

# LMI-based design of dynamic event-triggering mechanism for linear systems

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**Abstract**—This paper deals with the design of an event-triggering mechanism using local information for a linear system controlled by an observer-based feedback controller. The event-triggered control strategy is based on a new dynamic triggering mechanism, which is introduced through an internal dynamic variable. Sufficient conditions based on linear matrix inequalities are proposed to ensure the asymptotic stability of the closed loop together with the avoidance of Zeno behavior. A convex optimization problem leans on these conditions to determine the parameters of the event-trigger rule aiming at reducing the number of control updates. Discussion with respect to the simpler state feedback case is drawn.

## I. INTRODUCTION

The event-triggered control consists in devising event-triggering mechanisms leading only seldom control updates. In the context of event-triggered control, two objectives can be pursued: (1) Emulation: the controller is a priori predesigned and only the event-triggered rules have to be designed (see, for example, [11], [20], [14], [2], [8] and the references therein), or (2) Co-design: the joint design of the control law and the event-triggering conditions has to be performed (see, for example, [4], [15], [1], [10] [3], and the references therein). The current paper comes within the scope of the first case, that is, of the emulation context. Indeed, it is important to have in mind that control systems are connected to generic digital communication networks for implementation, transmission, coding or decoding. Then, event-triggered control strategies have been developed mainly to cope with communication, energy consumption and computation constraints (see, for example, [13], [11], [16], [4] and the references therein).

Note that the most of the papers above cited consider static event-triggered rules, differently from the dynamical case as addressed, for example, in [9], [7], [12], [6] and recently extended to the multi-agent systems case in [21]. In these papers, the idea consists in introducing an additional variable, which is indeed an internal dynamic variable allowing to augment the event-triggered rule with dynamics.

In the current paper, inspired by the approach proposed in [9], a new kind of event-triggering mechanism is developed in an emulation context, which can be seen as complementary to the approach developed in [5]. The originality of the approach relies on: 1) The class of dynamics considered for the internal variable, which extends that one proposed in [9]; 2) The event-triggering mechanism uses only local

information available for the control purpose; and 3) The resulting event-triggered law is free from Zeno behavior thanks to the use of an explicit tuning parameter, which plays the role of a minimal dwell time. This is also an alternative route to [6], in which a hybrid dynamical system framework is used. Furthermore, one can highlight that the event-triggered rules of [9] in the dynamic case, or in [19] in the static case, can be embedded in that one studied in this paper.

The paper is organized as follows. Section II describes first the system under consideration. Next the new dynamic triggering mechanism is presented in order to enrich the event generator algorithm managing the controller to decide when the control input has to be updated. Then, the control problem we intend to solve is formally stated. In Section III, the first result (Theorem 1) constitutes the backbone of the paper allowing to prove that the asymptotic stability of the closed-loop system with the proposed triggering mechanism is guaranteed. From this generic result, conditions based on linear matrix inequalities (LMIs) are proposed (Theorem 2). One of the salient facts relies on the dwell time explicitly appearing as a tuning parameter. Associated to these conditions, a simple optimization criterion is proposed to deal with the implicit objective of reducing the number of control updates by orienting the choice of the event-triggering mechanism parameters. In Section IV, two examples adapted from the literature illustrate the efficiency of the proposed approach. Finally, some concluding remarks and directions for potential future works are presented in Section V.

**Notation.**  $\mathbb{N}$ ,  $\mathbb{R}^n$ ,  $\mathbb{R}^{n \times m}$  denote respectively the sets of nonnegative integers,  $n$ -dimensional vectors and  $n \times m$  matrices. For any matrix  $A$ ,  $A^\top$  denotes its transpose and  $He\{A\} = A + A^\top$ . For any square matrix  $A$ ,  $trace(A)$  denotes its trace.  $diag(A_1; A_2)$  is a diagonal matrix with block diagonal matrices  $A_1$  and  $A_2$ . For two symmetric matrices of the same dimensions,  $A$  and  $B$ ,  $A > B$  means that  $A - B$  is symmetric positive definite.  $I$  and  $0$  stand respectively for the identity and the null matrix of appropriate dimensions. For a partitioned matrix, the symbol  $\star$  stands for symmetric blocks.  $\|\cdot\|$  stands for the Euclidean norm.  $|\cdot|$  stands for the absolute value.

