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An efficient solution to multi-objective control problems with LMI objectives

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Abstract

We revisit a technique for solving multi-objective control problems through affinely parameterizing the closed-loop system with the Youla parameterization and confining the search of the Youla parameter to finite-dimensional subspaces. It is pretty well-known how to solve such problems if the closed-loop specifications are formulated in terms of the solvability of linear matrix inequalities. However, all approaches proposed so far suffer from a substantial inflation of size of the resulting optimization problems if improving the approximation accuracy. On the basis of a novel state-space approach to solving static output feedback control problems by convex optimization for a specific class of plants, we reveal how the growth of the size of the optimization problems can be considerably reduced to arrive at more efficient algorithms. As an additional ingredient we discuss how to use the so-called mixed controller as a starting point for a genuine multi-objective design in order to improve the algorithms.

Keywords: Multi-objective control; Mixed control; Linear matrix inequalities; Observer-based controllers; Youla parameterization

1. Introduction

Notation. All plants are described in discrete time, and a (transfer) matrix is said to be stable if all its (poles) eigenvalues are contained in the open unit disk. For $A\in\mathbb{C}^{n\times m}$, $\|A\|_F$ denotes the Frobenius norm and $\|A\|$ the spectral norm of A. RH_∞ denotes the set of all proper and stable transfer matrices. This space is equipped either with the H_2-norm $\|T\|_2^2=(1/2\pi)\int_0^{2\pi}\|T(e^{it})\|_F\,dt$ or the H_∞-norm $\|T\|_\infty=\max_{t\in[0,2\pi]}\|T(e^{it})\|$.

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Suppose that a linear finite-dimensional time-invariant generalized plant (including all weightings) is described (with a minimal realization of McMillan degree n) as

$$\begin{pmatrix} z_1 \\ \vdots \\ z_q \\ y \end{pmatrix} = \left[\begin{array}{c|cccc} A & B_1 & \cdots & B_q & B \\ \hline C_1 & D_1 & \cdots & * & E_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_q & * & \cdots & D_q & E_q \\ C & F_1 & \cdots & F_q & 0 \end{array}\right] \begin{pmatrix} w_1 \\ \vdots \\ w_q \\ u \end{pmatrix}. \tag{1}$$

Here, u is the control input with m components, y is the measured output with k components, and the channels $w_j \to z_j$ from disturbance inputs to controlled outputs serve to specify robustness and/or performance objectives. If required, we collect the signals as $z = (z_1^{\mathrm{T}} \cdots z_q^{\mathrm{T}})^{\mathrm{T}}$ and $w = (w_1^{\mathrm{T}} \cdots w_q^{\mathrm{T}})^{\mathrm{T}}$.

A controller is any finite-dimensional linear time-invariant system

$$u = \left[\begin{array}{c|c} A_{\mathrm{c}} & B_{\mathrm{c}} \\ \hline B_{\mathrm{c}} & D_{\mathrm{c}} \end{array}\right] y \tag{2}$$

specified through the parameter matrices $A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}}, D_{\mathrm{c}}$.

The closed-loop system is denoted as

$$z = \mathscr{T} w = \left[\begin{array}{c|c} \mathscr{A} & \mathscr{B} \\ \hline \mathscr{C} & \mathscr{D} \end{array}\right] w,$$

where the channel $w_j \to z_j$ has the realization

$$z_j = \mathscr{T}_j w_j = \left[\begin{array}{c|c} \mathscr{A} & \mathscr{B}_j \\ \hline \mathscr{C}_j & \mathscr{D}_j \end{array}\right] w_j = \left[\begin{array}{cc|c} A + BD_{\mathrm{c}}C & BC_{\mathrm{c}} & B_j + BD_{\mathrm{c}}F_j \\ B_{\mathrm{c}}C & A_{\mathrm{c}} & B_{\mathrm{c}}F_j \\ \hline C_j \ + E_jD_{\mathrm{c}}C & E_jC_{\mathrm{c}} & D_j + E_jD_{\mathrm{c}}F_j \end{array}\right] w_j.$$

A controller is said to be stabilizing if $\mathscr{A}$ is stable. Stabilizing controllers exist iff (A, B) is stabilizable and (A, C) is detectable (with respect to the open unit disk); this is assumed from now on.

A typical multi-objective control problem imposes different specifications on different channels of the closed-loop system. In this paper we concentrate on those requirements that can be formulated in terms of the solvability of a linear matrix inequality (LMI). For pretty comprehensive lists of possible choices we refer to Boyd et al. [5], Scherer et al. [13] and Masubuchi et al. [10]. Only to simplify the exposition we confine ourselves to the discussion of a paradigm example that has received considerable attention in the literature [3,8], the so-called multi-objective H_2/H_∞ control problem. In this problem the goal is to keep bounds on the H_2-norm of, say, the first channel $w_1 \to z_1$ and on the H_∞-norm of, say, the second channel of the controlled system:

$$\|\mathscr{T}_1\|_2 < \gamma_1 \quad \text{and} \quad \|\mathscr{T}_1\|_\infty < \gamma_2. \tag{3}$$

The H_2-constraint is imposed to specify desired performance requirements on the controller such as reducing the asymptotic output variance against white noise inputs or the output energy against impulse inputs. The H_∞-constraint on $z_2 = \mathscr{T}_2 w_2$ guarantees that the controller is robustly stabilizing against uncertainties $w_2 = \Delta_2 z_2$ with $\Delta_2 \in \mathrm{RH}_\infty$ fulfilling the norm bound $\| \Delta_2 \|_\infty \| \mathscr{T}_2 \|_\infty < 1$. Both specifications can as well be interpreted as loop-shaping requirements. Note that we assume possibly necessary frequency-dependent weightings to be incorporated into the system description (1).

The goal might be to optimize performance under the constraint of keeping the controller robustly stabilizing against uncertainties of a fixed size; this amounts to minimizing γ_1 for a fixed γ_2 such that the stabilizing controller satisfies constraints (3). In order to determine the trade-off between the H_2- and H_∞-norm constraints, one rather minimizes the functional

$$\alpha_1\gamma_1 + \alpha_2\gamma_2 \tag{4}$$

for various real parameters α_1, α_2 over constraints (3) in order to (approximately) determine Pareto-optimal controllers [4,7].

For fixed α_1, α_2, let us consider from now on the following.

Multi-objective (MO) *control problem*: Minimize (4) over all stabilizing controllers and γ_1, γ_2 that satisfy (3).

