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A Tutorial on Positive Systems and Large Scale Control

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Abstract— **In this tutorial paper we first present some foundational results regarding the theory of positive systems. In particular, we present fundamental results regarding stability, positive realization and positive stabilization by means of state-feedback. Special attention is also paid to the system performance in terms of disturbance attenuation. Under the asymptotic stability assumption, such performance can be measured in terms of L_p-gain of the positive system. In the second part of the paper we propose some recent results about control synthesis by linear programming and semidefinite programming, under the positivity requirement on the resulting controlled system. These results highlight the value of positivity when dealing with large scale systems. Indeed, stability properties for these systems can be verified by resorting to linear (copositive) or diagonal Lyapunov functions that scale linearly with the system dimension, and such linear functions can be used also to design stabilizing feedback control laws. In addition, stabilization problems with disturbance attenuation performance can be easily solved by imposing special structures on the state feedback matrices. This is extremely valuable when dealing with large scale systems for which state feedback matrices are typically sparse, and their structure is a priori imposed by practical requirements.**

I. Introduction and Motivating Examples

In its broadest meaning, a *positive system* is simply a system whose describing variables can only take positive, or at least nonnegative, values. The class of natural and technological phenomena that can be naturally described by means of a positive system is far larger than one would expect at a first glance. Indeed, lots of physical quantities (concentrations, population levels, buffer sizes, queue lengths, charge levels, light intensity levels, prices, production quantities, etc.) are naturally constrained to be nonnegative, and any accurate mathematical model accounting for their dynamic evolution has to necessarily incorporate this constraint. Mathematical models with the positivity constraint on their describing variables have therefore flourished in several research areas, e.g., biology, ecology, physiology and pharmacology [17], [19], [37], [38], [39], [44], biomolecular and biochemical modelling [12], [13], [22], thermodynamics [12], [38], epidemiology [1], [40], [53], traffic and congestion modelling [12], [67], power systems [77], filtering and charge routing networks [6], [7], [10], econometrics [55], etc. In addition, since probabilities are nonnegative quantities, Markov chains [66], Hidden Markov chains and other probabilistic models represent special cases of positive systems.

In this tutorial paper we mainly focus our attention on positive linear state-space models, on which most of the research efforts in the last 40 years have focused, and hence in the following by a positive system we will always mean a positive linear state-space model. The merit of stimulating a systematic study of this class of systems, and hence of initiating what is nowadays known as "Positive System Theory", must be credited to David Luenberger, who published in 1979 a fundamental book [50] whose Chapter 6 was entirely devoted to positive linear systems (while Chapter 7 addressed Markov chains). Not unexpectedly, the first aspects that were investigated were asymptotic behavior (in particular, simple and asymptotic stability) and equilibrium points. Research on these topics heavily relies on (and benefits from) the well-know "Positive Matrix Theory" developed by Perron and Frobenius [35], [57] as well as by Karpelevich [46]. When moving a step further into classic System Theory problems, however, new mathematical tools had to be developed, mainly relying on graph theory and cone theory. In fact, in the nineties there was a long stream of research focusing on positive controllability and reachability [16], [19], [28], [32], [56], [64], [72], [73], observability of positive systems and their observers [4], [20], and positive realization [3], [9], [29], [30], [51], [56], [75].

The first part of this survey aims at recalling the fundamental results obtained about the stability and the state-feedback stabilization of positive systems, as well as on the positive realization problem. The interested reader is referred to the surveys [8], [9], [31] and to the fundamental books [33], [45] for a more complete account on these subjects, as well as for a more detailed list of references. Other subjects of intense research were robust stability [41], [42], [69] and positive stabilization [2], [21], [36], [63].

More recently, the interest has focused on quite different control problems, in particular, robust positive stabilization and system performance [2], [15], [25], the bounded real lemma [70] and the Kalman-Yakubovic-Popov lemma [60] for positive systems, decentralized and distributed control [26], [58], large scale positive systems and scalable control [27], [59]. These results have highlighted the deep impact of Positive System Theory in the study of large scale systems. Indeed, the use of nonnegative matrices for the study of large scale systems has long been recognized (see for example the surveys [54] and [65]). However, while criteria based on nonnegative matrices tend to be conservative when applied to generic linear systems, they are tight in the context of positive systems. There are two main reasons why positive systems are more tractable than general ones in a large scale setting:

1) Stability and performance analysis of standard linear systems require Lyapunov functions with a number

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of parameters that grows quadratically with the state dimension. This is not feasible when dealing with large scale systems. All the results for positive systems in this paper are based on Lyapunov functions with only linear growth. This makes a huge difference when the system size grows.
2) Standard linear system theory can only handle feedback laws taking the form $\mathbf{u} = K\mathbf{x}$, where the matrix K is dense. Again this leads to quadratic growth when the number of inputs is proportional to the number of states. For positive systems, sparse matrices K can be optimized with standard tools. This makes it possible to keep the complexity linear and manageable even when the input dimension grows with the state dimension. For positive systems, sparse matrices K can be optimized with standard tools, and this keeps the complexity linear and manageable.

During the the past decade, there has been a rapid growth of large scale control applications triggered by new information technology and internet-based services. This has triggered renewed interest in scalable control paradigms and positive systems [59], [76]. For example, hundreds of thousands of papers have been devoted to the analysis of consensus dynamics, which is a special form of positive system (see, in this respect, [71], [74] that specifically address the consensus problem under the positivity constraint on the agents' description).

In the second part of this paper, we will put particular emphasis on scalable control synthesis methods based on input-output gains. This is a problem formulation that has been standard practice in general multi-variable control theory for more than twenty years [78], but the results for positive systems are much more recent.

A recurrent example used to illustrate the results will be a simple model of a buffer network, i.e., a graph structure whose nodes behave as buffers. They are subject both to local inflow/outflow and to exchange with neighboring nodes. In formal terms, given a directed graph $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set of nodes and $\mathcal{E}$ is a set of edges, we will consider state-space models whose ith state variables are described by

$$\dot{x}_i = a_i x_i + \sum_{(i,j)\in\mathcal{E}} u_{ij} - \sum_{(j,i)\in\mathcal{E}} u_{ji} + w_i \qquad i \in \mathcal{V}. \tag{1}$$

See Figure 1 for an illustration. Here x_i represents the buffer content of node i and the input u_{ij} the flow from node j to node i. The term $a_i x_i$ describes natural decay (or growth) of the buffer content, while w_i describes the effect of local production or consumption.

In detail, the paper is organized as follows. Section II will introduce the fundamental notation and the (minimal) required background material. Positive systems and stability properties will be investigated in Section III, together with the positive system performance in terms of L_p-gains, $p \in (0, +\infty]$, upon assuming the positive system as an input-output map driven by an external disturbance. Positive realization will be the subject of Section IV, while positive state-feedback stabilization will be investigated in Section V. Control synthesis (with norm constraints) by linear programming and semidefinite programming will be investigated in Sections VI and VII, thus providing evidence for our previous claim regarding the pros of positivity when dealing with large scale systems. This aspect will be further explored in Section VIII, where some results about large scale H_∞ optimal control will be presented. Section IX will provide some extensions of the previous results to the general class of monotone systems, while bilinear positive systems will be addressed in Section X.

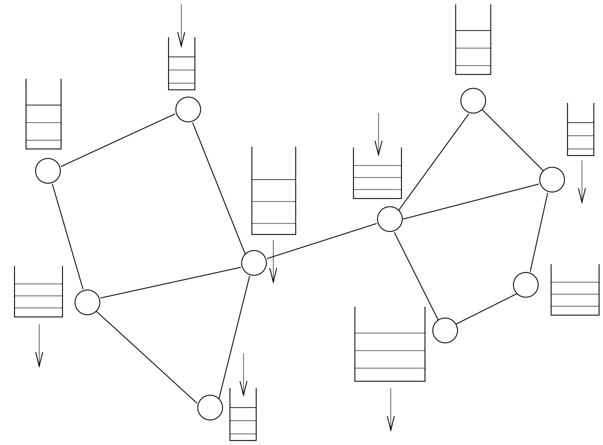

Fig. 1. Positive systems are commonly used to model dynamics of buffer networks (1). Each state represents the content of a buffer. Content can be transferred from one buffer to another via the network links. The content of a buffer can also change as a result of local production or consumption.

