Robotics - 34753

Introduction to Robotics

Matteo Fumagalli
Associate Professor
Automation and Control Group
Department of Electrical Engineering
DTU Lyngby, Building 326

Contributors:
Konstantinos Poulios
Nils Axel Andersen
Ole Ravn

Outline

- Introduction to the course
- Introduction to robotics
- Introduction to group assignment

Information Sheet

Matteo Fumagalli
Associate Professor
Department of Electrical and Photonics Engineering
Email: mafum@dtu.dk
Office: Building 326, room 120

Nils Axel Andersen
Associate Professor
Department of Electrical and Photonics Engineering
Email: nian@dtu.dk
Office: Building 326, room 016

Konstantinos Poulios
Associate Professor
Department of Civil and Mechanical Engineering
Email: kopo@dtu.dk
Office: Building 404, room 124

Ole Ravn
Professor
Department of Electrical and Photonics Engineering
Email: oravn@dtu.dk
Office: Building 326, room 012

Information Sheet

Teaching Assistants

Louise Mattelaer
s232831@student.dtu.dk

Kashish Verma
s230015@student.dtu.dk

Amin Mohseni
s222378@student.dtu.dk

Syllabus

Prerequisites

- 31300/ 31301/ 41671/ 41672/ 42672, knowledge of control theory corresponding to an introductory control course

Grading

- Written report (group) + Written 2-hours exam (individual MCQ)

Team work

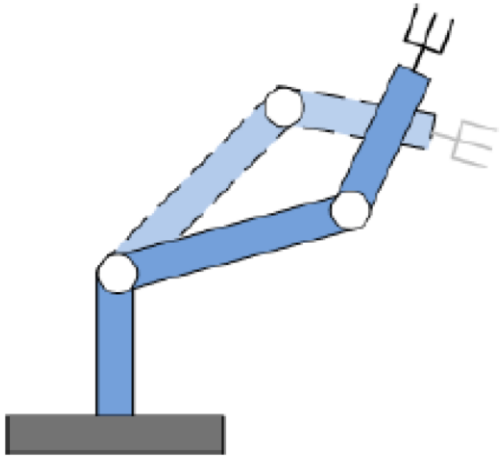
- 5 students form a group to work for the project assignment and exercise on real robot-arm

Textbook

- "ROBOT MODELING AND CONTROL", Mark W. Spong, Seth Hutchinson, M. Vidyasagar (first and second edition have the same relevant contents)