The prerequisite to apply our approach is the possibility to equivalently re-formulate constraints (3) in terms of the solvability of LMIs. In fact, $\mathscr{A}$ is stable and $\| \mathscr{T}_1 \|_2 < \gamma_1$ holds if and only if there exist symmetric $\mathscr{X}_1$ and Z that satisfy the inequalities

$$\begin{pmatrix} \mathscr{X}_1 & 0 & \mathscr{A}'\mathscr{X}_1 \\ 0 & \gamma_1 I & \mathscr{B}_1'\mathscr{X}_1 \\ \mathscr{X}_1\mathscr{A} & \mathscr{X}_1\mathscr{B}_1 & \mathscr{X}_1 \end{pmatrix} > 0, \qquad \begin{pmatrix} \mathscr{X}_1 & 0 & \mathscr{C}_1' \\ 0 & \gamma_1 I & \mathscr{D}_1' \\ \mathscr{C}_1 & \mathscr{D}_1 & Z \end{pmatrix} > 0, \quad \mathrm{tr}(Z) < \gamma_1. \tag{5}$$

Similarly, stability of $\mathscr{A}$ and $\| \mathscr{T}_2 \|_\infty < \gamma_2$ are equivalent to the existence of a symmetric $\mathscr{X}_2$ with

$$\left(\begin{array}{cc|cc} \mathscr{X}_2 & 0 & \mathscr{A}'\mathscr{X}_2 & \mathscr{C}_2' \\ 0 & \gamma_2 I & \mathscr{B}_2'\mathscr{X}_2 & \mathscr{D}_2' \\ \hline \mathscr{X}_2\mathscr{A} & \mathscr{X}_2\mathscr{B}_2 & \mathscr{X}_2 & 0 \\ \mathscr{C}_2 & \mathscr{D}_2 & 0 & \gamma_2 I \end{array}\right) > 0. \tag{6}$$

Recall that $\mathscr{A}, \mathscr{B}_j, \mathscr{C}_j, \mathscr{D}_j$ depend on the controller parameters. Hence, searching for both the controller parameters A_c, B_c, C_c, D_c and the matrices $\mathscr{X}_j$, Z to satisfy these inequalities turns out to be a non-convex (bilinear) matrix inequality problem and is, therefore, in general very hard to solve.

Only rather recently has it been clarified in this generality [13,10] how this non-convex problem can be transformed into a convex feasibility problem by imposing the extra (technical) constraint $\mathscr{X}_1 = \mathscr{X}_2$. The resulting problem is called the

Mixed control problem: Minimize (4) over all A_c, B_c, C_c, D_c and $\mathscr{X} = \mathscr{X}_1 = \mathscr{X}_2$, Z that satisfy (5)–(6).

The optimal value of the mixed control problem can be computed by solving suitable LMIs. Since the mixed problem results from the genuine multi-objective control problem by adjoining an extra constraint, the optimal value of the former is an *upper bound* of that of the latter.

This leads us to the topics of this paper. In Section 2 we provide a novel state-space solution of the genuine multi-objective control problem by static output feedback if the transfer matrix from u to y vanishes. It is clarified in Section 3 how this strong structural property is enforced by the Youla parameterization to solve multi-objective control problems by dynamic output-feedback for general plants. We also discuss the benefit of our approach by a direct comparison with that in [7]. In Section 4 we reveal how a mixed controller can be transformed in order to serve as a starting point for the Youla parameterization to potentially improve the algorithms. In Section 5, the proposed algorithm is summarized and illustrated by means of an academic example.

An abridged version of this paper has been published in [12].

2. MO control by static output-feedback

Let us consider in this section controllers of the form

$$u = Ny,$$

where N is a static gain. Even after neglecting one of the constraints in (3) and considering the single objective H_2- or H_∞-control problems, it not known how to compute the optimal value by solving a convex optimization problem, or whether this is possible at all.

Under the hypothesis that the controller enters the generalized plant (1) in an *affine* fashion, not only single-objective control problems but even multi-objective control problems by static-output feedback can be solved via convex optimization techniques.

As the main technical contribution of this article, we will reveal how a suitable parameter transformation allows to arrive at an *efficient* solution of these problems where efficiency is related to both the size and the number of variables of the resulting LMI problems.

Algebraically, the relevant property amounts to

$$C(zI-A)^{-1}B=0. \tag{7}$$

Under this hypothesis, we can choose specific state-space coordinates in the realization of (1) to arrive at

$$\left(\begin{array}{c|cc} A & B_j & B \\ \hline C_j & D_j & E_j \\ C & F_j & 0 \end{array}\right)=\left(\begin{array}{cc|cc} A_1 & \hat{A} & B_{j1} & \hat{B} \\ 0 & A_2 & B_{j2} & 0 \\ \hline C_{j1} & C_{j2} & D_j & E_j \\ 0 & \hat{C} & F_j & 0 \end{array}\right).$$

This implies that the closed-loop system matrix after static output feedback reads as

$$\begin{pmatrix} \mathscr{A} & \mathscr{B}_j \\ \mathscr{C}_j & \mathscr{D}_j \end{pmatrix}=\left(\begin{array}{cc|c} A_1 & \hat{A}+\hat{B}N\hat{C} & B_{j1}+\hat{B}NF_j \\ 0 & A_2 & B_{j2} \\ \hline C_{j1} & C_{j2}+E_jN\hat{C} & D_j+E_jNF_j \end{array}\right). \tag{8}$$

Remark. Since (A,B) and (A,C) are stabilizable, the matrix A itself must be already stable. Moreover, this representation clarifies why the closed-loop transfer matrices $\mathscr{T}_j$ depend affinely on the controller gain N.

Partition

$$\mathscr{X}_j=\begin{pmatrix} X_j & Z_j \\ Z_j' & Y_j \end{pmatrix}$$

according to $\mathscr{A}$. We observe that

$$\mathscr{X}_j\mathscr{A}=\begin{pmatrix} X_j & Z_j \\ Z_j' & Y_j \end{pmatrix}\begin{pmatrix} A_1 & \hat{A}+\hat{B}N\hat{C} \\ 0 & A_2 \end{pmatrix}$$

depends non-linearly on the unknowns X_j, Y_j, Z_j and N.