To conclude this Introduction, we would like to remark that an extensive research activity has been devoted to classes of systems that represent natural extensions of the class of positive systems here considered. In particular, we mention positive systems with delay, positive switched systems, positive two-dimensional (2D) systems, positive fractional systems, and monotone systems. While we will briefly consider monotone systems at the end of this tutorial, space constraints prevent us from addressing the other topics.

II. Notation and Background Material

Given $p \in \mathbb{Z}, p > 0$, we set $[1, p] := \{1, 2, \dots, p\}$. We denote by $\mathbf{e}_i$ the ith *canonical vector* in $\mathbb{R}^n$, and by $\mathbf{1}_n$ and $\mathbf{0}_n$ the n-dimensional vectors with all entries equal to 1 and 0, respectively. Given $A \in \mathbb{R}^{n\times n}$, we denote by $\sigma(A)$ the *spectrum* of A, i.e., the set of its eigenvalues, and by $\rho(A) := \max\{|\lambda| : \lambda \in \sigma(A)\}$ its *spectral radius*. A is *Schur* if $\lambda \in \sigma(A)$ implies $|\lambda| < 1$ (namely $\rho(A) < 1$), and it is *Hurwitz* if $\lambda \in \sigma(A)$ implies $\mathrm{Re}(\lambda) < 0$. The (i, j)th entry of a matrix A will be denoted by a_{ij}, and the ith entry of a vector $\mathbf{v}$ by v_i. Given n real numbers $d_1, d_2, \dots, d_n$, we denote by $\mathrm{diag}\{d_1, d_2, \dots, d_n\}$ the $n \times n$ diagonal matrix whose (i, i)th entry is d_i. Given a matrix M, $M^\top$ denotes its *transpose*, while M^* its *conjugate transpose*.

Given two positive integers m and n, a *sparsity structure* $\mathcal{S}$ in $\mathbb{R}^{m\times n}$ is a set of matrices in $\mathbb{R}^{m\times n}$ whose nonzero pattern is constrained, i.e.

$$\mathcal{S} = \{K \in \mathbb{R}^{m\times n} \mid k_{ij} = 0 \text{ for } (i,j) \notin \mathcal{E}\}, \tag{2}$$

for some given subset $\mathcal{E}$ of $[1,m]\times[1,n]$.

$\mathbb{R}_+$ is the semiring of nonnegative real numbers. A matrix (in particular, a vector) A with entries in $\mathbb{R}_+$ is called *nonnegative*, and if so we adopt the notation $A \geq 0$. If, in addition, A has at least one positive entry, the matrix is *positive* ($A > 0$), while if all its entries are positive, it is *strictly positive* ($A \gg 0$). Given a nonnegative matrix $A \in \mathbb{R}_+^{n\times n}$, Perron-Frobenius Theorem [52] ensures that $\rho(A)$ is always an eigenvalue of A, corresponding to a positive eigenvector. A *Metzler matrix* is a real square matrix, whose off-diagonal entries are nonnegative.

A set $\mathcal{K} \subset \mathbb{R}^n$ is said to be a *cone* provided that for every $\mathbf{x} \in \mathcal{K}$ and every $\alpha \in \mathbb{R}, \alpha > 0$, the vector $\alpha\mathbf{x}$ belongs to $\mathcal{K}$. If the cone $\mathcal{K} \subset \mathbb{R}^n$ contains an open ball of $\mathbb{R}^n$, then it is said to be *solid*; if $\mathcal{K} \cap -\mathcal{K} = \{0\}$, then $\mathcal{K}$ is said to be *pointed*. A cone is *convex* if $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{K}$ implies $\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in \mathcal{K}$ for every $\alpha \in \mathbb{R}, 0 \leq \alpha \leq 1$, and it is *closed* if it is a closed set of $\mathbb{R}^n$. A closed, convex, solid and pointed cone is a *proper cone*. A cone is said to be *polyhedral* if it coincides with the set of all nonnegative linear combinations of a finite family of vectors, i.e. $\mathcal{K} = \mathrm{Cone}(\mathbf{v}_1, \dots, \mathbf{v}_N) := \{\sum_{i=1}^N \alpha_i \mathbf{v}_i : \alpha_i \in \mathbb{R}, \alpha_i \geq 0\}$, for some $\mathbf{v}_1, \dots, \mathbf{v}_N \in \mathbb{R}^n$.

A symmetric matrix $P = P^\top \in \mathbb{R}^{n\times n}$ is said to be *positive definite* (*positive semidefinite*) and if so we adopt the notation $P \succ 0$ ($P \succeq 0$) if for every vector $\mathbf{x} \in \mathbb{R}\setminus\{0\}$ we have $\mathbf{x}^\top P\mathbf{x} > 0$ ($\mathbf{x}^\top P\mathbf{x} \geq 0$). A symmetric matrix $P = P^\top \in \mathbb{R}^{n\times n}$ is said to be *negative definite* (*negative semidefinite*) and if so we adopt the notation $P \prec 0$ ($P \preceq 0$) if $-P$ is positive definite (positive semidefinite).

Given a vector $\mathbf{x} \in \mathbb{R}_+^n$, for any $p \in (0,+\infty)$ we define its *vector p-norm* as $|\mathbf{x}|_p := (\sum_{i=1}^n x_i^p)^{\frac{1}{p}}$, while the *vector ∞-norm* is $|\mathbf{x}|_\infty := \max_{i\in[1,n]} |x_i|$. Given a matrix $M \in \mathbb{R}^{l\times m}$ and $p \in (0,+\infty]$, we define the *induced matrix p-norm* as

$$\|M\|_{p-\mathrm{ind}} := \sup_{\mathbf{x}:|\mathbf{x}|_p=1} |M\mathbf{x}|_p.$$

Given a function $f : \mathbb{R}_+ \to \mathbb{R}^n$, for every $p \in (0,+\infty)$ the *L_p-norm* of f is defined as

$$\|f\|_{L_p} := \left(\int_0^{+\infty} |f(t)|_p^p dt\right)^{\frac{1}{p}},$$

while for $p = +\infty$ we have

$$\|f\|_{L_\infty} := \mathrm{ess\ sup}_{t\geq 0}|f(t)|_\infty.$$

We denote by L_p^k the set of functions $f : \mathbb{R}_+ \to \mathbb{R}^k$ having finite L_p-norm. If $\mathcal{G}$ is an operator from L_p^m to L_p^r, $p \in (0,+\infty]$, its *L_p-gain* is defined as

$$\|\mathcal{G}\|_{L_p-L_p} := \sup_{\mathbf{w}:\|\mathbf{w}\|_{L_p}=1} \|\mathcal{G}\mathbf{w}\|_{L_p}.$$

Given a proper rational matrix $G(s) \in \mathbb{R}(s)^{r\times m}$, with no poles in the right half-plane, we define its *H_∞-norm* as

$$\|G\|_{H_\infty} := \sup_\omega \|G(i\omega)\|_{2-\mathrm{ind}}.$$

It is well known [78, Chapter 4] that if $\mathcal{G}$ is an operator from L_2^m to L_2^r, associated with a linear time-invariant asymptotically stable state-space model of transfer function $G(s)$, then $\|\mathcal{G}\|_{L_2-L_2} = \|G\|_{H_\infty}$.

III. Positive systems and stability

In general terms, a positive system is a system whose describing variables are constrained to take positive (or at least nonnegative) values. Most of the literature on positive systems, however, has focused on the specific class of (positive) linear state-space models described by the following equations

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{3a}$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t), \qquad t \in \mathbb{Z}_+, \tag{3b}$$

in the discrete-time case, and by the following equations

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{4a}$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t), \qquad t \in \mathbb{R}_+, \tag{4b}$$

in the continuous-time case. In these equations $\mathbf{x}$ represents the n-dimensional state variable, $\mathbf{u}$ the m-dimensional input variable and $\mathbf{y}$ the r-dimensional output variable.

