Dynamic output-feedback design for generalized Lyapunov inequalities

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Abstract

Differential Lyapunov matrix inequalities with matrix inertia constraints have recently emerged as a useful tool for studying properties of nonlinear systems such as k-contraction and p-dominance. However, efficient and systematic methods for designing controllers that ensure the closed-loop system satisfies these generalized Lyapunov inequalities remain underdeveloped. In this work, we propose solutions based on linear matrix inequalities (LMIs) for the design of linear dynamic output-feedback controllers addressing this challenge. Our results focus on partially linear systems whose nonlinearities satisfy a generic quadratic abstraction. Additionally, we introduce methods to handle the difficulties posed by matrix inertia constraints. Finally, we demonstrate the effectiveness and value of the newly proposed conditions by applying them within the frameworks of 2-contraction and extremum control for non-convex optimization.

Keywords: Generalized Lyapunov inequality, LMI, Controller design, Matrix inertia, Nonlinear systems, Contraction, k-contraction.

1. Introduction

Lyapunov matrix inequalities and their differential forms have proven extremely useful in studying asymptotic properties of control systems [19, 17, 26]. Their solution offers a natural quadratic Lyapunov function, which can be used to prove convergence of trajectories to unique ones and equilibrium points. Recently rediscovered, a generalized form of these Lyapunov inequalities appeared as an effective tool for studying complex behaviors of partially stable systems [37, 13, 10]. Here, the strict positivity constraint on the inequality solution is dropped and substituted with one on its number of positive, negative, and zero eigenvalues, i.e., on its matrix inertia. Unfortunately, while classical Lyapunov inequalities can be solved via semidefinite programming and linear matrix inequality (LMI) solvers [16, 14, 15], the inertia constraint does not allow for a straightforward application of existing methods. This complexity is exacerbated when the matrix inequality includes terms aimed at control design. Indeed, existing results must rely on projections, conditions on matrix inverses, or restrictive assumptions on the uncontrolled system to ensure a solution with proper inertia is found [11, 35].

In this work, we aim to propose easily solvable LMI conditions guaranteeing the satisfaction of differential Lyapunov-like inequalities under inertia constraints. We focus on a class of nonlinear systems whose nonlinear terms satisfy a quadratic abstraction. This type of system includes the wide range of Lur'e and Persidskii systems [12] as a special case. For such a class of systems, we start by providing conditions to recover solutions to generalized Lyapunov inequalities on the open-loop dynamics by building on [42]. Then, we extend the result to controlled systems. Inspired by [36, 16], we propose LMI conditions for the design of dynamic output-feedback controllers guaranteeing the satisfaction of a generalized Lyapunov inequality by the closed loop. Different from [35], our results do not rely on projections or conditions on matrix inverses that may be hard to include in the optimization problem. Additionally, different from [11] (which only considers the state-feedback case), we do not require the open-loop system to satisfy a generalized Lyapunov condition beforehand. As a second contribution, we discuss methods to impose a specific matrix inertia without requiring additional constraints on the open-loop dynamics. As a third contribution, we propose a result on controller design under multiple Lyapunov-like conditions with different inertia constraints. This is motivated by recent advances in methods to impose interesting properties such as k-contraction, which may require simultaneously solving multiple inequalities [10]. Therefore, the final contribution of the paper is to develop a framework for 2-contraction output-feedback design and apply it to the problem of extremum control with non-convex objectives.

Notation: $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{N} := \{0, 1, 2, \ldots\}$. $|\cdot|$ denotes the standard Euclidean norm. Given $x \in \mathbb{R}^n$,

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$y \in \mathbb{R}^m$, we set $\mathrm{col}(x, y) := (x^\top, y^\top)^\top$. The inertia of a matrix P is defined by the triplet of integers $\mathrm{In}(P) := (\pi_-(P), \pi_0(P), \pi_+(P))$, where $\pi_-(P)$, $\pi_+(P)$ and $\pi_0(P)$ denote the numbers of eigenvalues of P with negative, positive and zero real part, respectively, counting multiplicities. $A \succ 0$ (resp. $A \succeq 0$) denotes A being a positive definite (resp. positive semidefinite) matrix. We denote $\mathrm{tr}\, A$ as the trace of a matrix A. $\bar{\sigma}(\cdot), \underline{\sigma}(\cdot)$ stand for the maximum and minimum singular values of their arguments, respectively. For any matrix $Q \in \mathbb{R}^{n\times n}$ we denote $\lambda_i(Q)$ for $i \in \{1, \dots, n\}$ as its eigenvalues, ordered such that $\Re(\lambda_1(Q)) \geq \Re(\lambda_2(Q)) \geq \cdots \geq \Re(\lambda_n(Q))$. For a square matrix $A \in \mathbb{R}^{n\times n}$, we define $\mathrm{He}\{A\} := A^\top + A$.

2. Quadratic abstractions for generalized Lyapunov inequalities

2.1. Framework

From a general perspective, we consider continuous-time nonlinear systems of the form

$$\dot{x} = f(x)$$

where $x \in \mathbb{R}^{n_x}$ and $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ is a sufficiently smooth vector field. The main goal of this work is to derive a finite set of LMI-based sufficient conditions for solving generalized Lyapunov inequalities of the form

$$\frac{\partial f}{\partial x}(x)^\top P + P\frac{\partial f}{\partial x}(x) \prec 2\mu P, \quad \forall x \in \mathcal{X}, \tag{1}$$

with $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ being an arbitrary set, $\mu \in \mathbb{R}$ and $P = P^\top$ an invertible matrix with inertia $\mathrm{In}(P) = (p, 0, n_x - p)$ where $p \in \{0, \dots, n_x\}$. This objective is motivated by recent advances in k-contraction [10], p-dominance analysis [13] and systems that are invariant with respect to cones [34]. Verifying these properties necessitates satisfying inequalities of the form (1), yet practical and computationally efficient techniques for achieving this remain undeveloped.

To derive efficient techniques to validate (1), we restrict the class of system to ones whose vector field is represented by a partially linear function, that is

$$f(x) := \mathbf{A}x + \mathbf{D}\phi(v), \quad v = \mathbf{E}x, \tag{2}$$

where $v \in \mathbb{R}^{n_v}$ and $\phi : \mathbb{R}^{n_v} \to \mathbb{R}^{n_v}$ is a sufficiently smooth function. The generic matrices $\mathbf{A}, \mathbf{D}, \mathbf{E}$ will be specified in each section depending on our objective (i.e., system analysis or controller design). Naturally, under this structural assumption, (1) reduces to satisfying, for all $x \in \mathcal{X}$,

$$\left(\mathbf{A} + \mathbf{D}\frac{\partial \phi}{\partial x}(v)\right)^\top P + P\left(\mathbf{A} + \mathbf{D}\frac{\partial \phi}{\partial x}(v)\right) \prec 2\mu P. \tag{3}$$

We also restrict the class of nonlinearities to the one satisfying a quadratic differential constraint, denoted as a quadratic abstraction. More specifically, our results will be grounded on the existence of matrices $R = R^\top, Q = Q^\top$ and S such that

$$\begin{pmatrix} \mathrm{I}_{n_x} \\ \frac{\partial \phi}{\partial x}(v) \end{pmatrix}^\top \begin{pmatrix} R & S^\top \\ S & Q \end{pmatrix} \begin{pmatrix} \mathrm{I}_{n_x} \\ \frac{\partial \phi}{\partial x}(v) \end{pmatrix} \preceq 0 \tag{4}$$

holds for all $v \in \mathcal{V}$ where $\mathcal{V} := \{v \in \mathbb{R}^{n_v} : v = \mathbf{E}x, x \in \mathcal{X}\}$. We recall that such a class of nonlinearities includes common and interesting family of functions, such as (shifted) monotonic and differentially sector-bounded ones [42]. For more details on how to verify such a quadratic abstraction we refer to Section 2.5.

We are now ready to recall the first result related to (1). This result proposes a finite set of LMI conditions for autonomous systems, thus recovering the result in [42, Theorem 2]. Yet, in view of the results in Section 3, we allow the matrices in (4) to be decision variables.

2.2. System analysis

We start by studying conditions for uncontrolled systems. Hence, we consider nonlinear dynamics (2) with matrices

$$\mathbf{A} = A, \quad \mathbf{D} = D, \quad \mathbf{E} = E, \tag{5}$$

where $A \in \mathbb{R}^{n_x\times n_x}, D \in \mathbb{R}^{n_x\times n_v}$ and $E \in \mathbb{R}^{n_v\times n_x}$. With a minor reformulation, we now present the analysis result from the conference version of the paper.

Proposition 1. *[42, Theorem 2]: Consider the vector field f in (2) with (5) and suppose there exist matrices $R = R^\top, S, Q = Q^\top$, matrices $\Gamma_1, \Gamma_2 \in \mathbb{R}^{n_x\times n_x}, \Gamma_3 \in \mathbb{R}^{n_v\times n_x}$, a scalar $\beta \in \mathbb{R}$ and a nonsingular matrix $\Sigma = \Sigma^\top \in \mathbb{R}^{n_x\times n_x}$ with inertia $\mathrm{In}(\Sigma) = (p, 0, n_x - p)$ where $p \in \{1, \dots, n_x\}$ such that (6)[1] holds. Then, for all sufficiently smooth $\phi : \mathbb{R}^{n_v} \to \mathbb{R}^{n_v}$ such that (4) holds for all $v \in \mathcal{V}$, inequality (3) holds for all $x \in \mathcal{X}$ with $P = \Sigma$ and $\mu = \beta$.*

Notice that the matrices R, S and Q of the quadratic abstraction (4) appear linearly in inequality (6). Hence, if the nonlinearity ϕ is known, these matrices can be fixed before solving the inequality. However, if the nonlinearity ϕ is unknown, the affine relation allows R, S, Q to be unknown variables part of the solution to (6). As such, the solution will also provide a class of systems (nonlinearities) for which condition (1) holds.

We remark that [42, Theorem 2] is presented under the assumption of multiple quadratic abstractions being satisfied. This can be practical to describe common nonlinearities, e.g. (shifted) monotonic ones [42, Section III.B]. The same result can be recovered from Proposition 1. If $q > 1$ quadratic abstractions with relative matrices R_i, S_i, Q_i are known to hold, (4) holds with

$$R := \sum_{i=1}^{q} \alpha_i R_i \quad S := \sum_{i=1}^{q} \alpha_i S_i, \quad Q := \sum_{i=1}^{q} \alpha_i Q_i, \tag{7}$$

[1] Given at the top of the next page.

$$\Psi_1 := \begin{pmatrix} -2\beta\Sigma + \mathrm{He}\{\Gamma_1 A\} - R & \Sigma - \Gamma_1 + A^\top \Gamma_2^\top & \Gamma_1 D + A^\top \Gamma_3^\top - S^\top \\ \star & -\mathrm{He}\{\Gamma_2\} & \Gamma_2 D - \Gamma_3^\top \\ \star & \star & \mathrm{He}\{\Gamma_3 D\} - Q \end{pmatrix} \prec 0 \tag{6}$$

where $\alpha_i > 0$ for $i \in \{1, \dots, q\}$ are arbitrary positive weights. Due to the linearity of (6) in the abstraction matrices, these weights α_i can be imposed as variables to be solved in the optimization problem.

To further reduce conservativeness, one can consider (6) paired with

$$\begin{pmatrix} R & S^\top \\ S & Q \end{pmatrix} - \sum_{i=1}^{q} \alpha_i \begin{pmatrix} R_i & S_i^\top \\ S_i & Q_i \end{pmatrix} + \mathrm{He}\left\{ \begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \end{pmatrix} \begin{pmatrix} \frac{\partial \phi}{\partial x}(v) & -\mathrm{I}_{n_x} \end{pmatrix} \right\} \preceq 0 \tag{8}$$

for all $v \in \mathcal{V}$, where $\alpha_i > 0$, R, S, Q and Υ_1, Υ_2 are decision variables. Right and left multiplication of the above inequality by $\mathrm{col}(\mathrm{I}_{n_x}, \frac{\partial \phi}{\partial x}(v))$ shows that $i = 1, \dots, q$ known quadratic abstractions of the form (4) and (8) imply a quadratic abstraction in R, S, Q is also satisfied. The main advantage of (8) is the disjunction of the structure of matrices R, S, Q in (6) from the structure of the matrices R_i, S_i, Q_i of the known abstractions.

Remark 1. *Inequality* (6) *has a bilinear term generated by the variable* β*. Nonetheless,*(6) *can be solved as a generalized eigenvalue problem (GEVP) in* β*. Therefore,* β *can be found via iterative methods such as bisection.*

We now move to the analysis of controlled systems, which represents the first main contribution of the paper. More precisely, we focus on the problem of designing linear dynamic output-feedback controllers guaranteeing the closed-loop system satisfies an inequality of the form (1).

2.3. Control design

In this section, we focus on nonlinear controlled systems of the form

$$\begin{aligned} \dot{z} &= Az + D\phi(v) + Bu \\ y &= Cz + G\phi(v), \quad v = Ez \\ \dot{\xi} &= M\xi + Ny \\ u &= Ky + L\xi \end{aligned} \tag{9}$$

where $z \in \mathbb{R}^{n_z}$, $\xi \in \mathbb{R}^{n_\xi}$, $u \in \mathbb{R}^{n_u}$, $y \in \mathbb{R}^{n_y}$, $v \in \mathbb{R}^{n_v}$ and a smooth function $\phi : \mathbb{R}^{n_v} \to \mathbb{R}^{n_v}$. By defining the state in (2) as the extended state $x = \mathrm{col}(z, \xi)$, (9) can be reformulated as (2) with

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} A + BKC & BL \\ NC & M \end{pmatrix}, \\ \mathbf{D} &= \begin{pmatrix} D + BKG \\ NG \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} E & 0 \end{pmatrix}. \end{aligned} \tag{10}$$

The aim of this section is to design suitable matrices $M \in \mathbb{R}^{n_\xi \times n_\xi}$, $N \in \mathbb{R}^{n_\xi \times n_y}$, $K \in \mathbb{R}^{n_u \times n_y}$ and $L \in \mathbb{R}^{n_u \times n_\xi}$ such that (3) holds for some real number $\mu \in \mathbb{R}$, some symmetric matrix P with inertia $\mathrm{In}(P) = (p, 0, n_z + n_\xi - p)$ where $p \in \{1, \dots, n_z + n_\xi\}$ and for all x in some set $\mathcal{X} \subseteq \mathbb{R}^{n_z + n_\xi}$.