Let us now introduce the new variables

$$P_j:=\begin{pmatrix} Q_j & S_j \\ S_j' & R_j \end{pmatrix}=\begin{pmatrix} X_j^{-1} & -X_j^{-1}Z_j \\ -Z_j'X_j^{-1} & Y_j-Z_j'X_j^{-1}Z_j \end{pmatrix}.$$

The transformation

$$\begin{pmatrix} X & Z \\ Z' & Y \end{pmatrix}\to\begin{pmatrix} Q & S \\ S' & R \end{pmatrix}$$

maps the set of all positive-definite matrices into the set of all matrices whose diagonal blocks are positive definite. It is easily shown that this map is bijective, and that its inverse is given by

$$\begin{pmatrix} X_j & Z_j \\ Z_j' & Y_j \end{pmatrix}=\begin{pmatrix} Q_j^{-1} & -Q_j^{-1}S_j \\ -S_j'Q_j^{-1} & R_j+S_j'Q_j^{-1}S_j \end{pmatrix}.$$

The main reason for introducing these new variables is the factorization

$$\begin{pmatrix} Q_j & 0 \\ S_j' & I \end{pmatrix}\begin{pmatrix} X_j & Z_j \\ Z_j' & Y_j \end{pmatrix}=\begin{pmatrix} I & -S_j \\ 0 & R_j \end{pmatrix}.$$

If we define

$$\mathscr{Q}_j:=\begin{pmatrix} Q_j & 0 \\ S_j' & I \end{pmatrix},$$

we conclude that

$$\mathcal{Q}_j\mathcal{X}_j\mathcal{A}\mathcal{Q}_j' = \begin{pmatrix} A_1Q_j & A_1S_j - S_jA_2 + \hat{A} + \hat{B}N\hat{C} \\ 0 & R_jA_2 \end{pmatrix} =: \boldsymbol{A}(N, P_j).$$

Therefore, the block $\mathcal{X}_j\mathcal{A}$ depending non-linearly on the parameters X_j, Y_j, Z_j, N can be transformed by congruence into $\mathcal{Q}_j\mathcal{X}_j\mathcal{A}\mathcal{Q}_j'$ which indeed depends *affinely* on the new variables Q_j, S_j, R_j and on the original controller parameter N. The same structural property holds for the blocks

$$\mathcal{Q}_j\mathcal{X}_j\mathcal{Q}_j' = \begin{pmatrix} Q_j & 0 \\ 0 & R_j \end{pmatrix} =: \boldsymbol{X}(P_j),$$

$$\mathcal{Q}_j\mathcal{X}_j\mathcal{B}_j = \begin{pmatrix} B_{j1} + \hat{B}NF_j - S_jB_{j2} \\ R_jB_{j2} \end{pmatrix} =: \boldsymbol{B}_j(N, P_j),$$

$$\mathcal{C}_j\mathcal{Q}_j' = (C_{j1}Q_j \quad C_{j2} + E_jN\hat{C} + C_{j1}S_j) =: \boldsymbol{C}_j(N, P_j)$$

and

$$\mathcal{D}_j = D_j + E_jNF_j =: \boldsymbol{D}_j(N).$$

It remains to perform congruence transformations with $\mathrm{diag}(\mathcal{Q}_1, I, \mathcal{Q}_1)$ and $\mathrm{diag}(\mathcal{Q}_1, I, I)$ on (5) to get

$$\begin{pmatrix} \mathcal{Q}_1\mathcal{X}_1\mathcal{Q}_1' & 0 & * \\ 0 & \gamma_1 I & * \\ \mathcal{Q}_1\mathcal{X}_1\mathcal{A}\mathcal{Q}_1' & \mathcal{Q}_1\mathcal{X}_1\mathcal{B}_1 & \mathcal{Q}_1\mathcal{X}_1\mathcal{Q}_1' \end{pmatrix} > 0, \qquad \begin{pmatrix} \mathcal{Q}_1\mathcal{X}_1\mathcal{Q}_1' & 0 & * \\ 0 & \gamma_1 I & * \\ \mathcal{C}_1\mathcal{Q}_1' & \mathcal{D}_1 & Z \end{pmatrix} > 0, \qquad \mathrm{tr}(Z) < \gamma_1,$$

and with $\mathrm{diag}(\mathcal{Q}_2, I, \mathcal{Q}_2, I)$ on (6) to arrive at

$$\left(\begin{array}{cc|cc} \mathcal{Q}_2\mathcal{X}_2\mathcal{Q}_2' & 0 & * & * \\ 0 & \gamma_2 I & * & * \\ \hline \mathcal{Q}_2\ \mathcal{X}_2\mathcal{A}\mathcal{Q}_2' & \mathcal{X}_2\mathcal{B}_2 & \mathcal{Q}_2\mathcal{X}_2\mathcal{Q}_2' & 0 \\ \mathcal{C}_2\mathcal{Q}_2' & \mathcal{D}_2 & 0 & \gamma_2 I \end{array}\right) > 0$$

which results in

$$\begin{pmatrix} \boldsymbol{X}(P_1) & 0 & * \\ 0 & \gamma_1 I & * \\ \boldsymbol{A}(N, P_1) & \boldsymbol{B}_1(N, P_1) & \boldsymbol{X}(P_1) \end{pmatrix} > 0, \qquad \begin{pmatrix} \boldsymbol{X}(P_1) & 0 & * \\ 0 & \gamma_1 I & * \\ \boldsymbol{C}_1(N, P_1) & \boldsymbol{D}_1(N) & Z \end{pmatrix} > 0, \qquad \mathrm{tr}(Z) < \gamma_1, \tag{9}$$

$$\left(\begin{array}{cc|cc} \boldsymbol{X}(P_2) & 0 & * & * \\ 0 & \gamma_2 I & * & * \\ \hline \boldsymbol{A}(N, P_2) & \boldsymbol{B}_2(N, P_2) & \boldsymbol{X}(P_2) & 0 \\ \boldsymbol{C}_2(N, P_2) & \boldsymbol{D}_2(N) & 0 & \gamma_2 I \end{array}\right) > 0. \tag{10}$$

All these inequalities turn out to depend affinely on the variables Q_j, R_j, S_j, Z, N. We conclude that there exists a (stabilizing) static output-feedback controller N that renders (3) satisfied iff there exist P_1, P_2, Z and N that solve the LMIs (9) and (10).

Theorem 1. *Suppose that* $C(sI - A)^{-1}B = 0$. *Then the optimal value of the multi-objective control problem by static output feedback* $u = Ny$ equals *the minimal value of* (4) *if varying* P_1, P_2, Z, N, γ_1, γ_2 *over the LMI constraints* (9) *and* (10).

Table 1

	Size of LMI	Size of matrix variable
Standard approach	$2n*(km+1)+k_j+m_j$	$n*(km+1)$
Novel approach	$2n+k_j+m_j$	n
Difference	$2n*(km)$	$n*(km)$

Remarks. (1) The cost functional could also be chosen to depend affinely on any of the other variables involved in the LMIs.

(2) The controller gain N appears directly in the synthesis LMIs. Hence, one can without any trouble incorporate *structural requirements* on this controller gain. This reveals that one could even design *decentralized controllers* (with block-diagonal N) by convex optimization techniques.

(3) The synthesis inequalities (9) and (10) are only coupled via the controller gain N.