For systems (3) and (4) two notions of positivity have been defined [33], [45], [50]:

- *internal positivity*: for every nonnegative initial condition $\mathbf{x}(0)$ and every nonnegative input $\mathbf{u}(t), t \geq 0$, the associated state and output evolutions $\mathbf{x}(t)$ and $\mathbf{y}(t), t \geq 0$, remain nonnegative at every time instant t;
- *external positivity*: assuming zero initial condition $\mathbf{x}(0)$, for every nonnegative input $\mathbf{u}(t), t \geq 0$, the associated output evolution $\mathbf{y}(t), t \geq 0$, remains nonnegative at every time instant t.

In the discrete-time case internal positivity is equivalent to the fact that A, B, C and D are nonnegative matrices, while in the continuous-time case to the fact that A is a Metzler matrix, while B, C and D are nonnegative matrices. On the other hand, external positivity is equivalent to the nonnegativity of the system impulse response, which is expressed, in the discrete-time case, as

$$g(t) = \begin{cases} D, & t = 0, \\ CA^{t-1}B, & t \in \mathbb{Z}_+, t \geq 1, \end{cases} \tag{5}$$

and in the continuous-time case as

$$g(t) = Ce^{At}B\delta_{-1}(t) + D\delta(t), \tag{6}$$

where $\delta(t)$ is the Dirac impulse, while $\delta_{-1}(t)$ is the unitary step function. It is well known [33], [45], [50] that internal positivity ensures external positivity, while the converse is not true, and we will come back to this topic in Section IV.

In the following, by a *positive system* we will always mean an internally positive system. The first important property that has been investigated for this class of systems is, of course, stability in its various forms.

Definition 1: Systems (3) and (4) are said to be

- *asymptotically stable* (equivalently, exponentially stable) if, for every positive initial state $\mathbf{x}(0)$, the corresponding unforced state evolution converges to zero as t goes to $+\infty$;
- *simply stable* if, for every positive initial state $\mathbf{x}(0)$, the corresponding unforced state evolution remains (positive and) bounded.

It turns out that the fact that initial conditions are confined to belong to the positive orthant does not affect the standard characterizations of asymptotic stability and simple stability for linear state-space models. Indeed, [33], [45], [50] system (3) (system (4)) is asymptotically stable if and only if A is a positive Schur matrix (a Metzler Hurwitz matrix). On the other hand, system (3) (system (4)) is simply stable if and only if $\lambda \in \sigma(A)$ implies $|\lambda| \leq 1$ and when $|\lambda| = 1$ then λ is a simple root of the minimal annihilating polynomial of A ($\lambda \in \sigma(A)$ implies $\mathrm{Re}(\lambda) \leq 0$ and when $\mathrm{Re}(\lambda) = 0$ then λ is a simple root of the minimal annihilating polynomial of A). In the following we will focus our attention on asymptotic stability, since this is the property one typically wants a positive system to exhibit.

In addition to the standard characterizations available for Schur or Hurwitz matrices, positivity brings novel characterizations that we will comment on in some detail. Specifically, we have the following two propositions.

Proposition 1: [33], [45] For a positive matrix $A \in \mathbb{R}^{n\times n}$, the following properties are equivalent:

(1.1) A is Schur;
(1.2) There exists $\boldsymbol{\xi} \gg 0$ such that $A\boldsymbol{\xi} \ll \boldsymbol{\xi}$;
(1.3) There exists $\mathbf{z} \gg 0$ such that $\mathbf{z}^\top A \ll \mathbf{z}^\top$;
(1.4) There exists a diagonal $P \succ 0$ such that $A^\top PA - P \prec 0$;
(1.5) $(I_n - A)^{-1}$ exists and has nonnegative entries.

Proposition 2: [33], [43], [45][1] For a Metzler matrix $A \in \mathbb{R}^{n\times n}$, the following properties are equivalent:

(2.1) A is Hurwitz;
(2.2) There exists a $\boldsymbol{\xi} \gg 0$ such that $A\boldsymbol{\xi} \ll 0$;
(2.3) There exists a $\mathbf{z} \gg 0$ such that $\mathbf{z}^\top A \ll 0$;
(2.4) There exists a diagonal $P \succ 0$ such that $A^\top P + PA \prec 0$;
(2.5) $-A^{-1}$ exists and has nonnegative entries.

Conditions (1.2), (1.3) and (1.4) (as well as (2.2), (2.3) and (2.4)) in the above propositions have nice interpretations in terms of Lyapunov functions. Condition (1.4) in Proposition 1 corresponds to saying that for positive matrices the Schur property corresponds to the existence of a diagonal quadratic Lyapunov function, i.e., $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ where $P = P^\top \succ 0$ is diagonal, satisfying $\Delta V(\mathbf{x}) = V(A\mathbf{x}) - V(\mathbf{x}) < 0$ for every $\mathbf{x} \neq 0$. Similarly, condition (2.4) in Proposition 2 corresponds to saying that for Metzler matrices the Hurwitz property corresponds to the existence of a diagonal quadratic Lyapunov function , i.e., $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$, with $P = P^\top \succ 0$ diagonal, such that $\dot{V}(\mathbf{x}) = \nabla V(\mathbf{x})A\mathbf{x} < 0$ for every $\mathbf{x} \neq 0$.

[1]The characterizations in [43] were actually derived for M-matrices. A matrix M is an M-marix if and only if $-M$ is Metzler Hurwitz.

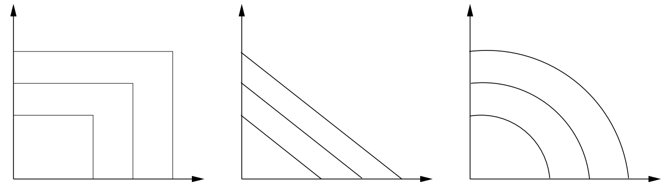

Fig. 2. Level curves of Lyapunov functions corresponding to the conditions (1.2), (1.3) and (1.4) in Proposition 1 or (2.2), (2.3) and (2.4) in Proposition 2.

To comment on the meaning of conditions (1.2) and (2.2), let us introduce a different kind of Lyapunov function that is very well-known and extensively used in the literature about positive systems because of its simplicity.

Definition 2: A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be *copositive* if $V(\mathbf{x}) > 0$ for every $\mathbf{x} > 0$. A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a *linear copositive function* if $V(\mathbf{x}) = \mathbf{z}^\top\mathbf{x}$, for some $\mathbf{z} \gg 0$.

Condition (1.3) in Proposition 1 and (2.3) in Proposition 2 correspond to saying that a positive matrix (a Metzler matrix) is Schur (Hurwitz) if and only if it admits a linear copositive Lyapunov function, i.e., there exists $V(\mathbf{x}) = \mathbf{z}^\top\mathbf{x}$, with $\mathbf{z} \gg 0$, such that $\Delta V(\mathbf{x}) < 0$ ($\dot{V}(\mathbf{x}) < 0$) for every $\mathbf{x} \neq 0$.

Finally, conditions (1.2) and (2.2) correspond to the existence of another special type of Lyapunov function, defined as

$$V(\mathbf{x}) = \max_{i\in[1,n]} \frac{x_i}{\xi_i},$$

that exhibits rectangular level curves. The level curves of these three Lyapunov functions are illustrated in Figure 2.

Asymptotic stability is concerned with the long term behavior of the unforced state trajectories of a system, and this is a fundamental property one needs to obtain, possibly by resorting to a state-feedback control action (see Section V). On the other hand, if the input $\mathbf{u}$ acting on the system represents a disturbance, it is natural to evaluate the system performance in terms of its capability to contain the effects of the disturbance $\mathbf{u}$ on the output $\mathbf{y}$. So, if we focus only on the forced dynamics of system (4), by assuming $\mathbf{x}(0) = 0$, and regard the system as an input-output operator $\mathcal{G}$ mapping input trajectories into output trajectories, it is natural to investigate the input-output performance of the system in terms of its L_p-gain. The L_p-gain of the system is defined as

$$\|\mathcal{G}\|_{L_p - L_p} = \sup_{\mathbf{u}:\|\mathbf{u}\|_{L_p}=1} \|g * \mathbf{u}\|_{L_p},$$

where $*$ represents the convolution product and $g(t)$ the system impulse response (see (5) and (6)). In the rest of this section we will focus on the continuous-time case and investigate the L_p-gain of an asymptotically stable positive system. To this end, we introduce the *transfer matrix*

$$G(s) := C(sI_n - A)^{-1}B + D \in \mathbb{R}^{r\times m}(s), \qquad (7)$$

of the system (4) (or, equivalently, of system (3)). Under the asymptotic stability assumption on the system, namely upon assuming that A is Metzler Hurwitz, quite remarkable results have been obtained [15], [25], [26], [58], [59].