## II. PROBLEM STATEMENT

### A. System description

Consider the following continuous-time linear plant

$$\begin{cases} \dot{x}_p(t) &= A_p x_p(t) + B_p u(t), \\ y_p(t) &= C_p x_p(t), \end{cases} \quad (1)$$

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where  $x_p(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$ ,  $y_p(t) \in \mathbb{R}^p$  are the state, the input and the output of the plant, respectively. Matrices  $A_p$ ,  $B_p$ ,  $C_p$  are constant and of appropriate dimensions. Pairs  $(A_p, B_p)$  and  $(C_p, A_p)$  are supposed to be controllable and observable, respectively.

We consider an observer-based feedback controller to stabilize the plant (1) and therefore to drive the output to zero. This controller is defined by:

$$\begin{cases} \dot{\hat{x}}(t) &= A_p \hat{x}(t) + B_p u(t) - L(y_p(t) - \hat{y}(t)), \\ \hat{y}(t) &= C_p \hat{x}(t), \\ u(t) &= K \hat{x}(t), \end{cases} \quad (2)$$

where  $\hat{x}(t) \in \mathbb{R}^n$  and  $\hat{y}(t) \in \mathbb{R}^p$  are the state and the output of the observer. Furthermore,  $L \in \mathbb{R}^{n \times p}$  and  $K \in \mathbb{R}^{m \times n}$  are the observer and controller gains, respectively.

By considering the continuous-time system described by (2), the control design is carried out according to the separation principle. In other words, the observer gain  $L$  is designed to make  $A_p + LC_p$  Hurwitz, which leads to the state estimation error dynamics given by

$$\dot{e}(t) = (A_p + LC_p)e(t), \quad (3)$$

where  $e(t) = x_p(t) - \hat{x}(t) \in \mathbb{R}^n$ , is globally asymptotically stable, i.e.  $\lim_{t \rightarrow \infty} e(t) = 0$ . Consequently, the estimation output error  $e_y(t) = C_p e(t)$  also asymptotically converges to zero, i.e.  $\lim_{t \rightarrow \infty} e_y(t) = 0$ . Furthermore, the dynamics of the observer is given by:

$$\dot{\hat{x}}(t) = (A_p + B_p K) \hat{x}(t) - L e_y(t). \quad (4)$$

### B. Control implementation

In this note the objective relies on the way to implement the control input  $u$ , taking into account that  $u$  is implemented through a sample-and-hold mechanism. It is not continuously updated or transmitted to the actuators. Indeed, it is updated at certain instants  $\{t_k\}_{k \in \mathbb{N}}$ , which form a sequence of strictly increasing positive scalars. Control action is held constant between two successive sampling instants ( $t_k$  and  $t_{k+1}$ ) through a zero order holder. Differently from classical digital control approaches, the sampling interval  $t_{k+1} - t_k$  is not assumed to be constant but can be seen as an additional control action.

Thus, the closed-loop system can be represented by

$$\begin{cases} \dot{x}_p(t) &= A_p x_p(t) + B_p u(t_k), \\ \dot{\hat{x}}(t) &= A_p \hat{x}(t) + B_p u(t_k) - L e_y(t), \\ u(t_k) &= K \hat{x}(t_k), \quad \forall t \in [t_k, t_{k+1}). \end{cases} \quad (5)$$

By using the same formulation as in [16] to define  $\delta(t)$ :

$$\delta(t) = \hat{x}(t_k) - \hat{x}(t), \quad (6)$$

the closed loop can be expressed in terms of  $\hat{x}$  and  $e$ , as follows:

$$\begin{cases} \dot{\hat{x}}(t) &= (A_p + B_p K) \hat{x}(t) + B_p K \delta(t) - L e_y(t), \\ \dot{e}(t) &= (A_p + LC_p) e(t), \quad \forall t \in [t_k, t_{k+1}). \end{cases} \quad (7)$$

**Remark 1:** It is important to note that  $\delta$  depends only on the observer variables and is therefore available at the

controller node. The function  $\delta$  corresponds to a measure of the difference between the continuous-time state observer value and its sampled and held version, which is currently used to implement the control.  $\circ$

To simplify the notation, consider the augmented state  $x = [\hat{x}^\top \ e^\top]^\top \in \mathbb{R}^{2n}$  and the available signals  $y = [\hat{x}^\top \ e_y^\top]^\top \in \mathbb{R}^{n+p}$ . Thus the closed loop under consideration reads:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\delta(t), \quad \forall t \in [t_k, t_{k+1}) \\ y(t) &= Cx(t) \end{aligned} \quad (8)$$

with

$$A = \begin{bmatrix} A_p + B_p K & -LC_p \\ 0 & A_p + LC_p \end{bmatrix}; B = \begin{bmatrix} B_p K \\ 0 \end{bmatrix} \quad (9)$$

$$C = \begin{bmatrix} I & 0 \\ 0 & C_p \end{bmatrix}.$$

### C. Problem formulation

In order to focus on the event-triggered implementation of the controller (2), we want to propose a new dynamic triggering mechanism to enrich the event generator algorithm managing the controller to decide when the control input has to be updated, based on the available information.