Let us finally stress the benefit of the presented approach over those that appear in the literature [11,7]. The set of all static gains $N\in\mathbb{R}^{m\times k}$ can be parameterized as $N=\sum_{v=1}^{km}\alpha_v N_v$ with km basis matrices N_v and with free parameters $\alpha_v\in\mathbb{R}$. If we define

$$T_{j,0}=\left[\begin{array}{cc|c} A_1 & \hat{A} & B_{j1}\\ 0 & A_2 & B_{j2}\\ \hline C_{j1} & C_{j2} & D_j\end{array}\right],\qquad T_{j,v}=\left[\begin{array}{c|c} A_1 & \hat{B}\\ \hline C_{j1} & E_j\end{array}\right]N_v\left[\begin{array}{c|c} A_2 & B_{j2}\\ \hline \hat{C} & F_j\end{array}\right]$$

and introduce the realization

$$T_{j,v}:=\left[\begin{array}{c|c} A_{j,v} & B_{j,v}\\ \hline C_{j,v} & D_{j,v}\end{array}\right],$$

we infer that the jth channel of the controlled system can be described as

$$\mathscr{T}_j=T_{j,0}+\sum_{v=1}^{km}\alpha_v T_{j,v}=\left[\begin{array}{cccc|c} A_{j,0} & & & 0 & B_{j,0}\\ & A_{j,1} & & & B_{j,1}\\ & & \ddots & & \vdots\\ 0 & & & A_{j,km} & B_{j,km}\\ \hline C_{j,0} & \alpha_1 C_{j,1} & \cdots & \alpha_{km}C_{j,km} & D_{j,0}+\sum_{v=1}^{km}\alpha_v D_{j,v}\end{array}\right]. \tag{11}$$

Due to this description inequalities (5) and (6) are convex in all the variables $\mathscr{X}_j, Z, \alpha_v$ without any parameter transformation. However, note that $A_{j,v}$ has generically the size n such that the size of the LMIs as well as the number of variables are, in particular for MIMO systems, considerably larger than those in our novel approach.

Let us be more specific and quantify the difference for the H_∞ constraint in (3) if $\mathscr{T}_j$ has dimension $k_j\times m_j$ (see Table 1):

Since the number of optimization variables grows quadratically with the size of the matrix variables, the parameter transformation suggested in this section can save a considerable amount of computation time.

3. MO control by dynamic output-feedback

Typically, in a general interconnection structure property (7) does not hold. It is, however, well known that the Youla parameterization [9] is the right tool to enforce this condition. Indeed, if one chooses K, L such that $A + BK$, $A + LC$ are stable, the set of all stabilizing controllers for (1) can be parameterized as

$$\begin{pmatrix} u \\ \hat{y} \end{pmatrix} = \left[\begin{array}{c|cc} A+BK+LC & -L & B \\ \hline K & 0 & I \\ -C & I & 0 \end{array}\right] \begin{pmatrix} y \\ \hat{u} \end{pmatrix}, \qquad \hat{u} = Q\hat{y}, \qquad Q \in \mathrm{RH}_\infty.$$

The corresponding closed-loop system then admits the description

$$\begin{pmatrix} z_j \\ \hat{y} \end{pmatrix} = \left[\begin{array}{cc|cc} A+BK & -BK & B_j & B \\ 0 & A+LC & B_j+LF_j & 0 \\ \hline C_j+E_jK & -E_jK & D_j & E_j \\ 0 & C & F_j & 0 \end{array}\right] \begin{pmatrix} w_j \\ \hat{u} \end{pmatrix} \tag{12}$$

with the Youla parameter $Q \in \mathrm{RH}_\infty$ entering as $\hat{u} = Q\hat{y}$. System (12) has the structure that is required to apply our previous results. We can conclude that the search for a constant Youla parameter to solve the MO control problem is amenable to the technique in Section 2 and can be reduced to an efficiently solvable LMI problem.

This particular case can be easily generalized to the search for the Youla parameter in an arbitrary *finite-dimensional subspace* of the infinite-dimensional space RH_∞. To be specific, let us represent the Youla parameter as

$$Q(z) = C_Q(zI - A_Q)^{-1}B_Q + D_Q = (C_Q \;\; D_Q) \left[\begin{array}{c|c} A_Q & B_Q \\ \hline I & 0 \\ 0 & I \end{array}\right].$$

If we fix A_Q (stable) and B_Q, the Youla parameter depends affinely on C_Q and D_Q. Moreover, connecting Q to (12) is the same as post-compensating (12) with

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline I & 0 \\ 0 & I \end{array}\right] \hat{y}$$

to obtain

$$\begin{pmatrix} z_j \\ y_1 \\ y_2 \end{pmatrix} = \left[\begin{array}{ccc|cc} A_Q & 0 & B_QC & B_QF_j & 0 \\ 0 & A+BK & -BK & B_j & B \\ 0 & 0 & A+LC & B_j+LF_j & 0 \\ \hline 0 & C_j+E_jK & -E_jK & D_j & E_j \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & C & F_j & 0 \end{array}\right] \begin{pmatrix} w_j \\ \hat{u} \end{pmatrix}, \tag{13}$$

and then applying the static output feedback control

$$\hat{u} = (C_Q \; D_Q) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{14}$$

Again, (13) obeys the crucial structural property to apply the results in Section 2 in order to efficiently reduce the search for C_Q, D_Q in the MO control problem to an LMI problem.

Let us finally recall that the specific choice

$$Q(z) = Q_0 + Q_1\frac{1}{z} + \cdots + Q_p\frac{1}{z^p} = \left[\begin{array}{cccccc|c} 0 & I & 0 & \cdots & 0 & & 0 \\ 0 & 0 & I & \cdots & 0 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & & \\ 0 & 0 & \cdots & 0 & I & & 0 \\ 0 & 0 & \cdots & 0 & 0 & & I \\ \hline Q_p & Q_{p-1} & \cdots & Q_2 & Q_1 & & Q_0 \end{array}\right] \tag{15}$$