Proposition 3: [15], [58] Suppose that $\mathcal{G}$ is the input-output operator of an asymptotically stable positive system. Then for $p = 1, 2$ and $+\infty$ we have

$$\|\mathcal{G}\|_{L_p - L_p} = \|G(0)\|_{p-\mathrm{ind}}. \qquad (8)$$

In particular, if $\mathcal{G}_1$ and $\mathcal{G}_2$ have transfer functions $G(s)$ and $G(s)^\top$ respectively, then

$$\|\mathcal{G}_1\|_{L_1 - L_1} = \|\mathcal{G}_2\|_{L_\infty - L_\infty}. \qquad (9)$$

Moreover, if the system is SISO, i.e. $r = m = 1$, then

$$\|\mathcal{G}\|_{L_p - L_p} = G(0), \qquad \forall\, p \in [1, +\infty].$$

Proposition 4: [15], [27] Given the positive system (4) and any $\gamma > 0$, the following facts are equivalent:

(4.1) A is Hurwitz and $\|\mathcal{G}\|_{L_\infty - L_\infty} < \gamma$;

(4.2) A is Hurwitz and $G(0)\mathbf{1}_m \ll \gamma\mathbf{1}_r$;

(4.3) There exists $\boldsymbol{\xi} \gg 0, \boldsymbol{\xi} \in \mathbb{R}^n$, such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} \boldsymbol{\xi} \\ \mathbf{1}_m \end{bmatrix} \ll \begin{bmatrix} 0 \\ \gamma\mathbf{1}_r \end{bmatrix}. \qquad (10)$$

Consequently, for an asymptotically stable positive system (4) the L_∞-gain can be found by solving a linear program:

$$\|\mathcal{G}\|_{L_\infty - L_\infty} = \min\{\gamma : (10) \text{ holds for some } \boldsymbol{\xi} \gg 0, \boldsymbol{\xi} \in \mathbb{R}^n\}.$$

Obviously Proposition 4 can be combined with (9) to also compute the L_1-induced gain of a positive system by linear programming. However, if the L_2-induced gain, also known as the H_∞-norm, of a positive MIMO system is of interest, then different methods are needed. The following theorem from [60] is a generalization of a result in [70].

Proposition 5: Given the positive system (4), assume that $A \in \mathbb{R}^{n\times n}$ is Hurwitz, and the pair (A, B) is controllable. Also, suppose that $M = M^\top \in \mathbb{R}^{(n+m)\times(n+m)}$ is a symmetric matrix with all nonnegative entries, except for the last m diagonal elements. Then the following statements are equivalent:

(5.1) For every $\omega \in [0, \infty)$ one has

$$\begin{bmatrix} (i\omega I_n - A)^{-1}B \\ I_m \end{bmatrix}^* M \begin{bmatrix} (i\omega I_n - A)^{-1}B \\ I_m \end{bmatrix} \preceq 0;$$

(5.2) $$\begin{bmatrix} -A^{-1}B \\ I_m \end{bmatrix}^\top M \begin{bmatrix} -A^{-1}B \\ I_m \end{bmatrix} \preceq 0;$$

(5.3) There exists a diagonal $P \succeq 0$ such that

$$M + \begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{bmatrix} \preceq 0;$$

(5.4) There exist $\mathbf{x}, \mathbf{p} \geq 0$, $\mathbf{u} \gg 0$ such that $A\mathbf{x} + B\mathbf{u} \leq 0$ and $\begin{bmatrix} \mathbf{x}^\top & \mathbf{u}^\top \end{bmatrix} M + \mathbf{p}^\top \begin{bmatrix} A & B \end{bmatrix} \leq 0$.

If all inequalities are replaced by strict ones, then the equivalences hold even without the controllability assumption.

In particular, Proposition 5 with

$$M = \begin{bmatrix} C^\top C & C^\top D \\ D^\top C & D^\top D - \gamma^2 I \end{bmatrix}$$

can be used to test if $G(s) = C(sI - A)^{-1}B + D$ has H_∞-norm smaller than γ.

IV. Positive Realization

One of the most challenging problems investigated in the context of positive systems is surely the *positive realization problem* [3], [9], [29], [30], [51], [56], [75], which can be stated as follows:

Given a proper rational matrix $G(s) \in \mathbb{R}(s)^{r\times m}$, under what conditions it admits a positive realization, namely it can be identified with the transfer matrix of a (continuous-time or discrete-time) positive state-space model?

Most of the literature on this subject has focused on the discrete-time case, so in the rest of the section we will consider this specific case. Under this assumption, the positive realization problem becomes:

Given a proper rational matrix $G(s) \in \mathbb{R}(s)^{r\times m}$, under what conditions there exist $N \in \mathbb{Z}, N > 0$, and nonnegative matrices $A \in \mathbb{R}_+^{N\times N}, B \in \mathbb{R}_+^{N\times r}, C \in \mathbb{R}_+^{r\times N}$ and $D \in \mathbb{R}_+^{r\times m}$ such that $G(s) = C(sI_N - A)^{-1}B + D$?

Clearly, a necessary condition for the existence of a solution to the positive realization problem is that $G(s)$ is the transfer matrix of an externally positive system, which means that the *Markov coefficients* G_i of $G(s)$, i.e., $G(s) = \sum_{i=0}^{+\infty} G_i s^{-i}$, are nonnegative matrices. However, as previously remarked, external positivity does not ensure internal positivity, and hence the nonnegativity of the Markov coefficients is not a sufficient condition. It is also worth mentioning that verification of external positivity for a given rational transfer function has proved to be NP-hard (see [14] for the discrete-time problem and [5] for the continuous-time one).

On the other hand, the positive realization problem is typically posed in contexts related, for instance, to compartmental systems in pharmacokinetics or biological applications, where external positivity is intrinsically guaranteed by the nature of the system under investigation. So, oftentimes it makes sense to assume that the nonnegativity of the Markov coefficients can be taken for granted.

Note that as the matrix D in every state-space realization is determined by G_0, the positive realization problem can always be restricted to strictly proper rational matrices. Finally, $G(s)$ admits a positive realization if and only if all its entries $g_{ij}(s), i \in [1, r], j \in [1, m]$, admit a positive realization. So, in the following we will restrict our attention to strictly proper scalar transfer functions.

The path to the problem solution started with three milestone contributions [48], [51], [56] in the early eighties, that characterized the existence of a positive realization in terms of the existence of a proper polyhedral cone that enjoys special properties. More specifically.

Theorem 1: Given a strictly proper rational function $G(s) \in \mathbb{R}(s)$, let $\Sigma = (F, g, H)$ be a minimal realization of $G(s)$ and let n be the dimension of Σ. Then $G(s)$ has a positive realization if and only if there exists a proper polyhedral cone $\mathcal{K} \subset \mathbb{R}^n$ such that

1) $\mathcal{K}$ is F-invariant, namely $F\mathcal{K} \subseteq \mathcal{K}$;
2) $g \in \mathcal{K}$;
3) $\mathcal{K} \subseteq \{\mathbf{x} \in \mathbb{R}^n : HF^{k-1}\mathbf{x} \geq 0, \forall\, k \in \mathbb{Z}, k \geq 1\}$.

If such a cone exists and $\mathcal{K} = \mathrm{Cone}(\mathbf{v}_1, \dots, \mathbf{v}_N)$, for some $\mathbf{v}_1, \dots, \mathbf{v}_N \in \mathbb{R}^n$, then there exists a positive realization of dimension N.

Based on this fundamental theoretical characterization, that however lacks of practical feasibility, several important results have been obtained. But, the final fundamental steps toward the solution of this difficult problem were taken only 15 years later in [3].

