Math for Energy

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1 Daughter nucleus abundance in n-generation decay chain

1.1 Formula Proposition

Where there 1 is one mole of the 0-th generation nucleus at $t = 0$ there are m_i moles of the $i-th$ generation nucleus at time t. We assume that the $i-th$ generation nucleus has a measurable e-folding time of λ_i. Additionally, r_i is the decay rate of the $i-th$ generation nucleus in moles/time.

$$\begin{aligned} m_0 &= e^{\frac{t}{-\lambda_0}} \\ m_{i>0} &= -\frac{1}{\lambda_0}\int_0^t e^{t(\frac{1}{-\lambda_0}+\frac{1}{-\lambda_1})} a_{i-1} dt \\ r_i &= -\frac{1}{\lambda_0} e^{\frac{t}{-\lambda_0}} a_i \\ a_i &= \prod_{k=1}^{i} (1 - e^{t\frac{1}{-\lambda_k}}) \end{aligned} \tag{1}$$

1.2 Formula Proof

At a given time t there are m_i moles of the $i-th$ daughter nucleus. For $i > 0$, there are ν_i daughter nuclei produced by the decay of the $i-1th$ generation in the decay chain, and this is the only source of any nuclei in the system. We assume there is no source of new 0-th generation nuclei. By the properties of radioactive decay

$$\begin{aligned} m_i &= \int_0^t P_i d\nu_i \\ \nu_i &= \int_0^t (1 - P_{i-1}) d\nu_{i-1} \\ \nu_0 &= 0 \quad \nu_1 = 1 - P_0 \\ \frac{d\nu_i}{dt} &= (1 - P_{i-1})\frac{d\nu_{i-1}}{dt} \forall i \in \mathbb{Z}, i > 1 \end{aligned} \tag{2}$$

Where the probability of a single nucleus with not decaying after a time t is

$$P_i = e^{-\frac{t}{\lambda_i}} \tag{3}$$

Thus we make the substitutions to write the following recursive formula

$$m_i = -\int_0^t P_i (1 - P_{i-1}) \frac{d\nu_{i-1}}{dt} dt \tag{4}$$

Assuming there is no source of new 0-th generation nuclei

$$m_1 = \int_0^t e^{\frac{t}{-\lambda_1}} d\nu_0 = \int_0^t e^{\frac{t}{-\lambda_1}} \frac{d}{dt}(1 - P_0)dt = -\int_0^t e^{\frac{t}{-\lambda_1}} \frac{dP_0}{dt}dt = \frac{1}{\lambda_0}\int_0^t e^{\frac{t}{-\lambda_1}} e^{\frac{t}{-\lambda_0}} dt = \frac{\lambda_1}{\lambda_1 + \lambda_0}(1 - e^{-t(\frac{1}{\lambda_0}+\frac{1}{\lambda_1})}) \tag{5}$$

For more insight on how to write the general case, we first compute using 2

$$\begin{aligned} m_2 &= \int_0^t P_2 d\nu_2 = \int_0^t e^{\frac{t}{-\lambda_2}} d\nu_2 \\ d\nu_2 &= (1 - P_1)\frac{d\nu_1}{dt}dt \\ m_2 &= \int_0^t e^{\frac{t}{-\lambda_2}}(1 - P_1)\frac{d\nu_1}{dt}dt = -\int_0^t e^{\frac{t}{-\lambda_2}}(1 - P_1)\frac{dP_0}{dt}dtdt \\ m_2 &= \frac{1}{\lambda_0}\int_0^t e^{\frac{t}{-\lambda_2}}(1 - e^{\frac{t}{-\lambda_1}})e^{\frac{t}{-\lambda_0}} dtdt = \frac{1}{\lambda_0}\int_0^t e^{t(\frac{1}{-\lambda_2}+\frac{1}{-\lambda_0})}(1 - e^{\frac{t}{-\lambda_1}})dtdt \\ m_2 &= \lambda_2(\lambda_1 \frac{e^{-t(\frac{1}{\lambda_2}+\frac{1}{\lambda_1}+\frac{1}{\lambda_0})} - 1}{\lambda_2\lambda_0 + \lambda_2\lambda_1 + \lambda_0\lambda_1} + \frac{1 - e^{-t(\frac{1}{\lambda_2}+\frac{1}{\lambda_0})}}{\lambda_0 + \lambda_2}) \end{aligned} \tag{6}$$

Using the results for dn_i from the properties of radioactive decay (2)

$$\frac{d\nu_{i+1}}{dt} = (1 - P_i)\frac{d\nu_i}{dt} = (1 - P_i)(1 - P_{i-1})\frac{d\nu_{i-1}}{dt} \tag{7}$$

Which suggests the following formula

$$\frac{d\nu_{i+1}}{dt} = \frac{d\nu_1}{dt}\prod_{k=2}^{i}(1 - P_k) = (1 - P_1)\frac{dP_0}{dt}\prod_{k=2}^{i}(1 - e^{t\frac{1}{-\lambda_k}}) = -\frac{1}{\lambda_0}e^{\frac{t}{-\lambda_0}}\prod_{k=1}^{i}(1 - e^{t\frac{1}{-\lambda_k}}) \tag{8}$$

We will now prove the following using induction

$$\frac{d\nu_i}{dt} = -\frac{1}{\lambda_0}e^{\frac{t}{-\lambda_0}}\prod_{k=1}^{i-1}(1 - e^{t\frac{1}{-\lambda_k}})\forall i \in \mathbb{Z}, i > 1 \tag{9}$$

Thus we have for the general case using the results for the daughter nuclei being produced (Equation 9) and the formula for the moles of the i-th generation nucleus (the top most equation in 2)

$$\begin{aligned} m_i &= \int_0^t P_i\frac{d\nu_i}{dt}dt = -\int_0^t e^{t\frac{1}{-\lambda_i}}\frac{1}{\lambda_0}e^{\frac{t}{-\lambda_0}}\prod_{k=1}^{i-1}(1 - e^{t\frac{1}{-\lambda_k}})dt \\ m_i &= -\frac{1}{\lambda_0}\int_0^t e^{t(\frac{1}{-\lambda_0}+\frac{1}{-\lambda_1})}\prod_{k=1}^{i-1}(1 - e^{t\frac{1}{-\lambda_k}})dt \end{aligned} \tag{10}$$

For the decay rate, we have

$$\begin{aligned} r_i &= \frac{d}{dt}\int_0^t 1 - P_i\frac{d\nu_i}{dt}dt = -\frac{1}{\lambda_0}\frac{d}{dt}\int_0^t(1 - e^{t\frac{1}{-\lambda_i}})e^{\frac{t}{-\lambda_0}}\prod_{k=1}^{i-1}(1 - e^{t\frac{1}{-\lambda_k}})dt \\ r_i &= -\frac{1}{\lambda_0}e^{\frac{t}{-\lambda_0}}\prod_{k=1}^{i}(1 - e^{t\frac{1}{-\lambda_k}}) \end{aligned} \tag{11}$$