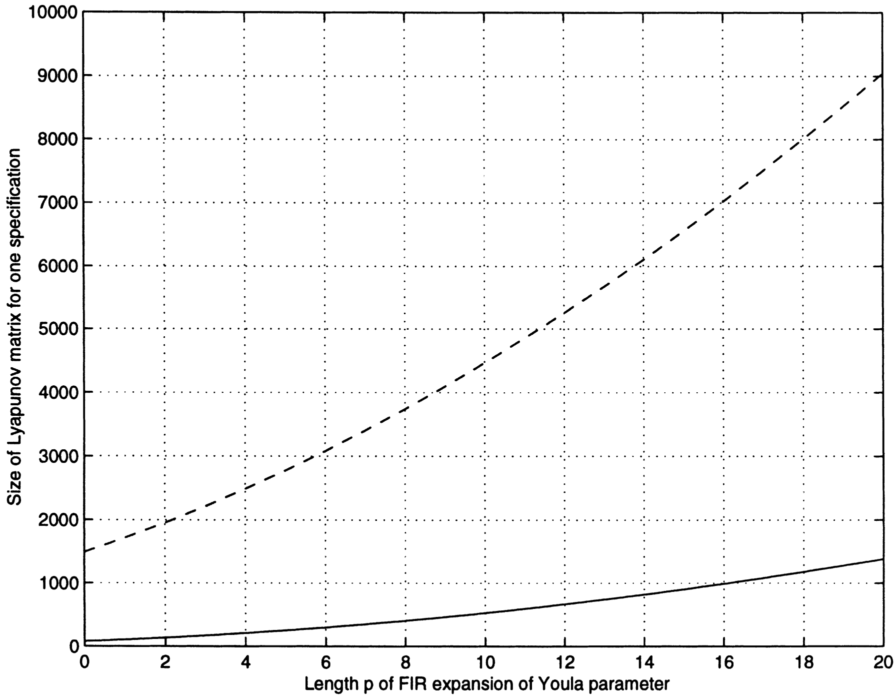

Fig. 1. Number of variables versus length of basis for previous (dashed) and novel (solid) approach.

with an FIR structure of the Youla parameter has the following favorable property: on letting p go to infinity, the optimal value of the problem with (15) converges to the optimal value of the original MO control problem [11,7].

Remark. Dually to what has been discussed, we could as well let A_Q, C_Q in $Q(z) = C_Q(zI - A_Q)^{-1}B_Q + D_Q$ be fixed and search over the parameters B_Q, D_Q. Since the size of A_Q influences the size of the resulting optimization problems, this latter remark is particularly important for the FIR structure (15). If Q of size $m \times k$ is tall ($m \geqslant k$), one should work with (15) and A_Q of size kp, and if Q is fat ($m < k$), the dual parameterization leads to the size mp for A_Q. In general, the smallest size of A_Q is hence given by $p * \min\{k, m\}$.

Recall again that McMillan degree of (13) determines the sizes of the LMIs and the number of variables in the resulting LMI solution of the MO control problem. This leads us to the fundamental benefit of our approach if compared to that in [7]. Indeed, realization (13) with (15) or its dual has, in our approach, the dimension

$$2n + p * \min\{k, m\},$$

whereas that in [7] (based on Kronecker calculus similar to what has been discussed at the end of Section 2) has the dimension

$$n + 2n * k * m + p * k * m * k_j.$$

We observe that, in particular for MIMO control problems, there can be a substantial gap between these dimensions. We reiterate that the number of both optimization variables and constraints depend quadratically on these realization sizes which makes it even more important for practical applications to keep them as small as possible

Let us illustrate the advantage for a pretty moderately sized plant of order 6 with 2 control inputs, 2 measured outputs, and 1 component in the performance output signal. In Fig. 1 we have plotted the number of variables required for one Lyapunov matrix versus the length p of the FIR representation.

4. Youla parameterization based on mixed controller

A central step in the design of dynamic MO controllers is the Youla parameterization. Instead of starting with an arbitrary stabilizing controller, it is expected – and will be demonstrated by means of an example – that one should use a mixed controller to define the Youla parameterization. However, controllers designed with the techniques in [13,10] in general do not admit an observer structure. Of course, it is well known that this causes no principal problem since the Youla parameterization can be based on an arbitrary stabilizing controller without any hypothesis on its structure [14]. By following the usual arguments in the frequency domain to derive the corresponding state-space formulae, one obtains the closed-loop parameterization

$$\begin{pmatrix} z_j \\ \hat{y} \end{pmatrix} = \left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_j & \tilde{B} \\ \hline \tilde{C}_j & D_j & E_j \\ \tilde{C} & F_j & 0 \end{array}\right] \begin{pmatrix} w_j \\ \hat{u} \end{pmatrix}, \qquad \hat{u} = Q\hat{y} \quad \text{with } \tilde{A} \text{ of size } 3n. \tag{16}$$

As shown in Section 3, observer-based controllers lead to such a representation of degree $2n$. We have already stressed that it is of considerable importance to keep this McMillan degree as small as possible. It was a big surprise to us that there seems to be no results available in the literature that allow to base the Youla parameterization on a mixed controller and, still, reduce the size of the realization in (16) to $2n$.

It has been demonstrated in [2,1] how to compute a state-coordinate change that transforms a general controller realization into one with an observer structure. Unfortunately, these papers give no conditions for such a transformation to exist, and it seems that it does not exist for an arbitrary stabilizing controller. We are in the specific situation that the set of all controller matrices that meet requirements (5) and (6) (for some bounds γ_j) is *open*. Hence, one could try to slightly perturb a given mixed controller and then show that one can indeed obtain an observer-based representation of this perturbed controller. To the best of our knowledge it seems to be a new result that this is *always* possible by an arbitrarily small perturbation of A_c.

Let us assume that controller (2) with A_c of size n stabilizes (1). Clearly, the closed-loop system can be obtained by interconnecting

$$\begin{pmatrix} z_j \\ y \end{pmatrix} = \left[\begin{array}{c|cc} A + BD_cC & B_j + BD_cF_j & B \\ \hline C_j + E_jD_cC & D_j + E_jD_cF_j & E_j \\ C & F_j & 0 \end{array}\right] \begin{pmatrix} w_j \\ u \end{pmatrix} \tag{17}$$

with the controller

$$u = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] y. \tag{18}$$

The intention is to show that (18) admits, after a state-coordinate change, the structure of an observer for (17). Hence, we are required to show that there exists a non-singular T and K, L with

$$T^{-1}A_cT = [A + BD_cC] + BK + LC, \qquad T^{-1}B_c = -L, \qquad C_cT = K. \tag{19}$$

We can eliminate K, L and observe that any such T satisfies the quadratic equation

$$A_cT - T[A + BD_cC] - TBC_cT + B_cC = 0. \tag{20}$$

Conversely, if (20) has a non-singular solution T, we arrive at (18) with $K := C_cT$ and $L := -T^{-1}B_c$. We conclude that (18) admits an observer structure if and only if (20) has a non-singular solution.