Hence, we want to adapt the event-triggered control strategy proposed in [19] and [9]. By considering that a dynamic event-triggering mechanism is introduced through an internal dynamic variable denoted  $\eta \in \mathbb{R}$ , the sampling instants are determined from the following logic:

$$t_{k+1} = \min\{t \geq t_k + T, \text{ s.t. } g(y(t), \delta(t), \eta(t)) \leq 0\}, \quad (10)$$

where the function  $g : \mathbb{R}^{2n+p+1} \rightarrow \mathbb{R}$  uses the vector of available information to the controller and  $T$  is a positive scalar. By definition (10) guarantees that the next sampling time will occur at least  $T$  time units ahead the last one.  $T$  can be then seen as a minimal dwell time. That allows to prevent Zeno solutions. Furthermore, for  $t \geq t_k + T$  the control is not updated before  $g(y(t), \delta(t), \eta(t)) \leq 0$ .

In the paper, we consider an impulsive dynamic of  $\eta$  of the following form:

$$\begin{cases} \dot{\eta}(t) &= f(y(t), \delta(t), \eta(t)), \\ &\forall t \in [t_k, t_k + T) \cup (t_k + T, t_{k+1}), \\ \eta(t_k + T) &= \max(0, \eta(t_k + T^-)), \end{cases} \quad (11)$$

for all  $k \in \mathbb{N}$ , where  $f : \mathbb{R}^{2n+p+1} \rightarrow \mathbb{R}$  and  $\eta(t_k + T^-)$  denotes the limit of  $\eta(t)$  when  $t$  approaches  $t_k + T$  from below. It is further assumed that the dynamic variable  $\eta$  is initialized to a value  $\eta(t_0) \geq 0$ .

The problem we intend to solve in this note can be summarized as follows:

**Problem 1:** Given a positive scalar  $T$ , devise functions  $f$ ,  $g$  such that the event-triggering mechanism defined as in (10) and (11), ensures the asymptotic stability of the closed-loop system (8), while reducing the number of events generated.

### III. EVENT-TRIGGERING MECHANISM

#### A. Theoretical conditions

To solve Problem 1, we need to design  $T$ ,  $f$  and  $g$  in order to ensure the asymptotic stability of the sampled-data system (5). Let us first propose a general formulation, inspired from [17], [18], which corresponds to extend Theorem 1 of [19] by taking into account the dynamic part of  $\eta$  in the same vein as in [9].

**Theorem 1:** *Given a positive scalar  $T$ , if there exist a function  $V : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ , scalars  $\epsilon_1 > 0$ ,  $\epsilon_2 > 0$ ,  $\epsilon_3 > 0$  and  $1 > \epsilon_4 > 0$  such that:*

- 1)  $\epsilon_1 \|x\|^2 \leq V(x) \leq \epsilon_2 \|x\|^2$ ,  $\forall x \in \mathbb{R}^{2n}$
- 2)  $\eta(t_k) \geq 0$  and  $\eta(t) \geq 0$ ,  
 $\forall t \in [t_k + T, t_{k+1}]$ ,  $\forall k \in \mathbb{N}$
- 3)  $\dot{V}(x(t)) + \dot{\eta}(t) \leq -\epsilon_3(V(x(t)) + \eta(t))$ ,  
 $\forall t \in [t_k, t_k + T) \cup (t_k + T, t_{k+1})$ ,  $\forall k \in \mathbb{N}$
- 4)  $V(x(t_k + T)) - V(x(t_k)) \leq -\epsilon_4 V(x(t_k))$ ,  $\forall k \in \mathbb{N}$

then, the closed-loop system (8) with the triggering mechanism (10) and (11) is asymptotically stable and the inter-event intervals are lower bounded by  $T$ .

*Proof:* Let us define the function  $\mathcal{W}(x, \eta) = V(x) + \eta$  and let  $k \in \mathbb{N}$ . The stability analysis is carried out considering the time intervals  $[t_k, t_k + T)$  and  $[t_k + T, t_{k+1})$ .