A general approach for the design of such linear gains appeared in [35]. However, the presence of projections, the necessity of verifying positivity conditions involving variable inverses and the lack of strategies for constraining the inertia make the result harder to implement in practice. An alternative and more naive strategy for solving (3) is to invoke Proposition 1 on (9). Nonetheless, the unstructured multipliers $\Gamma_1, \Gamma_2, \Gamma_3$ introduce bilinear terms in (6) due to the presence of M, N, K, L in (10).

Remark 2. *A way to remove the bilinear relations created by the direct use of Proposition 1 with* (10) *is to restrict the set of admissible solutions to structured multipliers with favorable properties, e.g., by forcing* $\Gamma_1 = \Gamma_2, \Gamma_3 = 0$*. This allows defining changes of variables separating the parameters in* $\mathbf{A}, \mathbf{D}$*. For instance, in the case of* (6) *applied on the closed loop* (2) *with* (10) *we can rewrite*

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ \mathrm{I}_{n_\xi} & 0 \end{pmatrix} \begin{pmatrix} M & N \\ L & K \end{pmatrix} \begin{pmatrix} 0 & \mathrm{I}_{n_y} \\ C & 0 \end{pmatrix}, \\ \mathbf{D} &= \begin{pmatrix} D \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ \mathrm{I}_{n_\xi} & 0 \end{pmatrix} \begin{pmatrix} N \\ K \end{pmatrix} G, \end{aligned}$$

and structure Γ_1 *such that* $\left(\begin{smallmatrix} 0 & \mathrm{I}_{n_y} \\ C & 0 \end{smallmatrix}\right) \Gamma_1^{-1} = 0$*, thus allowing the definition of a suitable change of variable for* G*. However, such a choice introduces further conservatism for two main reasons: i)* Γ_1 *needs to be structured, ii) differently from the scenario of positive definite* P *as in [32], we are not aware of results proving generality of the selection* $\Gamma_1 = \Gamma_2, \Gamma_3 = 0$ *for sign-indefinite invertible symmetric matrices* P*.*

Given the drawbacks mentioned above, we take inspiration from [36] and propose a coordinate change that transforms the control design problem into an LMI. Therefore, we restrict our design to the case $n_\xi = n_z$ (or, equivalently, $n_x = 2n_z$). Nonetheless, we remark that the result can be extended to different scenarios at the price of uniqueness of the recovered matrices [36]. The main result of the section thus represents an extension of [16] in two directions. First, we are not restricted to positive definite matrices P. Second, we embed general quadratic abstraction, encompassing monotonicity as a particular scenario.

We highlight that quadratic abstractions of the form (4) are typically defined over the open-loop system, while (9) involves the extended closed loop dynamics. We will then show that the extension of quadratic abstractions from open-loop to closed loop dynamics is non-unique and that this flexibility can be exploited during the design problem. Specifically, consider a set $\mathcal{Z} \in \mathbb{R}^{n_z}$ and suppose the nonlinearity ϕ satisfies, for some matrices $R_z = R_z^\top, Q_z = Q_z^\top$ and S_z,

$$\begin{pmatrix} \mathrm{I}_{n_z} \\ \frac{\partial \phi}{\partial z}(v) \end{pmatrix}^\top \begin{pmatrix} R_z & S_z^\top \\ S_z & Q_z \end{pmatrix} \begin{pmatrix} \mathrm{I}_{n_z} \\ \frac{\partial \phi}{\partial z}(v) \end{pmatrix} \preceq 0 \tag{11}$$

for all $v \in \mathcal{V}$ where $\mathcal{V} := \{v \in \mathbb{R}^{n_v} : v = Ez, z \in \mathcal{Z}\}$. The following result presents a family of quadratic abstractions satisfied by the extended system (9) with (10).

Lemma 1. *Consider* (9) *and let* (11) *hold for all $v \in \mathcal{V}$. Then, for all square symmetric $\mathbf{R}_{22} \preceq 0 \in \mathbb{R}^{n_z \times n_z}, \mathbf{S}_2 \in \mathbb{R}^{n_z \times n_v}$ and all $\mathbf{R}_{12} \in \mathbb{R}^{n_z \times n_\xi}$ such that*

$$(\mathrm{I}_{n_x} - \mathbf{R}_{22}\mathbf{R}_{22}^\dagger)\left(\mathbf{R}_{12} + \mathbf{S}_2^\top \frac{\partial \phi}{\partial z}(v)\right) = 0 \tag{12}$$

holds for all $v \in \mathcal{V}$ with $\mathbf{R}_{22}^\dagger$ such that $\mathbf{R}_{22}\mathbf{R}_{22}^\dagger\mathbf{R}_{22} = \mathbf{R}_{22}$, the quadratic abstraction (4) *holds with*

$$R = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12}^\top \\ \mathbf{R}_{12} & \mathbf{R}_{22} \end{pmatrix}, \qquad S = \begin{pmatrix} \mathbf{S}_1 & \mathbf{S}_2 \end{pmatrix}, \qquad Q = \mathbf{Q}, \tag{13a}$$

where

$$\begin{aligned} \mathbf{R}_{11} &= R_z + \mathbf{R}_{12}^\top \mathbf{R}_{22}^\dagger \mathbf{R}_{12}, \\ \mathbf{S}_1 &= S_z + \mathbf{S}_2 \mathbf{R}_{22}^\dagger \mathbf{R}_{12}, \\ \mathbf{Q} &= Q_z + \mathbf{S}_2 \mathbf{R}_{22}^\dagger \mathbf{S}_2^\top. \end{aligned} \tag{13b}$$

The proof of Lemma 1 is postponed to Appendix A.

Remark 3. *The change of variables $\mathbf{R}_{12} = \mathbf{R}_{22}\overline{\mathbf{R}}_{12}$, $\mathbf{S}_2 = \overline{\mathbf{S}}_2\mathbf{R}_{22}$ with arbitrary matrices $\overline{\mathbf{R}}_{12}, \overline{\mathbf{S}}_2$ allows avoiding the computation of the generalized inverse in* (13b) *and* (12). *However, such a choice introduces bilinear terms in* (13a). *Therefore, we now highlight some interesting scenarios for Lemma 1. The first appears when $\mathbf{R}_{22} \prec 0$. In this case, $\mathbf{R}_{22}$ is invertible (i.e., $\mathbf{R}_{22}^\dagger = \mathbf{R}_{22}^{-1}$) and* (12) *is always satisfied. Similarly, for the scenario $\mathbf{R}_{22} \preceq 0$, interesting cases appear when $\mathbf{R}_{12} = 0$ and $\mathbf{S}_2 = 0$ since* (12) *always holds,* (13b) *does not involve generalized inverse computation and it directly relates* (13a) *to* (11).

The degrees of freedom offered by Lemma 1 give rise to a useful coordinate change for controller design. We present such a variable change in the following theorem, which constitutes one of the main results of our paper.

Theorem 1. *Consider the vector field f in* (2) *with* (10) *and assume there exist matrices $\widehat{R}_{11} = \widehat{R}_{11}^\top, \widehat{R}_{12}, \widehat{R}_{22} = \widehat{R}_{22}^\top, \widehat{S}_1, \widehat{S}_2, \widehat{Q} = \widehat{Q}^\top$, a scalar $\beta \in \mathbb{R}$, symmetric matrices $X = X^\top, Y = Y^\top \in \mathbb{R}^{n_z \times n_z}$ and matrices $\widehat{K} \in \mathbb{R}^{n_u \times n_y}, \widehat{L} \in \mathbb{R}^{n_u \times n_z}, \widehat{M} \in \mathbb{R}^{n_z \times n_z}, \widehat{N} \in \mathbb{R}^{n_z \times n_y}$ such that:*

i) The matrix $\widehat{R} := \left(\begin{smallmatrix} \widehat{R}_{11} & \widehat{R}_{12}^\top \\ \star & \widehat{R}_{22} \end{smallmatrix}\right)$ is negative semidefinite;

ii) Inequality (14)[2] *holds;*

iii) The matrix $\Phi = \left(\begin{smallmatrix} Y & \mathrm{I}_{n_z} \\ \mathrm{I}_{n_z} & X \end{smallmatrix}\right)$ satisfies $\mathrm{In}(\Phi) = (p, 0, 2n_z - p)$, for some integer $p \in \{0, \ldots, 2n_z\}$.

Then, there exist square invertible matrices $U, V \in \mathbb{R}^{n_z \times n_z}$ such that $UV^\top = \mathrm{I}_{n_z} - XY$. Moreover, let

$$\Pi_1 := \begin{pmatrix} Y & \mathrm{I}_{n_z} \\ V^\top & 0 \end{pmatrix}, \quad \Pi_2 := \begin{pmatrix} \mathrm{I}_{n_z} & X \\ 0 & U^\top \end{pmatrix}. \tag{15}$$

Then, for all sufficiently smooth functions $\phi : \mathbb{R}^{n_v} \to \mathbb{R}^{n_v}$ such that inequalities (11) *and* (12) *hold for all $v \in \mathcal{V}$ with* (13b) *and*

$$\begin{aligned} \mathbf{R}_{11} &= \widehat{R}_{22}, & \mathbf{S}_1 &= \widehat{S}_2, \\ \mathbf{R}_{12} &= V^{-1}(\widehat{R}_{12}^\top - Y\widehat{R}_{22}), & \mathbf{S}_2 &= (\widehat{S}_1 - \widehat{S}_2 Y)V^{-\top}, \\ \mathbf{R}_{22} &= V^{-1}\left(\begin{smallmatrix} \mathrm{I}_{n_z} \\ -Y \end{smallmatrix}\right)^\top \widehat{R} \left(\begin{smallmatrix} \mathrm{I}_{n_z} \\ -Y \end{smallmatrix}\right) V^{-\top}, & \mathbf{Q} &= \widehat{Q}, \end{aligned} \tag{16}$$

the differential inequality (3) *holds for all $x \in \mathcal{X} = \mathcal{Z} \times \mathbb{R}^{n_z}$ with $P = \Pi_2 \Pi_1^{-1}$, $\mu = \beta$ and*

$$\begin{aligned} K &= \widehat{K}, \\ L &= (\widehat{L} - \widehat{K}CY)V^{-\top}, \\ N &= U^{-1}(\widehat{N} - XB\widehat{K}), \\ M &= U^{-1}\begin{pmatrix} \mathrm{I}_{n_z} \\ -X \end{pmatrix}^\top \begin{pmatrix} \widehat{M} & \widehat{N}C \\ B\widehat{L} & -A + B\widehat{K}C \end{pmatrix} \begin{pmatrix} \mathrm{I}_{n_z} \\ -Y \end{pmatrix} V^{-\top}. \end{aligned} \tag{17}$$

Moreover, $\mathrm{In}(P) = \mathrm{In}(\Phi) = (p, 0, 2n_z - p)$.

The proof of Theorem 1 is postponed to Appendix B.

Theorem 1 contains two linearizing transformations. The first change of variables in (17) removes the bilinear terms generated by the controller matrices in (9). This transformation is a minor generalization of the one in [36] to invertible yet sign-indefinite matrices P.

The second transformation of the quadratic abstraction matrices in (16) removes the bilinear terms generated by the matrices (11). This change of variables is one of the main novelties of Theorem 1 and, thus, it requires further discussion. First, note that sign definiteness of $\widehat{R}$, $\widehat{Q}$ imposed by item i) of Theorem 1 and (14) does not impose any sign definiteness constraint on R_z, Q_z derived from (13b) if $\mathbf{R}_{12}$ and $\mathbf{S}_2$ are nonzero. Second, (16) and (13b) hint at interesting variable selections. If matrices R_z, S_z, Q_z are not known, one can strengthen the constraint in item i) of Theorem 1 to obtain strict negative definiteness of $\mathbf{R}_{22}$. Aside from dropping the constraint (12), this allows recovering R_z, S_z, Q_z by means of standard inverses in (13b), see Remark 3. The following corollary formalizes this result.

Corollary 1. *Consider the vector field f in* (2) *with* (10) *and assume that items i),ii), iii) of Theorem 1 hold with $\widehat{R} \prec 0$. Then, there exist square invertible matrices $U, V \in \mathbb{R}^{n_z \times n_z}$ such that $UV^\top = \mathrm{I}_{n_z} - XY$. Moreover, let Π_1 and Π_2 be defined as in* (15). *Then, for all sufficiently smooth functions $\phi : \mathbb{R}^{n_v} \to \mathbb{R}^{n_v}$ such that inequality* (11) *holds*

[2] Given at the top of the next page.

$$\Psi_2 := \begin{pmatrix} \mathrm{He}\left\{AY + B\widehat{L}\right\} - 2\beta Y - \widehat{R}_{11} & A + B\widehat{K}C + \widehat{M}^\top - 2\beta\,\mathrm{I}_{n_z} - \widehat{R}_{12}^\top & D + B\widehat{K}G - \widehat{S}_1^\top \\ \star & \mathrm{He}\left\{XA + \widehat{N}C\right\} - 2\beta X - \widehat{R}_{22} & XD + \widehat{N}G - \widehat{S}_2^\top \\ \star & \star & -\widehat{Q} \end{pmatrix} \prec 0 \tag{14}$$

for all $v \in \mathcal{V}$ with (13b), (16) *and* $\mathbf{R}_{22}^\dagger = \mathbf{R}_{22}^{-1}$, *inequality* (3) *holds for all $x \in \mathcal{Z} \times \mathbb{R}^{n_z}$ with $P = \Pi_2\Pi_1^{-1}$, $\mu = \beta$ and* (17)*. Moreover,* $\mathrm{In}(P) = \mathrm{In}(\Phi) = (p, 0, 2n_z - p)$.

