Notes on Sector condition study

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Sector conditions are used to handle non-linearity in linear systems (activation functions in my case). From the book of Sophie I will take the basic concepts, firstly referred to the Dead-zone non linearity and then proceeding to the general case.

Defining the Dead-zone non-linearity:

$$\phi(v(t)) = sat(v(t)) - v(t) = \begin{cases} u_{max(i)} - v_{(i)} & \text{if } v_{(i)} > u_{max(i)} \\ 0 & \text{if } -u_{min(i)} \leq v_{(i)} \leq u_{max(i)} \\ -u_{min(i)} - v_{(i)} & \text{if } v_{(i)} < -u_{min(i)} \end{cases} \quad (1)$$

With this non-linearity it's possible to explicit the global sector conditions:

Lemma 1 *For all $v \in \mathbb{R}^n$, the non linearity $\phi(v)$ satisfies the following inequality:*

$$\phi(v)^\top T\,(\phi(v) + v) \leq 0 \quad (2)$$

for any diagonal positive matrix $T \in \mathbb{R}^{n\times n}$

Proof the sign of 2 will be discussed

- If $v > u_{max}$ we have $\phi(v) < 0$ and $\phi(v) + v = u_{max} > 0$ hence we have the overall product < 0
- if $v < -u_{min}$ we have $\phi(v) > 0$ and $\phi(v) + v = -u_{min} < 0$ hence we have the overall product < 0
- if $-u_{min} < v < u_{max}$ we have $\phi(v) = 0$ hence the product will always be 0

This proves that the product is always ≤ 0

The general form refers to the elements inside the set:

$$S(v - \omega, u_{min}, u_{max}) = \{v \in \mathbb{R}^n; \omega \in \mathbb{R}^n; -u_{min} \leq v - \omega \leq u_{max}\} \quad (3)$$

Then the following inequality is satisfied $\forall T \in \mathbb{R}^{n\times n}$ diagonal positive definite

$$\phi(v)^\top T\,(\phi(v) + \omega) \leq 0 \quad (4)$$

Proof By exploiting condition 3 the proof is analogous.

It is a powerful expression since we can substitute v with other expressions like for example the feedback law $v(t) = Kx(t)$ with a dynamic like $\dot{x} = Ax + B\ \text{sat}(Kx) = (A + BK)x + B\phi(Kx(t))$ with $\phi(Kx) = \text{sat}(Kx) - Kx$ obtaining something like

$$\phi(Kx)^\top T\,(\phi(Kx) + Gx) \leq 0$$

For every x in the polyhedral set:

$$S(K-G, u_{min}, u_{max}) = \{x \in \mathbb{R}^n; -u_{min} \leq (K-G)\, x \leq u_{max}\}$$

Like already seen for the global sector condition with S, T, R these conditions are useful to inject into the discussion of the Lyapunov function incremental sign since it's a term whose sign is always defined. Doing so we take into account the non linearity of the system and we can proceed with the LMI formulation.

$$V(x) = x^\top P x, P = P^\top > 0$$

$$\dot{V}(x) \leq \dot{V}(x) - 2\phi(Kx)^\top T(\phi(Kx) + Gx) \forall x \in \mathcal{E}(P, 1) \tag{5}$$

Note that $\mathcal{E}(P, 1)$ refers to a ellipsoid that is the current region of asymptotic stability (RAS). It is included in the LMI problem via a conditions similar to this:

$$\begin{bmatrix} W & WK^\top - Z^\top \\ \star & u_0^2 \end{bmatrix} \geq 0 \tag{6}$$

With $P = W^{-1}, Z = GW, S = T^{-1}$. Usually we pre and post multiply equation 5 by $\begin{bmatrix} P^{-1} & 0 \\ 0 & T^{-1} \end{bmatrix}$ and we change the variables we obtain something like this

$$\begin{bmatrix} x^\top \phi(Kx)^\top \end{bmatrix} \begin{bmatrix} W(A+BK)^\top + (A+BK)W & BS - WK^\top - Z^\top \\ \star & -2S \end{bmatrix} \begin{bmatrix} x \\ \phi(Kx) \end{bmatrix} < 0$$

Arcak paper discussion

The local sector constraints are considered with the offset to the equilibrium points $(\nu_*, \phi(\nu_*))$ for each single activation function:

