15 Stability Tests for Constrained Linear Systems

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Abstract. This paper is yet another demonstration of the fact that enlarging the design space allows simpler tools to be used for analysis. It shows that several problems in linear systems theory can be solved by combining Lyapunov stability theory with Finsler's Lemma. Using these results, the differential or difference equations that govern the behavior of the system can be seen as constraints. These *dynamic constraints*, which naturally involve the state derivative, are incorporated into the stability analysis conditions through the use of scalar or matrix Lagrange multipliers. No *a priori* use of the system equation is required to analyze stability. One practical consequence of these results is that they do not necessarily require a state space formulation. This has value in mechanical and electrical systems, where the inversion of the *mass* matrix introduces complicating nonlinearities in the parameters. The introduction of multipliers also simplify the derivation of robust stability tests, based on quadratic or parameter-dependent Lyapunov functions.

15.1 A Motivation from Lyapunov Stability

Consider the continuous-time linear time-invariant system described by the differential equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \tag{15.1}$$

where $x(t) : [0, \infty) \to \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. Define the quadratic form $V : \mathbb{R}^n \to \mathbb{R}$ as

$$V(x) := x^T P x, \tag{15.2}$$

where $P \in \mathbb{S}^n$. The symbol $(\cdot)^T$ denotes transposition and $\mathbb{S}^n$ denotes the space of the square and symmetric real matrices of dimension n. If

$$V(x) > 0, \quad \forall x \neq 0,$$

then matrix P is said to be *positive definite*. The symbol $X \succ 0$ ($X \prec 0$) is used to denote that the symmetric matrix X is *positive* (*negative*) *definite*.

The equilibrium point $x = 0$ of the system (15.1) is said to be (globally) *asymptotically stable* if

$$\lim_{t \to \infty} x(t) = 0, \quad \forall x(0) = x_0, \tag{15.3}$$

where $x(t)$ denotes a solution to the differential equation (15.1). If (15.3) holds, then, by extension, the system (15.1) is said to be asymptotically stable. A necessary and sufficient condition for the system (15.1) to be asymptotically stable is that matrix A be Hurwitz, that is, that all eigenvalues of A have negative real parts. According to Lyapunov stability theory, system (15.1) is asymptotically stable if there exists $V(x(t)) > 0$, $\forall x(t) \neq 0$ such that

$$\dot{V}(x(t)) < 0, \quad \forall \dot{x}(t) = Ax(t), \quad x(t) \neq 0. \tag{15.4}$$

That is, if there exists $P \succ 0$ such that the time derivative of the quadratic form (15.2) is negative *along all trajectories* of system (15.1). Conversely, it is well known that if the linear system (15.1) is asymptotically stable then there always exists $P \succ 0$ that renders (15.4) feasible. Notice that in (15.4), the time derivative $\dot{V}(x(t))$ is a function of the state $x(t)$ only, which implicitly assumes that the *dynamical constraint* (15.1) has been previously substituted into (15.4). This yields the equivalent condition

$$\dot{V}(x(t)) = x(t)^T \left(A^T P + PA\right) x(t) < 0, \quad \forall x(t) \neq 0.$$

Hence, asymptotic stability of (15.1) can be checked by using the following lemma.

Lemma 1 (Lyapunov). *The time-invariant linear system is asymptotically stable if, and only if,* $\exists\, P \in \mathbb{S}^n : P \succ 0, \quad A^T P + PA \prec 0.$

At this point, one might ask whether would it be possible to characterize the set defined by (15.4) without substituting (15.1) into (15.4)? The aim of this work is to provide an answer to this question. The recurrent idea is to analyze the feasibility of sets of inequalities subject to *dynamic equality constraints*, as (15.4), from the point of view of *constrained optimization*. By utilizing the well know Finsler's Lemma [9] it will be possible to characterize existence conditions for this class of problems without explicitly substituting the dynamic constraints. Equivalent conditions will be generated where the dynamic constraints appear weighted by *multipliers*, a standard expedient in the optimization literature. The method is conceptually simple, yet it seems that it has never been used with that purpose in the systems and control literature so far.

The advantage of substituting the dynamic constraints in the stability test conditions is the reduced size of the space on which one must search for a solution. In the context of the problem of Lyapunov stability this reduced space is characterized by the state $x(t)$. In contrast, the space composed of $x(t)$ and $\dot{x}(t)$ can be seen as an *enlarged* space. In this paper it will be shown that the use of larger search spaces for linear systems analysis provides better ways to explore the *structure* of the problems of interest. This will often lead to mathematically more tractable problems. Whereas working in a higher dimensional space requires the introduction of some extra variables to search for — which one might think at first sight as being a disadvantage, — this is frequently accompanied by substantial benefits. One example of a popular result that illustrates this is the use of the *Schur complement*, which is now widely employed in systems and control theory [2]. Consider the set defined by the quadratic form $f : \mathbb{R}^n \to \mathbb{R}$

$$f(x) := x^T \left(\mathcal{Q} - \mathcal{S}\mathcal{R}^{-1}\mathcal{S}^T\right) x, \quad f(x) > 0, \quad \forall x \neq 0,$$

where $\mathcal{Q} \in \mathbb{S}^n$, $\mathcal{S} \in \mathbb{R}^{n\times m}$ and $\mathcal{R} \in \mathbb{S}^m$, $\mathcal{R} \succ 0$. Using Schur's complement, one can test the existence of feasible solutions to the above set looking for a solution to the set defined by the *enlarged* quadratic form $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$

$$g(x,y) := \begin{pmatrix} x^T & y^T \end{pmatrix} \begin{bmatrix} \mathcal{Q} & \mathcal{S} \\ \mathcal{S}^T & \mathcal{R} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad g(x,y) > 0, \quad \forall \begin{pmatrix} x \\ y \end{pmatrix} \neq 0$$

An advantage of the higher dimensional form g is the fact that it is *linear* on the matrices $\mathcal{Q}$, $\mathcal{S}$ and $\mathcal{R}$, a property that does not appear in the original f. The authors believe that the technique that will be introduced in this work has the potential to show new directions to be explored in a several areas, such as decentralized control [15], fixed order dynamic output feedback control [18], integrating plant and controller design [12], singular descriptor systems [3]. In all these areas, the standard tests based on Lyapunov stability theory can be tough to manipulate. The introduction of a different perspective may reveal easier ways to deal with these difficult problems. Besides, several recent results can be given a broader and more consistent interpretation. For instance, the robust stability analysis results [11,7,4,14] and the *extended* controller and filter synthesis procedures [5,6,10] can be interpreted and generalized using these new tools.