Nonetheless, the matrices R_z, S_z, Q_z are often known in control design problems. In this scenario, one can fix $\mathbf{R}_{12} = 0$ and $\mathbf{S}_2 = 0$ to simplify embedding these known matrices into (14), see Remark 3. This can be obtained via a proper selection of $\widehat{R}_{12}$ and $\widehat{S}_1$ in (16), as formalized by the following proposition.

Proposition 2. *Consider the vector field f in* (2) *with* (10) *and let* (11) *hold for all $v \in \mathcal{V}$ with given matrices R_z, S_z, Q_z. Select $Z \preceq 0$ such that $Z \preceq R_z$ and assume there exist $\widehat{R}_{11} = \widehat{R}_{11}^\top$, a scalar β, matrices $X = X^\top, Y = Y^\top \in \mathbb{R}^{n_z\times n_z}$,matrices $\widehat{K} \in \mathbb{R}^{n_u\times n_y}, \widehat{L} \in \mathbb{R}^{n_u\times n_z}, \widehat{M} \in \mathbb{R}^{n_z\times n_z}, \widehat{N} \in \mathbb{R}^{n_z\times n_y}$ and a scalar $\alpha > 0$ such that:*

i) The matrix $\widehat{R}_Z := \begin{pmatrix} \widehat{R}_{11} & \alpha YZ \\ \star & \alpha Z \end{pmatrix}$ is negative semidefinite;

ii) Items ii) and iii) of Theorem 1 hold with $\widehat{R}_{12} = \alpha R_z Y$, $\widehat{R}_{22} = \alpha R_z$, $\widehat{S}_1 = \alpha S_z Y$, $\widehat{S}_2 = \alpha S_z$ and $\widehat{Q} = \alpha Q_z$.

Then, there exist square invertible matrices $U, V \in \mathbb{R}^{n_z\times n_z}$ such that $UV^\top = \mathrm{I}_{n_z} - XY$. Moreover, inequality (3) *holds for all $x \in \mathcal{Z} \times \mathbb{R}^{n_z}$ with $P = \Pi_2\Pi_1^{-1}$, $\mu = \beta$ and* (17)*, where Π_1 and Π_2 are defined as in* (15)*. Finally,* $\mathrm{In}(P) = \mathrm{In}(\Phi) = (p, 0, 2n_z - p)$.

The proof of Proposition 2 is postponed to Appendix C.

As discussed in Section 2.2, if $q > 1$ quadratic abstractions are known, the matrices R_z, S_z, Q_z in (11) can be defined as a linear combination with unknown weights as in (7). From Proposition 2, these weights can be found via an iterative procedure, as they generate bilinear terms in (14) due to the choice of $\widehat{R}_{12}$ and $\widehat{S}_1$.

Remark 4. *Setting $\mathbf{R}_{12} = 0$ and $\mathbf{S}_2 = 0$ simplifies the relation between* (11) *and the expressions in Lemma 1, but comes at the price of stricter conditions on Q_z induced by* (14)*. Differently from the scenario of arbitrary matrices, the constraints $\mathbf{R}_{12} = 0$ and $\mathbf{S}_2 = 0$ and* (14) *impose $Q_z \succ 0$. Nonetheless, common quadratic abstractions such as (shifted) monotonicity and differential sector boundedness satisfy the constraint $Q_z \succ 0$ [42]. Note that similar conclusions can be drawn for R_z if we impose $\mathbf{R}_{12} = 0$ and item i) of Theorem 1 holds. However, the use of the proxy matrix Z and the substitution of item i) of Theorem 1 with item i) of Proposition 2 circumvent the need of imposing sign-definiteness of R_z.*

Remark 5. *The matrix Z in Proposition 2 is a degree of freedom. Consequently, it can be obtained by solving the following optimization problem*

$$\begin{aligned} \min \quad & \epsilon \\ s.t. \quad & \begin{pmatrix} \epsilon\,\mathrm{I}_{n_z} & R_z - Z \\ R_z - Z & \mathrm{I}_{n_z} \end{pmatrix} \succeq 0,\ \epsilon \geq 0,\ Z \preceq 0,\ R_z \succeq Z. \end{aligned}$$

While the last two inequalities directly derive from the conditions on Z in Proposition 2, the first one (via a Schur complement) imposes $(R_z - Z)^\top(R_z - Z) \preceq \epsilon\,\mathrm{I}_{n_z}$. Therefore, ϵ acts as an upperbound on the square norm of $R_z - Z$, and its minimization allows directly deriving a matrix Z which is as close as possible to R_z.

The degree of freedom offered by $\alpha > 0$ may sometimes not be sufficient. To tackle this issue, item ii) of Proposition 2 can be modified to introduce more degrees of freedom in the choice of the variables $\widehat{R}_{12}$, $\widehat{R}_{22}$, $\widehat{S}_1$, $\widehat{S}_2$ and $\widehat{Q}$, as formalized by the following proposition.

Corollary 2. *Consider the vector field f in* (2) *with* (10) *and let* (11) *hold for all $v \in \mathcal{V}$ with given matrices R_z, S_z, Q_z. Select $Z \preceq 0$ such that $Z \preceq R_z$ and assume there exist $R_1 = R_1^\top$, a scalar β, matrices $X = X^\top, Y = Y^\top \in \mathbb{R}^{n_z\times n_z}$,matrices $\widehat{K} \in \mathbb{R}^{n_u\times n_y}, \widehat{L} \in \mathbb{R}^{n_u\times n_z}, \widehat{M} \in \mathbb{R}^{n_z\times n_z}, \widehat{N} \in \mathbb{R}^{n_z\times n_y}$, a positive scalar $\alpha > 0$, matrices Υ_1, Υ_2, Υ_3 and matrices $\widehat{R}_{11} = \widehat{R}_{11}^\top$, $\widehat{R}_{12}, \widehat{R}_{22} = \widehat{R}_{22}^\top$, $\widehat{S}_1$, $\widehat{S}_2$ and $\widehat{Q} = \widehat{Q}^\top$ such that:*

i) The matrix $\widehat{R}_Z := \begin{pmatrix} R_1 & \alpha YZ \\ \star & \alpha Z \end{pmatrix}$ is negative semidefinite;

ii) Items ii) and iii) of Theorem 1 hold with

$$\begin{pmatrix} \widehat{R}_{11} - R_1 & \widehat{R}_{12}^\top - \alpha Y R_z^\top & \widehat{S}_1^\top - \alpha Y S_z^\top \\ \star & \widehat{R}_{22} - \alpha R_z & \widehat{S}_2^\top - \alpha S_z^\top \\ \star & \star & \widehat{Q} - \alpha Q_z \end{pmatrix} + \mathrm{He}\left\{ \begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \\ \Upsilon_3 \end{pmatrix} \begin{pmatrix} \frac{\partial\phi}{\partial z}(v) & 0 & -\mathrm{I}_{n_z} \end{pmatrix} \right\} \preceq 0 \tag{18}$$

Then, there exist square invertible matrices $U, V \in \mathbb{R}^{n_z\times n_z}$ such that $UV^\top = \mathrm{I}_{n_z} - XY$. Moreover, inequality (3) *holds for all $x \in \mathcal{Z} \times \mathbb{R}^{n_z}$ with $P = \Pi_2\Pi_1^{-1}$, $\mu = \beta$ and* (17)*, where Π_1 and Π_2 are defined as in* (15)*. Finally,* $\mathrm{In}(P) = \mathrm{In}(\Phi) = (p, 0, 2n_z - p)$.

Inequality (18) and the positivity of α guarantee a quadratic abstraction in $\widehat{R}_{11}, \widehat{R}_{12}, \widehat{R}_{22}$, $\widehat{S}_1$, $\widehat{S}_2$ and $\widehat{Q}$ is satisfied, as it holds for the selection in item ii) of Proposition 2. This is easily seen by pre and post-multiplying (18) by the vector $\nu(z)^\top := \begin{pmatrix} \mathrm{I}_{n_z} & 0 & \frac{\partial\phi}{\partial z}(v)^\top \\ 0 & \mathrm{I}_{n_\xi} & 0 \end{pmatrix}$ and its transpose. This step also shows that item ii) of Corollary 2 recovers the one of Proposition 2 whenever the selection in item ii) of Proposition 2 is made. However, it reduces the conservativeness of the result as the optimization variables embedding the abstraction into (14) increase in dimension, from a single scalar α to full matrices. One possibility to reduce the infinite number of constraints imposed by (18) to a finite amount is to select all multipliers $\Upsilon_i = 0$, $i = 1, 2, 3$. This solution trades off conservativeness of the result for practical solvability. Another option is to rely on polytopic approaches and convex relaxation, by assuming the Jacobian of the nonlinearity ϕ lives in a polytope and by checking (18) only on its finitely many vertices.

Theorem 1 provides a family of controllers parameterized by U, V that generates a closed-loop system satisfying (3) for some matrix P with inertia $\mathrm{In}(P) = (p, 0, 2n_z - p)$. The degree of freedom offered by U, V can then be exploited to ensure additional properties. For instance, they can be selected to improve the conditioning of the matrix P in (B.1) and the recovered matrices in (17). This result is formalized in the following proposition.

Proposition 3. *Let $X = X^\top \in \mathbb{R}^{n_z \times n_z}, Y = Y^\top \in \mathbb{R}^{n_z \times n_z}$ be given. If there exist scalars, $\alpha > 0$, $\beta \geq 0$, $\gamma > 0$ and a matrix $\widehat{V} \succ 0$ such that*

$$\begin{pmatrix} -\mathbf{V} & \beta\overline{P} \\ \beta\overline{P} & -\mathbf{V} \end{pmatrix} \preceq 0, \qquad \begin{pmatrix} \mathbf{V} & \mathbf{V}\overline{P}^{-1} \\ \overline{P}^{-1}\mathbf{V} & \gamma\,\mathrm{I}_{2n_z} \end{pmatrix} \succeq 0, \tag{19a}$$

$$\begin{pmatrix} \mathbf{V} & \mathbf{V} \\ \mathbf{V} & \alpha\,\mathrm{I}_{2n_z} \end{pmatrix} \succeq 0, \qquad \begin{pmatrix} -\alpha & \beta \\ \beta & -\gamma \end{pmatrix} \preceq 0, \tag{19b}$$

with

$$\begin{aligned} \mathbf{V} &:= \begin{pmatrix} \mathrm{I}_{n_z} & 0 \\ 0 & \widehat{V} \end{pmatrix}, \\ \overline{P} &:= \begin{pmatrix} X & \mathrm{I}_{n_z} - XY \\ \mathrm{I}_{n_z} - YX & -(\mathrm{I}_{n_z} - YX)Y \end{pmatrix}, \\ \overline{P}^{-1} &:= \begin{pmatrix} Y & \mathrm{I}_{n_z} \\ \mathrm{I}_{n_z} & -(\mathrm{I}_{n_z} - XY)^{-1}X \end{pmatrix}, \end{aligned} \tag{20}$$

then $P = \Pi_2\Pi_1^{-1}$ with Π_1, Π_2 as in (15), $U = (\mathrm{I}_{n_z} - XY)\sqrt{\widehat{V}}^{-1}$ and $V = \sqrt{\widehat{V}}$, satisfies

$$\frac{\bar{\sigma}(P)}{\underline{\sigma}(P)} \leq \frac{\sqrt{\alpha\gamma}}{\beta}. \tag{21}$$

The proof of Proposition 3 is postponed to Appendix D.

2.4. Coupling LMIs

Some particular scenarios require solving multiple inequalities of the form (1) with relative matrices P with different inertia, e.g., [10]. While simultaneous resolution of multiple LMIs of the form (6) is a valuable option for unforced systems analysis, there is no trivial coupled formulation of multiple LMIs of the form (14) for controller design. This is due to the change of variables used to linearize the design problem in (16) and (3), which is related to the matrix P. In other words, each LMI would provide different transformed variables, yet they should all represent the same controller. This strong constraint often makes the problem intractable. A workaround to the issue is to combine both formulations (6) and (14). Indeed, one can use the LMI (14) to define proper controller variables, and the flexibility offered by the multipliers in (6) recover them in different LMI conditions. Such an approach is presented in the following result.

Theorem 2. *Consider the vector field f in (2) with (10) and suppose there exist matrices $\widehat{R}_{11} = \widehat{R}_{11}^\top, \widehat{R}_{12}, \widehat{R}_{22} = \widehat{R}_{22}^\top, \widehat{S}_1, \widehat{S}_2, \widehat{Q} = \widehat{Q}^\top$ a scalar $\beta \in \mathbb{R}$, symmetric matrices $X = X^\top, Y = Y^\top \in \mathbb{R}^{n_z \times n_z}$, matrices $\widehat{K} \in \mathbb{R}^{n_u \times n_y}, \widehat{L} \in \mathbb{R}^{n_u \times n_z}, \widehat{M} \in \mathbb{R}^{n_z \times n_z}, \widehat{N} \in \mathbb{R}^{n_z \times n_y}$, an integer $s \geq 0$, scalars β_i, matrices $\widehat{\Gamma}_1^i, \widehat{\Gamma}_2^i, \widehat{\Gamma}_3^i$ and invertible matrices $\widehat{\Sigma}_i = \widehat{\Sigma}_i^\top$ such that items i),ii) of Theorem 1 hold. Additionally, for each $i = 0, \ldots, s$ assume the following conditions hold*

i) $\mathrm{In}(\widehat{\Gamma}_2^i) = (p, 0, 2n_z - p)$;

ii) Inequality (22) *holds with matrices* $\widehat{R} = \begin{pmatrix} \widehat{R}_{11} & \widehat{R}_{12}^\top \\ \star & \widehat{R}_{22} \end{pmatrix}$ *and* $\widehat{S} = \begin{pmatrix} \widehat{S}_1 & \widehat{S}_2 \end{pmatrix}$,

iii) $\mathrm{In}(\widehat{\Sigma}_i) \neq \mathrm{In}(\widehat{\Sigma}_j)$ *for all* $j \neq i$.