Topics

Introduction to robotics



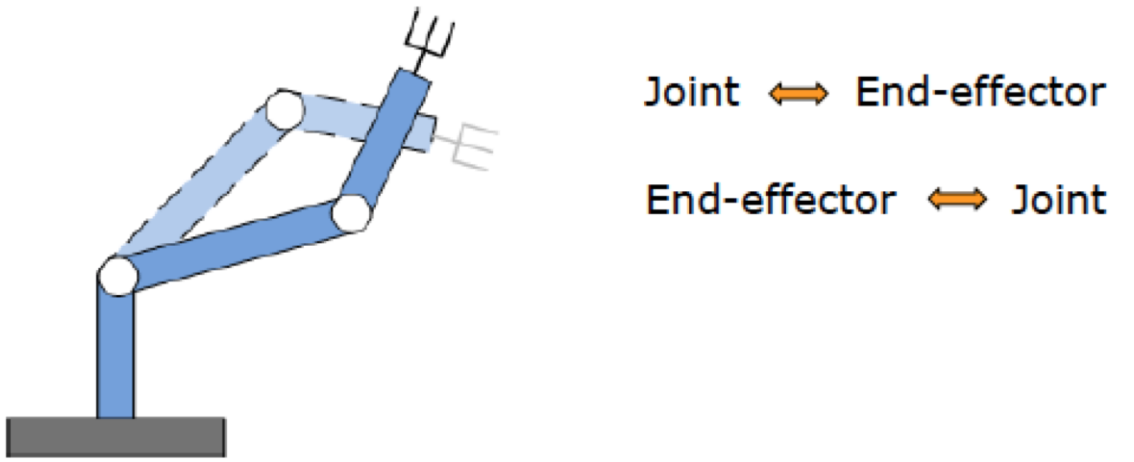
Joint ⟺ End-effector

End-effector ⟺ Joint

Topics

Introduction to robotics

Direct kinematics

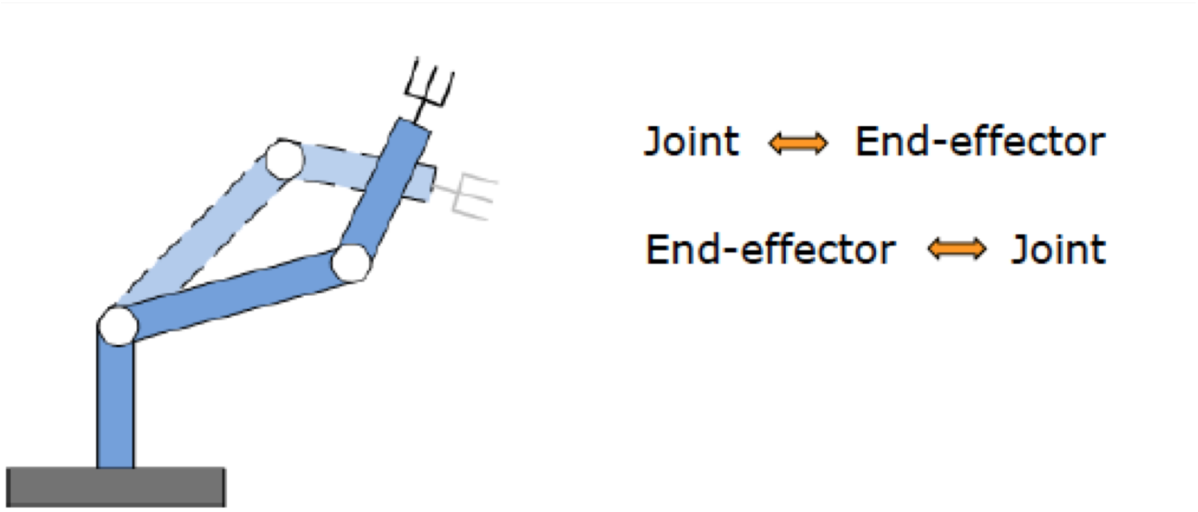


Topics

Introduction to robotics

Direct kinematics

Inverse Kinematics



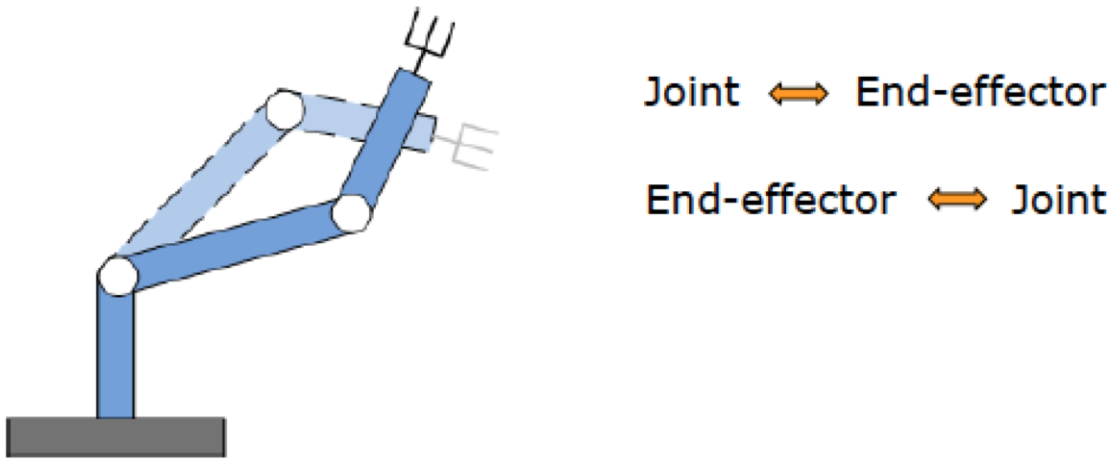
Topics

Introduction to robotics

Direct kinematics

Inverse Kinematics

Path planning and trajectory generation



Topics

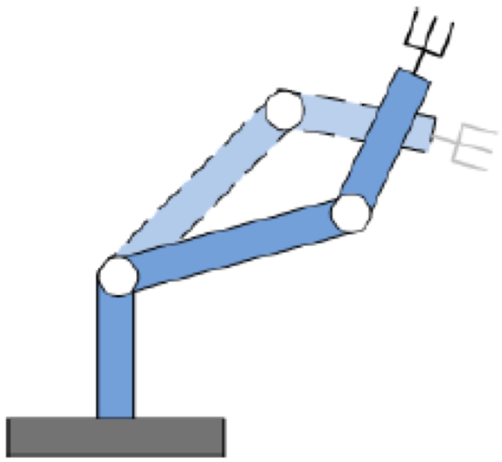
Introduction to robotics

Direct kinematics

Inverse Kinematics

Path planning and trajectory generation

Robot dynamics



Joint ⟺ End-effector

End-effector ⟺ Joint

Topics

Introduction to robotics

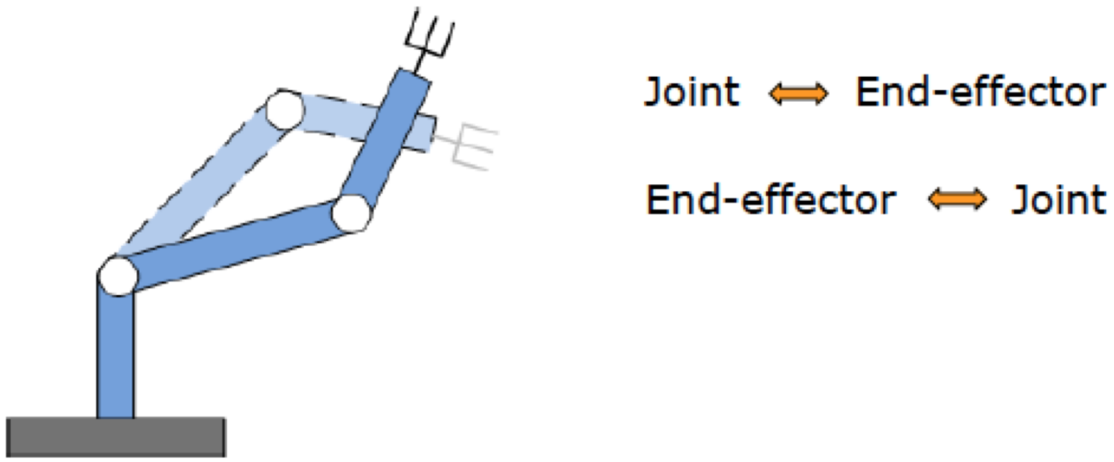
Direct kinematics

Inverse Kinematics

Path planning and trajectory generation

Robot dynamics

Robot control



Topics

Introduction to robotics (Matteo)

Direct kinematics (Konstantinos)

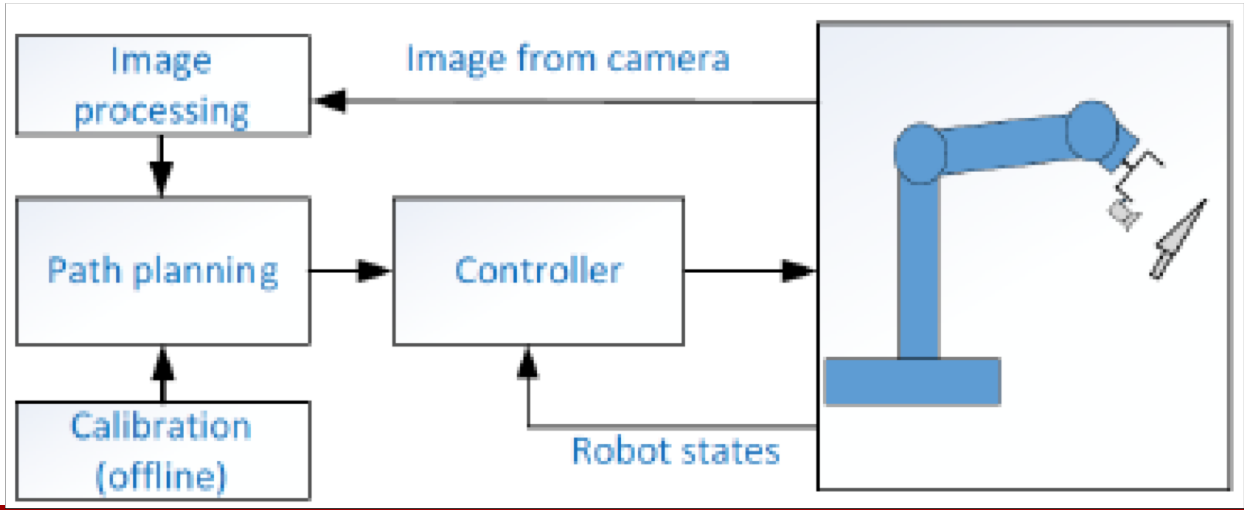
Inverse Kinematics (Konstantinos)

Path planning and trajectory generation (Konstantinos)

Robot dynamics (Konstantinos)

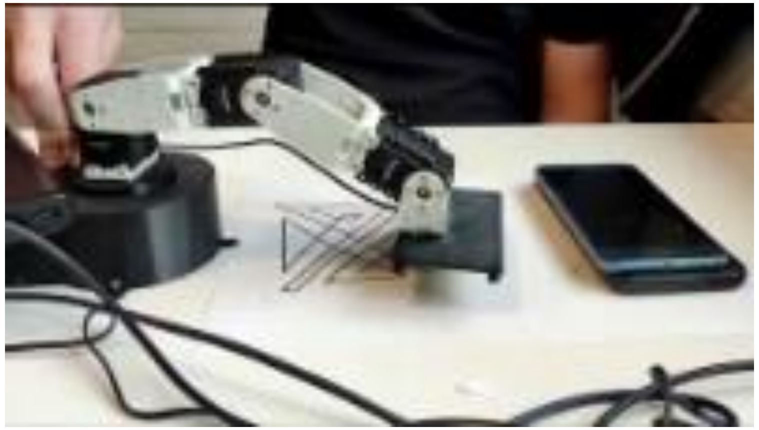
Robot control (Matteo)

Sensor systems in robotics (Matteo)