15.2 Lyapunov Stability Conditions with Multipliers

Consider the set of inequalities with dynamic constraints (15.4) arising from Lyapunov stability analysis of the linear time-invariant system (15.1). Define the quadratic form $\dot{V} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$\dot{V}(x(t), \dot{x}(t)) := x(t)^T P \dot{x}(t) + \dot{x}(t)^T P x(t), \tag{15.5}$$

which is the time derivative of the quadratic form (15.2) expressed as a function of $x(t)$ and $\dot{x}(t)$. Do not explicitly substitute $\dot{x}(t)$ in (15.5) using (15.1), and build the set

$$\dot{V}(x(t), \dot{x}(t)) < 0, \quad \forall \dot{x}(t) = Ax(t), \quad (x(t), \dot{x}(t)) \neq 0. \tag{15.6}$$

In the sequel, stability will be characterized by using (15.6) instead of (15.4). This replacement is possible even though (15.4) requires only that $x(t) \neq 0$ while (15.6) requires that $(x(t), \dot{x}(t)) \neq 0$. Utilizing an argument similar to the one found in [2], pp. 62–63, this equivalence between (15.4) and (15.6) can be proved by verifying that the set

$$\dot{V}(x(t), \dot{x}(t)) < 0, \quad \forall \dot{x}(t) = Ax(t), \quad x(t) = 0, \quad \dot{x}(t) \neq 0 \tag{15.7}$$

is empty. But from (15.5), it is not possible to make $\dot{V}(x(t), \dot{x}(t)) < 0$ with $x(t) = 0$, which shows that (15.7) is indeed empty. Moreover, $\dot{V}(x(t), \dot{x}(t))$ is never strictly negative for all $(x(t), \dot{x}(t)) \neq 0$ without the presence of the dynamic equality constraint (15.1).

The advantage of working with (15.6) instead of (15.4) is that the set of feasible solutions of (15.6) can be characterized using the following lemma, which is originally attributed to Finsler [9] (see also [19]).

Lemma 2 (Finsler). *Let $x \in \mathbb{R}^n$, $\mathcal{Q} \in \mathbb{S}^n$ and $\mathcal{B} \in \mathbb{R}^{m \times n}$ such that* $\operatorname{rank}(\mathcal{B}) < n$. *The following statements are equivalent:*

i) $x^T \mathcal{Q} x < 0, \quad \forall \mathcal{B} x = 0, \quad x \neq 0.$
ii) $\mathcal{B}^{\perp T} \mathcal{Q} \mathcal{B}^{\perp} \prec 0.$
iii) $\exists\, \mu \in \mathbb{R} : \mathcal{Q} - \mu \mathcal{B}^T \mathcal{B} \prec 0.$
iv) $\exists\, \mathcal{X} \in \mathbb{R}^{n \times m} : \mathcal{Q} + \mathcal{X}\mathcal{B} + \mathcal{B}^T \mathcal{X}^T \prec 0.$

Although Lemma 2 has been proven many times, a brief proof is given in Appendix A for completeness. In Lemma 2, statement *i*) is a *constrained* quadratic form, where the vector $x \in \mathbb{R}^n$ is confined to lie in the null-space of $\mathcal{B}$. In other words, vector x can be parametrized as $x = \mathcal{B}^{\perp} y$, $y \in \mathbb{R}^r$, $r := \operatorname{rank}(\mathcal{B}) < n$, where $\mathcal{B}^{\perp}$ denotes a basis for the null-space of $\mathcal{B}$. Statement *ii*) corresponds to explicitly substituting that information back into *i*), which then provides an *unconstrained* quadratic form in $\mathbb{R}^r$. Finally, items *iii*) and *iv*) give equivalent *unconstrained* quadratic forms in the original $\mathbb{R}^n$, where the constraint is taken into account by introducing multipliers. In *iii*) the multiplier is the scalar μ while in *iv*) it is the matrix $\mathcal{X}$. In this sense, the quadratic forms given in *iii*) and *iv*) can be identified as *Lagrangian* functions. Reference [13] explicitly identifies μ as a *Lagrange multiplier* and makes use of constrained optimization theory to prove a version of Lemma 2.

Finsler's Lemma has been previously used in the control literature mainly with the purpose of eliminating design variables in matrix inequalities. In this context, Finsler's Lemma is usually referred to as Elimination Lemma. Most applications move from statement *iv*) to statement *ii*), thus *eliminating* the variable (multiplier) $\mathcal{X}$. Several versions of Lemma 2 are available under different assumptions. A special case of item *iv*) served as the basis for the entire book [17], which shows that at least 20 different control problems can be solved using Finsler's Lemma.

Recalling that the requirement $V(x(t)) > 0$, $\forall x(t) \neq 0$ can be stated as $P \succ 0$, and rewriting (15.6) in the form

$$\begin{pmatrix} x(t)^T & \dot{x}(t)^T \end{pmatrix} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} < 0, \quad \forall \begin{bmatrix} A & -I \end{bmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = 0, \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} \neq 0,$$

it becomes clear that Lemma 2 can be applied to (15.6).

Theorem 1 (Linear System Stability). *The following statements are equivalent:*

i) The linear time-invariant system (15.1) is asymptotically stable.
ii) $\exists\, P \in \mathbb{S}^n : P \succ 0, \quad A^T P + PA \prec 0.$
iii) $\exists\, P \in \mathbb{S}^n, \mu \in \mathbb{R} : P \succ 0, \quad \begin{bmatrix} -\mu A^T A & \mu A^T + P \\ \mu A + P & -\mu I \end{bmatrix} \prec 0.$
iv) $\exists\, P \in \mathbb{S}^n, F, G \in \mathbb{R}^{n \times n} : P \succ 0, \quad \begin{bmatrix} A^T F^T + FA & A^T G^T - F + P \\ GA - F^T + P & -G - G^T \end{bmatrix} \prec 0.$

Proof. Item *i*) can be stated as $P \succ 0$ and (15.6). Lemma 2 can be used with

$$x \leftarrow \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}, \ \mathcal{Q} \leftarrow \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}, \ \mathcal{B}^T \leftarrow \begin{bmatrix} A^T \\ -I \end{bmatrix}, \ \mathcal{X} \leftarrow \begin{bmatrix} F \\ G \end{bmatrix}, \ \mathcal{B}^{\perp} \leftarrow \begin{bmatrix} I \\ A \end{bmatrix}.$$

and (15.6) to generate the inequalities given in items *ii*) to *iv*). ▲

It is a nice surprise that Finsler's Lemma has been able to generate item *ii)* of Theorem 1 which is exactly the standard Lyapunov stability condition given in Lemma 1. Items *iii)* and *iv)* are new stability conditions. Since A is a constant given matrix, all three conditions are LMI (Linear Matrix Inequalities) and the feasible sets of conditions *ii)*, *iii)* and *iv)* are convex sets (see [2] for details). Notice that the first block of the second inequality in condition *iii)* is $\mu A^T A \succ 0$, which implies that $\mu > 0$ and A is nonsingular. This agrees with the fact that Lyapunov stability requires that no eigenvalues of matrix A should lie on the imaginary axis.