Theorem 2: [3] Let $G(s) \in \mathbb{R}(s)$ be a strictly proper rational function, with nonnegative Markov coefficients. If $G(s)$ has a single pole of maximum modulus which is positive real and has arbitrary multiplicity, then $G(s)$ admits a (discrete-time) positive realization.

Theorem 3: [3] Let $G(s) \in \mathbb{R}(s)$ be a strictly proper rational function, with nonnegative Markov coefficients, and set

$$\rho := \max\{|\lambda| : \lambda \in \mathbb{C} \text{ is a pole of } G(s)\}.$$

If $\rho > 0$, $\lim_{i \to +\infty} \inf \dfrac{G_i}{\rho^i} > 0$, and each pole λ of $G(s)$ having modulus ρ satisfies the following conditions:

1) it is simple;
2) there exists $k \in \mathbb{Z}, k > 0$, such that $\lambda^k = \rho^k$;

then $G(s)$ admits a (discrete-time) positive realization.

The general (scalar) case, when poles of maximum modulus are not simple, was later addressed in [30]. To solve the problem, a procedure has been proposed that is based on the evaluation of the poles of maximum modulus of a finite sequence of strictly proper rational functions, whose Markov coefficients are suitable subsets of the Markov coefficients of the original $G(s)$. The procedure not only makes it possible to answer the question of whether a positive realization exists or not, but when a positive realization exists it shows how to explicitly derive one. We refer the interested reader to [30] and to the survey [9].

Once the problem of determining whether a given strictly proper transfer function $G(s) \in \mathbb{R}(s)$ admits a positive realization has been solved, the next natural question is: What is the minimal size that a positive realization of $G(s)$ may exhibit? By referring to the characterization given in Theorem 1, this amounts to looking for the polyhedral proper cone $\mathcal{K} \subset \mathbb{R}^n$ for which the number N of generating vectors $\mathbf{v}_1, \dots, \mathbf{v}_N$ is minimal. As a matter of fact, to the best of our knowledge this is still an open problem, and no major advancements have been obtained recently, so that the surveys [8], [9] still provide the state of the art in this subject. Apart from specific results, obtained for strictly proper rational functions of specific (McMillan) degrees or poles endowed with special properties (see, e.g., [11]), the main result available provides a lower bound on the size of a minimal positive realization. This result is based on the well-known Karpelevich Theorem [46], [52] that completely characterizes the regions of the complex plane where the eigenvalues of a positive matrix A with spectral radius ρ can be located. Specifically, by the symbol Θ_n^ρ we denote the region of the complex plane where the eigenvalues of an $n \times n$ nonnegative matrix with spectral radius ρ lie. The following result holds.

Proposition 6: Consider a strictly proper rational function $G(s)$, with nonnegative impulse response and minimal realization of order n, and assume that it has a real dominant pole $\rho > 0$. Then the minimal order of a positive realization of $G(s)$ is not less than $\max\{n, N\}$, where N is the minimal positive integer such that every pole p of $G(s)$ satisfies $p \in \Theta_N^\rho$. Moreover in every minimal positive realization the state matrix has ρ as nonnegative real dominant eigenvalue.

For additional results on the minimal positive realization problem, the interested reader is referred to [8], [9] and references therein.

V. State-feedback stabilization

As previously mentioned, asymptotic stability is a fundamental property one needs to ensure. In general, when dealing with positive systems, the goal of achieving stability by means of a state-feedback law $\mathbf{u}(t) = K\mathbf{x}(t)$, where $\mathbf{u}$ is now regarded as a control input, cannot be pursued at the cost of losing the positivity of the resulting feedback system. So, the standard stabilization problem is replaced by the problem of making the resulting state-space system both positive and asymptotically stable [2]. In this context, it is also possible to easily introduce constraints on the state-feedback matrices, and hence to pose the positive stabilization problem by assuming that the matrix K belongs to some sparsity structure $\mathcal{S}$ in $\mathbb{R}^{m\times n}$, rather than being an arbitrary matrix in $\mathbb{R}^{m\times n}$.

Positive stabilization problem: Given a sparsity structure $\mathcal{S} \subset \mathbb{R}^{m\times n}$ as in (2), determine if there exists a state feedback matrix $K \in \mathcal{S}$ such that the resulting feedback state-space model

$$\mathbf{x}(t+1) = (A + BK)\mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \tag{11}$$

in discrete-time or

$$\dot{\mathbf{x}}(t) = (A + BK)\mathbf{x}(t), \qquad t \in \mathbb{R}_+, \tag{12}$$

[2]For the sake of simplicity, in this section we consider only strictly proper positive systems, namely we assume $D = 0$. Consequently, the output equation is not affected by the state-feedback control and hence we omit it. In the general case, the output equation of the resulting feedback state-space model would be $\mathbf{y}(t) = (C + DK)\mathbf{x}(t)$, and hence we would need to include also the constraint $C + DK \geq 0$. This will be considered in the following sections.

in continuous-time, is positive and asymptotically stable.

Let us focus, again, on the continuous-time case. In this context the positive stabilization problem amounts to determining, if it exists, a matrix $K \in \mathcal{S}$ such that $A+BK$ is Metzler and Hurwitz. This problem was first investigated[3] in [36], and the existence of a solution was expressed in terms of the solvability of a family of Linear Matrix Inequalities (LMIs). We provide here the result given in Theorem 1 of [36], suitably rephrased in order to make a comparison with the subsequent characterizations more immediate.

Theorem 4: Given a continuous-time system (12) and a sparsity structure $\mathcal{S} \subset \mathbb{R}^{m\times n}$, the following facts are equivalent:

(4.1) The positive stabilization problem has a solution;

(4.2) There exist $Y \in \mathcal{S}$ and a positive diagonal matrix X, such that $AX+BY$ is Metzler and

$$(AX+BY)^\top + AX + BY \quad \prec \quad 0. \tag{13}$$

When so, a solution to the stabilization problem is obtained as $K = YX^{-1}$.

This problem was later investigated in [2], where the problem solution was converted into a Linear Programming (LP) problem.

Theorem 5: Given a continuous-time system (12) and a sparsity structure $\mathcal{S} \subset \mathbb{R}^{m\times n}$, the following facts are equivalent:

(5.1) The positive stabilization problem has a solution;

(5.1) There exist $Y \in \mathcal{S}$ and a positive diagonal matrix X, such that $AX+BY$ is Metzler and

$$(AX+BY)\mathbf{1}_n \quad \ll \quad 0. \tag{14}$$

When so, a solution to the stabilization problem is obtained as $K = YX^{-1}$.

It is worth noticing that Theorems 4 and 5 make use of two of the characterizations of Metzler Hurwitz matrices obtained in Proposition 2. Indeed, the former makes use of the fact that if $A+BK$ is Metzler Hurwitz then it admits a diagonal quadratic Lyapunov function, while the latter of the fact that it admits a linear copositive Lyapunov function.

From a computational point of view, the solution in terms of LP, even if equivalent from a theoretical viewpoint, is preferable due to its lower computational complexity. Even more, it is prone to be easily extended to cope with robust stabilization in the presence of polytopic uncertainties, stabilization with restricted sign controls and stabilization with bounded controls [2].

Alternative approaches to the positive stabilization problem have been proposed in [63] and [15]. The characterization derived in [63] is based on the construction of certain polytopes and on verifying whether a selection of their vertices can be used to construct a stabilizing state-feedback matrix. On the other hand, in [15] the problem of achieving by means of a state-feedback not only positivity and stability, but also certain L_1 and L_∞ performances, has been investigated. Also in this case, necessary and sufficient conditions for the existence of a solution have been expressed as LPs. We will address this aspect in Sections VI and VII.

Finally, the previous stabilization problem together with other versions of the positive stabilization problem have been investigated in [21] by focusing on the special case of single-input continuous-time positive systems.

[3] In fact, all the results about positive stabilization reported in this section were obtained without imposing any constraint on K, namely for $\mathcal{S} = \mathbb{R}^{m\times n}$, but their adaption to the case when K is constrained to belong to some sparsity structure $\mathcal{S}$ is immediate.