Practical Assignment

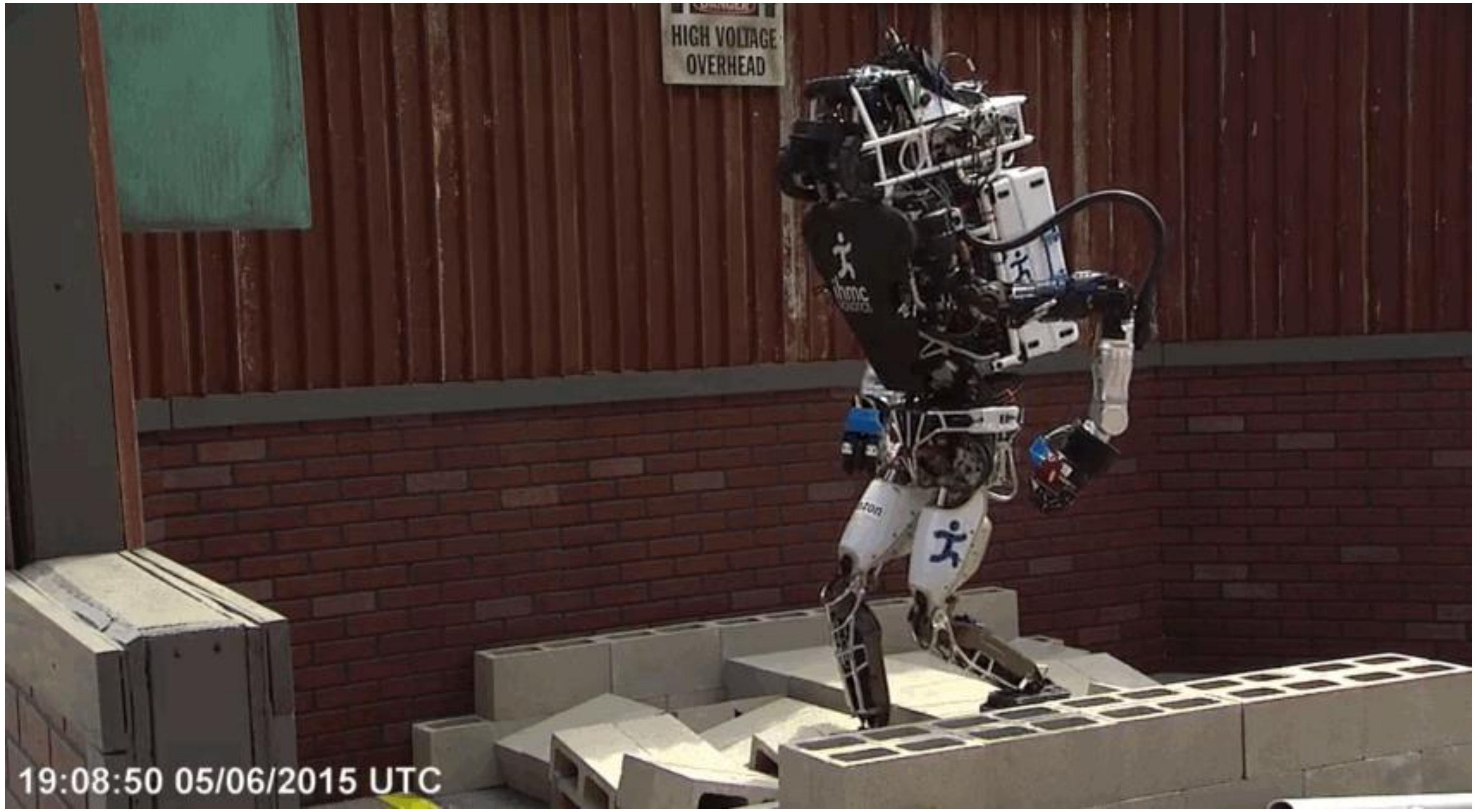
- **Purpose**
  - Practice the topics learned in the Robotics course 34753, on a realistic robotic application (on a real robotic system)
- **DTU CourseBot**
  - 4 DoF robot
- **Solve -problems**
  - Simulation, dynamics, kinematics, trajectory planning, control, real-life implementation



Schedule

03/09 Introduction to robotics
10/09 Robot kinematics I
17/09 Robot kinematics II & inverse kinematics
24/09 Velocity Kinematics & The Jacobian Matrix
01/10 Side track assignments opportunities
08/10 Trajectory Planning & Manipulator Dynamics
15/10 Robot control
12/10 (no lecture)
29/10 Sensor systems in robotics
05/11 Project assignment
12/11 Project assignment
19/11Project assignment
26/11 Advanced topics
03/12 Project assignment
05/12 Deadline for handing in the compulsory project assignment
16/12 Written exam

Break



Introduction to Robotics

What is a Robot?

Karel Capek

- Origin of the word robot
  - Czech word “robota” − labor
  - 1920 play by Karel Capek − Rossum’s Universal Robots

- Definition (no precise definition yet)
  - Robotics institute of American
    - “A robot (industrial robot) is a reprogrammable, multifunctional manipulator designed to move materials, parts, tools, or specialized devices, through variable programmed motions for the performance of a variety of tasks.”
  - European common market
    - “an independent acting and self controlling machine, equipped with specific tools to handle or machine and whose movements are programmable ….”

What is a Robot?

Hollywood's imagination

C3PO & R2-D2

Wall-E

Chappie

Terminator

What is a Robot?

- By general agreement, a robot is:

A **programmable machine that imitates the actions or appearance of an intelligent creature – usually a human.**

"Real steel", 2011

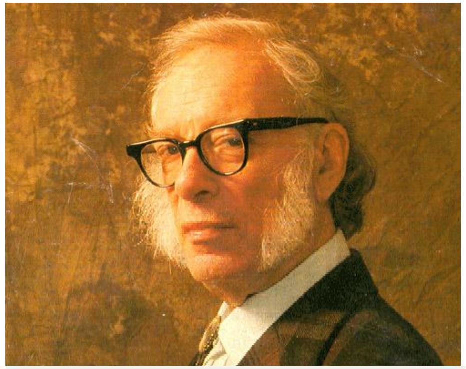
Sensing and perception

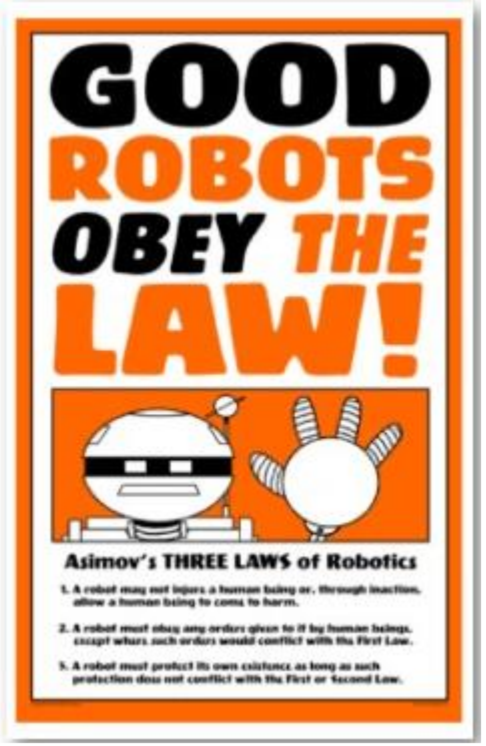
Carry out different tasks

Interact with human beings

The Three Laws of Robotics

- Asimov proposed three “Laws of Robotics”
- Law 1: A robot may not injure a human being or through inaction, allow a human being to come to harm
- Law 2: A robot must obey orders given to it by human beings, except where such orders would conflict with a higher order law
- Law 3: A robot must protect its own existence as long as such protection does not conflict with a higher order law





Types of Robots

Manipulator

https://www.youtube.com/watch?v=neWc5I9IdQ4

Types of Robots

Wheeled mobile robots

https://www.youtube.com/watch?v=BwG5yoTbX6c

Types of Robots

Legged robots

https://www.youtube.com/watch?v=bmNaLtC6vkU

‘BigDog runs at 4 mph (1.79 m/s), climbs slopes up to 35 degrees, walks across rubble, climbs muddy hiking trails, walks in snow and water, and carries 340 lb (154kg) load.’

Types of Robots

Humanoid

http://www.youtube.com/watch?v=Krl-YzdVZak

Types of Robots

Humanoid

https://www.youtube.com/watch?v=rVlhMGQgDkY

Types of Robots

Aerial robots

https://www.youtube.com/watch?v=6uh7zrIbHwM

Types of Robots

Underwater robots

https://www.youtube.com/watch?v=l3IlcUklC_s

Types of Robots

Manipulator

Aerial robots

Wheeled mobile robots

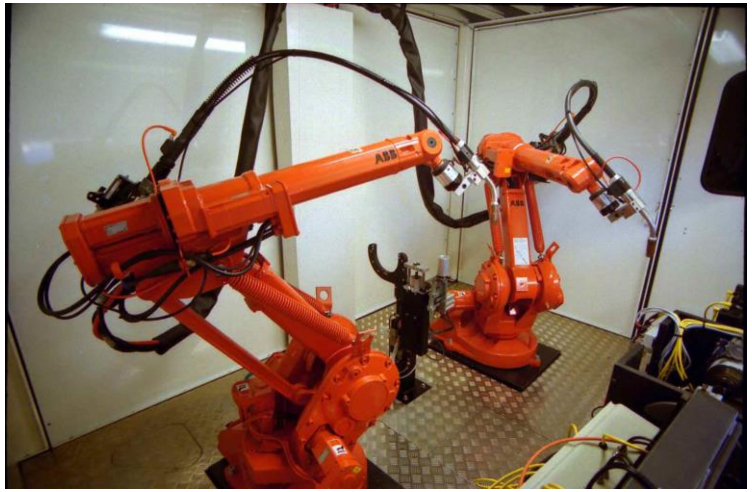
Humanoid

Underwater robots

Legged robots

Why use Robots?