Then, there exist square invertible matrices $U, V \in \mathbb{R}^{n_z \times n_z}$ such that $UV^\top = \mathrm{I}_{n_z} - XY$. Moreover, let Π_1 and Π_2 be defined as in (15). *Then, for all sufficiently smooth functions $\phi : \mathbb{R}^{n_v} \to \mathbb{R}^{n_v}$ such that inequality* (11) *and* (12) *hold for all $v \in \mathcal{V}$ with* (13b), (16) *inequality* (3) *holds for all $x \in \mathcal{Z} \times \mathbb{R}^{n_z}$ with* (17), *$P_i = \Pi_1^{-\top}\Sigma_i\Pi_1^{-1}$ and $\mu_i = \beta_i$ for each $i = 0, \ldots, s$ and $P_{s+1} = \Pi_2\Pi_1^{-1}$ and $\mu_{p+1} = \beta$. Moreover, $\mathrm{In}(P_{s+1}) = \mathrm{In}(\widehat{\Gamma}_2^i)$ for all $i = 1, \ldots, s$.*

The proof of Theorem 2 is postponed to Appendix E.

Theorem 2 can be adapted according to the knowledge (or lack thereof) of the matrices in the quadratic abstraction, similarly to Theorem 1 and Proposition 2. Unfortunately, while (6) offers the possibility of coupling multiple differential conditions of the form (1), it also introduces bilinear terms in (22). There are multiple ways of dealing with BMIs, such as iterative methods (e.g., [9]) or linearization by structuring multipliers, see Remark 2. However, this comes at the price of either loss of global convergence or increased conservativeness of the result.

Therefore, in Algorithm 1[3] we propose an iterative algorithm aimed at exploiting Theorem 2 to improve an initial solution computed by the separate application of Theorem 1 and Proposition 1. Such a separately computed initial condition provides a feasible starting point for the iterative algorithm. We identify with superscript $j \in \mathbb{N}$

[3]Given at the top of the next page.

$$\Psi_3^i := \begin{pmatrix} -2\beta_i\widehat{\Sigma}_i + \mathrm{He}\left\{\widehat{\Gamma}_1^i\widehat{A}\right\} - \widehat{R} & \widehat{\Sigma}_i - \widehat{\Gamma}_1^i\Phi + \widehat{A}^\top(\widehat{\Gamma}_2^i)^\top & \widehat{\Gamma}_1^i\widehat{D} + \widehat{A}^\top(\widehat{\Gamma}_3^i)^\top - \widehat{S}^\top \\ \star & -\mathrm{He}\left\{\widehat{\Gamma}_2^i\Phi\right\} & \widehat{\Gamma}_2^i\widehat{D} - \Phi(\widehat{\Gamma}_3^i)^\top \\ \star & \star & \mathrm{He}\left\{\widehat{\Gamma}_3^i\widehat{D}\right\} - \widehat{Q} \end{pmatrix} \prec 0 \tag{22}$$

$$\widehat{A} := \begin{pmatrix} AY+B\widehat{L} & A+B\widehat{K}C \\ \widehat{M} & XA+\widehat{N}C \end{pmatrix} \qquad \widehat{D} := \begin{pmatrix} D+B\widehat{K}G \\ XD+\widehat{N}G \end{pmatrix} \qquad \Phi := \begin{pmatrix} Y & \mathrm{I}_{n_z} \\ \mathrm{I}_{n_z} & X \end{pmatrix}$$

the values of the different matrices at iteration j. Termination constraints or optimization objectives can be included in the algorithm by selecting appropriate stopping conditions, see Section 4 as an example. Moreover, we recall that the terms μ_i in Algorithm 1 (lines 2,4,9) can be obtained via GEVPs, see Remark 1.

We conclude Section 2 by commenting on tools to verify quadratic abstractions of the form (4), and the possible extension of the results to more general dynamics.

2.5. Verifying quadratic abstractions

We highlight that (4) includes the derivative of ϕ with respect to the system state x rather than of the function input v. Nonetheless, simple computations show we can obtain inequalities of the form (4) starting from more easily verifiable ones, that is, quadratic differential constraints in $\frac{\partial\phi}{\partial v}$. Precisely, assume there exist square matrices $R_v = R_v^\top$, S_v, $Q_v = Q_v^\top$ such that

$$\begin{pmatrix} \mathrm{I}_{n_v} \\ \frac{\partial\phi}{\partial v}(v) \end{pmatrix}^\top \begin{pmatrix} R_v & S_v^\top \\ S_v & Q_v \end{pmatrix} \begin{pmatrix} \mathrm{I}_{n_v} \\ \frac{\partial\phi}{\partial v}(v) \end{pmatrix} \preceq 0, \tag{23}$$

holds for all $v \in \mathcal{V}$. Then, left and right multiplication of (23) by $\mathbf{E}^\top$ and $\mathbf{E}$, respectively, leads to

$$\begin{pmatrix} \mathrm{I}_{n_v} \\ \frac{\partial\phi}{\partial v}(v)\mathbf{E} \end{pmatrix}^\top \begin{pmatrix} \mathbf{E}^\top R_v\mathbf{E} & \mathbf{E}^\top S_v^\top \\ S_v\mathbf{E} & Q_v \end{pmatrix} \begin{pmatrix} \mathrm{I}_{n_v} \\ \frac{\partial\phi}{\partial v}(v)\mathbf{E} \end{pmatrix} \preceq 0,$$

which, by means of the relation $\frac{\partial\phi}{\partial x}(v) = \frac{\partial\phi}{\partial v}(v)\mathbf{E}$, reduces to (4) with $R = \mathbf{E}^\top R_v\mathbf{E}, S = S_v\mathbf{E}$ and $Q = Q_v$.

The derivation of results based on quadratic abstractions in $\frac{\partial\phi}{\partial x}$ simplifies their extension to different dynamics. For instance, this allows an extension of our framework to systems with implicit nonlinearities. The combination of contraction and stability properties of implicit dynamics and learning systems recently attracted significant attention [23, 22, 33, 27]. All the results in this paper can be adapted to such a scenario once quadratic abstractions of the form (4) are recovered. Hence, similarly to the previous scenario, we now propose a technical result allowing the derivation of quadratic abstractions of the form (4) starting from more easily verifiable ones (23) for implicit systems. We consider implicit nonlinear systems of the form

$$\begin{pmatrix} \dot{x} \\ v \end{pmatrix} = \begin{pmatrix} f(x,w) \\ g(x,w) \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{D} \\ \mathbf{E} & \mathbf{F} \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}, \quad w = \phi(v). \tag{24}$$

By implicit differentiation and the chain rule, we have

$$\left(\mathrm{I}_{n_v} - \frac{\partial\phi}{\partial v}(v)\mathbf{F}\right)\frac{\partial\phi}{\partial x}(v) = \frac{\partial\phi}{\partial v}(v)\mathbf{E}. \tag{25}$$

Lemma 2. *Consider system* (24) *and suppose* (23) *holds for all* $v \in \mathcal{V}$. *If there exist matrices* R, S, Q *such that*

$$\begin{pmatrix} \mathrm{I}_{n_x} & 0 \\ 0 & \mathrm{I}_{n_v} \\ 0 & \frac{\partial\phi}{\partial v}(v) \end{pmatrix}^\top \Xi \begin{pmatrix} \mathrm{I}_{n_x} & 0 \\ 0 & \mathrm{I}_{n_v} \\ 0 & \frac{\partial\phi}{\partial v}(v) \end{pmatrix} \preceq 0 \tag{26a}$$

with

$$\Xi := \begin{pmatrix} R - \mathbf{E}^\top R_v\mathbf{E} & S^\top - \mathbf{E}^\top S_v^\top & \mathbf{E}^\top S_v^\top \\ \star & Q - Q_v & Q_v \\ \star & \star & -Q_v \end{pmatrix} \tag{26b}$$

for all $v \in \mathcal{V}$, *then* (4) *holds for all* $v \in \mathcal{V}$.

The proof of Lemma 2 is postponed to Appendix F.

As a final remark, we highlight that the infinite set of LMIs defined by conditions of the form (23) and (26) can be easily verified for nonlinearities with favorable properties, for instance, ones whose Jacobian matrices live in a polytope. In such cases, it is convenient to obtain sufficient conditions for (23) which are affine in $\frac{\partial\phi}{\partial v}(v)$. In affine form, (23) reduces to a finite set of LMI evaluated at the vertices of the polytope by convex relaxation. The next lemma presents a possible affine sufficient condition for verifying (23).

Lemma 3. *Consider a smooth function* $\phi : \mathbb{R}^{n_v} \to \mathbb{R}^{n_v}$ *and square matrices* $R_v = R_v^\top$, S_v, $Q_v = Q_v^\top$. *If there exist matrices* Υ_1, Υ_2 *such that*

$$\begin{pmatrix} R_v & S_v^\top \\ S_v & Q_v \end{pmatrix} + \mathrm{He}\left\{\begin{pmatrix} \frac{\partial\phi}{\partial v}(v) \\ -\mathrm{I}_{n_v} \end{pmatrix}\begin{pmatrix} \Upsilon_1 & \Upsilon_2 \end{pmatrix}\right\} \preceq 0 \tag{27}$$

holds for all $v \in \mathcal{V}$, *then* (23) *holds.*

The result is shown by left and right multiplication by $\nu := \begin{pmatrix} \mathrm{I}_{n_v} & \frac{\partial\phi}{\partial v}(v)^\top \end{pmatrix}$ and its transpose. Similar results can be obtained for (26) and the kernel matrix $\begin{pmatrix} 0 & \frac{\partial\phi}{\partial v}(v) & -\mathrm{I}_{n_v} \end{pmatrix}$. Note that (if R_v, S_v, Q_v are left as decision variables) LMIs (27) on the vertices of the polytope can be solved together with (6), (14) or (22) to ensure the nonlinearity ϕ satisfies the recovered quadratic abstraction.

Up to now, we discussed how to solve inequalities of the form (1) under inertia constraints by means of LMI techniques. However, we did not provide tools for imposing such a desired matrix inertia. The next section is then dedicated to such a topic.

Algorithm 1 Iterative algorithm for output-feedback design with multiple conditions of the form (3)

$j \leftarrow 0$
Compute $\mu_{p+1}^j, P_{p+1}^j, K^j, L^j, M^j, N^j$ via Theorem 1
$(K, L, M, N) \leftarrow (K^j, L^j, M^j, N^j)$
Compute μ_i^j, P_i^j via Proposition 1 with (10) for $i = 1, \dots, p$ ▷ Solve p LMIs (6) at once
$(\mu_i, P_i) \leftarrow (\mu_i^j, P_i^j)$ for $i = 1, \dots, p+1$
while not stopping condition **do**
    Compute $\widehat{\Gamma}_1^i, \widehat{\Gamma}_2^i, \widehat{\Gamma}_3^i$ for $i = 1, \dots, p$ via Theorem 2 ▷ Solve p LMIs (22) at once
    $j \leftarrow j + 1$
    Compute $\mu_i^j, P_i^j, K^j, L^j, M^j, N^j$ via Theorem 2 for $i = 1, \dots, p+1$ ▷ Solve (14) and p LMIs (22) at once
    $(\mu_i, P_i, K, L, M, N) \leftarrow (\mu_i^j, P_i^j, K^j, L^j, M^j, N^j)$
end while

3. On the inertia constraint

One of the main complexities in the application of Proposition 1 and Theorem 1 (or Theorem 2) is the inclusion of the inertia constraints. For $p = 0$, these constraints are easily imposed by substituting them with positivity constraints on Σ and Φ (or $\widehat{\Gamma}_2^i$). However, no straightforward solutions exist for $p > 0$. Therefore, we now present solvable sufficient conditions to impose bounds on matrix inertia.

For system analysis (see Proposition 1) inertia can be imposed rather straightforwardly. The following lemma is an immediate consequence of points (2) and (3) of [10, Lemma 8].

Lemma 4. *Consider a set $\mathcal{X} \subseteq \mathbb{R}^{n_x}$, a vector field $f : \mathcal{X} \to \mathbb{R}^{n_x}$ and a constant $\mu \in \mathbb{R}$. Then, any symmetric non-singular matrix $P \in \mathbb{R}^{n_x \times n_x}$ satisfying*

$$\frac{\partial f}{\partial x}(x)^\top P + P \frac{\partial f}{\partial x}(x) \prec 2\mu P, \quad \forall x \in \mathcal{X},$$

also satisfies $\mathrm{In}(P) = \mathrm{In}(-\frac{\partial f}{\partial x}(x) + \mu I)$ *for all $x \in \mathcal{X}$.*

Lemma 4 suggests possible algorithmic steps for applying Proposition 1 on a set $\mathcal{X}$ under the inertia constraint $\mathrm{In}(\Sigma) = (p, 0, n_x - p)$. More specifically, two steps are required by such an algorithm:

1. Find (if possible) a $\mu \in \mathbb{R}$ such that $\mathrm{In}(-\frac{\partial f}{\partial x}(x) + \mu I) = (p, 0, n_x - p)$ for all $x \in \mathcal{X}$.
2. Solve (6) with $\beta = \mu$ (if possible).

This procedure disentangles the inequality solution from inertia constraint by imposing it on the term $\frac{\partial f}{\partial x}(x) + \mu I$, which indirectly affects the solution itself. If this process is completed, the solution of (6) satisfies $\mathrm{In}(\Sigma) = (p, 0, n_x - p)$ by Lemma 4, even if such a constraint is not directly included in (6). This is the most common strategy for fixing inertia in existing literature, see e.g. [10, 13, 8].

Unfortunately, the applicability of such a strategy is rather limited. Clear examples of such limitations appear in the controller design problem. Indeed, different controllers can make inequality (3) solvable with matrices P of different inertia under the same choice of μ. In other words, for a fixed parameter β the solution (X, Y) to (14) is not unique because of the degrees of freedom offered by the controller parameters. These solutions can generate matrices Φ with different inertia. Similar effects appear for system analysis whenever R, S, Q in (4) are left as variables to be estimated. Motivated by the aforementioned drawbacks, we now propose alternative methods for constraining matrix inertia even in the controlled framework.

We start by proposing studying soft inertia constraints, namely, constraints on the minimum number of positive eigenvalues and on the maximum number of negative ones. The result is formalized in the following lemma, which proposes a convex method to constrain from above the number of negative eigenvalues of a symmetric matrix.