VI. Control synthesis by linear programming

We consider, now, a more general version of the state-space models introduced in Section III, since it includes both the control input and the external disturbance. As in the previous section, we will focus our attention on the continuous-time case and hence on the model:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) + E\mathbf{w}(t), \tag{15a}$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t) + F\mathbf{w}(t), \qquad t \in \mathbb{R}_+, \tag{15b}$$

where $\mathbf{x}$ represents the n-dimensional state variable, $\mathbf{u}$ the m-dimensional control variable, $\mathbf{w}$ the q-dimensional disturbance, and $\mathbf{y}$ the r-dimensional output variable. Note that we will not introduce any positivity assumption on the real matrices A, B, C, D, E and F. Indeed, we will impose the positivity on the resulting feedback system, but not on the original one. By referring to the previous state-space model (15), in the next sections we will extend the previous stabilization methods in order to optimize the performance of the resulting feedback system in terms of input-output gains. All the results will be illustrated by examples dealing with buffer networks.

If we introduce the state-feedback law $\mathbf{u}(t) = K\mathbf{x}(t)$, the resulting feedback system becomes:

$$\dot{\mathbf{x}}(t) = (A+BK)\mathbf{x}(t) + E\mathbf{w}(t), \tag{16a}$$

$$\mathbf{y}(t) = (C+DK)\mathbf{x}(t) + F\mathbf{w}(t), \ t \in \mathbb{R}_+. \tag{16b}$$

Note that the only input acting on system (16) is the disturbance $\mathbf{w}$. We denote by

$$G_K(s) := (C+DK)[sI-(A+BK)]^{-1}E + F \in \mathbb{R}^{r\times q}(s) \tag{17}$$

the transfer matrix of system (16). If we assume zero initial condition, i.e., $\mathbf{x}(0) = 0$, such a system can be seen as an operator $\mathcal{G}_K$ from the input disturbance $\mathbf{w}$ to the output $\mathbf{y}$.

A fundamental problem in the literature on robust control is the minimization of gain from disturbance to error. The purpose of this section and the next one is to demonstrate how this can be done by extending the ideas of Section V. We start with the optimization of L_∞-gain, which can be reduced to linear programming along the lines of Proposition 4 and Theorem 5.

Theorem 6: [15] Consider a continuous-time system (15), a sparsity structure $\mathcal{S}$ and a positive real number γ. If E and

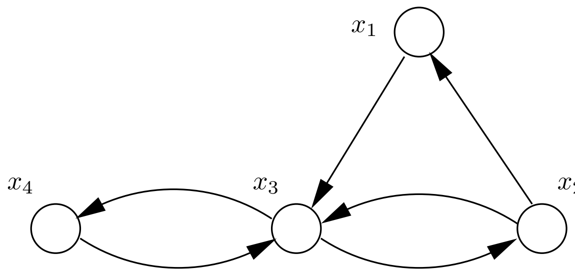

Fig. 3. A directed graph representing the buffer network in Example 1. The arrow from node 2 to node 1 corresponds to the term $\kappa_{12}x_2$, which indicates a flow from buffer 2 to buffer 1 proportional to the buffer level x_2.

F are nonnegative matrices, then the following conditions are equivalent:

(6.1) There exists a matrix $K \in \mathcal{S}$ such that $C + DK$ is nonnegative, $A + BK$ is Metzler Hurwitz and $\|\mathcal{G}_K\|_{L_\infty - L_\infty} < \gamma$.

(6.2) There exist a positive diagonal matrix X and a matrix $Y \in \mathcal{S}$ such that $\bar{A} := AX + BY$ is Metzler, $\bar{C} := CX + DY$ is nonnegative and

$$\bar{A}\mathbf{1}_n + E\mathbf{1}_q \ll 0$$
$$\bar{C}\mathbf{1}_n + F\mathbf{1}_q \ll \gamma\mathbf{1}_n.$$

If X and Y satisfy (6.2), then (6.1) holds for $K = YX^{-1}$.

Example 1: Consider the buffer network as described in (1), with one input $u_{ij} = \kappa_{ij}x_j$ for every edge in the directed graph illustrated in Figure 3:

$$\begin{cases} \dot{x}_1 = -x_1 - \kappa_{31}x_1 + \kappa_{12}x_2 + w_1 \\ \dot{x}_2 = -2x_2 - \kappa_{12}x_2 - \kappa_{32}x_2 + \kappa_{23}x_3 + w_2 \\ \dot{x}_3 = -3x_3 + \kappa_{31}x_1 + \kappa_{32}x_2 - \kappa_{23}x_3 - \kappa_{43}x_3 + \kappa_{34}x_4 \\ \quad + w_3 \\ \dot{x}_4 = -4x_4 + \kappa_{43}x_3 - \kappa_{34}x_4 + w_4. \end{cases}$$

Our problem is to find (nonnegative) feedback gains κ_{ij}, to minimize the influence of the disturbances w_i on states and inputs u_{ij}. To do this, notice that the buffer network can be described as in (16a) for

$$A = \mathrm{diag}\{-1, -2, -3, -4\} \tag{18}$$

$$B = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \tag{19}$$

$$K = \mathrm{diag}\left\{\kappa_{31}, \begin{bmatrix}\kappa_{12} \\ \kappa_{32}\end{bmatrix}, \begin{bmatrix}\kappa_{23} \\ \kappa_{43}\end{bmatrix}, \kappa_{34}\right\} \tag{20}$$

$$E = I_4 \tag{21}$$

and hence the transfer matrix from $\mathbf{w}$ to $(\mathbf{x}, \mathbf{u})$, where $\mathbf{u}$ is the vector $\mathbf{u} = \begin{bmatrix} u_{31} & u_{12} & u_{32} & u_{23} & u_{43} & u_{34} \end{bmatrix}^\top$, can be expressed as

$$G_K(s) = \begin{bmatrix} I_4 \\ K \end{bmatrix} \left[sI_4 - (A + BK)\right]^{-1}.$$

This amounts to assuming $\mathbf{y} = \begin{bmatrix}\mathbf{x} \\ \mathbf{u}\end{bmatrix}$ and hence

$$C = \begin{bmatrix} I_n \\ 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \quad F = 0. \tag{22}$$

Hence Theorem 6 can be used to minimize the L_∞-gain of the input-output operator from $\mathbf{w}$ to $\mathbf{y}$. With

$$X = \mathrm{diag}\{\xi_1, \xi_2, \xi_3, \xi_4\}$$
$$Y = \mathrm{diag}\left\{\mu_{31}, \begin{bmatrix}\mu_{12} \\ \mu_{32}\end{bmatrix}, \begin{bmatrix}\mu_{23} \\ \mu_{43}\end{bmatrix}, \mu_{34}\right\}$$

the linear programming problem becomes to minimize γ subject to the constraints

$$\begin{aligned} -\xi_1 - \mu_{31} + \mu_{12} + 1 &\le 0 \\ -2\xi_2 - \mu_{12} - \mu_{32} + \mu_{23} + 1 &\le 0 \\ -3\xi_3 + \mu_{31} + \mu_{32} - \mu_{23} - \mu_{43} + \mu_{34} + 1 &\le 0 \\ -4\xi_4 + \mu_{43} - \mu_{34} + 1 &\le 0 \\ 0 \le \xi_i &\le \gamma \\ 0 \le \mu_{ij} &\le \gamma \end{aligned}$$

where, to ensure the Metzler property of $A + BK$, we imposed that also the ξ_i's are nonnegative.

VII. Synthesis by Semidefinite Programming

This section and the next one are devoted to minimization of the L_2-gain, a subject which in the literature on robust control is known as H_∞-optimization. There is a very rich mathematical theory associated with H_∞-norm, and corresponding control synthesis methods are either based on Riccati equations or on semidefinite programming. It should therefore come as no surprise that Theorem 6 has an analogue for L_2-gain optimization, which uses semidefinite rather than linear programming.