The multipliers μ, F and G represent extra degrees of freedom that can be used, for instance, for robust analysis or controller synthesis. In some cases, not all degrees of freedom introduced by the multipliers are really necessary, and it can be useful to constrain the multipliers. Notice that constraining a multiplier is usually less conservative than constraining the Lyapunov matrix (see [5]). Some constraints on the matrix multiplier can be enforced without loss of generality. For instance, the proof of Lemma 2 given in Appendix A shows that $\mathcal{X}$ can always be set to $-(\mu/2)\mathcal{B}^T$ without loss of generality. Besides this "trivial" choice, some more elaborated options might be available. For example, choosing the variables in item *iv)* to be

$$F = F^T = P, \quad G = \epsilon I,$$

introduces no conservativeness in the sense that there will always exist a sufficiently small ϵ that will enable the proof of stability. This behavior is similar to the one exhibited by the stability condition developed in [11]. In fact, item *iv)* is a particular case of [11], which has been obtained as an application of the positive-real lemma.

The introduction of *extra variables*, here identified as Lagrange multipliers, is the core of the recent works [11,7,4], which investigate robust stability conditions using parameter dependent Lyapunov functions. A link with these results is provided by considering that matrix A in system (15.1) is not precisely known but that all its possible values lie on a convex and bounded polyhedron $\mathcal{A}$. This polyhedron is described as the *unknown* convex combination of N given extreme matrices $A_i \in \mathbb{R}^{n\times n}$, $i = 1, \dots, N$, through

$$\mathcal{A} := \left\{ A(\xi) : A(\xi) = \sum_{i=1}^{N} A_i \xi_i, \quad \xi \in \Xi \right\},$$

where

$$\Xi := \left\{ \xi = (\xi_1, \dots, \xi_N) : \sum_{i=1}^{N} \xi_i = 1, \quad \xi_i \geq 0, \, i = 1, \dots, N \right\}. \tag{15.8}$$

If all matrices in $\mathcal{A}$ are Hurwitz then system (15.1) is said to be robustly stable in $\mathcal{A}$. The following theorem can be derived from Theorem 1 as an extension.

Theorem 2 (Robust Stability). *If at least one of the following statements is true:*

i) $\exists\, P \in \mathbb{S}^n : P \succ 0, \quad A_i^T P + P A_i \prec 0, \forall i = 1, \dots, N,$
ii) $\exists\, F, G \in \mathbb{R}^{n\times n}, P_i \in \mathbb{S}^n, i = 1, \dots, N :$

$$P_i \succ 0, \quad \begin{bmatrix} A_i^T F^T + F A_i & A_i^T G^T - F + P_i \\ G A_i - F^T + P_i & -G - G^T \end{bmatrix} \prec 0, \forall i = 1, \dots, N,$$

then the linear time-invariant system (15.1) is robustly stable in $\mathcal{A}$.

Proof. Assume that *i)* holds. Evaluate the convex combination of the second inequality in *i)* to obtain

$$P \succ 0, \quad A(\xi)^T P + PA(\xi) \prec 0, \, \forall \xi \in \Xi,$$

which imply robust stability in $\mathcal{A}$ according to item *ii)* in Theorem 1.

Now assume that *ii)* holds. The convex combination of the inequalities in *ii)* provide

$$P(\xi) \succ 0, \quad \begin{bmatrix} A(\xi)^T F^T + FA(\xi) & A(\xi)^T G^T - F + P(\xi) \\ GA(\xi) - F^T + P(\xi) & -G - G^T \end{bmatrix} \prec 0, \, \forall \xi \in \Xi,$$

where $P(\xi) \in \mathbb{S}^n$ is the affine (time-invariant) parameter dependent Lyapunov function

$$P(\xi) := \sum_{i=1}^{N} P_i \xi_i \succ 0.$$

The above inequalities imply robust stability in $\mathcal{A}$ according to item *iv)* of Theorem 1. ▲

Theorem 2 illustrates how the degrees of freedom obtained with the introduction of the Lagrange multipliers can be explored in order to generate less conservative robust stability tests. Notice that although the items *ii)* and *iv)* of Theorem 1 are equivalent statements, their robust stability versions provided in Theorem 2 have different properties. The Lyapunov function used in the robust stability condition *i)* is quadratic [1] while the one used in item *ii)* is parameter dependent [8]. Robust versions of all results presented in this paper can be derived using the same reasoning.

15.3 Discrete-time Lyapunov Stability

The methodology described so far can be adapted to cope with stability of discrete-time linear time-invariant systems given by the difference equation

$$x_{k+1} = Ax_k, \quad x_0 \text{ given.} \tag{15.9}$$

In this case, if the same quadratic Lyapunov function (15.2) is used, asymptotic stability is characterized as the existence of $V(x_k) > 0$, $\forall x_k \neq 0$ such that

$$V(x_{k+1}) - V(x_k) < 0, \quad \forall x_{k+1} = Ax_k, \quad x_k \neq 0. \tag{15.10}$$

As before, the above set is not appropriate for the application of Lemma 2. Instead, the enlarged set

$$V(x_{k+1}) - V(x_k) < 0, \quad \forall x_{k+1} = Ax_k, \quad (x_k, x_{k+1}) \neq 0, \tag{15.11}$$

is considered. As for continuous-time systems, (15.10) and (15.11) can be shown to be equivalent since the set

$$V(x_{k+1}) - V(x_k) < 0, \quad \forall x_{k+1} = Ax_k, \quad x_k = 0, \quad x_{k+1} \neq 0. \tag{15.12}$$

is empty. Indeed, the first inequality in (15.12) is never satisfied with $x_k = 0$ since $V(x_{k+1}) > 0$ for all $x_{k+1} \neq 0$. The following theorem is the discrete-time counterpart of Theorem 1.

