New analytic rotations for bifactor modeling and metric invariance in Exploratory Factor Analysis

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Only two kinds of factor rotation?

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A SIMPLE GENERAL PROCEDURE FOR ORTHOGONAL ROTATION

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A SIMPLE GENERAL METHOD FOR OBLIQUE ROTATION

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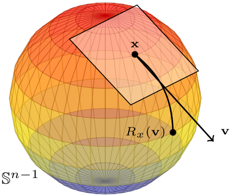
The gradient projection method



Rotated (Λ) = Unrotated $(\mathbf{U})$ $\times$ Rotation matrix $(\mathbf{X})$

$$\begin{pmatrix} \mathbf{0.37} & 0.07 & 0.00 \\ \mathbf{0.57} & 0.01 & 0.02 \\ \mathbf{0.53} & -0.03 & -0.02 \\ -0.02 & \mathbf{0.47} & -0.03 \\ 0.01 & \mathbf{0.44} & 0.01 \\ 0.01 & \mathbf{0.57} & 0.01 \\ -0.05 & 0.00 & \mathbf{0.52} \\ -0.02 & 0.00 & \mathbf{0.56} \\ 0.04 & 0.00 & \mathbf{0.63} \end{pmatrix} = \begin{pmatrix} 0.04 & -0.26 & -0.27 \\ 0.09 & -0.32 & -0.46 \\ 0.04 & -0.27 & -0.46 \\ -0.07 & -0.38 & 0.28 \\ -0.01 & -0.37 & 0.23 \\ -0.02 & -0.48 & 0.31 \\ 0.51 & 0.02 & 0.09 \\ 0.56 & 0.01 & 0.07 \\ 0.63 & -0.01 & 0.03 \end{pmatrix} \begin{pmatrix} -0.13 & -0.06 & 1 \\ 0.55 & -0.83 & 0.01 \\ 0.82 & 0.56 & 0.10 \end{pmatrix}$$

Orthogonal Projection

$$P(\mathbf{G}) = \mathbf{G} - 0.5\left(\mathbf{X}\mathbf{G}^\top + \mathbf{X}^\top\mathbf{G}\right)\mathbf{X}$$

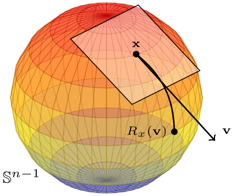
$$\mathbf{X}^\top\mathbf{X} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Oblique Projection

$$P(\mathbf{G}) = \mathbf{G} - \mathbf{X}\mathrm{diag}(\mathbf{X}^\top\mathbf{G})$$

$$\mathbf{X}^\top\mathbf{X} = \begin{pmatrix} 1 & .35 & .44 \\ .35 & 1 & .14 \\ .44 & .14 & 1 \end{pmatrix}$$

The Partially Oblique Manifold



$$\text{Rotated } (\Lambda) \quad = \quad \text{Unrotated } (\mathbf{U}) \times \text{Rotation matrix } (\mathbf{X})$$

$$\begin{pmatrix} \mathbf{0.37} & 0.07 & 0.00 \\ \mathbf{0.57} & 0.01 & 0.02 \\ \mathbf{0.53} & -0.03 & -0.02 \\ -0.02 & \mathbf{0.47} & -0.03 \\ 0.01 & \mathbf{0.44} & 0.01 \\ 0.01 & \mathbf{0.57} & 0.01 \\ -0.05 & 0.00 & \mathbf{0.52} \\ -0.02 & 0.00 & \mathbf{0.56} \\ 0.04 & 0.00 & \mathbf{0.63} \end{pmatrix} = \begin{pmatrix} 0.04 & -0.26 & -0.27 \\ 0.09 & -0.32 & -0.46 \\ 0.04 & -0.27 & -0.46 \\ -0.07 & -0.38 & 0.28 \\ -0.01 & -0.37 & 0.23 \\ -0.02 & -0.48 & 0.31 \\ 0.51 & 0.02 & 0.09 \\ 0.56 & 0.01 & 0.07 \\ 0.63 & -0.01 & 0.03 \end{pmatrix} \begin{pmatrix} -0.13 & -0.06 & 1 \\ 0.55 & -0.83 & 0.01 \\ 0.82 & 0.56 & 0.10 \end{pmatrix}$$

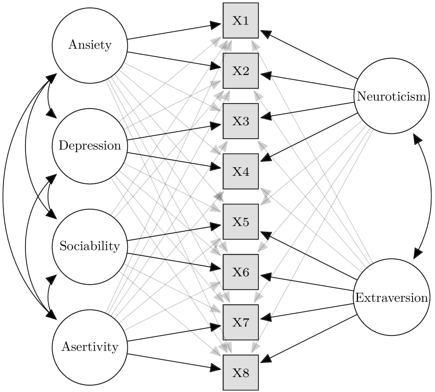
Partially Oblique Projection

$$P(\mathbf{G}) = \mathbf{G} - \mathbf{X}\left(I \circ \mathbf{V}\left(\frac{\mathbf{V}^\top \mathbf{Q} \mathbf{V}}{\mathbf{1}\mathbf{d}^\top + \mathbf{d}\mathbf{1}^\top}\right)\mathbf{V}^\top\right)$$

$$\mathbf{X}^\top \mathbf{X} = \begin{pmatrix} 1 & \mathbf{0} & .44 \\ \mathbf{0} & 1 & .14 \\ .44 & .14 & 1 \end{pmatrix}$$

Exploratory Bi-factor Model with Multiple General factors

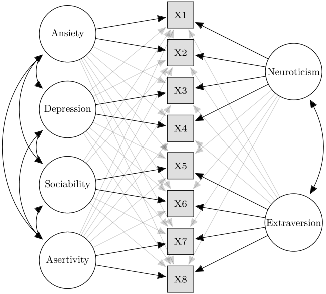
- [✓] Disentangle the variance due to the general and specific factors.
- [✓] Use any rotation criteria that you want (i.e., target, oblimin, etc.).