Lemma 5. *Consider a symmetric matrix $P \in \mathbb{R}^{n_x \times n_x}$ and a constant $p \in \{0, \dots, n_x - 1\}$. If there exist a constant $z \in \mathbb{R}$ and a symmetric matrix X such that*

$$\begin{aligned} X &\succeq 0 \\ z\,\mathrm{I} + X + P &\succeq 0 \\ (p+1)z + \mathrm{tr}\, X &\preceq 0, \end{aligned} \tag{28}$$

then

$$\pi_+(P) \geq n_x - p, \qquad \pi_-(P) \leq p. \tag{29a}$$

Moreover,

$$\Re(\lambda_{n_x - p}(P)) \geq -\sum_{i = n_x - p + 1}^{n_x} \Re(\lambda_i(P)). \tag{29b}$$

The proof of Lemma 5 is postponed to Appendix G.

For analysis purposes, one can combine (6) and (28) with $P = \Sigma$ to impose an upper-bound the number of negative eigenvalues allowed in Σ. Similarly, for controller design, one can combine (14) and (28) with $P = \Phi$ to constrain the inertia of Φ.

In many practical applications imposing an upperbound on the number of negative eigenvalues is enough, as the aim is to obtain a sufficiently small number of negative eigenvalues in some matrix (or even the minimum one). Nonetheless, some situations may require imposing exact (i.e., hard) inertia constraints. A possibility to impose exact inertia is the addition of a constraint bounding the number of negative eigenvalues from below. This choice comes at the price of convexity of the final problem, as shown by the following lemma.

Lemma 6. *[18, Lemma 3]: The following two statements are equivalent:*

1. $\pi_-(P) \geq p$.
2. *There exists a matrix* $Q = Q^\top \succeq 0$ *with* $\operatorname{rank}\{Q\} \leq n_x - p$ *such that* $P - Q \prec 0$.

In plain words, Lemma 6 shows that a lower bound on the number of negative eigenvalues of a symmetric matrix is equivalent to a rank constraint on an auxiliary variable. This transformation makes the optimization problem lose its convexity. Nonetheless, there exist multiple techniques for solving linear matrix inequalities with rank constraints, e.g.[21, 31].

Remark 6. *The number of negative eigenvalues can also be lower-bounded by inverting the sign of* P *in* (28)*. Then, by Lemma 5, we have*

$$\pi_-(P) = \pi_+(-P) \geq n_x - p,\ \pi_+(P) = \pi_-(-P) \leq p \quad \text{(30a)}$$

and, since $\Re(\lambda_{p+1}(P)) = -\Re(\lambda_{n_x-p}(-P))$ *and* $\sum_{i=1}^{p} \Re(\lambda_i(P)) = -\sum_{i=n_x-p+1}^{n_x} \Re(\lambda_i(-P))$*, the following bound holds*

$$\Re(\lambda_{p+1}(P)) \leq -\sum_{i=1}^{p} \Re(\lambda_i(P)). \quad \text{(30b)}$$

Unfortunately, while pairing two conditions (28) *to impose the correct inertia (one for* P *and one for* $-P$*) seems a valuable option, the set of solutions to the combined constraints on the real part of the eigenvalues is always empty. Indeed, the two bounds on the number of negative eigenvalues must intersect only at equality to strictly impose the desired matrix inertia. In other words, if we aim to impose* $1 \leq k \leq n_x$ *negative eigenvalues, we must select* $p = k$ *in the first set of inequalities (in* P*) and* $p = n_x - k$ *in the second one (in* $-P$*). From* (29b) *and* (30b) *we obtain*

$$-\sum_{i=n_x-k+2}^{n_x} \Re(\lambda_i(P)) \leq \sum_{i=n_x-k}^{n_x-k+1} \Re(\lambda_i(P)) \leq -\sum_{i=1}^{n_x-k-1} \Re(\lambda_i(P)),$$

which cannot hold due to (30a) *and* (29a)*.*

Combining the conditions in Lemma 5 and Lemma 6 it is possible to impose hard constraints on the inertia of a symmetric matrix P. However, such a strategy may increase conservativeness or introduce numerical issues, due to (29b). More specifically, if the constant p is large and the matrix P has few positive eigenvalues, the sum in (G.4) may lead to large positive eigenvalues and small negative ones. These drawbacks can be avoided by replacing Lemma 5 with Lemma (6), namely, by imposing a secondary rank constraint on the opposite matrix $-P$. The result is formalized in the following corollary.

Corollary 3. *Consider a symmetric matrix* $P \in \mathbb{R}^{n_x \times n_x}$ *and a constant* $p \in \{0, \ldots, n_x\}$*. The following two statements are equivalent:*

1. $\operatorname{In}(P) = (p, 0, n_x - p)$.
2. *There exist a pair of symmetric matrices* $Q_1 = Q_1^\top \geq 0$ *and* $Q_2 = Q_2^\top \geq 0$ *with*

$$\operatorname{rank}\{Q_1\} \leq p, \quad \operatorname{rank}\{Q_2\} \leq n_x - p$$

such that

$$-Q_1 \prec P \prec Q_2$$

We now move to the last section of the paper, aimed at showcasing the practical potential of the proposed conditions in the context of partial stabilization.

4. Application: 2-contraction for extremum control with non-convex costs

4.1. Preliminaries on 2-contraction

The notion of 2-contraction recently received significant attention due to its potential in controller design for multistable systems and removal of chaotic behaviors [10, 3, 4, 29]. In plain words, for an arbitrary nonlinear system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^{n_x}, \quad (31)$$

2-contraction implies that the area of any surface of initial conditions in $\mathcal{X}$ is exponentially shrinking along the system dynamics. More details on this geometrical interpretation can be found in [10, 38]. Intuitively, if any area between initial conditions exponentially shrinks to zero, the asymptotic behavior of the system converges to a subspace of dimension strictly smaller than 2. An immediate consequence is that no solution can converge to a limit cycle or a chaotic attractor. Thus, any bounded solution of a 2-contractive system of the form (31) converges to an equilibrium point, which may not be unique. This fact was originally proven in [25], and is summarized in the following lemma.

Lemma 7. *[25, Theorem 2.5]: Assume that system* (31) *is* 2*-contractive in a compact and forward invariant set* $\mathcal{X} \subset \mathbb{R}^{n_x}$*. Then, each solution of system* (31) *initialized in* $\mathcal{X}$ *converges to an equilibrium point, which may be different for each solution.*

Sufficient conditions based on generalized Lyapunov conditions recently appeared as a valuable tool to study and impose 2-contraction [41, 10]. In this section, we aim to combine them with the results of Sections 2 and 3 to design a linear output-feedback controller making the closed loop 2-contractive. We now recall these sufficient conditions.

Theorem 3. *[10, Theorem 5]: Let $\mathcal{X} \subset \mathbb{R}^{n_x}$ be a compact forward invariant set. Suppose there exist two symmetric matrices $P_0, P_1 \in \mathbb{R}^{n_x \times n_x}$ of respective inertia $(0,0,n_x)$ and $(1,0,n_x-1)$, and $\mu_0, \mu_1 \in \mathbb{R}$ such that for all $x \in \mathcal{X}$*

$$\frac{\partial f}{\partial x}(x)^\top P_0 + P_0 \frac{\partial f}{\partial x}(x) \prec 2\mu_0 P_0, \tag{32a}$$
$$\frac{\partial f}{\partial x}(x)^\top P_1 + P_1 \frac{\partial f}{\partial x}(x) \prec 2\mu_1 P_1, \tag{32b}$$
$$\mu_1 + \mu_0 < 0, \tag{32c}$$

Then, system (31) is 2-contractive on $\mathcal{X}$.

We highlight that different sufficient conditions for 2-contraction have also been developed exploiting mathematical objects known as matrix compounds [7, 38, 2, 3]. These results have been used for state-feedback design for 2-contraction [4] or analysis of 2-contraction for Lur'e system [30]. Nonetheless, conditions based on matrix compounds usually explode in dimension, they may destroy advantageous structural properties of the system and hinder the development of general controllers. For this reason, the combination of Theorem 3 with the framework proposed in this paper is, to the best of our knowledge, the first computationally viable option for designing output-feedback controllers for 2-contraction.

4.2. The extremum control problem

Consider a controlled linear system of the form

$$\dot{\zeta} = \overline{A}\zeta + \overline{B}u, \quad v = \overline{E}\zeta \tag{33}$$

were $\zeta \in \overline{\mathcal{Z}} \subset \mathbb{R}^{n_\zeta}$ is the state of the system, $u \in \mathbb{R}^{n_u}$ is the input vector and $v \in \mathbb{R}$ is a (not necessarily measurable) linear combination of states. Consider a smooth objective function $V : \mathcal{V} \to \mathbb{R}$ with at least one extremum and $\mathcal{V} := \{v \in \mathbb{R} : v = \overline{E}\zeta, \zeta \in \overline{\mathcal{Z}}\}$. We aim to design a feedback policy such that a signal v stabilizes to an extremum (maximum or minimum) of V. Moreover, we assume its gradient $\frac{\partial V}{\partial v}(v)$ satisfies a known quadratic abstraction of the form (23) with $\phi = \frac{\partial V}{\partial v}(v)$. Classical functions falling into this category are convex (concave) ones, since convexity (concavity) implies a monotonic gradient $\frac{\partial V}{\partial v}(v)$ and monotonicity can be described through differential quadratic abstractions, see e.g. [42, Section III.B]. However, quadratic abstractions of the form (23) are not limited to monotonic functions and cover a much wider class of (possibly non-convex) objectives.

An intuitive method for solving the aforementioned problem is to consider the extended system

$$\begin{aligned} \dot{\eta} &= \frac{\partial V}{\partial v}(\overline{E}\zeta) \\ \dot{\zeta} &= \overline{A}\zeta + \overline{B}u \\ y &= \begin{bmatrix} \eta \\ \frac{\partial V}{\partial x}(\overline{E}\zeta) \end{bmatrix}, \end{aligned} \tag{34}$$

and design a dynamic output-feedback controller to stabilize (34). Since at equilibria the integral action imposes $\dot{\eta} = 0$, we derive $\frac{\partial V}{\partial v}(\overline{E}\zeta) = 0$, thus implying the system has reached an extremum of the function $V(\overline{E}\zeta)$. We highlight that the considered problem has its similarities with the problem of unconstrained extremum control as presented in [5] and recently reproposed in [28, 24, 20]. Nonetheless, different from all these works, we are not restricted to convex objective functions.

Designing a stabilizing controller for (34) presents some challenges. First, the function V may have multiple extrema, therefore, system (34) may have multiple equilibrium points. Second, if the function V is unknown, the equilibrium points of (34) may be unknown. An effective way of overcoming these obstacles is to design a controller that guarantees a 2-contractive closed-loop system. Then, even if the function V is non-convex and has multiple extrema, convergence to at least one of them is guaranteed by Lemma 7. Moreover, since η integrates the gradient of V (and not its opposite), the closed loop will behave similarly to a gradient-ascent algorithm with momentum.

In this section, we use the results of this paper to design a dynamic output controller for the extended system (34) that guarantees 2-contraction via Theorem 3 and Algorithm 1. In other words, by letting $z = \mathrm{col}(\eta, \zeta)$ in (34), we aim to design matrices M, N, K, L in (9) such that the closed-loop system (2), (10) with $x = \mathrm{col}(z, \xi)$ satisfies (32). This solution is comparable to a first-order method. Moreover, it does not require complete knowledge of the objective function, since the only requirement for Algorithm 1 to be applicable is the knowledge of the quadratic abstraction (23) satisfied by the gradient function $\frac{\partial V}{\partial v}(v)$. Differently put, the cost function V does not need to be known for the design or the application (if the gradient is measurable). Moreover, the obtained controller will guarantee convergence to extrema of any objective functions belonging to the family defined by the selected quadratic abstraction (23). Finally, such a behavior can be proven to be robust to slowly space-varying perturbations. Indeed, generalized Lyapunov conditions offer some robustness to perturbations with sufficiently small Jacobian norms, as shown by the following lemma.

Lemma 8. *Consider a smooth vector field $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ and suppose that (1) holds for some $\mu \in \mathbb{R}$, a symmetric and invertible matrix $P \in \mathbb{R}^{n_x \times n_x}$ and a compact set $\mathcal{X} \subset \mathbb{R}^{n_x}$. Then, there exist a positive constant $\Delta^\star > 0$ and $\bar{\mu} > \mu$ such that*

$$\frac{\partial \bar{f}}{\partial x}(x)^\top P + P \frac{\partial \bar{f}}{\partial x}(x) \prec 2\bar{\mu} P, \quad \forall x \in \mathcal{X}, \tag{35}$$

holds for any perturbed vector field $\bar{f}(x) = f(x) + \delta(x)$ with $|\frac{\partial \delta}{\partial x}(x)| < \Delta^\star$.

The proof is postponed to Appendix H.

Therefore, Lemma 8 shows that the controlled system will also stabilize extrema of objective functions whose gradient is "sufficiently close" (in the sense of Lemma 8) to the ones of functions in the family described by the quadratic abstraction.

4.3. Numerical example

Consider a system of the form (33) with matrices

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \overline{E} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Let $\mathcal{V} = [-1.5, 1.5]$ and consider a family of (possibly non-convex) objective functions $V : \mathcal{V} \to \mathbb{R}$ satisfying

$$-0.6 < \frac{\partial^2 V}{\partial v^2}(v) < 0.6, \quad \forall v \in \mathcal{V}. \tag{36}$$

Constraint (36) implies that any V in this family of objectives satisfies a differential sector condition, which can be represented as a quadratic abstraction of the form (11), see [42, Section III.C] for more details. The extended system (34) can be written in the closed-loop form (9) with an extended state $z = \mathrm{col}(\eta, \zeta) \in \mathbb{R}^3$, the following matrices:

$$A = \begin{bmatrix} 0 & \mathbf{0}_2^\top \\ \mathbf{0}_2 & \overline{A} \end{bmatrix} - 0.6DE, \quad B = \begin{bmatrix} 0 \\ \overline{B} \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - 0.6GE, \quad E = \begin{bmatrix} 0 & \overline{E} \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

with $\mathbf{0}_2 = \mathrm{col}(0, 0)$, and the following nonlinearity

$$\phi(v) = \frac{\partial V}{\partial v}(v) + 0.6. \tag{37}$$

By (36), the nonlinearity (37) satisfies a quadratic abstraction of the form (11) for all $v \in \mathcal{V}$ with $R_z = 0$, $S_z = E$, $Q_z = 2$. Therefore, the matrices in the quadratic abstraction (11) are known, $Q_z > 0$ and we can design the controller in line 2 of Algorithm 1 via Proposition 2. Moreover, since $R_z = 0$ we can select $Z = R_z$ in Proposition 2 and avoid the minimization in Remark 5. Since we need to solve simultaneously two inequalities of the form (3), one with a positive definite matrix P_0 and a second one with a matrix P_1 with inertia $\mathrm{In}(P_1) = (1, 0, 5)$, we combine Proposition 2 with Theorem 2 in line 9 of Algorithm 1. For solving (32b) with correct inertia, we constraint the inertia of the matrices in Proposition 2 via Lemma 5. Finally, we obtain the matrices U, V required for the control matrices (17) through the conditioning strategy proposed in Proposition 3.