Theorem 3 (Discrete-time Linear System Stability). *The following statements are equivalent:*

i) The linear time-invariant system (15.9) is asymptotically stable.
ii) $\exists P \in \mathbb{S}^n : P \succ 0, \quad A^T P A - P \prec 0.$
iii) $\exists P \in \mathbb{S}^n, \mu \in \mathbb{R} : P \succ 0, \quad \begin{bmatrix} -\mu A^T A - P & \mu A^T \\ \mu A & -\mu I + P \end{bmatrix} \prec 0.$
iv) $\exists P \in \mathbb{S}^n, F, G \in \mathbb{R}^{n \times n} : P \succ 0, \quad \begin{bmatrix} A^T F^T + FA - P & A^T G^T - F \\ GA - F^T & P - G - G^T \end{bmatrix} \prec 0.$

Proof. This lemma follows as an application of Lemma 2 with

$$x \leftarrow \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}, \; \mathcal{Q} \leftarrow \begin{bmatrix} -P & 0 \\ 0 & P \end{bmatrix}, \; \mathcal{B}^T \leftarrow \begin{bmatrix} A^T \\ -I \end{bmatrix}, \; \mathcal{X} \leftarrow \begin{bmatrix} F \\ G \end{bmatrix},$$

on (15.11). ▲

As in the continuous-time case, it is possible to constrain the multipliers without introducing conservatism. For instance, the choice

$$F = 0, \quad G = G^T = P,$$

in *iv)* produces

$$\begin{bmatrix} -P & A^T P \\ PA & -P \end{bmatrix} \prec 0,$$

whose Schur complement is exactly *ii)*. Indeed, this particular choice of multipliers recovers the stability condition given in [4]. In this form, stability and also H_2 and H_∞ norm minimization problems involving synthesis of linear controllers and filters can be handled as LMI using linearizing change-of-variables [5,6,10]. Finally, it is interesting to notice that, as expected, the discrete-time stability conditions do not require that A be nonsingular. Indeed, the first block of the second inequality in item *iii)* can now be satisfied with a singular matrix A.

15.4 Handling Input/Output Signals

At this point, a natural question is if the method introduced in this paper can be used to handle systems with inputs and outputs. For instance, consider the linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \quad x(0) = 0, \\ z(t) &= Cx(t) + Dw(t). \end{aligned} \tag{15.13}$$

In the presence of inputs, there is no sense in talking about stability of system (15.13) without characterizing the input signal $w(t)$. Thus, assume that the signal $w(t) : [0, \infty) \to \mathbb{R}^m$ is a piecewise continuous function in $\mathcal{L}_2$, that is,

$$\|w\|_{\mathcal{L}_2} := \left(\int_0^\infty w(\tau)^T w(\tau) \, d\tau \right)^{1/2} < \infty.$$

The system (15.13) will be said to be $\mathcal{L}_2$ stable if the output signal $z(t) \in \mathbb{R}^p$ is also in $\mathcal{L}_2$ for all $w(t) \in \mathcal{L}_2$. This condition can be checked, for instance, by evaluating the $\mathcal{L}_2$ to $\mathcal{L}_2$ gain

$$\gamma_\infty := \sup_{w(t)\in\mathcal{L}_2} \frac{\|z\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}}.$$

For a linear and time-invariant stable system (15.13) it can be shown that

$$\|H_{wz}(s)\|_\infty := \sup_{\omega\in\mathbb{R}} \|H_{wz}(j\omega)\|_2 = \gamma_\infty,$$

where $H_{wz}(s)$ is the transfer function from the input $w(t)$ to the output $z(t)$, and $\|(\cdot)\|_2$ denotes the maximum singular value of matrix $(\cdot)$.

Now, define the same Lyapunov function $V(x(t)) > 0$, $\forall x(t) \neq 0$, considered in Section 15.2, and the modified Lyapunov stability conditions

$$\begin{aligned} &\dot V(x(t),\dot x(t)) < 0, \quad \gamma^2 w(t)^T w(t) \le z(t)^T z(t), \\ &\forall (x(t),\dot x(t),w(t),z(t)) \text{ satisfying (15.13)}, \quad (x(t),\dot x(t),w(t),z(t)) \neq 0, \end{aligned} \tag{15.14}$$

defined for a given $\gamma > 0$. These inequalities appear in the stability analysis of system (15.13) under the feedback

$$w(t) := \Delta(t) z(t), \quad \|D\|_2 < \gamma, \quad \forall \Delta(t) : \|\Delta(t)\|_2 \le \gamma^{-1}.$$

Following the same steps as in [2], pp. 62–63, the S-procedure can be used to generate the equivalent condition[1,2]

$$\begin{aligned} &\dot V(x(t),\dot x(t)) < \gamma^2 w(t)^T w(t) - z(t)^T z(t), \\ &\forall (x(t),\dot x(t),w(t),z(t)) \text{ satisfying (15.13)}, \quad (x(t),\dot x(t),w(t),z(t)) \neq 0. \end{aligned} \tag{15.15}$$

Hence, when the above conditions are satisfied it is possible to conclude that

$$0 < V(x(t)) = \int_0^t \dot V(x(\tau),\dot x(\tau))\, d\tau < \int_0^t \gamma^2 w(\tau)^T w(\tau) - z(\tau)^T z(\tau)\, d\tau,$$

which is valid for all $t > 0$. In particular, taking $t \to \infty$,

$$\|z\|_{\mathcal{L}_2}^2 < \gamma^2 \|w\|_{\mathcal{L}_2}^2,$$

which implies that $\gamma > \gamma_\infty$. In other words, feasibility of (15.15) yields an upper-bound to $\|H_{wz}(s)\|_\infty$. For the linear system (15.13), it is known that

$$\gamma_\infty = \inf \gamma : (15.15).$$

Therefore, if (15.15) is feasible for some $0 < \gamma < \infty$ then it is possible to conclude that the system (15.13) is $\mathcal{L}_2$ stable. Moreover, the conditions (15.15) also guarantee

[1] In this particular case, the S-procedure produces a necessary and sufficient equivalent test. This result can also be seen as a version of Finsler's Lemma where the constraint is a quadratic form (see [2], pp. 23–24).