The main goal is to show that a small perturbation of A_c guarantees that (20) has a non-singular solution. For that purpose we choose $A_c(0) := [A + BD_cC] + BC_c - B_cC$ to conclude that (20) with A_c replaced by $A_c(0)$ has the solution $T = I$. This motivates to introduce the analytic family

$$A_c(z) := zA_c + (1 - z)([A + BD_cC] + BC_c - B_cC), \quad z \in \mathbb{C}$$

and consider the parameter-dependent quadratic equation

$$A_c(z)T(z) - T(z)[A + BD_cC] - T(z)BC_cT(z) + B_cC = 0. \tag{21}$$

Recall that we are interested in solving this equation in a neighborhood of $z = 1$, and that we know the solution I for $z = 0$ by construction. Let us now prove that the exceptional set z for which (21) does not have a non-singular solution is at most discrete (and hence finite in a compact neighborhood of $z = 1$).

The crucial technical fact is found in Theorem 20.4.1 of Gohberg et al. [6]. This result implies that there exists a function $T(z)$ that is analytic on $\mathbb{C}\backslash S$ with some discrete set S, and that satisfies the quadratic equation (21) as well as the interpolation condition $T(0)=I$. (The solution $T(0)=I$ admits an extension that is analytic off a discrete set.) This implies that the exceptional set for which (21) has no solution is at most discrete. The same holds if we consider non-singular solutions since $\det(T(z))$ is analytic and not identically zero ($\det(T(0)) = 1$) on the connected set $\mathbb{C}\backslash S$ such that its set of zeros is at most discrete.

Therefore, in any neighborhood of $z = 1$ and on the line $[0, 1]$ there exist (infinitely many) parameters for which (21) indeed has a non-singular solution. This proves that any controller (2) for (1) admits, after possibly a slight perturbation of A_c, a representation with an observer structure.

Theorem 2. *Given $A_c \in \mathbb{R}^{n\times n}$, B_c, C_c, D_c and $\varepsilon > 0$, there exists an $\hat{A}_c \in \mathbb{R}^{n\times n}$ with $\| \hat{A}_c - A_c \| < \varepsilon$ such that*

$$\left[\begin{array}{c|c} \hat{A}_c & B_c \\ \hline C_c & 0 \end{array}\right] = \left[\begin{array}{c|c} A + BD_cC + BK + LC & -L \\ \hline K & 0 \end{array}\right]$$

for some matrices K and L.

Remark. It is easily seen that (20) has a non-singular solution iff there exists a subspace $\mathscr{S}$ of dimension n that is

$$\begin{pmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{pmatrix} \text{-invariant and complementary to } \operatorname{im}\begin{pmatrix} I \\ 0 \end{pmatrix}, \ \operatorname{im}\begin{pmatrix} 0 \\ I \end{pmatrix}.$$

Let the columns of the matrix $\binom{S_1}{S_2}$ form a basis of any such subspace. Due to the complementarity conditions, both S_1 and S_2 are non-singular, and it is straightforward to verify that $T = S_2S_1^{-1}$ satisfies (20) due to the invariance property. In the same way as for symmetric Riccati equations, T can hence be computed in a numerically stable fashion on the basis of a Schur decomposition of the matrix $\mathscr{A}$ [2,1].

5. Summary of the algorithm with example

To summarize, we suggest the following algorithm to efficiently compute a sequence of controllers such that the corresponding sequence of costs converges monotonically to the optimal value of the MO control problem:

- Compute a mixed controller and transform it (possibly after a slight perturbation) into an observer structure as shown in the proof of Theorem 2.
- For $p=0, 1, 2, \ldots$, choose realization (15) with fixed A_Q, B_Q and $C_Q = (Q_p \cdots Q_1)$, $D_Q = Q_0$, and solve the multi-objective control problem for the system (13) with static output feedback (14) using the technique described in Section 2.

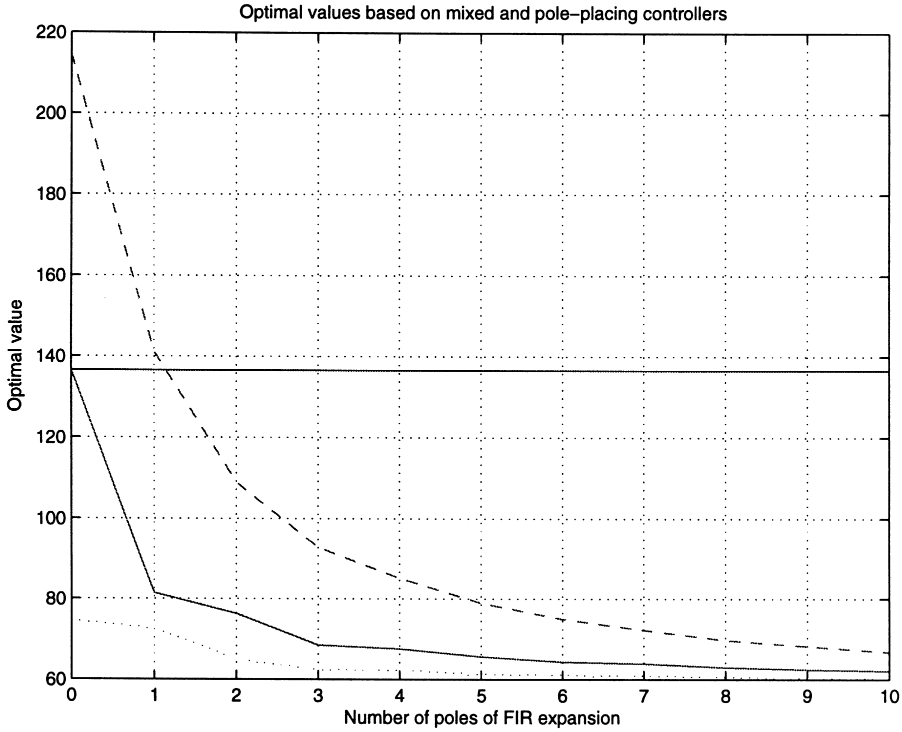

Fig. 2. Optimal value of $\| \mathscr{T}_1 \|_\infty + \| \mathscr{T}_2 \|_\infty$ versus length of FIR expansion with mixed controller (solid) and two pole-placing controllers (dashed and dotted).