Algorithm 1 under the aforementioned procedure outputs the controller

$$M = \begin{bmatrix} -39.9 & -0.001 & -0.126 \\ 0.74 & -2.96 & -3.15 \\ 0.56 & -1.13 & -8.35 \end{bmatrix}, \quad N = \begin{bmatrix} -324.07 & -7.27 \\ 5.38 & -3.29 \\ 4.01 & -2.86 \end{bmatrix},$$
$$K = \begin{bmatrix} 4.48 & -2.63 \end{bmatrix}, \qquad L = \begin{bmatrix} 0.62 & -2.09 & -4.56 \end{bmatrix},$$

The numerical values of the remaining matrices are not presented due to space constraints.

To validate the extremum control strategy, we first consider the following objective function

$$V(v) = 0.01v + 0.05v^2 - 0.025v^4. \tag{38}$$

For all values $v \in [-1.5, 1.5]$, (38) satisfies the bound (36). Additionally, the function presents one local maximum at $v \approx -0.945$ and a global one at $v \approx 1.047$. This function also presents a local minimum at $v \approx -0.101$. The graph of this function is depicted in Figure 2.

Under the obtained controller, numerical simulations confirm that the closed-loop system evolves in a compact and forward invariant set $\mathcal{X}$ with $Ex \in [-1.5, 1.5]$. Therefore, since the cost function (38) satisfies the bound (36), the closed-loop system is 2-contractive by Theorem 3. Consequently, by Lemma 7, trajectories initiated in $\mathcal{X}$ converge to one equilibrium point. The closed-loop system with cost function (38) presents three equilibrium points. The linearized system around each equilibrium point confirms that the two equilibrium points related to the maxima of V are locally asymptotically stable, while the other is unstable. This behavior can be seen in Figure 1.

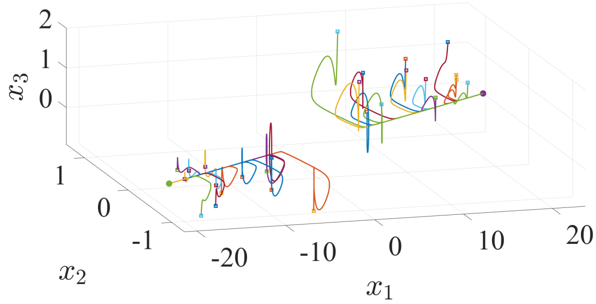

Figure 1: Evolution of 26 trajectories of the closed-loop system with the cost function (38). The squares depict the initial conditions. The points depict the asymptotically stable equilibrium of the system. The states of the controller have been obviated.

We highlight that, differently from some gradient-based algorithms, the convergence of the closed-loop trajectory to one maximum or the other depends both on the initial value of $V(Ex)$ and the initial condition of the closed-loop system, as shown in Figure 2. Indeed, even if both trajectories are initialized at the same value of Ex, one converges to the local maximum while the other crosses it.

To verify the claim of the controller being able to stabilize extrema of an arbitrary function satisfying (36), in Table 1 we present the point of convergence of the closed-loop system for different non-convex objective functions, alongside their extrema. The first function is non-convex in the considered domain due to the presence of one global minimum and one global maximum. The second function in Table 1[4] has a unique maximum but is only pseudo-convex. Since the function has a unique maximum, convergence to it is guaranteed by Lemma 7. The last three functions in Table 1 are the Ricker wavelet, the Morlet wavelet and a shifted Griewank function, respectively, that present multiple local minimum and maximum around their global maximum at zero (or near zero).

All simulations share the same initial condition, and in all of them the closed-loop system converges to one of the

[4] Given at the top of the next page.

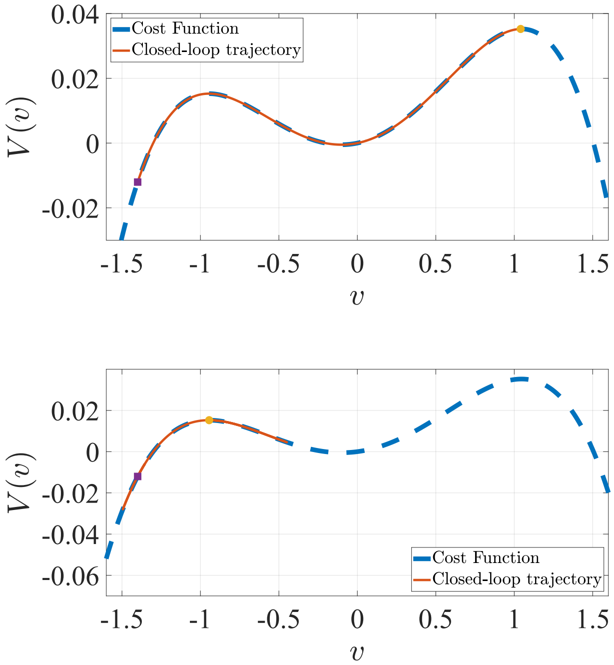

Figure 2: Graph of the cost function (38) and evolution of $V(Ex)$ of two different trajectories of the closed-loop system. The purple square depicts the initial conditions and the yellow circle the converging equilibrium point. The top figure has an initial condition $x = \begin{bmatrix}0 & -1.2 & 0 & 0 & 0 & 0\end{bmatrix}^\top$, while the bottom figure at $x = \begin{bmatrix}-10 & -1.2 & -0.8 & 174.13 & 3.7 & 0.97\end{bmatrix}^\top$.

extrema. In particular, in all simulations the closed loop stabilizes at the global maximum of the given function, even if initialized close to a local maximum/minimum. For instance, for the last function in Table 1, the system crosses multiple local maximums/minimums before reaching the global maximum. Nonetheless, we stress that this behavior does not imply the closed-loop system is guaranteed to converge to the global maximum. Indeed, Lemma 7 only guarantees convergence to one of the extrema of the cost function and we refer to Figure 2 for an example of this convergence to a local extremum.

5. Conclusion

In this paper, we presented computationally efficient methods for designing dynamic output-feedback controllers that ensure the resulting closed-loop system satisfies a generalized Lyapunov inequality. Assuming partially linear dynamics with nonlinearities that meet a quadratic abstraction, we derived LMI-based controller design conditions. We explored options for selecting the degrees of freedom within the LMIs and extended our approach to controllers that ensure the satisfaction of multiple inequalities. To address both soft and hard matrix inertia constraints, we proposed convex and non-convex criteria. We then applied our results in the context of extremum control and k-contraction. Promising future research directions include investigating semidefinite relaxations of the rank constraints involved in imposing matrix inertia and analyzing the inertia distribution in the solutions to the generalized Lyapunov inequalities. Future work will also explore discrete-time extensions, building on the recent studies [40, 8].

Appendix A. Proof of Lemma 1

Expanding the product, (11) yields

$$R_z + \frac{\partial\phi}{\partial x}(v)^\top S_z + S_z^\top \frac{\partial\phi}{\partial z}(v) + \frac{\partial\phi}{\partial z}(v)^\top Q_z \frac{\partial\phi}{\partial z}(v) \preceq 0.$$

Therefore, by (13b) we obtain

$$\overbrace{\mathbf{R}_{11} + \frac{\partial\phi}{\partial z}(v)^\top \mathbf{S}_1 + \mathbf{S}_1^\top \frac{\partial\phi}{\partial z}(v) + \frac{\partial\phi}{\partial z}(v)^\top \mathbf{Q} \frac{\partial\phi}{\partial z}(v)}^{\Theta} \\ - (\mathbf{R}_{12} + \mathbf{S}_2^\top \frac{\partial\phi}{\partial z}(v))^\top \mathbf{R}_{22}^\dagger (\mathbf{R}_{12} + \mathbf{S}_2^\top \frac{\partial\phi}{\partial z}(v)) \preceq 0.$$

Since $\mathbf{R}_{22} \preceq 0$ and (12) holds, by generalized Schur complement [39, Theorem 1.20] the above inequality is equivalent to

$$\begin{pmatrix} \Theta & \mathbf{R}_{12}^\top + \frac{\partial\phi}{\partial z}(v)^\top \mathbf{S}_2 \\ \mathbf{R}_{12} + \mathbf{S}_2^\top \frac{\partial\phi}{\partial z}(v) & \mathbf{R}_{22} \end{pmatrix} \preceq 0. \tag{A.1}$$

Note now that $\frac{\partial\phi}{\partial x}(v) = \begin{pmatrix}\frac{\partial\phi}{\partial z}(v) & 0\end{pmatrix}$. Therefore, (4) with (13) reads

$$\begin{pmatrix} \mathrm{I}_{n_z} & 0 \\ 0 & \mathrm{I}_{n_\xi} \\ \frac{\partial\phi}{\partial z}(v) & 0 \end{pmatrix}^\top \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12}^\top & \mathbf{S}_1^\top \\ \star & \mathbf{R}_{22} & \mathbf{S}_2^\top \\ \star & \star & \mathbf{Q} \end{pmatrix} \begin{pmatrix} \mathrm{I}_{n_z} & 0 \\ 0 & \mathrm{I}_{n_\xi} \\ \frac{\partial\phi}{\partial z}(v) & 0 \end{pmatrix} \preceq 0. \tag{A.2}$$

Straightforward computation shows the above relation is equivalent to (A.1), thus concluding the proof.

Appendix B. Proof of Theorem 1

The proof is grounded on the change of coordinates introduced in [36]. Therefore, before proving the result, we recall this variable change.

Without loss of generality and in view of the invertibility of P in (3), consider the partitions

$$P = \begin{pmatrix} X & U \\ U^\top & \overline{X} \end{pmatrix} \quad P^{-1} = \begin{pmatrix} Y & V \\ V^\top & \overline{Y} \end{pmatrix} \tag{B.1}$$

where $X = X^\top \in \mathbb{R}^{n_z \times n_z}, Y = Y^\top \in \mathbb{R}^{n_z \times n_z}$, U, V are square matrices and $\overline{X}, \overline{Y}$ are some matrices with no relevance for this proof. As previously mentioned, this decomposition is always possible by means of the assumption $n_\xi = n_z$. Following [36, Section IV.B], we exploit the transformation (15). By introducing the change of variables

$$\begin{aligned} \widehat{K} &:= K, \\ \widehat{L} &:= KCY + LV^\top, \\ \widehat{N} &:= XBK + UN, \\ \widehat{M} &:= X(A + BKC)Y + UNCY + (XBL + UM)V^\top, \end{aligned} \tag{B.2}$$

Function	Extrema	Convergence point
$-0.1e^{-v^2}v\cos(e^{-v^2}-0.5))$	-0.73 (glob. max) 0.73 (glob. min)	-0.73
$0.1v+0.125v^2-0.125v^4$	0.86 (glob. max)	0.86
$1.23e^{-2v^2}(1-4v^2)$	0 (glob. max) ± 0.87 (loc. min.)	0
$0.1e^{\frac{-v^2}{2}}\cos(5v)+0.1$	0 (glob. max) $-2.24, \pm 1.82, \pm 1.21, \pm 0.6$ (loc. max/loc. min)	0
$-0.005(1+((10v-1)^2)/4000-\cos(10v-1))$	0.1 (glob. max) 9 loc. min/ loc. max in $\mathcal{V}$	0.1

Table 1: Convergence of the closed-loop system for different cost functions. The first column depicts the function $V(v)$ considered. The second column depicts the extrema of this function. The last column is the point of convergence of the closed-loop system. In all the simulations, the initial condition of the systems has been $z = \mathrm{col}(\eta, \zeta) = \begin{bmatrix} 0 & -1.2 & 1 \end{bmatrix}$, $\xi = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$.

we obtain the relations

$$\begin{aligned}
\Pi_1^\top P\mathbf{A}\Pi_1 &= \begin{pmatrix} AY+B\widehat{L} & A+B\widehat{K}C \\ \widehat{M} & XA+\widehat{N}C \end{pmatrix}, \\
\Pi_1^\top P\mathbf{D} &= \begin{pmatrix} D+B\widehat{K}G \\ XD+\widehat{N}G \end{pmatrix}, \\
P\Pi_1 &= \Pi_2, \\
\Pi_1^\top P\Pi_1 &= \begin{pmatrix} Y & \mathrm{I}_{n_z} \\ \mathrm{I}_{n_z} & X \end{pmatrix}.
\end{aligned} \tag{B.3}$$

Note that if U and V are square and invertible, all design variables can be uniquely recovered from (B.2) going top-to-bottom, thus obtaining (17). Moreover, from (B.3) and the fact that V is invertible, Π_1 is invertible and one can recover $P = \Pi_2\Pi_1^{-1}$.