[2] As in [2], p. 63, the function $\dot V(x(t),\dot x(t))$ is homogeneous in P so that the scalar introduced with the S-procedure can be set to 1 without loss of generality.

that (15.13) is *internally* asymptotically stable. When the state space realization of system (15.13) is minimal, i.e., controllable and observable, both notions of stability coincide. If minimality does not hold, then system (15.13) might be $\mathcal{L}_2$ stable but not internally asymptotically stable[3], in which case the set (15.15) is empty.

A generalized version of (15.15) can be obtained by considering constraints on the input and on the output signals in the form

$$\begin{pmatrix} z(t)^T & w(t)^T \end{pmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} \geq 0,$$

where $Q \in \mathbb{S}^p$, $R \in \mathbb{S}^m$, $S \in \mathbb{R}^{p \times m}$. After applying the S-procedure this constraint yields the inequality

$$\dot{V}(x(t), \dot{x}(t)) < -\begin{pmatrix} z(t)^T & w(t)^T \end{pmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}, \tag{15.16}$$
$$\forall (x(t), \dot{x}(t), w(t), z(t)) \text{ satisfying (15.13)}, \quad (x(t), \dot{x}(t), w(t), z(t)) \neq 0,$$

The following theorem comes from using Finsler's Lemma on (15.16).

Theorem 4 (Integral Quadratic Constraint). *The following statements are equivalent:*

i) The set of solutions to (15.16) with $P \succ 0$ is not empty.
ii) $\exists\, P \in \mathbb{S}^n : P \succ 0$,

$$\begin{bmatrix} A^T P + PA + C^T QC & PB + C^T S + C^T QD \\ B^T P + S^T C + D^T QC & R + S^T D + D^T S + D^T QD \end{bmatrix} \prec 0.$$

iii) $\exists\, P \in \mathbb{S}^n, \mu \in \mathbb{R} : P \succ 0$,

$$\begin{bmatrix} -\mu\left(A^T A + C^T C\right) & \mu A^T + P & \mu C^T & -\mu\left(A^T B + C^T D\right) \\ \mu A + P & -\mu I & 0 & \mu B \\ \mu C & 0 & Q - \mu I & S + \mu D \\ -\mu\left(B^T A + D^T C\right) & \mu B^T & S^T + \mu D^T & R - \mu\left(B^T B + D^T D\right) \end{bmatrix} \prec 0.$$

iv) $\exists\, P \in \mathbb{S}^n, F_1, G_1 \in \mathbb{R}^{n \times n}, F_2, G_2 \in \mathbb{R}^{n \times p}, H_1 \in \mathbb{R}^{p \times n}, J_1 \in \mathbb{R}^{m \times n}, H_2 \in \mathbb{R}^{p \times p}, J_2 \in \mathbb{R}^{m \times p} : P \succ 0, \quad \mathcal{H} + \mathcal{H}^T \prec 0$, *where*

$$\mathcal{H} := \begin{bmatrix} F_1 A + F_2 C & -F_1 & -F_2 & F_1 B + F_2 D \\ G_1 A + G_2 C + P & -G_1 & -G_2 & G_1 B + G_2 D \\ H_1 A + H_2 C & -H_1 & (1/2)Q - H_2 & H_1 B + H_2 D \\ J_1 A + J_2 C & -J_1 & S^T - J_2 & (1/2)R + J_1 B + J_2 D \end{bmatrix}.$$

Proof. Assign

$$x \leftarrow \begin{pmatrix} x(t) \\ \dot{x}(t) \\ z(t) \\ w(t) \end{pmatrix}, \mathcal{Q} \leftarrow \begin{bmatrix} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ 0 & 0 & Q & S \\ 0 & 0 & S^T & R \end{bmatrix}, \mathcal{B}^T \leftarrow \begin{bmatrix} A^T & C^T \\ -I & 0 \\ 0 & -I \\ B^T & D^T \end{bmatrix}, \mathcal{X} \leftarrow \begin{bmatrix} F_1 & F_2 \\ G_1 & G_2 \\ H_1 & H_2 \\ J_1 & J_2 \end{bmatrix},$$

and apply Lemma 2 on (15.16). ▲

[3] Some uncontrollable or unobservable mode of (15.13) may not be asymptotically stable.

Several well known results can be generated as particular cases of Theorem 4. For instance, (15.16) reduces to (15.15) with the choice

$$Q = I, \quad R = -\gamma^2 I, \quad S = 0.$$

With these matrices, as expected, item *ii)* of Theorem 4 reduces to the standard *bounded-real lemma.* The choice

$$Q = 0, \quad R = 0, \quad S = -I,$$

produces the *positive-real lemma.* Items *iii)* and *iv)* can be seen as new equivalent statements of these well known results.

It is interesting to notice that the introduction of the new signal $z(t)$ brings an extra '$-I$' term into $\mathcal{B}$. Thus preserving an identity full row rank block inside matrix $\mathcal{B}$, that can be used to compute a straightforward $\mathcal{B}^\perp$.

15.5 Analysis of Systems Described by Transfer Functions

So far Finsler's Lemma has been used to generate stability conditions for systems given in state space form. In this section, it will be used on linear time-invariant systems described by transfer functions. For simplicity, consider a second-order SISO system represented by the transfer function

$$H_{wz}(s) = \frac{b(s)}{a(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}. \tag{15.17}$$

The results to be presented can be generalized to cope with transfer functions of higher order. Asymptotic stability of this transfer function can be analyzed by considering the second order differential equation

$$\ddot{x}(t) + a_1 \dot{x}(t) + a_0 x(t) = 0, \quad (\dot{x}(0), x(0)) = (\dot{x}_0, x_0). \tag{15.18}$$

The stability of this equation can be probed by the quadratic Lyapunov function

$$V(x(t)) := x(t)^T P x(t), \quad P := \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \succ 0,$$

and the associated stability conditions

$$\dot{V}(x(t), \dot{x}(t)) < 0, \quad \forall (x(t), \dot{x}(t), \ddot{x}(t)) \neq 0 \text{ satisfying (15.18)}. \tag{15.19}$$

Arguments similar to the ones used in Section 15.2 can be used to show that the above conditions and $P \succ 0$ fully characterize the stability of (15.18) or, equivalently, of the transfer function (15.17).

