U} : \mathcal{H} \to \mathcal{H}$ a linear operator. $\hat{U}$ is a **Unitary operator** if

$$\hat{U}\hat{U}^{\dagger} = \hat{U}^{\dagger}\hat{U} = \hat{I}$$

where $\hat{I}$ is the identity operator.

Unitary Operators (2/2)

Unitary operators are key elements in the formulation of quantum mechanics and, consequently, in the development of quantum algorithms, because they preserve the inner product between vectors:

Let $\hat{U}$ be a Unitary operator and $|\psi\rangle = \alpha|p\rangle + \beta|q\rangle$, where $\alpha, \beta \in \mathbb{C}$ and $||\alpha||^2 + ||\beta||^2 = 1 \Rightarrow$

$$\hat{U}|\psi\rangle = |\psi\rangle'$$

where $|\psi\rangle' = \alpha'|p\rangle + \beta'|q\rangle$ and $||\alpha'||^2 + ||\beta'||^2 = 1$

Outer product

We can also use the Dirac notation to compute vectors. Let $|\psi\rangle, |a\rangle \in \mathcal{H}_1$ and $|\phi\rangle \in \mathcal{H}_2$ then the *outer product* is the linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ defined by

$$(|\phi\rangle\langle\psi|)|a\rangle \equiv (\langle\psi|a\rangle)|\phi\rangle$$

As it may be expected, the summation of outer products is also a linear operator.

Example - Outer product

For example, let us define the Hadamard operator

$$\hat{H} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

The action of $\hat{H}$ on ket $|0\rangle$ is given by

$$\begin{aligned}
\hat{H}|0\rangle &= \left(\frac{1}{\sqrt{2}}|0\rangle\langle 0| + \frac{1}{\sqrt{2}}|0\rangle\langle 1| + \frac{1}{\sqrt{2}}|1\rangle\langle 0| - \frac{1}{\sqrt{2}}|1\rangle\langle 1|\right)|0\rangle \\
&= \frac{\langle 0|0\rangle}{\sqrt{2}}|0\rangle + \frac{\langle 1|0\rangle}{\sqrt{2}}|0\rangle + \frac{\langle 0|0\rangle}{\sqrt{2}}|1\rangle - \frac{\langle 1|0\rangle}{\sqrt{2}}|1\rangle \\
&= \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}|\mathbf{0}\rangle + \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}|\mathbf{1}\rangle
\end{aligned}$$

Exercise 01 - Outer product

Let $\hat{\sigma}_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$ and $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$. Compute $\hat{\sigma}_y|\psi\rangle$.

Answer to Exercise 01 - Outer product

Let $\hat{\sigma}_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$ and $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$. Compute $\hat{\sigma}_y|\psi\rangle$.

$$
\begin{aligned}
\hat{\sigma}_y|\psi\rangle &= (-i|0\rangle\langle 1| + i|1\rangle\langle 0|)(\tfrac{\sqrt{3}}{2}|0\rangle + \tfrac{i}{2}|1\rangle) \\
&= \frac{-\sqrt{3}i\langle 1|0\rangle}{2}|0\rangle - \frac{i^2\langle 1|1\rangle}{2}|0\rangle + \frac{\sqrt{3}i\langle 0|0\rangle}{2}|1\rangle + \frac{i^2\langle 0|1\rangle}{2}|1\rangle \\
&= \tfrac{1}{2}|0\rangle + \tfrac{\sqrt{3}i}{2}|1\rangle
\end{aligned}
$$

Exercise 02 - Outer product

How would you write $\hat{H} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$ and $\hat{\sigma}_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$ in matrix notation using the conventional column vector representation of the computational basis?

Answer to Exercise 02 - Outer product

Since

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1,\ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (0,\ 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$|1\rangle\langle 0| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1,\ 0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$|1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0,\ 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Answer to Exercise 02 - Outer product

Consequently,

$$\hat{H} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

can be written in matrix form as

$$\begin{aligned} H &= \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right) \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \\ &= \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix} \end{aligned}$$

Answer to Exercise 02 - Outer product

As for

$$\hat{\sigma}_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$$

it can be written as follows:

$$\begin{aligned} \sigma_y &= -i\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{aligned}$$

Exercise 03 - Outer product

Product of outer products.

Compute

$$(|0\rangle\langle 0|)(|0\rangle\langle 0|)$$

and

$$(|1\rangle\langle 1|)(|1\rangle\langle 1|)$$

Answer to Exercise 03 - Outer product

Matrix approach. Let us remember that

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1,\ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$|1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0,\ 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So,

$$(|0\rangle\langle 0|)(|0\rangle\langle 0|) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0|$$

and

$$(|1\rangle\langle 1|)(|1\rangle\langle 1|) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1|$$

Therefore, $(|\mathbf{0}\rangle\langle \mathbf{0}|)(|\mathbf{0}\rangle\langle \mathbf{0}|) = |\mathbf{0}\rangle\langle \mathbf{0}|$ and $(|\mathbf{1}\rangle\langle \mathbf{1}|)(|\mathbf{1}\rangle\langle \mathbf{1}|) = |\mathbf{1}\rangle\langle \mathbf{1}|$

Exercise 04 - Outer product

Dagger operator on outer products.

Compute

$$(|0\rangle\langle 0|)^\dagger$$

and

$$(|1\rangle\langle 1|)^\dagger$$

Answer to Exercise 04 - Outer product

Since

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$|1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

then

$$(|0\rangle\langle 0|)^\dagger = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^t \right)^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0|$$

and

$$(|1\rangle\langle 1|)^\dagger = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^t \right)^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1|$$

Therefore,

$$\mathbf{(|0\rangle\langle 0|)^\dagger = |0\rangle\langle 0|} \text{ and } \mathbf{(|1\rangle\langle 1|)^\dagger = |1\rangle\langle 1|}$$