- Increase product quality
  – Superior Accuracies (thousands of an inch, wafer-handling: microinch)
  – Repeatable precision → consistency of products
- Increase efficiency
  – Work continuously without fatigue
  – Need no vacation



Welding Robot


The SCRUBMATE Robot

Why use Robots?

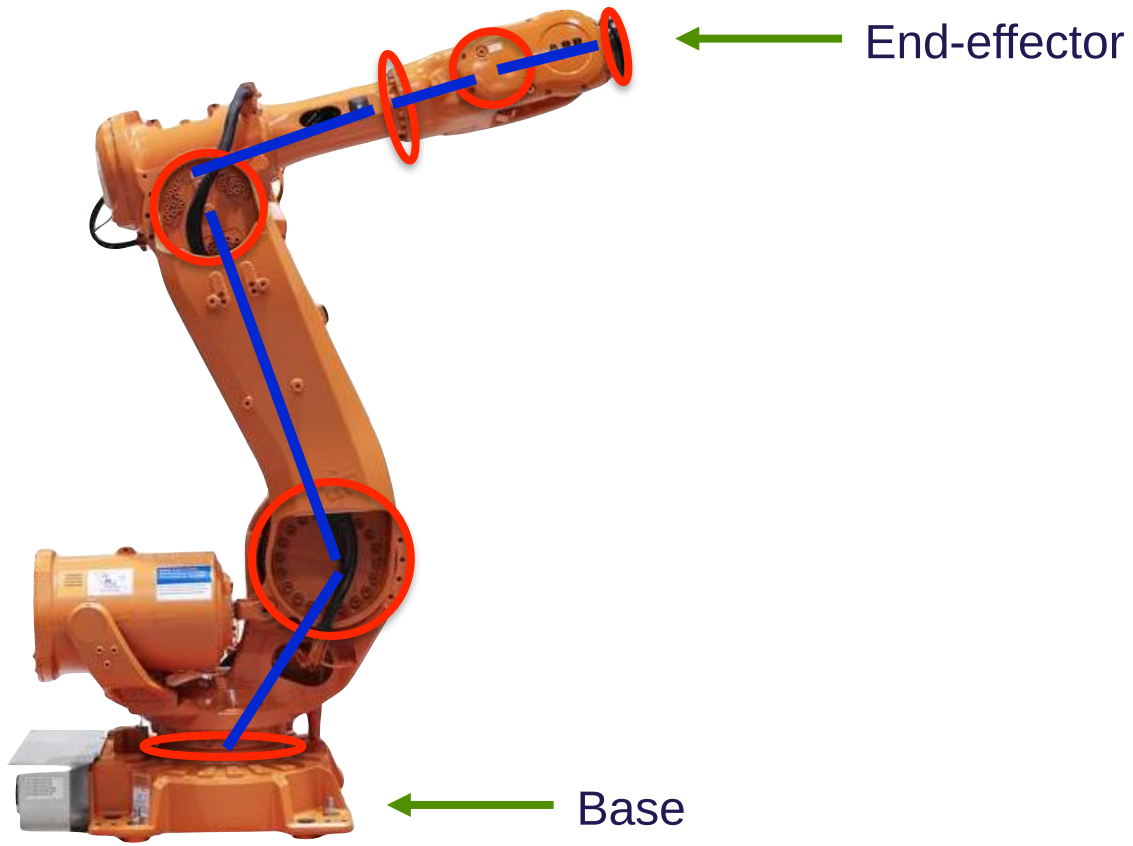
- Increase safety
  - Operate in dangerous environment
  - Need no environmental comfort air conditioning, noise protection, etc

Decontaminating Robot

Cleaning the main circulating pump housing in the nuclear power plant

Anatomy of a Robotic Arm

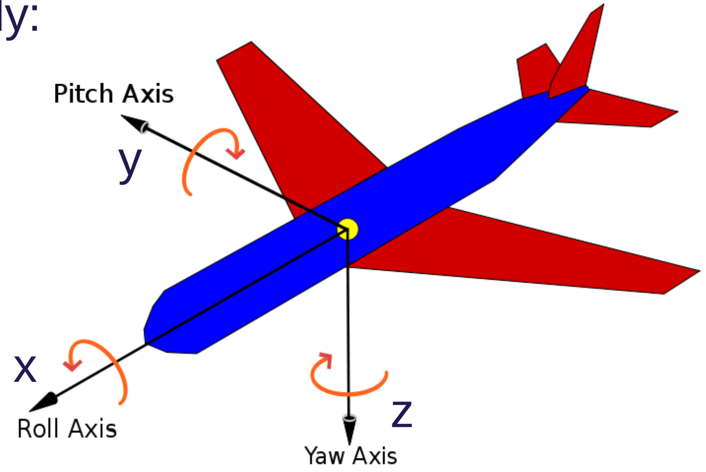
- Base
- End-effector
- Links
- Joints



Degrees of Freedom (DOF)

- Definition:
  A DOF of a mechanical system is the number of independent parameters that define its configuration

- The general case, for an unrestricted rigid body:
  - 3 directional DOF: (x,y,z)
  - 3 rotational DOF: (roll, pitch, yaw)

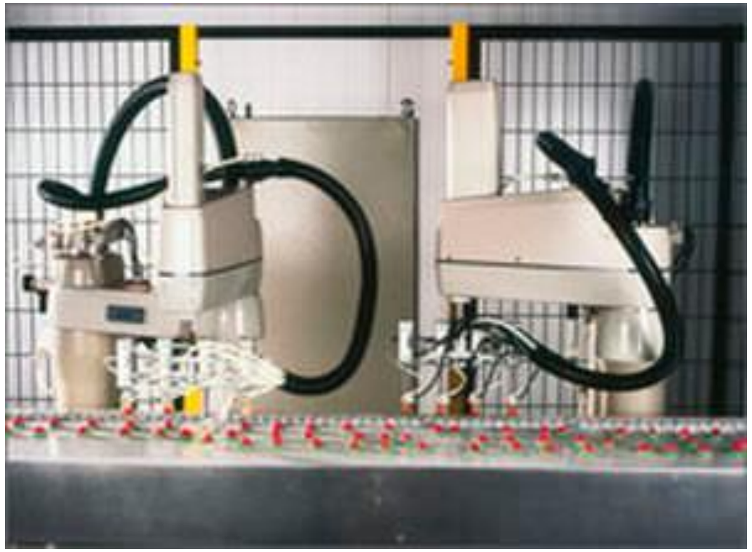


Choosing the Right Robot

- Workspace
  - Must reach a number of fixtures and workpieces
  - Margin around fixtures and workpieces in order to avoid collisions
  - Consider the shape and singularities
- Load capacity
- Speed
- Repeatability and accuracy
- External interfaces

Degrees of Freedom (DOF)

- General positioning and orienting requires 6 degrees of freedom (DOF).
- Tasks with rotationally symmetric tool requires only 5 DOF (welding, grinding, polishing etc.)
- Positioning parts on planar surface (pick and place) requires only 4 DOF (X,Y,Z,roll).



Kinematic Configuration and Workspace

A **configuration** of a manipulator is a complete specification of the location of every point on the manipulator. The set of all possible configurations is called the **configuration space**.

An object is said to have **n** degrees-of-freedom (DOF) if its configuration can be minimally specified by n parameters. Thus, the number of DOF is equal to the dimension of the configuration space.