Theorem 5 (Transfer Function Stability). *The following statements are equivalent:*

i) The linear time-invariant system (15.17) is asymptotically stable.

ii) $\exists P \in \mathbb{S}^2 : P \succ 0, \quad A^T P + PA \prec 0$, *where*

$$A := \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix},$$

iii) $\exists P \in \mathbb{S}^2, \mu \in \mathbb{R} : P \succ 0, \quad \mathbf{U}(P) - \mu\, \mathbf{a}\mathbf{a}^T \prec 0$, *where*

$$\mathbf{U}(P) := \begin{bmatrix} 0 & p_1 & p_2 \\ p_1 & 2p_2 & p_3 \\ p_2 & p_3 & 0 \end{bmatrix}, \quad \mathbf{a} := \begin{bmatrix} a_0 \\ a_1 \\ 1 \end{bmatrix},$$

iv) $\exists P \in \mathbb{S}^2, \mathbf{f} \in \mathbb{R}^{3\times 1} : P \succ 0, \quad \mathbf{U}(P) + \mathbf{f}\mathbf{a}^T + \mathbf{a}\mathbf{f}^T \prec 0$.

Proof. Set

$$x \leftarrow \begin{pmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \end{pmatrix}, \quad \mathcal{Q} \leftarrow \mathbf{U}(P), \quad \mathcal{B}^T \leftarrow \mathbf{a}, \quad \mathcal{X} \leftarrow \mathbf{f}$$

and apply Lemma 2 on (15.19) with $P \succ 0$. ▲

Item *ii)* of Theorem 5 recovers exactly the standard Lyapunov stability test that would have been obtained if item *ii)* of Theorem 1 had been applied to the *companion* state space realization

$$\begin{pmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}. \tag{15.20}$$

On the other hand, items *iii)* and *iv)* are *polynomial* stability conditions. Notice that they differ from the *state-space* conditions *iii)* and *iv)* given by Theorem 1 for (15.20).

The input/output results of Section 15.4 can also be generalized to cope with transfer functions. Consider again the simple second-order SISO system (15.17), and define the dynamic constraints

$$\begin{aligned} \ddot{x}(t) + a_1\dot{x}(t) + a_0 x(t) = w(t), \qquad (\dot{x}(0), x(0)) = (0,0), \\ z(t) = b_2\ddot{x}(t) + b_1\dot{x}(t) + b_0 x(t). \end{aligned} \tag{15.21}$$

The analog of the integral quadratic performance conditions (15.16) can be shown to be given by

$$\begin{aligned} &\dot{V}(x(t), \dot{x}(t), \ddot{x}(t)) < -\left(z(t)^T \; w(t)^T\right) \begin{bmatrix} q & s \\ s & r \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}, \\ &\forall (x(t), \dot{x}(t), \ddot{x}(t), w(t), z(t)) \neq 0 \text{ satisfying (15.21)}, \end{aligned} \tag{15.22}$$

where $q, s, r \in \mathbb{R}$. The form of the dynamic constraint (15.21) deserves some comments. First, it is based on the *phase-variable* canonical realization [16], where the transfer function (15.17) is implemented via

$$H_{wz}(s) = \frac{Z(s)}{W(s)}, \quad Z(s) = b(s)\xi(s), \quad a(s)\xi(s) = W(s).$$

Second, in standard state space methods, the second equation (output equation) of (15.21) must have the term $\ddot{x}$ substituted from the first equation. This yields the standard phase-variable canonical form

$$\begin{aligned} \ddot{x}(t) + a_1\dot{x}(t) + a_0x(t) &= w(t), \qquad (\dot{x}(0), x(0)) = (0,0), \\ z(t) &= b_2w(t) + c_1\dot{x}(t) + c_0x(t). \end{aligned} \tag{15.23}$$

where $c_i := (b_i - b_2a_i)$, $i = 0, 1$. Finsler's Lemma can handle both (15.21) and (15.23) without further ado.

Theorem 6 (Transfer Function Integral Quadratic Constraint). *The following statements are equivalent:*

i) The set of solutions to (15.22) with $P \succ 0$ is not empty.
ii) $\exists\, P \in \mathbb{S}^2 : P \succ 0,$

$$\begin{bmatrix} A^TP + PA + qC^TC & PB + sC^T + qb_2C^T \\ B^TP + sC + qb_2C & r + 2sb_2 + qb_2^2 \end{bmatrix} \prec 0,$$

where

$$A := \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C := \begin{bmatrix} c_0 & c_1 \end{bmatrix},$$

iii) $\exists\, P \in \mathbb{S}^2 : P \succ 0, \quad \mathbf{U}(P) + r\mathbf{a}\mathbf{a}^T + s\mathbf{a}\mathbf{b}^T + s\mathbf{b}\mathbf{a}^T + q\mathbf{b}\mathbf{b}^T \prec 0,$ *where*

$$\mathbf{U}(P) := \begin{bmatrix} 0 & p_1 & p_2 \\ p_1 & 2p_2 & p_3 \\ p_2 & p_3 & 0 \end{bmatrix}, \quad \mathbf{a} := \begin{bmatrix} a_0 \\ a_1 \\ 1 \end{bmatrix}, \quad \mathbf{b} := \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix},$$

iv) $\exists\, P \in \mathbb{S}^2, \mu \in \mathbb{R} : P \succ 0,$

$$\begin{bmatrix} \mathbf{U}(P) - \mu\left(\mathbf{a}\mathbf{a}^T + \mathbf{b}\mathbf{b}^T\right) & \mu\mathbf{b} & \mu\mathbf{a} \\ \mu\mathbf{b}^T & q - \mu & s \\ \mu\mathbf{a}^T & s & r - \mu \end{bmatrix} \prec 0,$$

v) $\exists\, P \in \mathbb{S}^2, \mathbf{f}_1, \mathbf{f}_2 \in \mathbb{R}^{3\times 1}, g_1, g_2, h_1, h_2 \in \mathbb{R} : P \succ 0,$

$$\begin{bmatrix} \begin{pmatrix} \mathbf{U}(P) + \mathbf{f}_1\mathbf{b}^T + \mathbf{f}_2\mathbf{a}^T \\ +\mathbf{b}\mathbf{f}_1^T + \mathbf{a}\mathbf{f}_2^T \end{pmatrix} & g_1\mathbf{b} + g_2\mathbf{a} - \mathbf{f}_1 & h_1\mathbf{b} + h_2\mathbf{a} - \mathbf{f}_2 \\ g_1\mathbf{b}^T + g_2\mathbf{a}^T - \mathbf{f}_1^T & q - 2g_1 & s - g_2 - h_1 \\ h_1\mathbf{b}^T + h_2\mathbf{a}^T - \mathbf{f}_2^T & s - g_2 - h_1 & r - 2h_2 \end{bmatrix} \prec 0.$$