Theorem 7: [70] Consider a continuous-time system (15), a sparsity structure $\mathcal{S}$ and a positive real number γ. If E and F are nonnegative matrices, then the following conditions are equivalent:

(7.1) There exists a matrix $K \in \mathcal{S}$ such that $C + DK$ is nonnegative, $A + BK$ is Metzler Hurwitz and $\|G_K\|_{H_\infty} < \gamma$.

(7.2) There exist a positive diagonal matrix X and a matrix $Y \in \mathcal{S}$ such that $\bar{A} := AX + BY$ is Metzler, $\bar{C} := CX + DY$ is nonnegative and

$$\begin{bmatrix} \bar{C}^\top \bar{C} + \bar{A} + \bar{A}^\top & \bar{C}^\top F + E \\ F^\top \bar{C} + E^\top & F^\top F - \gamma^2 I \end{bmatrix} \prec 0. \tag{23}$$

If X and Y satisfy (7.2), then (7.1) holds for $K = YX^{-1}$.

At first sight, it may look like condition (23) is poorly scalable, since it involves all problem data in a single inequality. However, for sparse matrices, the following result makes it possible to break the inequality into pieces and hence to apply distributed algorithms.

Theorem 8: [60] An $n \times n, n \ge 3$, symmetric Metzler matrix with m nonzero entries above the diagonal is negative semidefinite if and only if it can be written as a sum of m negative semidefinite matrices, each of which has only four nonzero entries.

Example 2: We consider the same example as in the previous section, with just a different norm. Hence, for A, B, C, D, E and F given by (18)-(19) and (21)-(22), we want to find a matrix K as in (20) making $A+BK$ Metzler and minimizing

$$\|G_K\|_{H_\infty} = \left\| \begin{bmatrix} I \\ K \end{bmatrix} [sI - (A+BK)]^{-1} \right\|_{H_\infty}.$$

Applying Theorem 7, and keeping in mind the expressions of the matrices A, B, C, D, E, F and K, equation (23) takes the form

$$\begin{bmatrix} X^2 + Y^\top Y + AX + BY + (AX+BY)^\top & I \\ I & -\gamma^2 I \end{bmatrix} \prec 0.$$

Equivalently

$$X^2 + Y^\top Y + AX + BY + (AX+BY)^\top + \gamma^{-2} I \prec 0$$

and more explicitly

$$\begin{bmatrix} d_1 & \mu_{12} & \mu_{31} & 0 \\ \mu_{12} & d_2 & \mu_{23}+\mu_{32} & 0 \\ \mu_{31} & \mu_{23}+\mu_{32} & d_3 & \mu_{34}+\mu_{43} \\ 0 & 0 & \mu_{34}+\mu_{43} & d_4 \end{bmatrix} \prec 0$$

where $\xi_i \geq 0$, $\mu_{ij} \geq 0$ and the diagonal elements are

$$\begin{aligned} d_1 &= \xi_1^2 + \gamma^{-2} - 2\xi_1 - 2\mu_{31} \\ d_2 &= \xi_2^2 + \gamma^{-2} - 4\xi_2 - 2\mu_{12} - 2\mu_{32} \\ d_3 &= \xi_3^2 + \gamma^{-2} - 6\xi_3 - 2\mu_{23} - 2\mu_{43} \\ d_4 &= \xi_4^2 + \gamma^{-2} - 8\xi_4 - 2\mu_{34}. \end{aligned}$$

Given that H_∞-optimal controllers can also be computed by standard Riccati equations, it is natural to compare the gains achievable by the two methods. This allows us to determine how restrictive the demand for closed loop positivity is. As it will be shown in the next section, there is a large class of buffer network control problems for which the positivity condition makes no difference at all in terms of achievable performance.

VIII. Large Scale H_∞ optimal control

Consider, first, the following problem motivated by disturbance rejection in buffer networks:

Given a directed graph with nodes $\mathcal{V}$ and edges $\mathcal{E}$, consider the buffer network whose ith node updates according to the following equation

$$\dot{x}_i = a_i x_i + \sum_{(i,j)\in\mathcal{E}} (u_{ij} - u_{ji}) + w_i \qquad i \in \mathcal{V}. \tag{24}$$

Find feedback gains κ_{ij} such that by assuming $u_{ij} = \kappa_{ij} x_j$ $A+BK$ is Metzler and the H_∞-norm of the transfer function from the disturbance $\mathbf{w}$ to the controlled output $(\mathbf{x}, \mathbf{u})$, where $\mathbf{u}$ is the input vector (whose specific definition depends on the network structure), is minimized. Suppose that all nodes are stable, i.e. $a_i < 0$. Then we will see below that an optimal control law is given by

$$u_{ij} = x_i/a_i - x_j/a_j. \tag{25}$$

The closed loop system from $\mathbf{w}$ to $\mathbf{x}$ is a positive system and the control law is decentralized in the sense that control action on the edge (i,j) is entirely determined by the states in node i and node j. The formula for the optimal control law follows from the following result of [49].

Theorem 9: Suppose that A is symmetric (in particular, diagonal) and Hurwitz. Consider

$$\min_K \left\| \begin{bmatrix} I \\ K \end{bmatrix} [sI - (A+BK)]^{-1} \right\|_{H_\infty}$$

where minimization is done over all K such that $A+BK$ is Hurwitz. Then the minimum is attained by $K_* = B^\top A^{-1}$. The minimal value of the norm is $\sqrt{\|(A^2 + BB^\top)^{-1}\|}$.

The theorem identifies a rare but important class of systems for which decentralized controllers are known to achieve the same H_∞-performance as the best centralized ones. Moreover, the combination of (1) and the control law (25) gives a closed loop positive system, so this is a case where closed loop positivity can be attained without sacrificing performance.

It is interesting to compare Theorem 9 with Theorem 7. It is straightforward to verify using completion of squares that $X = -A$, $Y = -B^\top$ solves (23) whenever a solution exists. However, this solution does not necessarily satisfy the other constraints of Theorem 7, so there is only a partial overlap between the two theorems.

Combining (1) with the control law (25) is insufficient for many practical applications. One reason is that proportional controllers like (25) are unable to remove static errors in presence of constant disturbances. Motivated by this fact, an extension to controllers with integral action was derived in [62].

IX. Extensions to monotone systems

All real control systems come with nonlinearities, and buffer networks are no exception. On the contrary, proper handling of (nonlinear) buffer capacity constraints is often a main concern. It is therefore interesting to note that positive systems have natural extensions to nonlinear "monotone" systems [68]. Just like positivity, also monotonicity is a term that has different meanings in different contexts, but here we use it to denote a system $\dot{\mathbf{x}} = f(\mathbf{x})$ that preserves a partial ordering of the states (see Figure 4). In other words, a dynamical system is said to be monotone if its linearizations are positive systems. For example, if (24) is connected to the saturated control law

$$u_{ij} = \mathrm{sat}(x_i/a_i - x_j/a_j), \tag{26}$$

where $\mathrm{sat}(x) = \min\{\max\{x, -1\}, 1\}$, then the closed loop system becomes

$$\dot{x}_i = a_i x_i + 2 \sum_{(i,j)\in\mathcal{E}} \mathrm{sat}(x_i/a_i - x_j/a_j) + w_i \quad i \in \mathcal{V}$$

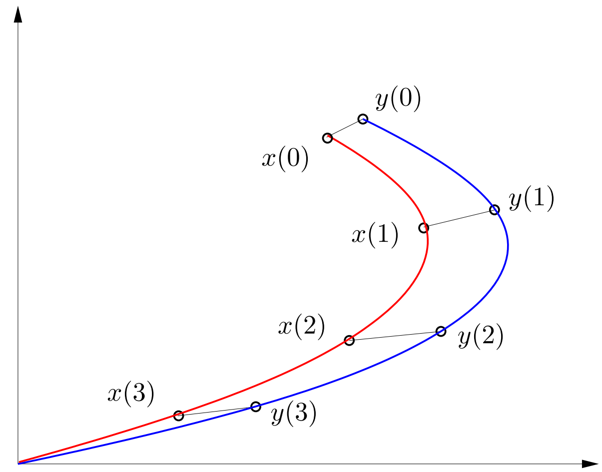

Fig. 4. A monotone dynamical system is a system that preserves a partial ordering of the states. For example, the system is monotone if, given two of its trajectories, say $\mathbf{y}(t)$ and $\mathbf{x}(t), t \geq 0$, then $\mathbf{y}(t) - \mathbf{x}(t)$ is a nonnegative vector for $t > 0$ whenever this is true for $t = 0$.

