Ideas on RNN + Dynamic event-triggered control

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1 Introduction

2 Description of the system

Consider the plant under consideration

$$\begin{array}{rl} x_p^+ = & A_p x_p + B_p u_p \\ y = & C_p x_p \end{array} \tag{1}$$

and the RNN defined as follows:

$$\begin{array}{rl} \begin{pmatrix} \xi^+ \\ u_p \\ v \end{pmatrix} & = \begin{pmatrix} A_c & B_{c1} & B_{c2} \\ C_{c1} & D_{c11} & D_{c12} \\ C_{c2} & D_{c21} & D_{c22} \end{pmatrix} \begin{pmatrix} \xi \\ w \\ y \end{pmatrix} \\ w & = \phi(v) \end{array} \tag{2}$$

where ϕ is the activation function, decentralized, memoryless and such that $\phi(0) = 0$.

We can write the closed loop (1) and (2) in a compact form by defining the extended state $x = \begin{pmatrix} x_p \\ \xi \end{pmatrix}$:

$$\begin{array}{rl} \begin{pmatrix} x^+ \\ u_p \\ v \end{pmatrix} & = \begin{pmatrix} A & B \\ K_1 & K_2 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \\ w & = \phi(v) \end{array} \tag{3}$$

with

$$\begin{array}{l} A = \begin{pmatrix} A_p + B_p D_{c12} C_p & B_p C_{c1} \\ B_{c2} C_p & A_c \end{pmatrix}; B = \begin{pmatrix} B_p D_{c11} \\ B_{c1} \end{pmatrix} \\ K_1 = \begin{pmatrix} D_{c12} C_p & C_{c1} \end{pmatrix}; K_2 = D_{c11} \\ C_1 = \begin{pmatrix} D_{c22} C_p & C_{c2} \end{pmatrix}; D_1 = D_{c21} \end{array} \tag{4}$$

Remark 1 *From (3) and (4, the connection between v and w is affected by an algebraic loop and therefore we need to assume the well-posedness of this algebraic loop. Note that it is not the case in [4], where $D_{21} = 0$.*

3 Event-triggered state-feedback control law

We want to consider a dynamical event-triggered control in order to reduce the computational cost associated with the network evaluation. We can introduce an event-triggering mechanism at the output of each layer, to decide whether or not to update the current outputs w. Inspired by the results proposed in [5], we consider memory variables χ_k and s_k in $\mathbb{N}$, which denote the value of w^i employed at time k[1]:

$$\begin{pmatrix}\chi\\ s\end{pmatrix} = \begin{cases} \begin{pmatrix} w \\ k,\end{pmatrix} & \text{if the memory is updated,} \\ \begin{pmatrix}\chi_{k-1}\\ s_{k-1}\end{pmatrix}, & \text{otherwise.}\end{cases} \tag{5}$$

Hence, one gets:

$$\begin{aligned} u_p &= K_1 x + K_2 \chi, \ \forall k \\ v &= C_1 x + D_1 \chi \\ \chi_0 &= w_0 = \phi(v_0), \ s_0 = 0 \end{aligned} \tag{6}$$

The rule to determine if the value of χ_k has to be updated or not is chosen as a dynamic discrete-time event-triggering rule, defined by

$$s^+ = \min_{m\in\mathbb{N}} \{ m \geq s+1 \mid \psi(w_m, \chi) \geq \rho\eta_m \}, \tag{7}$$

where the triggering condition is parameterized by:

- A quadratic triggering function, ψ given by

$$\psi(w,\chi) = \begin{bmatrix} w\\ \chi\end{bmatrix}^\top \begin{bmatrix}\Psi_1 & \Psi_2\\ \Psi_2^\top & \Psi_3\end{bmatrix}\begin{bmatrix} w\\ \chi\end{bmatrix}, \ \forall (w,\chi)\in\mathbb{R}^{n_x}\times\mathbb{R}^{n_x}, \tag{8}$$

  with $\Psi = \begin{bmatrix}\Psi_1 & \Psi_2\\ \Psi_2^\top & \Psi_3\end{bmatrix}$ being a symmetric matrix satisfying

$$\begin{bmatrix} I\\ I\end{bmatrix}^\top \begin{bmatrix}\Psi_1 & \Psi_2\\ \Psi_2^\top & \Psi_3\end{bmatrix}\begin{bmatrix} I\\ I\end{bmatrix} = \Psi_1 + \Psi_2 + \Psi_2^\top + \Psi_3 \preceq 0. \tag{9}$$

- The triggering variable η_k in $\mathbb{R}$ is driven by the following discrete-time dynamic equation:

$$\eta^+ = (\lambda+\rho)\eta - \psi(w,\chi), \ \forall \eta_0 \geq 0, \tag{10}$$

  where λ and ρ are such that $\rho \geq 0$ and $(\lambda+\rho)\in[0,1)$.

Remark 2 *If we choose $\rho = 0$ in (7) makes the dynamic event-triggering rule a static one. Furthermore, when $\chi = w$, (9) ensures $\psi(w,w) \leq 0$ for all w, which means that the event-triggered condition is not violated just after a control update.*

[1]Note that $\chi = \chi_k$ and $s = s_k$. We keep only k when considering the previous value at instant $k-1$ to avoid confusion.

As in [3], [2], [6], the variable η has to be guaranteed to be nonnegative for all k in $\mathbb{N}$. Then Lemma 3 in [5] can be used.

The closed loop including the event-triggered variable is defined by:

$$\begin{aligned} \begin{pmatrix} x^+ \\ \eta^+ \\ u_p \\ v \end{pmatrix} &= \begin{pmatrix} A & 0 & B \\ 0 & \lambda+\rho & 0 \\ K_1 & 0 & K_2 \\ C_1 & 0 & D_1 \end{pmatrix} \begin{pmatrix} x \\ \eta \\ \chi \end{pmatrix} \\ w &= \phi(v) \end{aligned} \tag{11}$$

The problem we intend to solve can be summarized as follows.