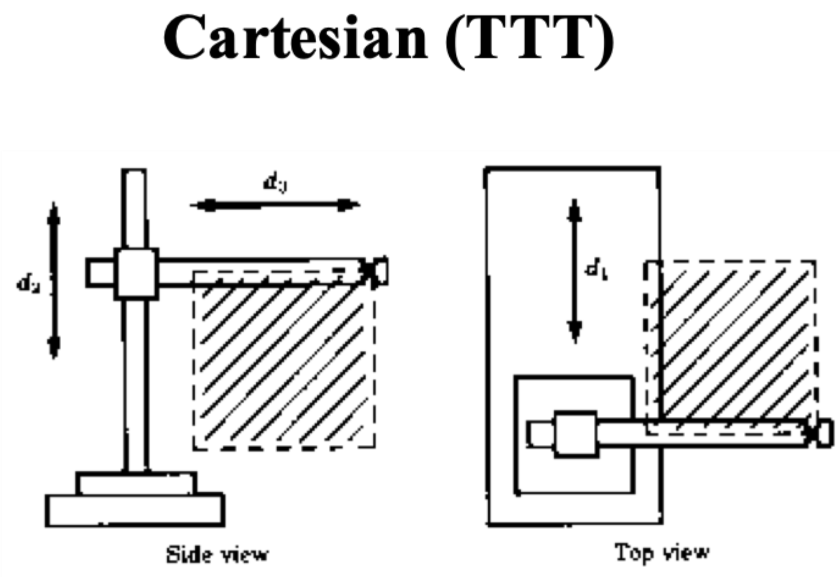
The **workspace** of a manipulator is the total volume swept out by the end-effector as the manipulator executes all possible motions. The workspace is constrained by the geometry of the manipulator as well as mechanical constraints on the joints.

Reachable workspace: entire set of points reachable by the manipulator

Dexterous workspace: points that the manipulator can reach with an arbitrary orientation of the end-effector.

Kinematic Configurations

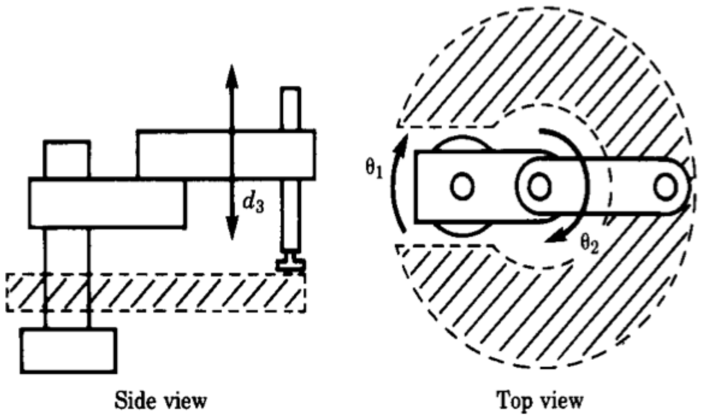
- The type of joints before the wrist determine the difference
- Common kinematic configurations:

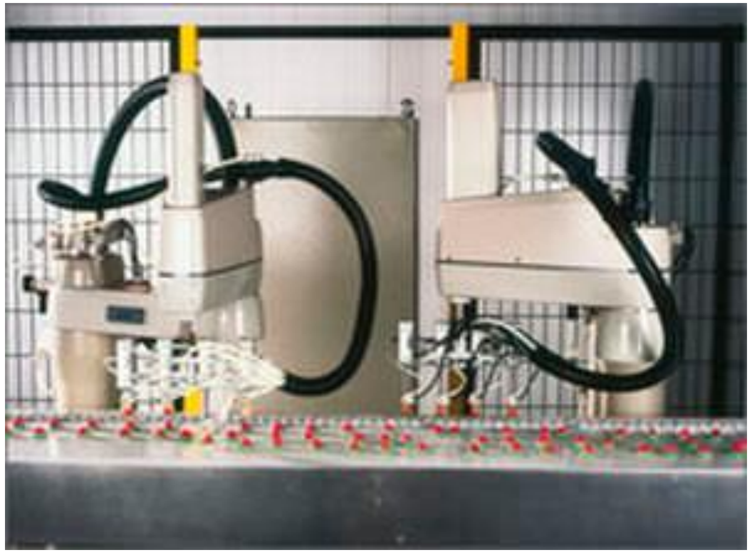


Kinematic Configurations

- The type of joints before the wrist determine the difference
- Common kinematic configurations:

Scara (RRT)

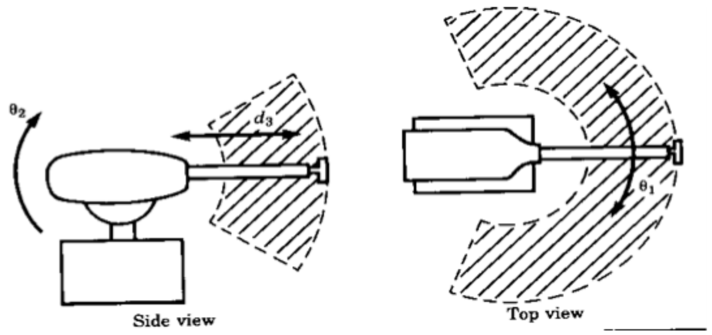




Kinematic Configurations

- The type of joints before the wrist determine the difference
- Common kinematic configurations:

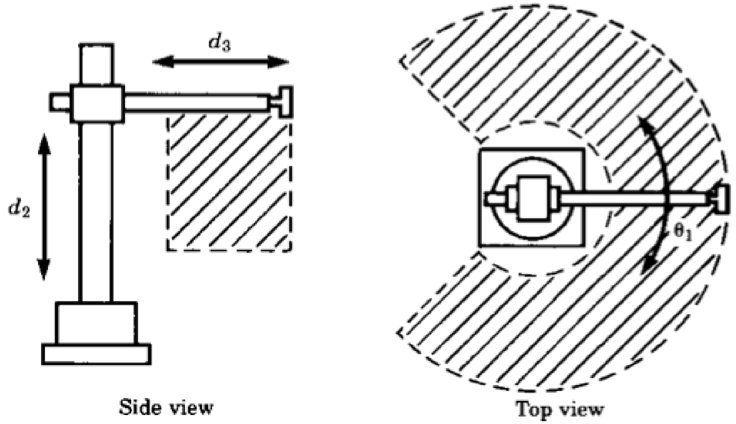
Spherical (RRT)



Kinematic Configurations

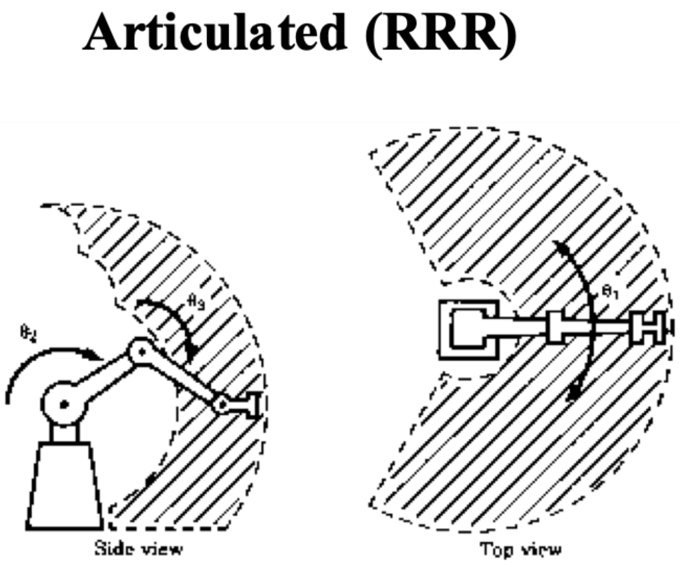
- The type of joints before the wrist determine the difference
- Common kinematic configurations:

Cylindrical (RTT)