We are now ready to prove Theorem 1. We start by proving the first result of the theorem, namely, the existence of square matrices U, V such that $UV^\top = \mathrm{I}_{n_z} - XY$. Note that if item iii) holds, then $\Phi = \left(\begin{smallmatrix} Y & \mathrm{I}_{n_z} \\ \mathrm{I}_{n_z} & X \end{smallmatrix}\right)$ is invertible. Therefore, there exists a symmetric matrix $\Phi^{-1} = \left(\begin{smallmatrix} \Phi_1 & \Phi_2^\top \\ \Phi_2 & \Phi_3 \end{smallmatrix}\right)$ such that $\Phi\Phi^{-1} = \Phi^{-1}\Phi = \mathrm{I}_{2n_z}$. The expansion of the product $\Phi\Phi^{-1} = \mathrm{I}_{2n_z}$ leads to

$$\begin{aligned}
\Phi_1 = -X\Phi_2, \quad \Phi_3 = -Y\Phi_2^\top, \\
\mathrm{I}_{n_z} = Y\Phi_1 + \Phi_2 = (\mathrm{I}_{n_z} - YX)\Phi_2, \\
\mathrm{I}_{n_z} = \Phi_2^\top + X\Phi_3 = (\mathrm{I}_{n_z} - XY)\Phi_2^\top.
\end{aligned} \tag{B.4}$$

Similarly, the expansion of the product $\Phi^{-1}\Phi = \mathrm{I}_{2n_z}$, combined with the above relations leads, to

$$\begin{aligned}
X\Phi_2 = \Phi_2^\top X, \quad Y\Phi_2^\top = \Phi_2 Y \\
\mathrm{I}_{n_z} = \Phi_1 Y + \Phi_2^\top = \Phi_2^\top(\mathrm{I}_{n_z} - XY), \\
\mathrm{I}_{n_z} = \Phi_2 + \Phi_3 X = \Phi_2(\mathrm{I}_{n_z} - YX).
\end{aligned} \tag{B.5}$$

By combining the second (resp. third) line in (B.4) and the third (resp. second) line of (B.5), we deduce that Φ_2 (resp. $\Phi_2^\top$) is the inverse of $\mathrm{I}_{n_z} - YX$ (resp. $\mathrm{I}_{n_z} - XY$). Thus, $\mathrm{I}_{n_z} - YX$ and $\mathrm{I}_{n_z} - XY$ are non-singular. Consequently, there always exist square invertible matrices U, V such that $UV^\top = \mathrm{I}_{n_z} - XY$.

We now move to the second result of Theorem 1. Consider the left-hand side of (14). Under the choice $P = \Pi_2\Pi_1^{-1}$, by (10), (15), (B.2) and (B.3) it is equivalent to

$$\Psi_2 = \Omega - \begin{pmatrix} \widehat{R} & \widehat{S}^\top \\ \star & \widehat{Q} \end{pmatrix},$$

where

$$\Omega := \begin{pmatrix} \mathrm{He}\left\{\Pi_1^\top P\mathbf{A}\Pi_1\right\} - 2\beta\Pi_1^\top P\Pi_1 & \Pi_1^\top P\mathbf{D} \\ \mathbf{D}^\top P\Pi_1 & 0 \end{pmatrix} \tag{B.6}$$

and

$$\widehat{R} = \begin{pmatrix} \widehat{R}_{11} & \widehat{R}_{12}^\top \\ \star & \widehat{R}_{22} \end{pmatrix}, \quad \widehat{S} = \begin{pmatrix} \widehat{S}_1 & \widehat{S}_2 \end{pmatrix}.$$

Since item i) of Theorem 1 ensures negative semidefiniteness of $\mathbf{R}_{22}$ in (16), if (11) and (12) hold for all $v \in \mathcal{V}$ under the selections (13b) and (16), Lemma 1 guarantees (4) holds with (13a). Note that under selection (16) we have

$$\widehat{R} = \Pi_1^\top \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12}^\top \\ \star & \mathbf{R}_{22} \end{pmatrix} \Pi_1, \quad \widehat{S} = \begin{pmatrix} \mathbf{S}_1 & \mathbf{S}_2 \end{pmatrix} \Pi_1.$$

Hence, the following inequality also holds for all $v \in \mathcal{V}$

$$\begin{pmatrix} \mathrm{I}_{2n_z} \\ \frac{\partial\phi}{\partial x}(v)\Pi_1 \end{pmatrix}^\top \begin{pmatrix} \widehat{R} & \widehat{S}^\top \\ \star & \widehat{Q} \end{pmatrix} \begin{pmatrix} \mathrm{I}_{2n_z} \\ \frac{\partial\phi}{\partial x}(v)\Pi_1 \end{pmatrix} \preceq 0. \tag{B.7}$$

Let $\nu := \mathrm{col}(\mathrm{I}_{2n_z}, \frac{\partial\phi}{\partial x}(v)\Pi_1)$. Then, (14) and (B.7) imply $\nu^\top\Omega\nu \preceq \nu^\top\Psi_2\nu \prec 0$. Consider now the extremities of this last inequality. By (B.6), their relation can be equivalently written as

$$\Pi_1^\top \begin{pmatrix} \mathrm{I}_{2n_z} \\ \frac{\partial\phi}{\partial x}(v) \end{pmatrix}^\top \begin{pmatrix} \mathrm{He}\left\{P\mathbf{A}\right\} - 2\beta P & P\mathbf{D} \\ \star & 0 \end{pmatrix} \begin{pmatrix} \mathrm{I}_{2n_z} \\ \frac{\partial\phi}{\partial x}(v) \end{pmatrix} \Pi_1 \prec 0.$$

Since Π_1 is invertible, this implies

$$\begin{pmatrix} \mathrm{I}_{2n_z} \\ \frac{\partial\phi}{\partial x}(v) \end{pmatrix}^\top \begin{pmatrix} \mathrm{He}\left\{P\mathbf{A}\right\} - 2\beta P & P\mathbf{D} \\ \star & 0 \end{pmatrix} \begin{pmatrix} \mathrm{I}_{2n_z} \\ \frac{\partial\phi}{\partial x}(v) \end{pmatrix} \prec 0.$$

Expansion of the left-hand side product recovers (3) with $\mu = \beta$. Finally, equivalence in inertias $\mathrm{In}(\Phi) = \mathrm{In}(P)$ follows from (B.3) and [6, Lemma 1], thus concluding the proof.

Appendix C. Proof of Proposition 2

In view of Theorem 1, the result is proven if the selection $\widehat{R}_{12} = \alpha R_z Y$, $\widehat{R}_{22} = \alpha R_z$, $\widehat{S}_1 = \alpha S_z Y$, $\widehat{S}_2 = \alpha S_z$ and $\widehat{Q} = \alpha Q_z$ under (11) allows deriving an inequality of the form (B.7) via Lemma 1.

Note that if $\mathbf{R}_{12} = 0$ and $\mathbf{S}_2 = 0$, (12) is satisfied independently of $\mathbf{R}_{22}$. Moreover, if (11) holds and $\alpha > 0$, any quadratic abstraction of the form (11) with matrices $\alpha R_z, \alpha S_z, \alpha Q_z$ also holds. Therefore, by Lemma 1, if (11) holds and $\alpha > 0$, the extended inequality (A.2) holds for arbitrary $\mathbf{R}_{22} \preceq 0$ and $\mathbf{R}_{11} = \alpha R_z$, $\mathbf{R}_{12} = 0$, $\mathbf{S}_1 = \alpha S_z$, $\mathbf{S}_2 = 0$ and $\mathbf{Q} = \alpha Q_z$. We now show that items i) and ii) of Proposition 2 recover this quadratic abstraction. Consider item i) of Proposition 2. If $\widehat{R}_Z \preceq 0$ holds, $\widehat{R}_{11} \preceq 0$ and by generalized Schur complement (with $ZZ^\dagger Z = Z$) we have

$$\widehat{R}_{11} - \alpha YZY \preceq 0.$$

Since $Z \preceq R_z$, we have $-YZY \succeq -YR_zY$ and consequently

$$\widehat{R}_{11} - \alpha Y R_z Y \preceq \widehat{R}_{11} - \alpha YZY \preceq 0.$$

Note that under the selection $\widehat{R}_{12} = \alpha R_z Y$, $\widehat{R}_{22} = \alpha R_z$ the central portion of $\mathbf{R}_{22}$ in (16) reads

$$\begin{pmatrix} \mathrm{I}_{n_z} \\ -Y \end{pmatrix}^\top \begin{pmatrix} \widehat{R}_{11} & \alpha Y R_z \\ \alpha R_z Y & \alpha R_z \end{pmatrix} \begin{pmatrix} \mathrm{I}_{n_z} \\ -Y \end{pmatrix} = \widehat{R}_{11} - \alpha Y R_z Y \preceq 0.$$

Therefore, $\mathbf{R}_{22} \preceq 0$. Consider now (16) under the selection in item ii) of Proposition 2. Since $\widehat{R}_{12} = \alpha R_z Y = \widehat{R}_{22} Y$, $\widehat{S}_1 = \alpha S_z Y = \widehat{S}_2 Y$, we have $\mathbf{R}_{11} = \widehat{R}_{22} = \alpha R_z$, $\mathbf{R}_{12} = 0$, $\mathbf{S}_1 = \widehat{S}_2 = \alpha S_z$, $\mathbf{S}_2 = 0$ and $\mathbf{Q} = \widehat{Q} = \alpha Q_z$. Hence, Lemma 1 shows that inequality (B.7) holds under the given selection, thus concluding the proof.

Appendix D. Proof of Proposition 3

Consider the left inequality in (19a). Since $\mathbf{V} \succ 0$, a Schur complement on the bottom right entry ensures

$$\beta^2 \overline{P} \mathbf{V}^{-1} \overline{P} - \mathbf{V} \preceq 0.$$

Since $\widehat{V} \succ 0$, it can be uniquely decomposed as $\widehat{V} = \sqrt{\widehat{V}}\sqrt{\widehat{V}}$ with $\sqrt{\widehat{V}} \succ 0$. Therefore, left and right multiplication of the above inequality by $\mathrm{blkdiag}(\mathrm{I}_{n_z}, \sqrt{\widehat{V}}^{-1}) = \sqrt{\mathbf{V}}^{-1}$ yields

$$\sqrt{\mathbf{V}}^{-1} \overline{P} \mathbf{V}^{-1} \overline{P} \sqrt{\mathbf{V}}^{-1} \preceq \frac{1}{\beta^2} \lambda \, \mathrm{I}_{2n_z}. \tag{D.1}$$

Consider now the right inequality in (19a). Left and right multiplication by $\mathrm{blkdiag}(\overline{P}\mathbf{V}^{-1}, \mathrm{I}_{n_z})$ and its transpose yields

$$\begin{pmatrix} \overline{P}\mathbf{V}^{-1}\overline{P} & \mathrm{I}_{2n_z} \\ \mathrm{I}_{2n_z} & \gamma \, \mathrm{I}_{2n_z} \end{pmatrix} \succeq 0.$$

Consequently, a Schur complement on the bottom right entry gives

$$\overline{P}\mathbf{V}^{-1}\overline{P} - \frac{1}{\gamma} \mathrm{I}_{2n_z} \succeq 0.$$

Therefore, similarly to (D.1), left and right multiplication by $\sqrt{\mathbf{V}}^{-1}$ yields

$$\sqrt{\mathbf{V}}^{-1} \overline{P} \mathbf{V}^{-1} \overline{P} \sqrt{\mathbf{V}}^{-1} - \frac{1}{\gamma} \mathbf{V}^{-1} \succeq 0. \tag{D.2}$$

Let us move to the left inequality in (19b). Left and right multiplication by $\mathrm{blkdiag}(\mathbf{V}^{-1}, \mathrm{I}_{2n_z})$ yields

$$\begin{pmatrix} \mathbf{V}^{-1} & \mathrm{I}_{2n_z} \\ \mathrm{I}_{2n_z} & \alpha \, \mathrm{I}_{2n_z} \end{pmatrix} \succeq 0.$$

Then, via Schur complement on the bottom right entry and (D.2) we obtain

$$\sqrt{\mathbf{V}}^{-1} \overline{P} \mathbf{V}^{-1} \overline{P} \sqrt{\mathbf{V}}^{-1} \succeq \frac{1}{\gamma} \mathbf{V}^{-1} \succeq \frac{1}{\alpha\gamma} \mathrm{I}_{2n_z}. \tag{D.3}$$

Combining (D.1) and (D.3), if (19) hold then

$$\frac{1}{\alpha\gamma} \mathrm{I}_{2n_z} \preceq \sqrt{\mathbf{V}}^{-1} \overline{P} \mathbf{V}^{-1} \overline{P} \sqrt{\mathbf{V}}^{-1} \preceq \frac{1}{\beta^2} \lambda \, \mathrm{I}_{2n_z}. \tag{D.4}$$

Note that the set of $\mathbf{V}$ satisfying (D.4) is non-empty if the right inequality in (19b) holds. Indeed, a Schur complement ensures $\beta^2 \leq \alpha\gamma$. Consider now $P = \Pi_1^{-1}\Pi_2$ and notice that

$$\Pi_1^{-1} = \begin{pmatrix} 0 & V^{-\top} \\ \mathrm{I}_{n_z} & -YV^{-\top} \end{pmatrix}.$$

By selecting $V = \sqrt{\widehat{V}}$ and $U = (\mathrm{I}_{n_z} - XY)\sqrt{\widehat{V}}^{-1}$, we obtain

$$P = \sqrt{\mathbf{V}}^{-1} \overline{P} \sqrt{\mathbf{V}}^{-1}.$$

Therefore, (D.4) implies

$$\frac{1}{\alpha\gamma} \mathrm{I}_{2n_z} \preceq PP \preceq \frac{1}{\beta^2} \mathrm{I}_{n_z}.$$

Since P is symmetric and real, the eigenvalues of PP are the square of the singular values of P. Consequently, the above inequality implies

$$\bar{\sigma}(P)^2 \leq \frac{1}{\beta^2}, \qquad \underline{\sigma}(P)^2 \geq \frac{1}{\alpha\gamma},$$

thus proving the result.