Factor Correlation Matrix

$$\begin{pmatrix} 1 & \mathbf{.30} & 0 & 0 & 0 & 0 \\ \mathbf{.30} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \mathbf{.30} & \mathbf{.30} & \mathbf{.30} \\ 0 & 0 & \mathbf{.30} & 1 & \mathbf{.30} & \mathbf{.30} \\ 0 & 0 & \mathbf{.30} & \mathbf{.30} & 1 & \mathbf{.30} \\ 0 & 0 & \mathbf{.30} & \mathbf{.30} & \mathbf{.30} & 1 \end{pmatrix}$$

Estimation with the `bifactor` package

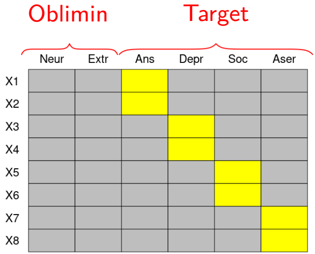



```
efast(Sigma, nfactors = 6, estimator = "ULS",
      rotation = "target", Target = Target,
      projection = "poblq", oblq_factors = c(2, 4))

rotate(Lambda, rotation = "target", Target = Target,
       projection = "poblq", oblq_factors = c(2, 4))
```

Mixed rotation criteria

[✓] Use Oblimin for the general factors and Target for the specific factors.




```
efast(Sigma, nfactors = 6, estimator = "ULS", Target = Target,
      projection = "poblq", oblq_factors = c(2, 4),
      rotation = c("oblimin", "target"), blocks = list(1:2, 3:6))
```

Multitrait-Multimethod analysis

$$\begin{pmatrix}
\textbf{Trait} & \textbf{Trait} & \textbf{Trait} & \textbf{Method} & \textbf{Method} \\
\mathbf{0.37} & 0.07 & 0.00 & \mathbf{0.56} & 0.00 \\
\mathbf{0.57} & 0.01 & 0.02 & \mathbf{0.56} & 0.00 \\
\mathbf{0.53} & 0.03 & 0.02 & 0.00 & \mathbf{0.51} \\
\mathbf{0.49} & 0.02 & 0.03 & 0.00 & \mathbf{0.43} \\
0.02 & \mathbf{0.47} & 0.03 & \mathbf{0.61} & 0.00 \\
0.01 & \mathbf{0.44} & 0.01 & \mathbf{0.45} & 0.00 \\
0.01 & \mathbf{0.57} & 0.01 & 0.00 & \mathbf{0.39} \\
0.08 & \mathbf{0.34} & 0.08 & 0.00 & \mathbf{0.55} \\
0.05 & 0.00 & \mathbf{0.52} & \mathbf{0.47} & 0.00 \\
0.02 & 0.00 & \mathbf{0.56} & \mathbf{0.67} & 0.00 \\
0.04 & 0.00 & \mathbf{0.63} & 0.00 & \mathbf{0.44} \\
0.06 & 0.07 & \mathbf{0.58} & 0.00 & \mathbf{0.50}
\end{pmatrix}$$

Loading Matrix

$$\begin{pmatrix}
 & \textbf{Trait} & \textbf{Trait} & \textbf{Trait} & \textbf{Method} & \textbf{Method} \\
\textbf{Trait} & 1 & \mathbf{.33} & \mathbf{.45} & 0 & 0 \\
\textbf{Trait} & \mathbf{.33} & 1 & \mathbf{.25} & 0 & 0 \\
\textbf{Trait} & \mathbf{.45} & \mathbf{.25} & 1 & 0 & 0 \\
\textbf{Method} & 0 & 0 & 0 & 1 & \mathbf{.44} \\
\textbf{Method} & 0 & 0 & 0 & \mathbf{.44} & 1
\end{pmatrix}$$

Factor Correlation Matrix

```
efast(Sigma, nfactors = 5, estimator = "ULS",
      rotation = "target", Target = Target,
      projection = "poblq", oblq_factors = c(3, 2))
```

Model identification in CFA

[✗] `warning: covariance matrix of latent variables is not positive definite`

[✓] Solution: Parameterize the covariance matrix of latent variables as $\mathbf{X}^\top \mathbf{X}$ and optimize it on the partially oblique manifold.

$$\begin{pmatrix} 1 & \mathbf{.33} & \mathbf{.45} & 0 & \mathbf{.68} \\ \mathbf{.33} & 1 & \mathbf{.25} & 0 & \mathbf{.77} \\ \mathbf{.45} & \mathbf{.25} & 1 & \mathbf{.56} & 0 \\ 0 & 0 & \mathbf{.56} & 1 & \mathbf{.44} \\ \mathbf{.68} & \mathbf{.77} & 0 & \mathbf{.44} & 1 \end{pmatrix}$$

Factor correlation matrix optimized
on the Partially Oblique manifold

Summary

Applications of the Partially Oblique manifold:

[✓] Uncorrelate the general and specific factors in Exploratory Bi-factor analysis with either one or multiple general factors.

[✓] Uncorrelate trait and method factors in multitrait-multimethod analysis.

[✓] Identify CFA models where the correlations between the latent factors are, at least, positive semi-definite.

[✓] Similar rotations can be developed to obtain a method for metric invariance in EFA.

Thank you

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https://github.com/Marcosjnez/bifactor

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