Let $\alpha_\phi, \beta_\phi, \underline{\nu}, \overline{\nu}, \nu_* \in \mathbb{R}^{n_\phi}$ be given with $\alpha_\phi \leq \beta_\phi$, $\underline{\nu} \leq \nu_* \leq \overline{\nu}$, $\omega_* := \phi(\nu_*)$. Assuming ϕ satisfies the offset local sector $[\alpha_\phi, \beta_\phi]$ around (ν_*, ω_*) element wise for all $\nu_\phi \in [\underline{\nu}, \overline{\nu}]$. If $\lambda \in \mathbb{R}^{n_\phi}$ with $\lambda \geq 0$ then:

$$\begin{bmatrix} \nu_\phi - \nu_* \\ \omega_\phi - \omega_* \end{bmatrix}^\top \Psi_\phi^\top M_\phi(\lambda) \Psi_\phi \begin{bmatrix} \nu_\phi - \nu_* \\ \omega_\phi - \omega_* \end{bmatrix} \geq 0 \;\; \forall \nu_\phi \in [\underline{\nu}, \overline{\nu}],\; \omega_\phi = \phi(\nu_\phi) \tag{7}$$

where

$$\Psi_\phi := \begin{bmatrix} diag(\beta_\phi) & -I_\phi \\ -diag(\alpha_\phi) & I_\phi \end{bmatrix}$$

and

$$M_\phi(\lambda) := \begin{bmatrix} 0_{n_\phi} & diag(\lambda) \\ diag(\lambda) & 0_{n_\phi} \end{bmatrix}$$

By putting into explicit form this product we obtain:

$$\sum_{i=1}^{n_\phi} \lambda_i \left(\Delta\omega_i - \alpha_i \Delta\nu_i\right) \cdot \left(\beta_i \Delta\nu_i - \Delta\omega_i\right)$$

Which is the offset local sector condition applied for each activation function in the NN.

Another interesting thing is the computation of vectors $\underline{\nu}, \overline{\nu}$. Essentially we choose $\underline{\nu}^1, \overline{\nu}^1$ with ν_* in between. Then we compute $[\underline{\omega}^1, \overline{\omega}^1]$ from the output of $\omega^1 = \phi^1(\nu^1)$ that can be used to compute the bounds $\underline{\nu}^2, \overline{\nu}^2$ and so on. To sum up everything is dependent on the initial choice of $\underline{\nu}^1, \overline{\nu}^1$. This is important for the later estimation of the ROA: decreasing $(\overline{\nu}^1 - \nu_*^1)$ if beneficial for the ROA estimation but restricts the region where ROA inner approximations lie. The opposite if we increase $(\overline{\nu}^1 - \nu_*^1)$, it has been chosen to parametrize as δ this quantity and choose δ that leads to the larges inner approximation.

Static ETM paper discussion

Appendix

- Lipschitz constant of NN: specifies how much the output of the network can change with respect to changes in the input. It is Lipschitz continuous if $\exists L \geq 0$ such that $\forall x_1, x_2$:

$$||f(x_1) - f(x_2)|| \leq L||x_1 - x_2||$$

- General global sector condition: Let $\alpha \leq \beta$, the function $\phi : \mathbb{R} \to \mathbb{R}$ lies in the global sector $[\alpha, \beta]$ if:

$$(\phi(\nu) - \alpha\nu) \cdot (\beta\nu - \phi(\nu)) \geq 0 \quad \forall \nu \in \mathbb{R}$$

  Note how this condition can be brought back to eq4 that is the sector condition $[0, -1]$ by imposing $\alpha = 0, \beta = -1$:

$$\phi(\nu) \cdot (-\nu - \phi(\nu)) \geq 0 \to \phi(\nu) \cdot (\nu + \phi(\nu)) \leq 0$$

  For the one dimensional case, brought to multidimensional case with T diagonal and positive definite

- Offset local sector: Let $\alpha, \beta, \underline{\nu}, \overline{\nu}, \nu_*$ be given with $\alpha \leq \beta$ and $\underline{\nu} \leq \nu_* \leq \overline{\nu}$. The function $\phi : \mathbb{R} \to \mathbb{R}$ satisfies the offset local sector $[\alpha, \beta]$ around the point $(\nu_*, \phi(\nu_*))$ if

$$(\Delta\phi(\nu) - \alpha\Delta\nu) \cdot (\beta\Delta\nu - \Delta\phi(\nu)) \geq 0 \quad \forall \nu \in [\underline{\nu}, \overline{\nu}]$$

  where $\Delta\phi(\nu) := \phi(\nu) - \phi(\nu_*)$ and $\Delta\nu := \nu - \nu_*$