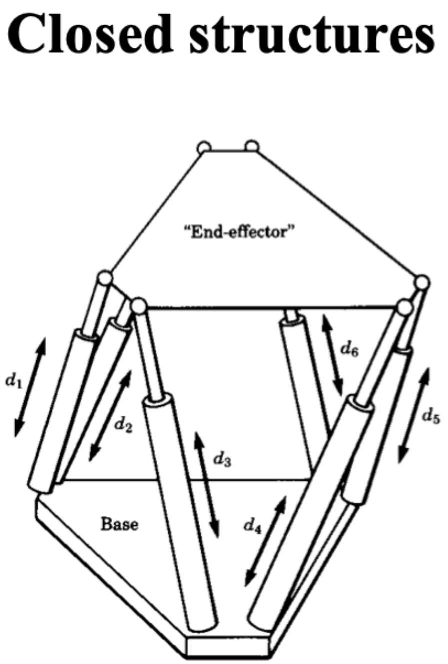
Kinematic Configurations

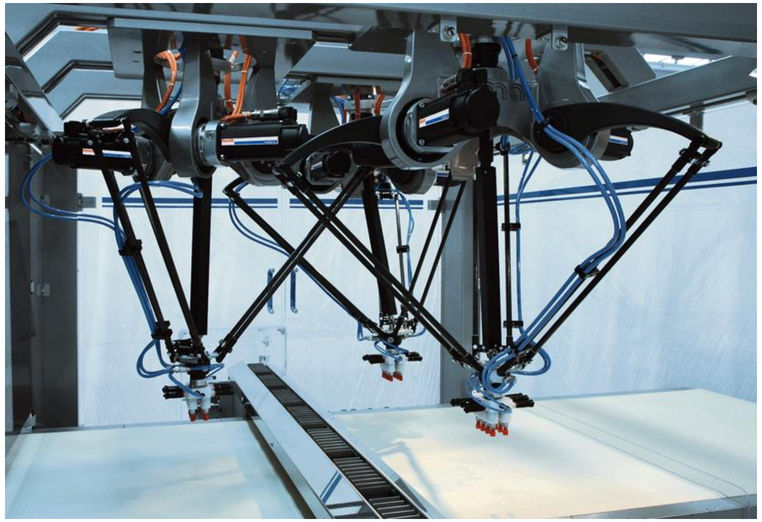
- The type of joints before the wrist determine the difference
- Common kinematic configurations:



Kinematic Configurations

- The type of joints before the wrist determine the difference
- Common kinematic configurations:



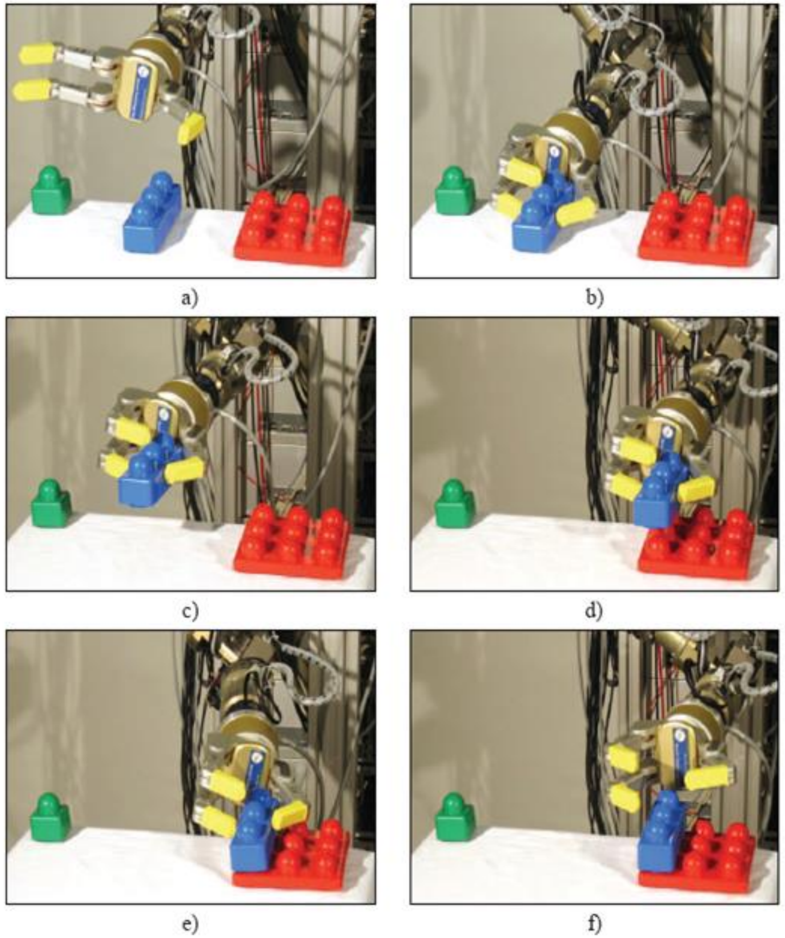


Industrial Robot – PUMA (1978)

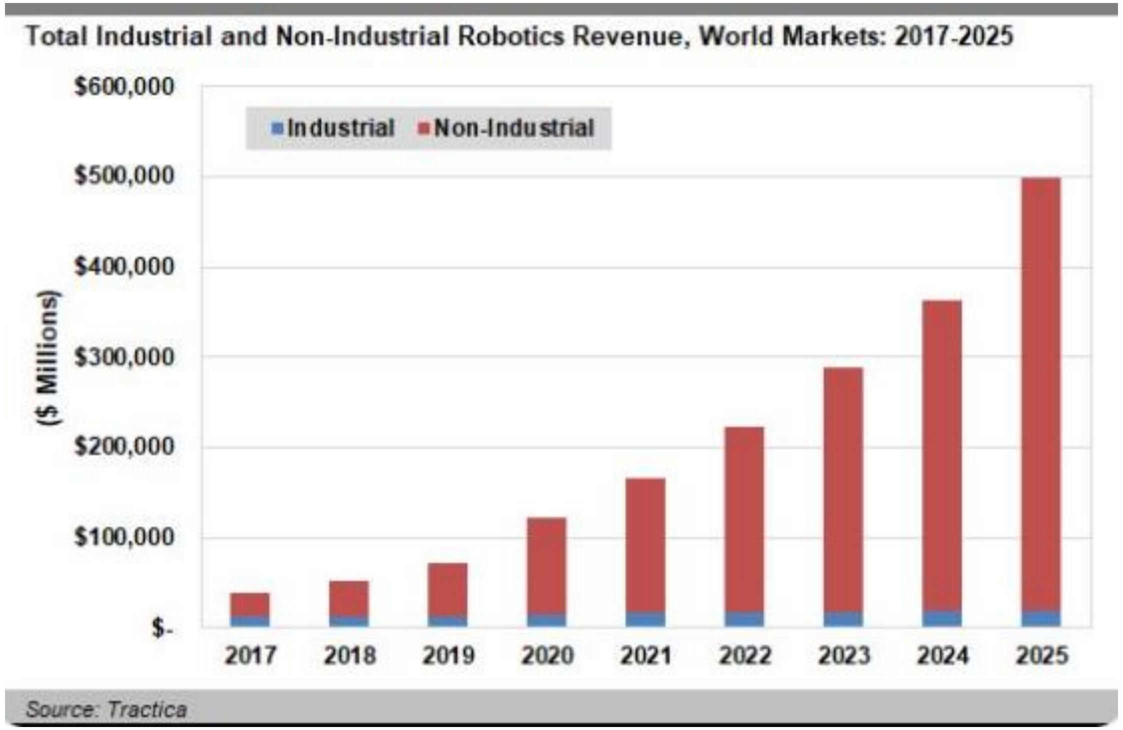
How are They Used?

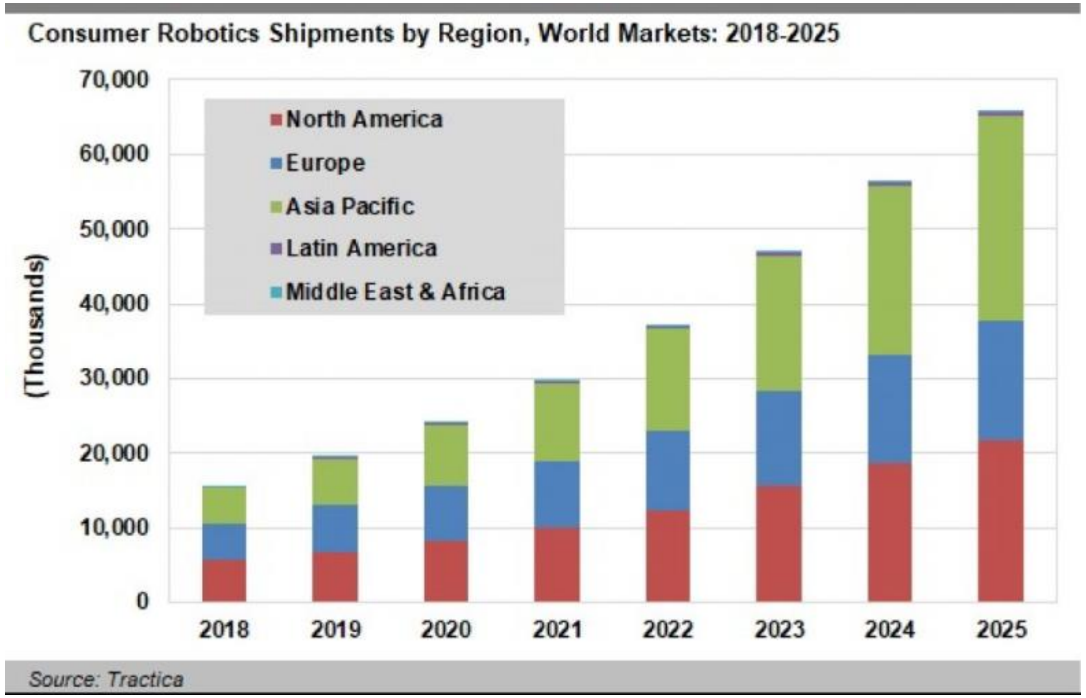
- Industrial robots
  - – 70% welding and painting
  - – 20% pick and place
  - – 10% others