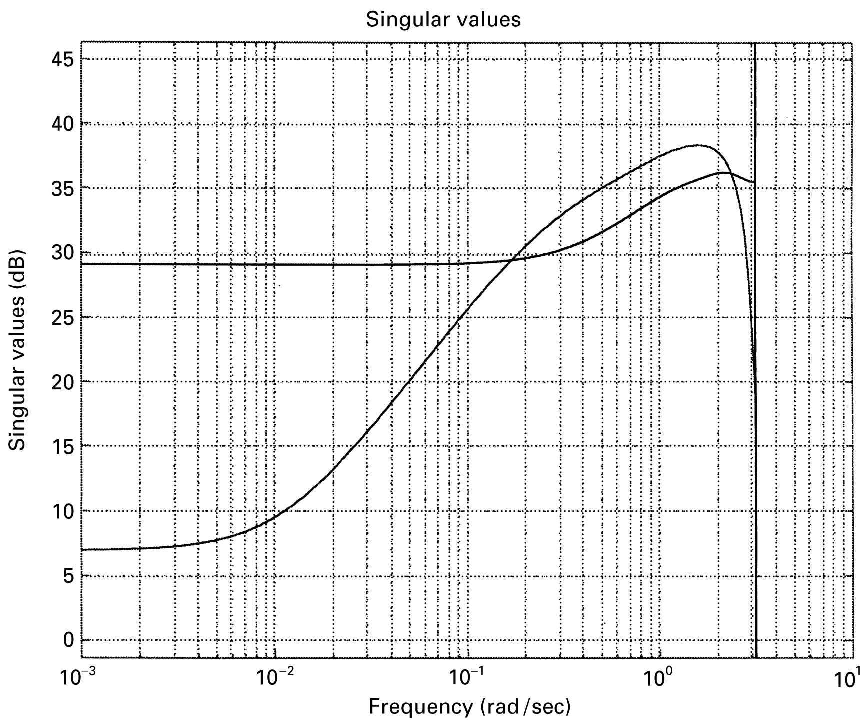

Fig. 3. Magnitude plots of $\mathscr{T}_1$, $\mathscr{T}_2$ for mixed design with objective $\|\mathscr{T}_1\|_\infty + \| \mathscr{T}_2 \|_\infty$.

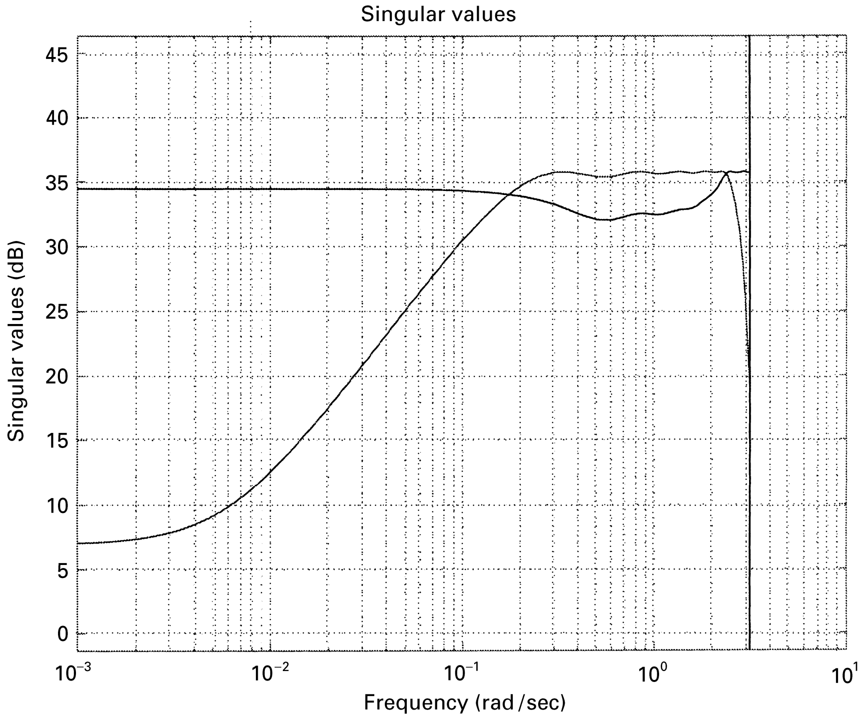

Fig. 4. Magnitude plots of $\mathscr{T}_1$, $\mathscr{T}_2$ for multi-objective design with objective $\|\mathscr{T}_1\|_\infty + \|\mathscr{T}_2\|_\infty$ and with FIR expansion length 10.

For reasons of illustration we apply this algorithm to the problem of minimizing

$$\|\mathscr{T}_1\|_\infty + \|\mathscr{T}_2\|_\infty$$

over all controllers that stabilize the (open-loop unstable) system

$$\begin{pmatrix} z_1 \\ z_2 \\ y \end{pmatrix} = \left[\begin{array}{cccc|ccc} 0.5 & 1 & 1.5 & 1 & 1 & 0 & 0 \\ -1 & 3 & 2.1 & 2 & 0 & 0 & 0 \\ 1 & -1 & -0.6 & 1 & 0 & 1 & 0 \\ -2 & 2 & -1 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \begin{pmatrix} w_1 \\ w_2 \\ u \end{pmatrix}.$$

A mixed design leads to the upper bound 136.5 (straight line in Fig. 2) with controller K_{mix}. Performing the Youla parameterization with K_{mix} and searching the Youla parameter (15) for $p = 0, \ldots, 10$ leads to the controllers K_p with the solid optimal value curve in Fig. 2. If we base the Youla parameterization on an arbitrary stabilizing controller (designed by pole-placement), we arrive at the dashed curve in Fig. 2 – the algorithm performs far worse. However, by playing with the closed-loop poles, we could find a pole-placing controller that leads to the dotted curve in Fig. 2 which is, actually, better suited than the mixed controller. Note, however, that mixed controllers form a *systematic* starting point for the algorithm suggested in this article. By comparing the magnitudes of the two performance channels for the controllers K_{mix} (Fig. 3) and K_{10} (Fig. 4), the additional freedom in the multi-objective design is exploited to flatten the pronounced peaks that appear for K_{mix}.

For reasons of comparison we plot the results in Figs. 5–7 by performing the same computations for the cost $\|\mathscr{T}_1\|_2 + \|\mathscr{T}_2\|_\infty$. As expected, one can clearly observe that the multi-objective design only flattens the frequency response of $\mathscr{T}_2$ whereas that of $\mathscr{T}_1$ is reduced in an average sense as measured by the H_2-norm.