If  $\eta(t_k + T^-) < 0$ , then by (11) we have  $\eta(t_k + T) = 0$ . Therefore we have

$$\begin{aligned} \mathcal{W}(x(t_k + T), \eta(t_k + T)) &= V(x(t_k + T)) \\ &\leq (1 - \epsilon_4)V(x(t_k)) \\ &\leq (1 - \epsilon_4)\mathcal{W}(x(t_k), \eta(t_k)) \end{aligned} \quad (12)$$

where the first and second inequalities come from items 4) and 2), respectively. If  $\eta(t_k + T^-) \geq 0$ , then by (11) we have  $\eta(t_k + T) = \eta(t_k + T^-)$ . Therefore, we have

$$\begin{aligned} \mathcal{W}(x(t_k + T), \eta(t_k + T)) &= \mathcal{W}(x(t_k + T), \eta(t_k + T^-)) \\ &\leq e^{-\epsilon_3 T} \mathcal{W}(x(t_k), \eta(t_k)) \end{aligned} \quad (13)$$

where the inequality comes from item 3).

Item 3) also implies that

$$\mathcal{W}(x(t_{k+1}), \eta(t_{k+1})) \leq \mathcal{W}(x(t_k + T), \eta(t_k + T)),$$

which together with (12), (13) gives for all  $k \in \mathbb{N}$ ,

$$\mathcal{W}(x(t_{k+1}), \eta(t_{k+1})) \leq \rho \mathcal{W}(x(t_k), \eta(t_k)) \quad (14)$$

where  $\rho = \max(1 - \epsilon_4, e^{-\epsilon_3 T}) < 1$ . Moreover, from items 1) and 2), we have for all  $k \in \mathbb{N}$ ,

$$\begin{aligned} \epsilon_1 \|x(t_k)\|^2 + |\eta(t_k)| &\leq \mathcal{W}(x(t_k), \eta(t_k)) \\ &\leq \epsilon_2 \|x(t_k)\|^2 + |\eta(t_k)|. \end{aligned} \quad (15)$$

From (6) and (8), one can easily show that there exists  $\alpha > 1$  such that

$$\|x(t)\|^2 \leq \alpha \|x(t_k)\|^2, \quad \forall t \in [t_k, t_k + T), \quad \forall k \in \mathbb{N}.$$

Then, for all  $k \in \mathbb{N}$  and for all  $t \in [t_k + T, t_{k+1})$

$$\begin{aligned} \epsilon_1 \|x(t)\|^2 &\leq V(x(t)) \leq W(x(t), \eta(t)) \\ &\leq W(x(t_k + T), \eta(t_k + T)) \\ &\leq W(x(t_k), \eta(t_k)) \\ &\leq \epsilon_2 \|x(t_k)\|^2 + |\eta(t_k)| \end{aligned} \quad (16)$$

where inequalities come from items 1), 2) and 3), (12) and (13) and finally (15). Hence, letting  $\beta = \max(\alpha, \frac{\epsilon_2}{\epsilon_1})$  we get

$$\|x(t)\|^2 \leq \beta \|x(t_k)\|^2 + \frac{1}{\epsilon_1} |\eta(t_k)|, \quad \forall t \in [t_k, t_{k+1}), \quad \forall k \in \mathbb{N}. \quad (17)$$

Then, asymptotic stability follows from (14), (15) and (17). The lower bound on the inter-event time, as  $t_{k+1} - t_k \geq T$ , is guaranteed by the triggering rule (10). ■

**Remark 2:** Theorem 1 generalizes to the dynamic event-triggering laws Theorem 1 provided in [19], which is dedicated to the case of static event-triggering mechanisms. ○

#### B. Numerical design of event-triggering mechanism

In this section, we present numerical techniques to design functions  $f$ ,  $g$  such that the assumptions of Theorem 1 hold. We focus on functions  $f$  and  $g$  of the following form:

$$f(y, \delta, \eta) = \begin{bmatrix} y \\ \delta \\ \sqrt{|\eta|} \end{bmatrix}^\top Q \begin{bmatrix} y \\ \delta \\ \sqrt{|\eta|} \end{bmatrix} \quad (18)$$

where  $Q \in \mathbb{R}^{(2n+p+1) \times (2n+p+1)}$  is a symmetric matrix, and

$$g(y, \delta, \eta) = \eta. \quad (19)$$

In order to address Problem 1, we need to design  $Q$ , and the function  $V$  complying with the items of Theorem 1. The following result states matrix inequality conditions in this direction.