- Research focus on
  - – Manipulator control
  - – End-effector design
    - Compliance device
    - Dexterous robot hand
  - – Visual and force feedback



Industrial and Non-Industrial Robots Forecast





Service Robots

floor cleaning

human guider

entertainment

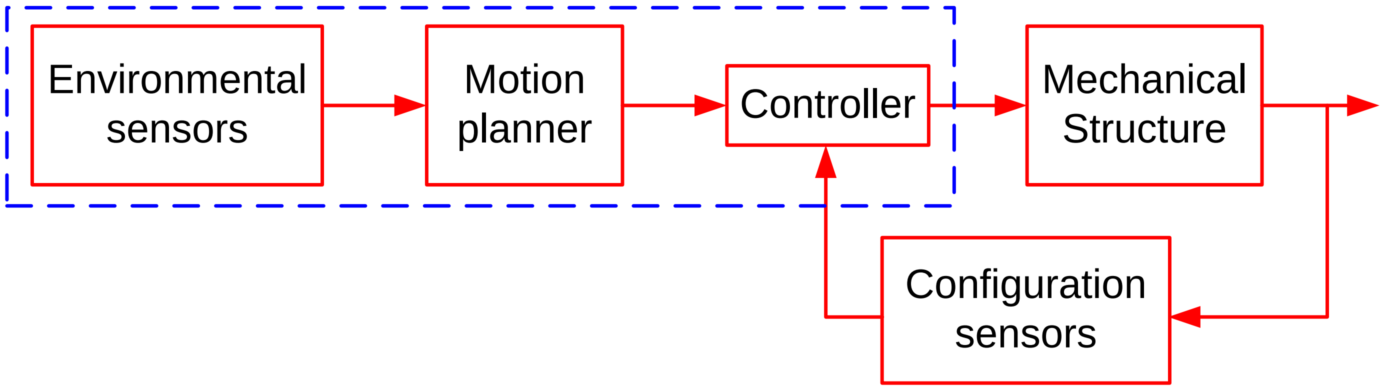
lawn-mower

vacuum cleaner

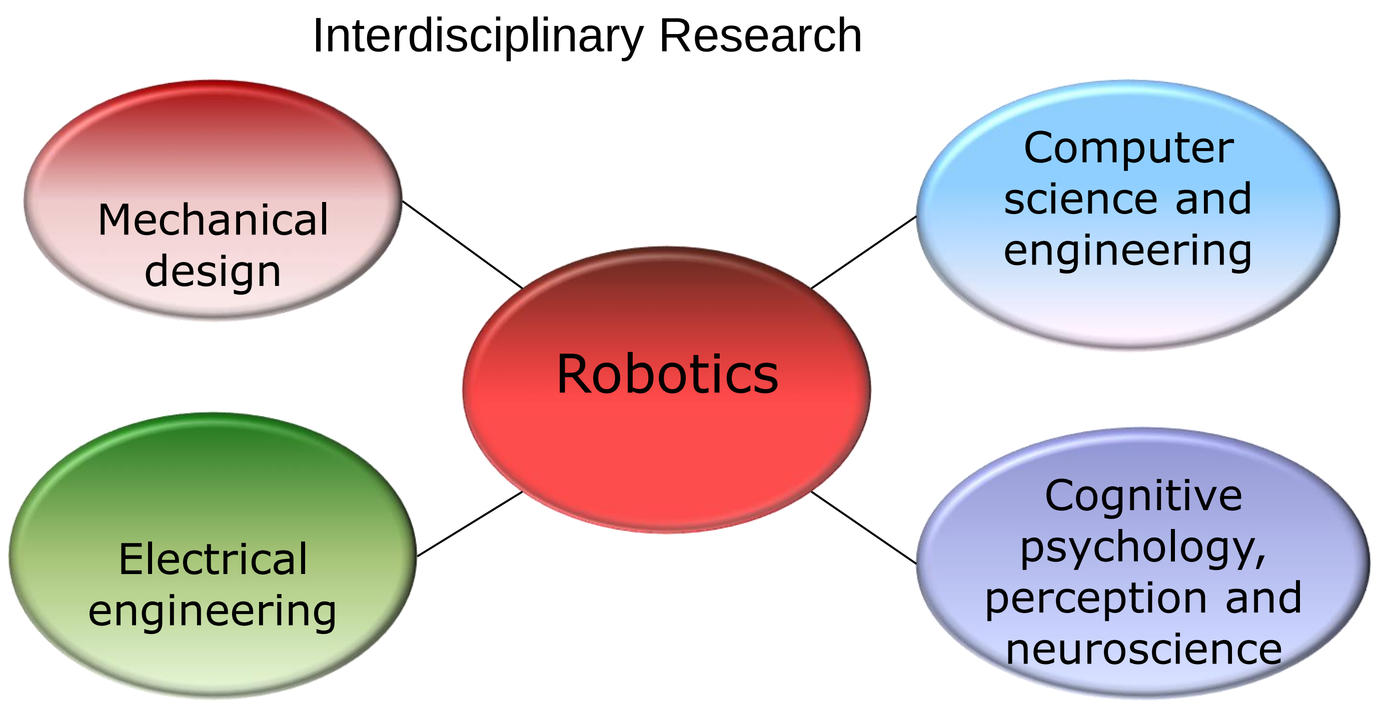
tennis ball collector

Architecture of Robotic Systems

- Mechanical Structure
  - Kinematics model
  - Dynamics model
- Actuators: electrical, hydraulic, pneumatic, artificial muscle
- Computation and controllers
- Sensors
- Communications
- User interface



Summary



- Open problems
  - Manipulation, Locomotion
  - Control, Navigation
  - Human-Robot Interaction
  - Learning & Adaptation (AI)

Robotics – 34753

Robot Kinematics I

Konstantinos Poulios
Associate Professor
Department of Civil and Mechanical Engineering
DTU Lyngby, building 404 / room 124

Robot Kinematics I – Lecture Overview

1. **Robot Manipulators**
   - Revolute/prismatic Joints
   - Robot grippers
   - Robot configurations
   - Workspace

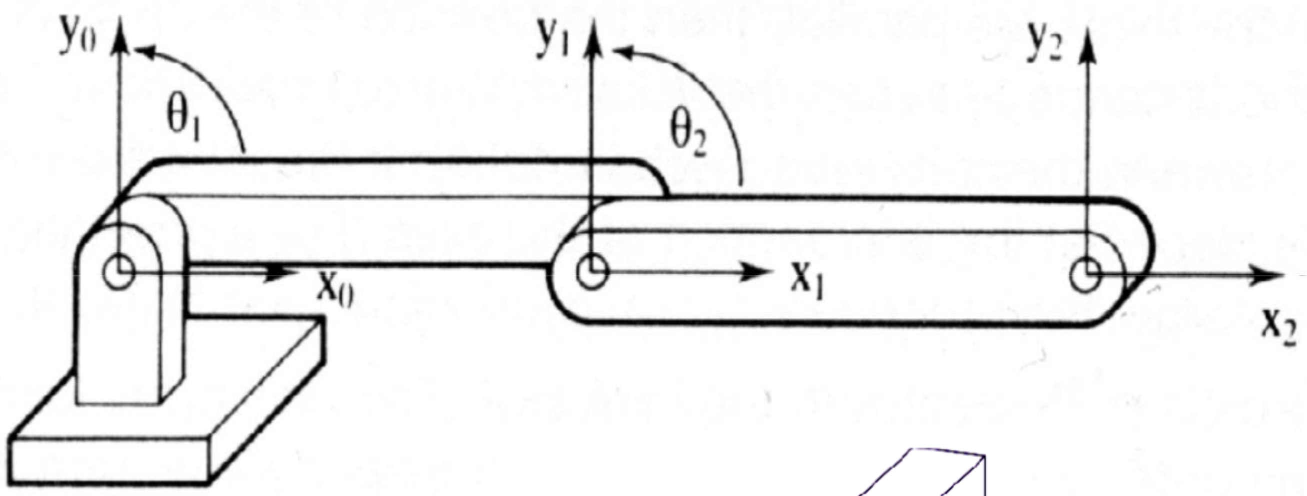
2. **Kinematics**
   - Position representation
   - Rotation/Orientation representation
   - Compositions of rotations
   - Rotation parametrization – Euler angles
   - Homogeneous transformation matrices