Proof. Items *ii)* to *v)* have been generated applying Lemma 2 on (15.22) with $P \succ 0$, the dynamic constraint (15.21) and

$$x \leftarrow \begin{pmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \\ z(t) \\ w(t) \end{pmatrix}, \quad \mathcal{Q} \leftarrow \begin{bmatrix} \mathbf{U}(P) & 0 & 0 \\ 0 & q & s \\ 0 & s & r \end{bmatrix}, \quad \mathcal{B}^T \leftarrow \begin{bmatrix} \mathbf{b} & \mathbf{a} \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathcal{X} \leftarrow \begin{bmatrix} \mathbf{f}_1 & \mathbf{f}_2 \\ g_1 & g_2 \\ h_1 & h_2 \end{bmatrix}.$$

Items *ii)* and *iii)* have been generated with item *ii)* of Lemma 2 using

$$ii) : \mathcal{B}^{\perp} \leftarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a0 & -a1 & 1 \\ c_0 & c_1 & b_2 \\ 0 & 0 & 1 \end{bmatrix}, \quad iii) : \mathcal{B}^{\perp} \leftarrow \begin{bmatrix} I \\ \mathbf{b}^T \\ \mathbf{a}^T \end{bmatrix}.$$

which are two possible choices for the null-space basis of $\mathcal{B}$. ▲

It is interesting to notice that the same conditions obtained in Theorem 6 are generated if Lemma 2 is applied to (15.23) with

$$\mathcal{B} \leftarrow \begin{bmatrix} c_0 & c_1 & 0 & -1 & b_2 \\ a_0 & a_1 & 1 & 0 & -1 \end{bmatrix}.$$

In fact, it is straightforward to verify that this matrix and matrix $\mathcal{B}$ used in the proof of Theorem 6 have the same range space, hence they share the same null space. Notice that there is also some freedom in the choice of $\mathcal{B}^{\perp}$. This freedom has been used to generate items *ii)* and *iii)* of Theorem 6. While *ii)* is the standard integral quadratic constraint condition generated for the state space representation of (15.23), item *iii)* is a new condition where the coefficient vectors **a** and **b** are not involved in any product with the Lyapunov matrix P.

Both Theorem 5 and 6 can be generalized to cope with higher order transfer functions by appropriately augmenting the vectors **a**, **b** and **f**. Extensions to general MIMO systems with m inputs and p outputs are also straightforward by considering

$$H_{wz}(s) = Z(s)W(s)^{-1}, \quad Z(s) = N(s)\xi(s), \quad D(s)\xi(s) = W(s). \tag{15.24}$$

This factorization can be obtained as $H_{wz}(s) = N(s)D(s)^{-1}$, that is, by computing $N(s)$ and $D(s)$ as right coprime polynomial factors of $H_{wz}(s)$. From (15.24), one can compute matrices A and B so that

$$\begin{aligned} Ax(t) &= w(t), \\ z(t) &= Bx(t), \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times n}$ and $x(t) \in \mathbb{R}^n$ is a vector containing the *state* $\xi(t)$ and the appropriate time derivatives. Another possible generalization of these results is for systems described by higher order *vector* differential equations as, for instance, *vector second-order systems* in the form

$$\begin{aligned} M\ddot{x}(t) + D\dot{x}(t) + Kx(t) &= Bw(t), \\ z(t) &= P\ddot{x}(t) + Q\dot{x}(t) + Rx(t). \end{aligned} \tag{15.25}$$

Robust versions of Theorems 5 and 6 would be able to provide stability conditions that enables one to take into account uncertainties on *all matrices* of (15.25), including the mass matrix M.

15.6 Some Non-Standard Applications

The ability to define extra signals and derive stability conditions directly involving these signals opens some new possibilities. For instance, consider an stability problem that is similar to the one discussed in [11,7]. Characterize the stability of the discrete-time linear time-invariant system given by

$$x_{k+1} = ABx_k, \quad x_0 \text{ given} \tag{15.26}$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{m\times n}$. With the introduction of the auxiliary signal $y_k \in \mathbb{R}^m$ it is possible to rewrite this system in the equivalent form

$$\begin{aligned} x_{k+1} &= Ay_k, \quad x_0 \text{ given},\\ y_k &= Bx_k. \end{aligned} \tag{15.27}$$

Following the same steps as in Section 15.3, asymptotic stability of this system can then be characterized in the enlarged space of (x_k, x_{k+1}, y_k) as the existence of a quadratic Lyapunov function $V(x_k) > 0$, $\forall x_k \neq 0$ such that

$$V(x_{k+1}) - V(x_k) < 0, \quad \forall (x_k, x_{k+1}, y_k) \neq 0 \text{ satisfying (15.27)} \tag{15.28}$$

The following theorem comes after applying Finsler's Lemma on (15.28).

Theorem 7 (**AB Linear System Stability**). *The following statements are equivalent:*

i) The linear time-invariant system (15.26) is asymptotically stable.
ii) $\exists P \in \mathbb{S}^n : P \succ 0, \quad B^T A^T P A B - P \prec 0.$
iii) $\exists P \in \mathbb{S}^n, \mu \in \mathbb{R} : P \succ 0, \quad \begin{bmatrix} -\mu B^T B - P & 0 & \mu B^T \\ 0 & -\mu I + P & \mu A \\ \mu B & \mu A^T & -\mu A^T A - \mu I \end{bmatrix} \prec 0.$
iv) $\exists P \in \mathbb{S}^n, F_1, G_1 \in \mathbb{R}^{n\times n}, F_2, G_2 \in \mathbb{R}^{n\times m}, H_1 \in \mathbb{R}^{m\times n}, H_2 \in \mathbb{R}^{m\times m} : P \succ 0, \quad \mathcal{H} + \mathcal{H}^T \prec 0$, *where*

$$\mathcal{H} := \begin{bmatrix} F_2 B - (1/2)P & -F_1 & F_1 A - F_2 \\ G_2 B & (1/2)P - G_1 & G_1 A - G_2 \\ H_2 B & -H_1 & H_1 A - H_2 \end{bmatrix} \prec 0.$$