**Theorem 2:** *Given the controller and observer gains  $K$  and  $L$  and a positive scalar  $T$ , if there exist symmetric positive definite matrix  $P \in \mathbb{R}^{2n \times 2n}$ , a symmetric matrix  $Q \in \mathbb{R}^{(2n+p+1) \times (2n+p+1)}$ ,  $\epsilon_3 > 0$ ,  $1 > \epsilon_4 > 0$  such that*

$$\begin{bmatrix} He\{PA\} + C^\top Q_1 C + \epsilon_3 P & PB + C^\top Q_2 & C^\top Q_3 \\ \star & Q_4 & Q_5 \\ \star & \star & Q_6 + \epsilon_3 \end{bmatrix} \leq 0 \quad (20)$$

$$\begin{bmatrix} -P & M(T)^\top P \\ \star & -(1 - \epsilon_4)P \end{bmatrix} \leq 0 \quad (21)$$

with

$$\begin{aligned} Q &= \begin{bmatrix} Q_1 & Q_2 & Q_3 \\ \star & Q_4 & Q_5 \\ \star & \star & Q_6 \end{bmatrix} \\ M(T) &= \exp(A_c T) + \int_0^T \exp(A_c s) ds B \begin{bmatrix} I_n & 0 \end{bmatrix} \\ A_c &= \begin{bmatrix} A_p & -LC_p \\ 0 & A_p + LC_p \end{bmatrix}, \end{aligned} \quad (22)$$

then assumptions of Theorem 1 hold with  $V(x) = x^\top P x$  and the asymptotic stability of the closed-loop system (8) is guaranteed. Furthermore, the inter-sampling times are lower bounded by  $T$ .

*Proof:* Consider the function  $V(x) = x^\top P x$ , with  $P = P^\top > 0$ . Item 1) of Theorem 1 is satisfied with  $\epsilon_1 = \lambda_{\min}(P)$  and  $\epsilon_2 = \lambda_{\max}(P)$ .

By (11), we have  $\eta(t_k + T) \geq 0$ . From continuity of  $\eta$  on  $[t_k + T, t_{k+1}]$ , (10) and (19) give that  $\eta(t) \geq 0$  for all  $t \in [t_k + T, t_{k+1}]$ . Together with  $\eta(t_0) \geq 0$ , this guarantees the satisfaction of item 2) of Theorem 1.

By computing the time-derivative of  $\mathcal{W}(x, \eta) = V(x) + \eta$ , for any  $t \in [t_k, t_k + T) \cup (t_k + T, t_{k+1})$ , along the closed-loop system (8) with the dynamics of  $\eta$  defined by (11) one gets:

$$\begin{aligned}\dot{\mathcal{W}}(x, \eta) &= \dot{V}(x) + \dot{\eta} \\ &= x^\top (A^\top P + PA)x + 2x^\top PB\delta + f(y, \delta, \eta)\end{aligned}$$

Using the form of  $f$  given by (18) and the partition of matrix  $Q$  as defined in (22), one can write  $\dot{\mathcal{W}}$  in a more compact form as follows:

$$\dot{\mathcal{W}} = \begin{bmatrix} x \\ \delta \\ \sqrt{|\eta|} \end{bmatrix}^\top \Phi \begin{bmatrix} x \\ \delta \\ \sqrt{|\eta|} \end{bmatrix} - \epsilon_3 x^\top P x - \epsilon_3 |\eta|$$

where  $\Phi$  corresponds to the matrix in the left-hand side of inequality (20). Thus, the satisfaction of relation (20) ensures that

$$\begin{bmatrix} x \\ \delta \\ \sqrt{|\eta|} \end{bmatrix}^\top \Phi \begin{bmatrix} x \\ \delta \\ \sqrt{|\eta|} \end{bmatrix} \leq 0$$

That means that

$$\begin{aligned}\dot{\mathcal{W}} &\leq -\epsilon_3 x^\top P x - \epsilon_3 |\eta| \\ &\leq -\epsilon_3 (x^\top P x + \eta)\end{aligned}$$

and therefore item 3) in Theorem 1 is satisfied.

In order to prove that item 4) of Theorem 1 holds, one solves (8) over the interval  $[t_k, t_k + T]$  and it follows:

$$x(t_k + T) = M(T)x(t_k)$$

with  $M(T)$  defined in (22). Hence, one gets

$$\begin{aligned}V(x(t_k + T)) - V(x(t_k)) \\ = x(t_k)^\top (M(T)^\top P M(T) - P)x(t_k),\end{aligned}$$

If relation (21) holds, then  $V(x(t_k + T)) - V(x(t_k)) = x(t_k)^\top (M(T)^\top P M(T) - P)x(t_k) \leq -\epsilon_4 x(t_k)^\top P x(t_k)$ . That means that item 4) of Theorem 1 holds.