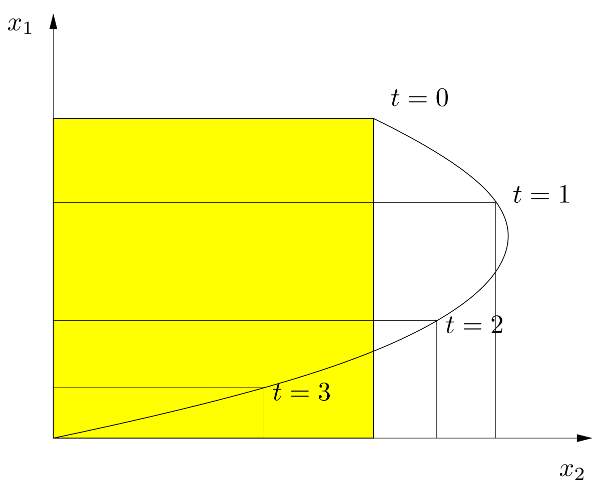

Fig. 5. The main idea behind Theorem 10 is very simple. All trajectories starting in the yellow box at $t = 0$ will be confined, due to monotonicity, to smaller and smaller rectangles as $t \to \infty$. The boundaries of these rectangles can be used to define a Lyapunov function of the form $V(\mathbf{x}) = \max\{V_1(x_1), \ldots, V_n(x_n)\}$.

which is a monotone system. Many attractive features of positive systems generalize to monotone systems. The existence of simpler and more scalable stability certificates is one of them, as illustrated by the following result from [24].

Theorem 10: Consider a monotone system $\dot{\mathbf{x}} = f(\mathbf{x})$ with a globally asymptotically stable equilibrium at $\mathbf{x} = 0$. Suppose that f is locally Lipschitz and that the system leaves the compact set $X \subset \mathbb{R}_+^n$ invariant. Then there exist strictly increasing functions $V_k : \mathbb{R}_+ \to \mathbb{R}_+$ for $k = 1, \ldots, n$ such that $V(\mathbf{x}) = \max\{V_1(x_1), \ldots, V_n(x_n)\}$ satisfies

$$\frac{d}{dt}V(\mathbf{x}) = -V(\mathbf{x})$$

for all the state trajectories $\mathbf{x}(t), t \in \mathbb{R}_+$, included in X.

The proof idea for Theorem 10 is illustrated in Figure 5.

A control system of the form

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

with $f \in C^1$ is said to be a *convex monotone system* if f is convex, while $\partial f/\partial \mathbf{x}$ is Metzler and $\partial f/\partial \mathbf{u}$ is nonnegative at every point $(\mathbf{x}, \mathbf{u})$. This non-linear generalization of linear positive systems retains another important property of positive systems, namely convexity in the dependence on initial conditions [61]:

Theorem 11: If $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$, $\mathbf{x}(0) = \mathbf{a}$, is a convex-monotone system with a unique solution $\mathbf{x}(t) = \phi_t(\mathbf{a}, \mathbf{u})$, then each component of $\phi_t(\mathbf{a}, \mathbf{u})$ is a convex function of $(\mathbf{a}, \mathbf{u})$.

The convexity property is of course very useful in the numerical computation of optimal trajectories and can sometimes also yield analytical results. An interesting instance will be studied in the next section.

X. Bilinear Positive Systems

Before concluding the paper, we will briefly discuss systems of the form

$$\dot{\mathbf{x}}(t) = \left(A + \sum_i u_i(t) D^i\right)\mathbf{x}(t), \tag{27}$$

where A is Metzler, while $D^1, \ldots, D^m$ are diagonal matrices. Such models are useful in the study of combination therapies of diseases such as HIV and cancer, where A describes the mutation dynamics without drugs, while $D^1, \ldots, D^m$ model the effects of drug doses $u_1(t), \ldots, u_m(t)$ on the mutant concentrations $x_k(t)$. We assume that all variables $u_i(t)$ take values in the same set U.

For fixed values of u_i, the model (27) is a linear positive system. However, due to the multiplication of $\mathbf{u}$ and $\mathbf{x}$, the state generally has a nonlinear dependence on $\mathbf{u}$. Such systems are generally very complicated to analyze, but in this case the positivity properties help a lot.

Theorem 12: [18], [61] Given a Metzler matrix A, let $\mathbf{x}(t)$ be the solution of

$$\dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^m u_i(t) D^i\right)\mathbf{x}(t)$$

where $\mathbf{x}(0) = \mathbf{a} > 0$ and $D^1, \ldots, D^m$ are diagonal matrices. Then $\log x_k(t)$ is a convex function of $(\mathbf{a}, \mathbf{u})$.

The theorem follows from the fact that the change of variables $z_k(t) = \log x_k(t)$ yields a convex monotone system:

$$\dot{z}_k(t) = \sum_{k,j} a_{kj}\exp(z_j - z_k) + \sum_i u_i(t) D^i_k.$$

The same transformation can also be used to address several other synthesis problems for this system. For example, for a bilinear state-space model

$$\dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^m u_i(t) D^i\right)\mathbf{x}(t) + B\mathbf{w}(t) \tag{28a}$$

$$\mathbf{y}(t) = C\mathbf{x}(t), \qquad t \in \mathbb{R}_+, \tag{28b}$$

where A is Metzler, B and C are nonnegative, $D^1, \ldots, D^m$ are diagonal, and $\mathbf{w}$ is a disturbance, [23] proved that the square of the H_2-norm of the input-output map from $\mathbf{w}$ to $\mathbf{y}$ is a convex function of $\mathbf{u}$ and also that the H_∞-norm is a convex function of $\mathbf{u}$. Moreover, and this is important for large scale problems, not only convexity but also sparsity can be exploited, just like we did earlier in Section VI. This is illustrated by the following formulation of conditions for minimization of the L_∞-gain (L_1-optimal control):

Theorem 13: [79] Consider the positive bilinear system (28), and assume that A is Metzler, C is nonnegative and $D^1, \ldots, D^m$ are diagonal. Let γ be a positive real number, then the following conditions are equivalent:

(13.1) There exist $u_1, \ldots, u_m \in U$ such that $\|\mathcal{G}_u\|_{L_\infty - L_\infty} < \gamma$, where $\mathcal{G}_u$ is the operator corresponding to the linear and time-invariant input-output map defined by (28) once the input variables $u_i(t)$ take the constant values u_i.

(13.2) There exist $u_1, \ldots, u_m \in U$ and $z_1, \ldots, z_n$ such that $\sum_j c_{lj} e^{z_j} \leq \gamma$ and

$$\sum_j a_{kj} e^{z_j - z_k} + \sum_i u_i D_k^i + e^{-z_k} \leq 0$$

for all k and l.

All previous inequalities are convex in $(\mathbf{z}, \mathbf{u})$ and if A and C are sparse, each inequality only involves a small number of terms.

XI. Conclusions

In this tutorial paper we have first presented some foundational results regarding stability, L_p-gain, state-feedback stabilization and positive realization of positive systems. Subsequently, we have addressed the state-feedback stabilization with norm constraints and imposed sparsity structure on the feedback gain matrix. This highlighted how the techniques available for positive systems scale well with dimension. As a result, positive systems have a noteworthy advantage over standard systems when dealing with large scale systems. Extensions of these results to the classes of monotone systems and bilinear positive systems have also been briefly discussed.

While a number of system theoretic problems for positive systems have reached complete maturity, we believe that there are still several exciting open problems that deserve attention. In this tutorial we have highlighted what, in our opinion, is one of the most promising currently, namely the exploitation of positivity properties when dealing with large scale systems. In spite of the long history, it is clear that a large number of important and useful results still remain to be discovered. For example, extensions have recently been made to time-varying systems [47], not to mention the exciting new results on differentially positive (non-linear) systems [34]. Together with an ever growing number of applications where positivity plays a major role, this suggests that many of the most important results are yet to come.

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