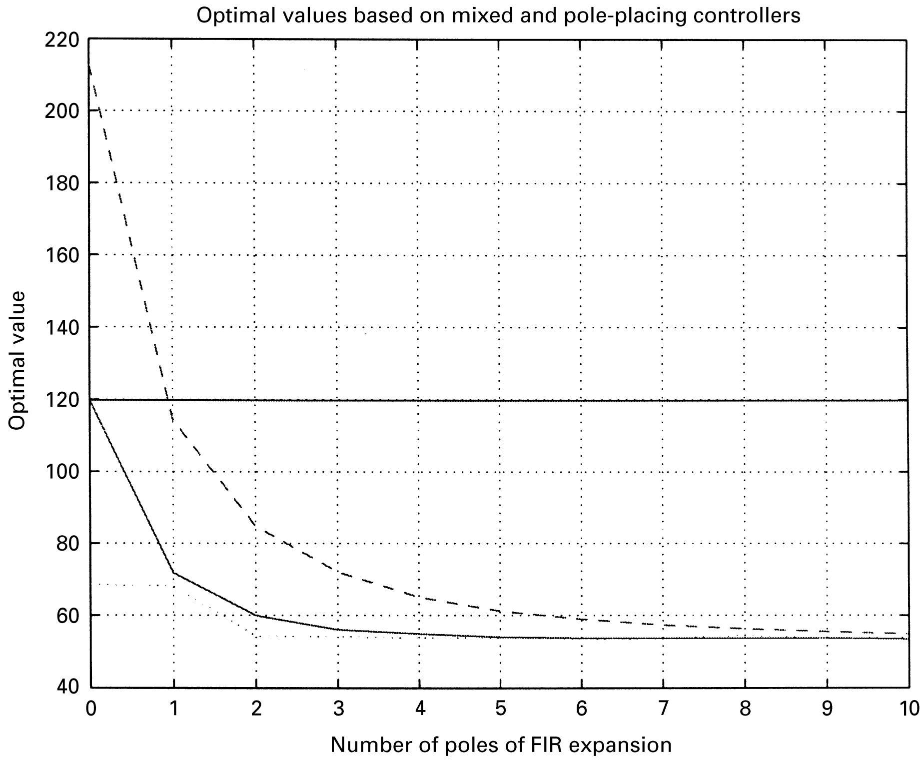

Fig. 5. Optimal value of $\| \mathscr{T}_1 \|_2 + \| \mathscr{T}_2 \|_\infty$ versus length of FIR expansion with mixed controller (solid) and two pole-placing controllers (dashed and dotted).

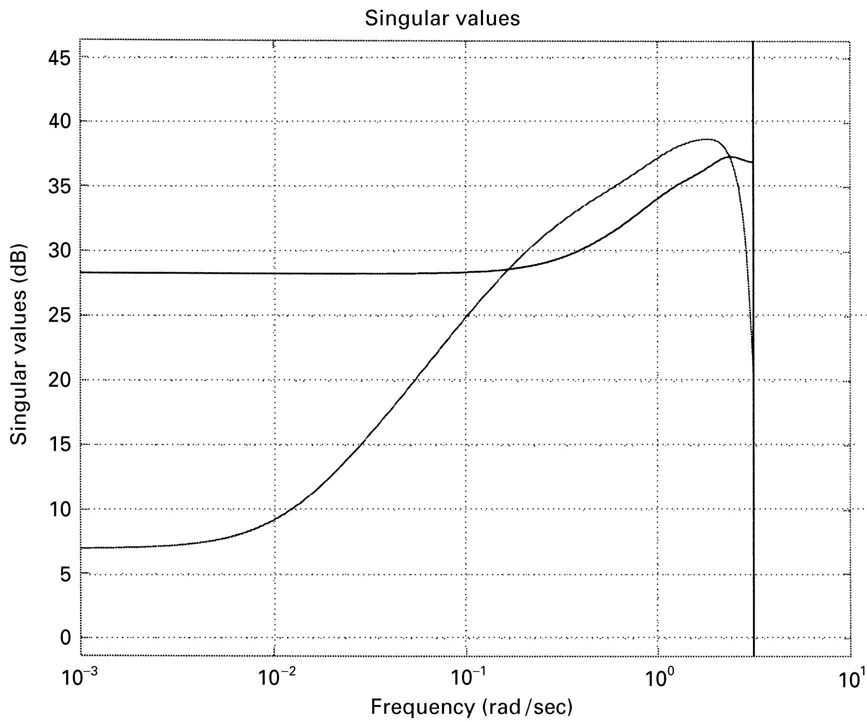

Fig. 6. Magnitude plots $\mathscr{T}_1$ and $\mathscr{T}_2$ for mixed design with objective $\|\mathscr{T}_1\|_2 + \| \mathscr{T}_2 \|_\infty$.

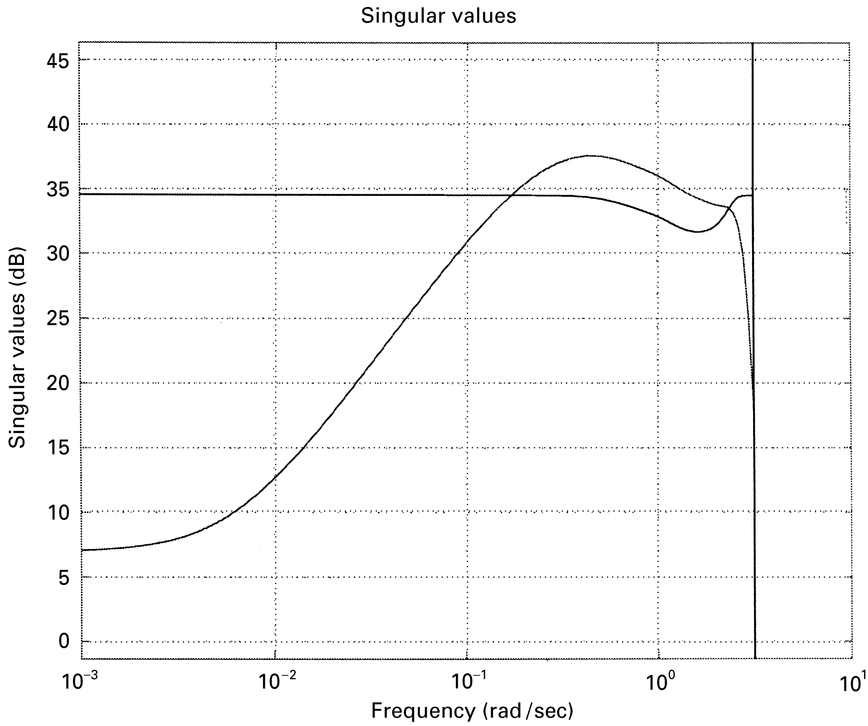

Fig. 7. Magnitude plots of $\mathscr{T}_1$, $\mathscr{T}_2$ for multi-objective design with objective $\|\mathscr{T}_1\|_2 + \|\mathscr{T}_2\|_\infty$ and with FIR expansion length 10.

6. Conclusions

We have suggested an algorithm in order to efficiently solve the multi-objective H_2/H_∞ control problem. Compared to previous approaches, we have revealed how to use a mixed controller as a starting point for the algorithm, and how to considerably reduce the size of the resulting LMI problems if increasing the approximation accuracy. As an auxiliary step we have provided a novel state-space approach to solve static output feedback control problem by convex optimization if the transfer matrix from control inputs to measured outputs vanishes. All the results in this paper generalize in a straightforward fashion to multi-objective control problems for more than two channels if the desired closed-loop specifications admit LMI representations.

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