Hence, Theorem 1 applies and the rest of the result follows. ■

**Remark 3:** The case of static event-triggering mechanisms similarly to that one studied in [19] corresponds to remove  $\eta$  and consider the event-triggering rule (10) with

$$g(y, \delta, \eta) = g(y, \delta) = \begin{bmatrix} y \\ \delta \end{bmatrix}^\top R \begin{bmatrix} y \\ \delta \end{bmatrix}$$

with

$$R = \begin{bmatrix} R_\epsilon^{-1} & 0 \\ \star & -K^\top R_\delta K \end{bmatrix} \in \mathbb{R}^{(2n+p) \times (2n+p)} \quad (23)$$

○

**Remark 4:** The main differences between our Theorems 1 and 2 and Theorem 1 in [6] are: 1) The parameter  $T$  directly appears from the sampled system on the interval  $[t_k, t_k + T]$  and is involved in the decreasing of the Lyapunov function on the interval  $[t_k, t_k + T]$  without other parameters to fix; 2) In the dynamic of  $\eta$  there is a quadratic term in  $\sqrt{|\eta|}$  and matrix  $Q$  is only symmetric; 3) The considered Lyapunov function is different. ○

### C. Particular case: state feedback case

Let us consider the particular case where the state  $x_p$  is available and therefore no observer is needed. Consider the closed-loop system:

$$\dot{x}_p(t) = (A_p + B_p K)x_p(t) + B_p K \delta(t), \forall t \in [t_k, t_{k+1}) \quad (24)$$

with  $\delta(t) = x_p(t_k) - x_p(t)$ , and the dynamics of  $\eta$  is defined by

$$f(x_p, \delta, \eta) = \begin{bmatrix} x_p \\ \delta \\ \sqrt{|\eta|} \end{bmatrix}^\top Q \begin{bmatrix} x_p \\ \delta \\ \sqrt{|\eta|} \end{bmatrix} \quad (25)$$

where  $Q \in \mathbb{R}^{(2n+1) \times (2n+1)}$  is a symmetric matrix. One can handle this case with the framework of Theorem 2 as follows.

**Corollary 1:** Given the controller gain  $K$  and a positive scalar  $T$ , if there exist symmetric positive definite matrix  $P \in \mathbb{R}^{n \times n}$ , a symmetric matrix  $Q \in \mathbb{R}^{(2n+1) \times (2n+1)}$  (partitioned adequately as in (22)),  $\epsilon_3 > 0$ ,  $1 > \epsilon_4 > 0$  such that

$$\begin{bmatrix} He\{PA_K\} + Q_1 + \epsilon_3 P & PB_p K + Q_2 & Q_3 \\ \star & Q_4 & Q_5 \\ \star & \star & Q_6 + \epsilon_3 \end{bmatrix} \leq 0 \quad (26)$$

$$\begin{bmatrix} -P & M_K(T)^\top P \\ \star & -(1 - \epsilon_4)P \end{bmatrix} \leq 0 \quad (27)$$

with  $A_K = A_p + B_p K$  and

$$M_K(T) = \exp(A_p T) + \int_0^T \exp(A_p s) ds B_p K \quad (28)$$

then assumptions of Theorem 1 hold with  $V(x_p) = x_p^\top P x_p$  and the asymptotic stability of the closed-loop system (24) is guaranteed. Furthermore, the inter-sampling times are lower bounded by  $T$ .

It is interesting to have a quick focus on the case of state feedback case as studied in [9]. Indeed, the dynamics of  $\eta$  used in [9] can be recovered from (25) by considering the following matrix

$$Q = \begin{bmatrix} \sigma Q_0 & -PB_p K & 0 \\ \star & 0 & 0 \\ \star & \star & -\lambda \end{bmatrix} \in \mathbb{R}^{(2n+1) \times (2n+1)}, \quad (29)$$

where  $Q_0 \in \mathbb{R}^{n \times n}$  is an arbitrary symmetric positive definite matrix such that  $(A_p + B_p K)^\top P_0 + P_0 (A_p + B_p K) = -Q_0$ , with  $P_0 = P_0^\top > 0$  (see relations (15) and (20) in [9]). For this case relation (26) of Corollary 1 simply reads:

$$\begin{bmatrix} He\{PA_K\} + \sigma Q_0 + \epsilon_3 P & 0 \\ \star & -\lambda + \epsilon_3 \end{bmatrix} < 0 \quad (30)$$

The interest of the slightly modified version of Corollary 1 (that is replacing (26) by (30)) with respect to results in Section III in [9] resides in the following facts.