Appendix E. Proof of Theorem 2

If conditions for Theorem 1 hold, there exists a matrix $P = \Pi_2 \Pi_1^{-1}$ such that (14) holds. Moreover, if (22) holds, its central block and Theorem 1 imply $\mathrm{In}(\widehat{\Gamma}_2) = \mathrm{In}(\Phi) = \mathrm{In}(P)$ [37]. Consider now (22) and notice that for each $i = 0, \ldots, s$ we have

$$\Psi_3^i = \Omega_i - \begin{pmatrix} \widehat{R} & 0 & \widehat{S}^\top \\ \star & 0 & 0 \\ \star & \star & \widehat{Q} \end{pmatrix},$$

where Ω_i is a symmetric matrix with elements

$$\Omega_i(1,1) = -2\beta_i\widehat{\Sigma}_i + \mathrm{He}\left\{\widehat{\Gamma}_1^i \left(\begin{smallmatrix} AY+B\widehat{L} & A+B\widehat{K}C \\ \widehat{M} & XA+\widehat{N}C \end{smallmatrix}\right)\right\},$$

$$\Omega_i(1,2) = \widehat{\Sigma}_i - \widehat{\Gamma}_1^i \left(\begin{smallmatrix} Y & \mathrm{I}_{n_z} \\ \mathrm{I}_{n_z} & X \end{smallmatrix}\right) + \left(\begin{smallmatrix} AY+B\widehat{L} & A+B\widehat{K}C \\ \widehat{M} & XA+\widehat{N}C \end{smallmatrix}\right)^\top (\widehat{\Gamma}_2^i)^\top,$$

$$\Omega_i(1,3) = \widehat{\Gamma}_1^i \left(\begin{smallmatrix} D+B\widehat{K}G \\ XD+\widehat{N}G \end{smallmatrix}\right) + \widehat{A}^\top (\widehat{\Gamma}_3^i)^\top,$$

$$\Omega_i(2,2) = -\mathrm{He}\left\{\widehat{\Gamma}_2^i \left(\begin{smallmatrix} Y & \mathrm{I}_{n_z} \\ \mathrm{I}_{n_z} & X \end{smallmatrix}\right)\right\},$$

$$\Omega_i(2,3) = \widehat{\Gamma}_2^i \left(\begin{smallmatrix} D+B\widehat{K}G \\ XD+\widehat{N}G \end{smallmatrix}\right) - \left(\begin{smallmatrix} Y & \mathrm{I}_{n_z} \\ \mathrm{I}_{n_z} & X \end{smallmatrix}\right)(\widehat{\Gamma}_3^i)^\top,$$

$$\Omega_i(3,3) = \mathrm{He}\left\{\widehat{\Gamma}_3^i \left(\begin{smallmatrix} D+B\widehat{K}G \\ XD+\widehat{N}G \end{smallmatrix}\right)\right\},$$

where $\Omega_i(a,b)$ stands for the block of Ω_i at position (a,b). By following the same initial steps as in the proof of Theorem 1, under the conditions of Theorem 1 and (16) we have

$$\begin{pmatrix} \mathrm{I}_{2n_z} \\ \Pi_1^{-1}\frac{\partial f}{\partial x}(x,w)\Pi_1 \\ \frac{\partial \phi}{\partial x}(v)\Pi_1 \end{pmatrix}^\top \begin{pmatrix} \widehat{R} & 0 & \widehat{S}^\top \\ \star & 0 & 0 \\ \star & \star & \widehat{Q} \end{pmatrix} \begin{pmatrix} \mathrm{I}_{2n_z} \\ \Pi_1^{-1}\frac{\partial f}{\partial x}(x,w)\Pi_1 \\ \frac{\partial \phi}{\partial x}(v)\Pi_1 \end{pmatrix} \preceq 0.$$

Let $\nu := \mathrm{col}(\mathrm{I}_{2n_z}, \Pi_1^{-1}\frac{\partial f}{\partial x}(x,w)\Pi_1, \frac{\partial \phi}{\partial x}(v)\Pi_1)$. Then, we have $\nu^\top\Omega_i\nu \preceq \nu^\top\Psi_3^i\nu \prec 0$. Therefore, by (B.3), $\nu^\top\Omega_i\nu \prec 0$ implies

$$\Pi_1^\top \begin{pmatrix} \mathrm{I}_{2n_z} \\ \frac{\partial f}{\partial x}(x,w) \\ \frac{\partial \phi}{\partial x}(v) \end{pmatrix}^\top \bar{\Omega}_i \begin{pmatrix} \mathrm{I}_{2n_z} \\ \frac{\partial f}{\partial x}(x,w) \\ \frac{\partial \phi}{\partial x}(v) \end{pmatrix} \Pi_1 \prec 0,$$

where

$$\bar{\Omega}_i(1,1) = -2\beta\Pi_1^{-\top}\widehat{\Sigma}_i\Pi_1^{-1} + \mathrm{He}\left\{\Pi_1^{-\top}\widehat{\Gamma}_1^i\Pi_2^\top\mathbf{A}\right\},$$

$$\bar{\Omega}_i(1,2) = \Pi_1^{-\top}\widehat{\Sigma}_i\Pi_1^{-1} + \Pi_1^{-\top}\widehat{\Gamma}_1^i\Pi_2^\top + \mathbf{A}^\top\Pi_2(\widehat{\Gamma}_2^i)^\top\Pi_1^{-1},$$

$$\bar{\Omega}_i(1,3) = \Pi_1^{-\top}\widehat{\Gamma}_1^i\Pi_2^\top\mathbf{D} + \mathbf{A}^\top\Pi_2(\widehat{\Gamma}_3^i)^\top,$$

$$\bar{\Omega}_i(2,2) = -\mathrm{He}\left\{\Pi_1^{-\top}\widehat{\Gamma}_2^i\Pi_2^\top\right\},$$

$$\bar{\Omega}_i(2,3) = \Pi_1^{-\top}\widehat{\Gamma}_2^i\Pi_2^\top\mathbf{D} - \Pi_2(\widehat{\Gamma}_3^i)^\top,$$

$$\bar{\Omega}_i(3,3) = \mathrm{He}\left\{\widehat{\Gamma}_3^i\Pi_2^\top\right\}.$$

By invertibility of Π_1, following the same steps as in the proof of Proposition 1 the closed loop is proven to satisfy (1) with $P = \Pi_1^{-\top}\widehat{\Sigma}_i\Pi_1^{-1}$, $\Gamma_1 = \Pi_1^{-\top}\widehat{\Gamma}_1^i\Pi_2^\top$, $\Gamma_2 = \Pi_1^{-\top}\widehat{\Gamma}_2^i\Pi_2^\top$, $\Gamma_3 = \widehat{\Gamma}_3^i\Pi_2^\top$ for each $i = 0,\dots,s$. Consequently, the simultaneous satisfaction of the conditions for Theorem 1 and s conditions of the form (22) ensures $s+1$ differential inequalities (1) are satisfied by the closed loop, thus concluding the proof.

Appendix F. Proof of Lemma 2

We start by rewriting the left-hand side of (23) to the scenario of implicit functions. By (25), we have

$$\left(\begin{smallmatrix} \mathrm{I}_{n_x} & 0 & 0 \\ 0 & \mathrm{I}_{n_v} & -\mathrm{I}_{n_v} \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathrm{I}_{n_x} & 0 \\ 0 & \mathrm{I}_{n_v} \\ 0 & \frac{\partial \phi}{\partial v}(v)\mathbf{F} \end{smallmatrix}\right)\begin{pmatrix} \mathrm{I}_{n_x} \\ \frac{\partial \phi}{\partial x}(v) \end{pmatrix} = \begin{pmatrix} \mathrm{I}_{n_x} \\ \frac{\partial \phi}{\partial v}(v)\mathbf{E} \end{pmatrix}.$$

Therefore, right and left multiplication of the left-hand side of (23) by $\mathbf{E}$ and its transpose, combined with the above relation, yields

$$\begin{pmatrix} \mathrm{I}_{n_x} \\ \frac{\partial \phi}{\partial v}(v)\mathbf{E} \end{pmatrix}^\top \Lambda \begin{pmatrix} \mathrm{I}_{n_x} \\ \frac{\partial \phi}{\partial v}(v)\mathbf{E} \end{pmatrix} = \begin{pmatrix} \mathrm{I}_{n_x} \\ \frac{\partial \phi}{\partial x}(v) \end{pmatrix}^\top \nu^\top\bar{\Xi}\nu \begin{pmatrix} \mathrm{I}_{n_x} \\ \frac{\partial \phi}{\partial x}(v) \end{pmatrix}. \tag{F.1}$$

where

$$\Lambda := \begin{pmatrix} \mathbf{E}^\top R_v\mathbf{E} & \mathbf{E}^\top S_v^\top \\ \star & Q_v \end{pmatrix}, \quad \nu := \begin{pmatrix} \mathrm{I}_{n_x} & 0 \\ 0 & \mathrm{I}_{n_v} \\ 0 & \frac{\partial \phi}{\partial v}(v)\mathbf{F} \end{pmatrix},$$
$$\bar{\Xi} := \begin{pmatrix} \Lambda & \begin{pmatrix} -\mathbf{E}^\top S_v^\top \\ -Q_v \end{pmatrix} \\ \star & Q_v \end{pmatrix}. \tag{F.2}$$

Consider now (23). By right and left multiplication by $\mathbf{E}$ and its transpose, (F.1) and (F.2), the quadratic abstraction (23) implies

$$\begin{pmatrix} \mathrm{I}_{n_x} \\ \frac{\partial \phi}{\partial x}(v) \end{pmatrix}^\top \nu^\top\bar{\Xi}\nu \begin{pmatrix} \mathrm{I}_{n_x} \\ \frac{\partial \phi}{\partial x}(v) \end{pmatrix} \preceq 0. \tag{F.3}$$

Note that (26b) equivalently reads

$$\Xi = \begin{pmatrix} R & S^\top & 0 \\ S & Q & 0 \\ 0 & 0 & 0 \end{pmatrix} - \bar{\Xi}.$$

Therefore, for all $v \in \mathcal{V}$, (26a) implies

$$\begin{pmatrix} R & S^\top \\ S & Q \end{pmatrix} = \nu^\top \begin{pmatrix} R & S^\top & 0 \\ S & Q & 0 \\ 0 & 0 & 0 \end{pmatrix} \nu \preceq \nu^\top\Xi\nu. \tag{F.4}$$

The proof is concluded by combining (F.3) and (F.4).

Appendix G. Proof of Lemma 5

Consider any symmetric matrix $Q \in \mathbb{R}^{n_x \times n_x}$. The proof is based on the following equality [1, Theorem 6].

$$\begin{aligned} \sum_{i=1}^{p+1} \lambda_i(Q) = \min \quad & \gamma \\ \text{s.t.} \quad & (p+1)z + \mathrm{tr}\, X \preceq \gamma \\ & z\,\mathrm{I} + X - Q \succeq 0 \\ & X \succeq 0. \end{aligned} \tag{G.1}$$

Then, for any matrix $Q \in \mathbb{R}^{n_x \times n_x}$, constant $p \in \{0,\dots,n-1\}$ and constant $\gamma \in \mathbb{R}$ such that

$$\begin{aligned} X &\succeq 0 \\ z\,\mathrm{I} + X - Q &\succeq 0 \\ (p+1)z + \mathrm{tr}\, X &\preceq \gamma, \end{aligned} \tag{G.2}$$

we necessarily have that

$$\sum_{i=1}^{p+1} \lambda_i(Q) \leq \gamma. \tag{G.3}$$

In other words, for any feasible solution of (G.2), the constant γ is bounding the sum of $p+1$ eigenvalues with largest real part of Q. Notice that (28) is (G.2) with $\gamma = 0$ and $Q = -P$. Combining this fact with (G.3) we have that

$$\sum_{i=1}^{p+1} \lambda_i(-P) = -\sum_{i=n-p}^{n} \lambda_i(P) \leq 0,$$

which implies

$$\sum_{i=n-p}^{n} \lambda_i(P) \geq 0. \tag{G.4}$$

The eigenvalue ordering (see Notation) combined with the bound (G.4) implies that $\lambda_i(P) \geq 0$ for all $i \in \{1, \ldots, n-p\}$, thus, $\pi_+(P) \geq n-p$. Now, since $\pi_+(P) + \pi_-(P) \leq n$, the bound on the positive eigenvalues imply that $\pi_-(P) \leq p$. Moreover, it implies $\Re(\lambda_{n-p}(P)) \geq -\sum_{i=n-p+1}^{n} \Re(\lambda_i(P))$.

Appendix H. Proof of Lemma 8

Since $\mathcal{X}$ is a compact and bounded set, this inequality implies the existence of a constant $\varepsilon > 0$ such that

$$\frac{\partial f}{\partial x}(x)^\top P + P\frac{\partial f}{\partial x}(x) \preceq 2\mu P - \varepsilon \, \mathrm{I}_{n_x}, \quad \forall x \in \mathcal{X} \subsetneq \mathbb{R}^{n_x}.$$

Let $\bar{f}(x) = f(x) + \delta(x)$ with a perturbation δ such that $|\frac{\partial \delta}{\partial x}(x)| < \Delta$ for some positive real $\Delta > 0$. Then,

$$\frac{\partial \bar{f}}{\partial x}(x)^\top P + P\frac{\partial \bar{f}}{\partial x}(x) \preceq 2\mu P - \varepsilon \, \mathrm{I}_{n_x} + \frac{\partial \delta}{\partial x}(x)^\top P + P\frac{\partial \delta}{\partial x}(x),$$

for all $x \in \mathcal{X}$. By Young's inequality, the following holds

$$\frac{\partial \delta}{\partial x}(x)^\top P + P\frac{\partial \delta}{\partial x}(x) \preceq \gamma P + \frac{1}{\gamma}\frac{\partial \delta}{\partial x}(x)^\top \frac{\partial \delta}{\partial x}(x)$$

for any $\gamma > 0$. Therefore

$$\frac{\partial \bar{f}}{\partial x}(x)^\top P + P\frac{\partial \bar{f}}{\partial x}(x) \preceq 2\bar{\mu} P - (\varepsilon - \frac{\Delta}{\gamma}) \, \mathrm{I}_{n_x}, \quad \forall x \in \mathcal{X} \subsetneq \mathbb{R}^{n_x},$$

where $\bar{\mu} = \mu + \frac{\gamma}{2}$. By selecting γ small enough such that $\mathrm{In}(\frac{\partial \bar{f}}{\partial x}(x) - \bar{\mu}\,\mathrm{I}) = \mathrm{In}(\frac{\partial f}{\partial x}(x) - \mu\,\mathrm{I})$ for all $x \in \mathcal{X}$, if $\Delta < \varepsilon\gamma := \Delta^*$, the perturbed dynamics satisfy (35) thus concluding the proof.

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