Problem 1 *Design the event-triggering parameters (Ψ, ρ, λ) such that the regional stability of the closed loop is ensured, while the computational cost associated with the evaluation of the recurrent neural network.*

Maybe the co-design could be discussed by taking inspiration of the results I had with Alex regarding data-driven and the way to deal with unknown matrices A and B.

4 Quadratic abstraction

In order to develop constructive condition for solve the emulation problem, we consider the following assumption to handle the activation function ϕ.

Assumption 1 *There exist $S \in \mathbb{S}^s_+$, $T \in \mathbb{R}^{s\times p}$, and $R \in \mathbb{S}^p_-$ such that, for all $v_1, v_2 \in \mathbb{R}^s$,*

$$\begin{bmatrix} v_1 - v_2 \\ \phi(v_1) - \phi(v_2) \end{bmatrix}^\top \begin{bmatrix} S & T \\ \star & R \end{bmatrix} \begin{bmatrix} v_1 - v_2 \\ \phi(v_1) - \phi(v_2) \end{bmatrix} \geq 0. \tag{12}$$

Assumption 1 is related to the notion of incremental sector boundedness and was, for example, considered in [9], [7], [8], [1]. Such an Assumption allows to consider several kind of nonlinear activation function ϕ, as for example Lure function.

Discussion regarding the equilibrium point of the closed loop. Suppose $x^\star, v^\star, \ldots = 0$. The existence of $x^\star, v^\star, \ldots$ can be also handled.

5 Emulation conditions

One considers the Lyapunov function

$$V(x, \eta) = x^\top P x + \eta \tag{13}$$

with $P = P^\top > 0$. Function V is well defined and is positive definite from the hypothesis on η and the positive definiteness of P. Then, one computes the forward increment of V along the trajectories of closed loop, using Assumption 1.

One gets:

$$\Delta V = V(x^+, \eta^+) - V(x, \eta) = (x^+)^\top P x^+ - x^\top P x + \eta^+ - \eta$$

and from Assumption 1:

$$\Delta V \leq \Delta V + \tau \begin{bmatrix} v - v^\star \\ \phi(v) - \phi(v^\star) \end{bmatrix}^\top \begin{bmatrix} S & T \\ \star & R \end{bmatrix} \begin{bmatrix} v - v^\star \\ \phi(v) - \phi(v^\star) \end{bmatrix}$$

with $\tau \geq 0$. Let us denote the right-hand term of the previous inequality by $\mathcal{L}(x, \eta)$. By replacing x^+ and η^+ by their expression, it follows:

$$\begin{aligned} \mathcal{L}(x,\eta) &= (Ax + B\chi)^\top P (Ax + \chi) - x^\top P x + (\lambda + \rho - 1)\eta \\ &\quad - \begin{bmatrix} w \\ \chi \end{bmatrix}^\top \begin{bmatrix} \Psi_1 & \Psi_2 \\ \Psi_2^\top & \Psi_3 \end{bmatrix} \begin{bmatrix} w \\ \chi \end{bmatrix} + \tau \begin{bmatrix} v \\ \phi(v) \end{bmatrix}^\top \begin{bmatrix} S & T \\ \star & R \end{bmatrix} \begin{bmatrix} v \\ \phi(v) \end{bmatrix} \\ &\leq (Ax + B\chi)^\top P (Ax + \chi) - x^\top P x - \tfrac{(1-\lambda)}{\rho} \begin{bmatrix} w \\ \chi \end{bmatrix}^\top \begin{bmatrix} \Psi_1 & \Psi_2 \\ \Psi_2^\top & \Psi_3 \end{bmatrix} \begin{bmatrix} w \\ \chi \end{bmatrix} \\ &\quad + \tau \begin{bmatrix} v \\ \phi(v) \end{bmatrix}^\top \begin{bmatrix} S & T \\ \star & R \end{bmatrix} \begin{bmatrix} v \\ \phi(v) \end{bmatrix} \end{aligned} \tag{14}$$

By replacing v and w by their expressions and by defining the augmented state $\varsigma = (x^\top \;\; \chi^\top \;\; \phi(v)^\top)$ the right-hand term in (14)can be written as:

$\mathcal{L}(x, \eta) \leq -\varsigma^\top M \varsigma$

with

$$M = \begin{bmatrix} P - \tau C_1^\top S C_1 & -\tau C_1^\top S D_1 & -\tau C_1^\top T & A^\top \\ \star & \frac{(1-\lambda)}{\rho}\Psi_3 - \tau D_1^\top S D_1 & \frac{(1-\lambda)}{\rho}\Psi_2^\top - \tau D_1^\top T & B^\top \\ \star & \star & -\tau R + \frac{(1-\lambda)}{\rho}\Psi_1 & 0 \\ \star & \star & \star & P^{-1} \end{bmatrix} \tag{15}$$

If $M > 0$, then $\mathcal{L}(x, \eta) < 0$ and therefore $\Delta V < 0$.

- One can state a theorem with such a condition, for which we handle the presence of P and P^{-1}, with different trick.
- We can consider Assumption 1 in a local context, that is condition (12) hold for $v \in \mathcal{V}$, with $\mathcal{V}$ for example defined as a polytope or as $\mathcal{V} = \left\{ v, \begin{bmatrix} 1 \\ v \end{bmatrix}^\top \begin{bmatrix} N_1 & N_2 \\ \star & N_3 \end{bmatrix} \begin{bmatrix} 1 \\ v \end{bmatrix} \preceq 0 \right\}$.
- We can add the problem of constant reference tracking.

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