- The condition (27) provides a direct way to prevent Zeno solutions thanks to the use of the tunable parameter  $T$ , avoiding the calculations of Proposition 3.2 in [9];
- We can compute at the same time the adequate parameters  $P$ ,  $\lambda$  and  $\sigma$  ( $Q_0$  being given as in [9]);
- We can drop the condition  $(A_p + B_p K)^T P_0 + P_0 (A_p + B_p K) = -Q_0$  since we can replace  $\sigma Q_0$  by  $Q_1$  in (30) and therefore search at the same time for  $P$ ,  $Q_1$  and  $\lambda$ .

#### D. Optimization issues

It is interesting to note that the conditions of Theorem 2 are linear matrix inequalities (LMIs) as long as  $K$ ,  $L$  and  $T$ ,  $\epsilon_3$ ,  $\epsilon_4$  are fixed. Hence, by considering that the gains  $K$  and  $L$  have been obtained from classical design techniques, and that  $T$ ,  $\epsilon_3$ ,  $\epsilon_4$  are a priori fixed to ensure a desired minimal dwell time and convergence rate, conditions (20) and (21) are linear in the decision variables  $P$  and  $Q_i$ ,  $i = 1, \dots, 6$ . Let us emphasize that due to the fact that  $A_p + B_p K$  and  $A_p + LC_p$  are Hurwitz matrices, there always exist positive scalars  $T$ ,  $\epsilon_3$ ,  $\epsilon_4$ , small enough, such that conditions (20) and (21) are feasible. The objective behind the design of the decision variables is to reduce the amount of control updates. Note that  $T$  is a tuning parameter.

Intuitively, a way to reduce the number of control updates is to minimize the conservatism on the estimate of the decay rate given by condition 3) in Theorem 1, or equivalently to maximize the derivative of  $\eta$ . Numerically, this means that we want the matrix  $\Phi$  in the left-hand side of inequality (20) to be as close as possible to zero. The following LMI optimization scheme can then be considered:

$$\begin{aligned} \min \theta \\ \text{subject to } (20), (21), \Phi \geq -\theta I_{3n+1} \\ \text{and } Q_1 > 0, P > I_{2n}, \text{trace}(Q_1) \leq 1 \end{aligned} \quad (31)$$

The constraints  $P > I_{2n}$  and  $\text{trace}(Q_1) \leq 1$  are added for well conditioning purposes. The addition of the constraint of positive definiteness of  $Q_1$  allows to enforce the negative definiteness of  $He\{PA\}$ , preventing exponentially unstable continuous dynamics and therefore leading to more elegant inter-sample transients.

Moreover, by definition of the event-triggering mechanism, Zeno behavior is avoided because  $T$  imposes an explicit minimum inter-event time. The role of  $T$  relies on the expected average sampling rate of the event-triggered implementation. An interesting way consists of obtaining the maximal admissible value of  $T$  by iteratively increasing  $T$  and solving problem (31).

Note that in the case when the used observer is fast, it may be preferable to minimize the conservatism of condition 3) after the observer has converged. In that case, the following

LMI optimization scheme can be considered:

$$\begin{aligned} \min \theta \\ \text{subject to } (20), (21), D^T \Phi D \geq -\theta I_{3n+1} \\ \text{and } Q_1 > 0, P > I_{2n}, \text{trace}(Q_1) \leq 1 \end{aligned} \quad (32)$$

where

$$D = \text{diag}\left(\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}; I_n; 0\right) \in \mathbb{R}^{3n+1}$$

**Remark 5:** By considering the state feedback case, and more especially the case treated in [9] and commented in the previous Section III-C, the problem of optimization (31) can be written as follows:

$$\begin{aligned} \min \theta \\ \text{subject to } (26), (27), \Psi \geq -\theta I_{2n+1} \\ \text{and } Q_1 > 0, P > I_n, \text{trace}(Q_1) \leq 1 \end{aligned} \quad (33)$$

where  $\Psi$  is the matrix of the left-hand side of inequality (26). In this case one can also obtain the maximal admissible value of  $T$  by iteratively increasing  $T$  and solving problem (33).  $\circ$

#### IV. ILLUSTRATIVE EXAMPLES

In this section, we illustrate the proposed approach through two examples borrowed from [19] and from [9], respectively.