Proof. Define

$$x \leftarrow \begin{pmatrix} x_k \\ x_{k+1} \\ y_k \end{pmatrix}, \ \mathcal{Q} \leftarrow \begin{bmatrix} -P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathcal{B}^T \leftarrow \begin{bmatrix} 0 & B^T \\ -I & 0 \\ A^T & -I \end{bmatrix}, \ \mathcal{X} \leftarrow \begin{bmatrix} F_1 & F_2 \\ G_1 & G_2 \\ H_1 & H_2 \end{bmatrix}.$$

and apply Lemma 2 on (15.28). ▲

The stability result in [7] is a particular case of item *iv)* of Theorem 7 where the matrices A and B are assumed to be square and where the multipliers are set to

$$\begin{bmatrix} F_1 & F_2 \\ G_1 & G_2 \\ H_1 & H_2 \end{bmatrix} = \begin{bmatrix} G & G \\ (1/2)P & H \\ (1/2)A^T P & H \end{bmatrix}, \quad G, H \in \mathbb{R}^{n\times n}.$$

Since this multiplier is a function of the system matrix A, the robustness analysis in [7] assumes that A is known.

One advantage of dealing with (15.27) instead of (15.26) is that robust stability tests — either using quadratic or parameter-dependent Lyapunov function — for systems with *multiplicative* uncertainty in the form

$$x_{k+1} = A(\xi)B(\xi)x_k, \quad \xi \in \Xi,$$

where Ξ is defined in (15.8), become readily available through item *iv*) of Theorem 7. Analogously, *fractional* or more involved uncertainty models can be taken care with no more effort. It is nice surprise that LMI robust stability conditions can be derived for uncertain models with complicated uncertainty structures such as

$$E(\xi)x_{k+1} = A(\xi)C(\xi)^{-1}B(\xi)x_k, \quad \xi \in \Xi,$$

by simply considering (15.28) and the yet *linear* dynamic constraint

$$\begin{aligned} Ex_{k+1} &= Ay_k, \\ Cy_k &= Bx_k, \end{aligned}$$

where $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{m\times m}$, and $E \in \mathbb{R}^{n\times n}$. Notice that the above system contains as a particular case the class of linear (nonsingular) descriptor systems.The subject of singular descriptor systems is slightly more involved and will be addressed in a separate paper.

Counterparts of these results for continuous-time systems can be obtained as well. However, notice that in this case the dimension m should be necessarily greater or equal than n, since a singular dynamic matrix is never asymptotically stable in the continous-time sense.

15.7 Conclusion

In this paper Lyapunov stability theory has been combined with Finsler's Lemma providing new stability tests for linear time-invariant systems. In a new procedure, the dynamic differential or difference equations that characterize the system are seen as constraints, which are naturally incorporated into the stability conditions using Finsler's Lemma. In contrast with standard state space methods, where stability analysis is carried in the space of the state vector, the stability tests are generated in the enlarged space containing both the state and its time derivative. This accounts for the flexibility of the method, that does not necessarily rely on state space representations. Stability conditions involving the coefficients of transfer functions representing linear systems are derived using this technique. Systems with inputs and outputs can be treated as well. Alternative new formulations of stability analysis tests with integral quadratic constraints, which contain the bounded-real lemma and the positive-real lemma as special cases, are provided for systems described by transfer functions or in state space. The philosophy behind the generation of these new stability tests can be summarized as follows:

1. Identify the Lyapunov stability inequalities (quadratic forms) in the enlarged space.
2. Identify the dynamic constraints in the enlarged space.

3. Apply Finsler's Lemma to incorporate the dynamic constraints into the stability conditions.

The dynamic constraints are incorporated into the stability conditions via three main processes: a) evaluating the null space of the dynamic constraints, b) using a scalar Lagrange multiplier or c) using a matrix Lagrange multiplier. These multipliers bring extra degrees of freedom that can be explored to derive robust stability tests. Quadratic stability or parameter-dependent Lyapunov functions can be used to test robust stability.

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A Proof of Lemma 2 (Finsler's Lemma)

i) $\Leftrightarrow$ *ii)*: All x such that $\mathcal{B}x = 0$ can be written as $x = \mathcal{B}^{\perp}y$. Consequently, *i)* $\Rightarrow$ $y^T\mathcal{B}^{\perp T}\mathcal{Q}\mathcal{B}^{\perp}y < 0$, for all $y \neq 0 \Rightarrow \mathcal{B}^{\perp T}\mathcal{Q}\mathcal{B}^{\perp} \prec 0$. Conversely, assuming that the first part of *ii)* holds, multiply $\mathcal{B}^{\perp T}\mathcal{Q}\mathcal{B}^{\perp}$ on the right by any $y \neq 0$ and on the left by y^T to obtain $y^T\mathcal{B}^{\perp T}\mathcal{Q}\mathcal{B}^{\perp}y < 0 \Rightarrow$ *i)*.

iii), *iv)* $\Rightarrow$ *ii)*: Multiply *ii)* or *iii)* on the right by $\mathcal{B}^{\perp}$ and on the left by $\mathcal{B}^{\perp T}$ so as to obtain *ii)*.

ii) $\Rightarrow$ *iii)*: Assume that *ii)* holds. Partition $\mathcal{B}$ in the full rank factors $\mathcal{B} = \mathcal{B}_l\mathcal{B}_r$, define $\mathcal{D} := \mathcal{B}_r^T\left(\mathcal{B}_r\mathcal{B}_r^T\right)^{-1}\left(\mathcal{B}_l^T\mathcal{B}_l\right)^{1/2}$ and apply the congruence transformation

$$\begin{bmatrix}\mathcal{D}^T \\ \mathcal{B}^{\perp T}\end{bmatrix}\left(\mathcal{Q} - \mu\mathcal{B}^T\mathcal{B}\right)\left[\mathcal{D}\ \mathcal{B}^{\perp}\right] = \begin{bmatrix}\mathcal{D}^T\mathcal{Q}\mathcal{D} - \mu I & \mathcal{D}^T\mathcal{Q}\mathcal{B}^{\perp} \\ \mathcal{B}^{\perp T}\mathcal{Q}\mathcal{D} & \mathcal{B}^{\perp T}\mathcal{Q}\mathcal{B}^{\perp}\end{bmatrix} \prec 0.$$

Since the second diagonal block is negative definite by assumption, a sufficiently large μ exists so that the whole matrix is negative definite.

iii) $\Rightarrow$ *iv)*: Choose $\mathcal{X} = -(\mu/2)\mathcal{B}^T$. ▲

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