**Example 1:** By considering the example studied in [19] in which we forget the tracking problem, the system data is defined by:

$$\begin{aligned} A_p &= \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}, B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_p = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ L &= \begin{bmatrix} -42 \\ -444 \end{bmatrix}, K = \begin{bmatrix} -11.4222 & -6.9135 \end{bmatrix}. \end{aligned} \quad (34)$$

Since the observer time scale is an order of magnitude faster than the controller time scale, we synthesize a dynamic event-triggering mechanism by solving problem (32) with  $T = 0.05$ ,  $\epsilon_3 = 0.1$ ,  $\epsilon_4 = 1 - e^{-\epsilon_3 T}$ . The corresponding closed-loop behavior of the system is shown on Figure 1. One can see that the rate of control updates is high at the beginning of the execution, and tends to decrease after the observer has converged. This is the expected behavior when solving problem (32). If we had solved (31) instead, the contrary behavior would have been observed: large inter-sampling times at the beginning, which tend to destabilize the system and smaller inter-sampling time after the observer has converged. Of course, such a behavior is not desirable.

**Example 2:** Consider the example studied in [9] for which we address the state feedback case. The system data is given by:

$$A_p = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, K = \begin{bmatrix} 1 & -4 \end{bmatrix}. \quad (35)$$

We synthesize a dynamic event-triggering mechanism by solving problem (33) with  $T = 0.05$ ,  $\epsilon_3 = 0.99$ ,  $\epsilon_4 = 1 - e^{-\epsilon_3 T}$ . The corresponding closed-loop behavior of the system is shown on Figure 2. Interestingly, the synthesized matrix  $Q$  has the particular structure shown in (29). It is also noticeable that the average inter-sampling time is similar to the maximal average inter-sampling time reported in [9].

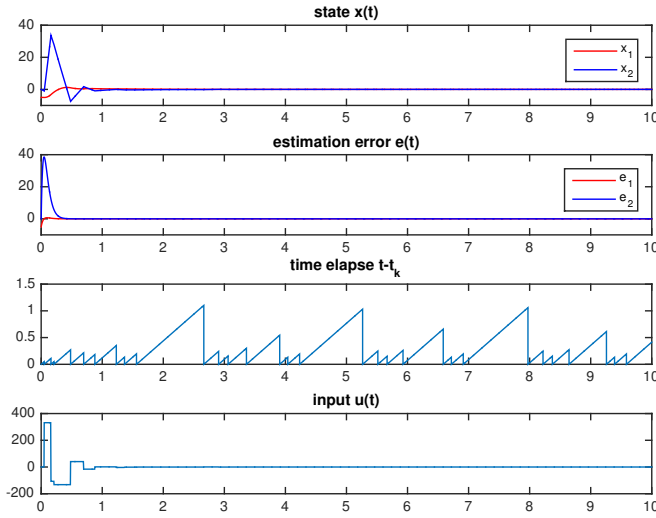


Fig. 1: Closed-loop behavior for system (34), with dynamic event-triggering mechanism synthesized using (32).

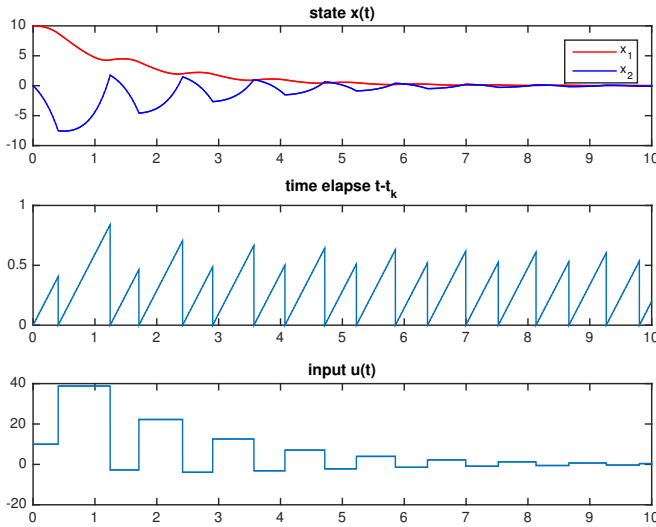


Fig. 2: Closed-loop behavior for system (35), with dynamic event-triggering mechanism synthesized using (33).

## V. CONCLUSION

This paper tackled to devise a new dynamic triggering mechanism based on the available information issued from the plant and the observer-based controller. The proposed strategy ensures that Zeno effect is avoided if the sufficient LMI conditions are attended with a tuning positive parameter  $T$ . Furthermore, a simple optimization criterion allows to design the parameters of the event-triggered law with the implicit objective of reducing the number of control updates.

These results open the door for future research directions. In particular, the co-design problem, that is the design of both the controller and observer gains together with the parameters of the event-triggering mechanism should be carried out. Furthermore, we envision to study what happens with the dynamic event-triggering mechanism when the control

system is affected by uncertainty.

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