Robot Joints

Joints

revolute joint

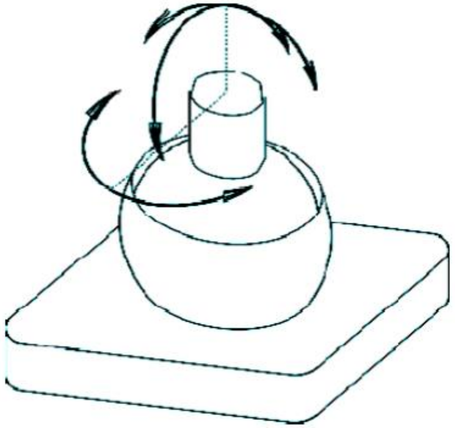
1 DOF – variable θ



spherical joint

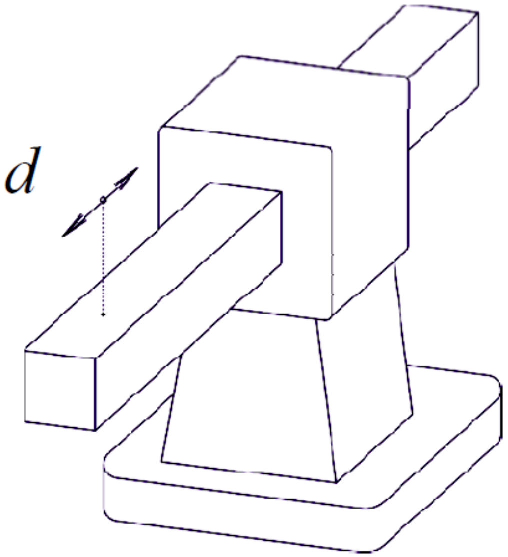
(not considered in the course)

3 DOF – variables θ_1, θ_2, θ_3

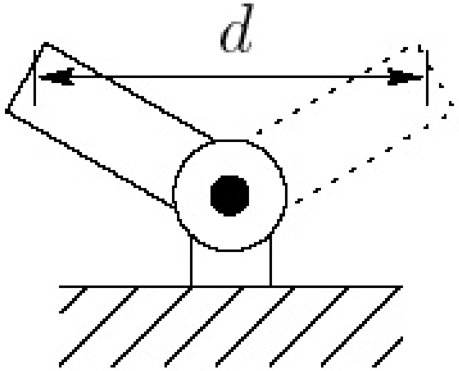


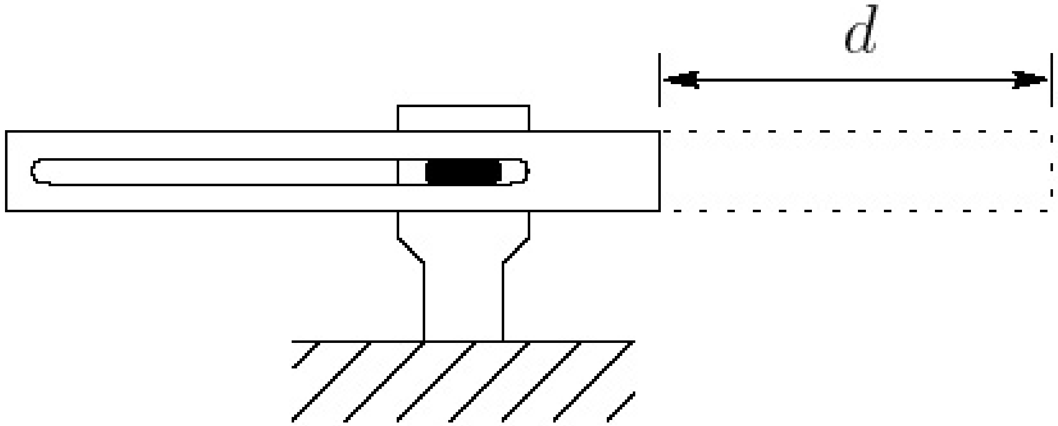
prismatic joint

1 DOF – variable d



Revolute vs Prismatic Joints





- Prismatic link: covers distance **equal to** the length of the link
- Rotational link: covers distance **twice as** the length of the link

Revolute joint:

- Compact
- Simple construction
- Even transmission of constraint forces
- Easy to lubricate
- Rather insensitive to dirt
- Difficult to produce linear motion (in limited space)

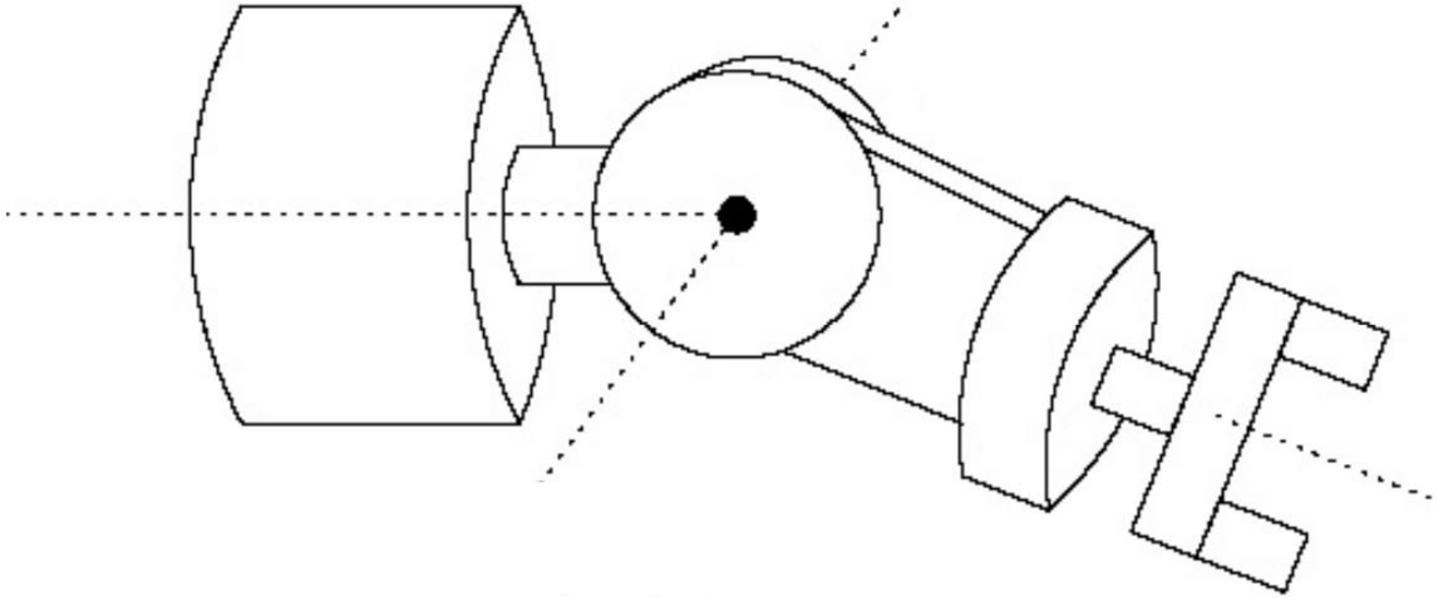
https://rozum.com/

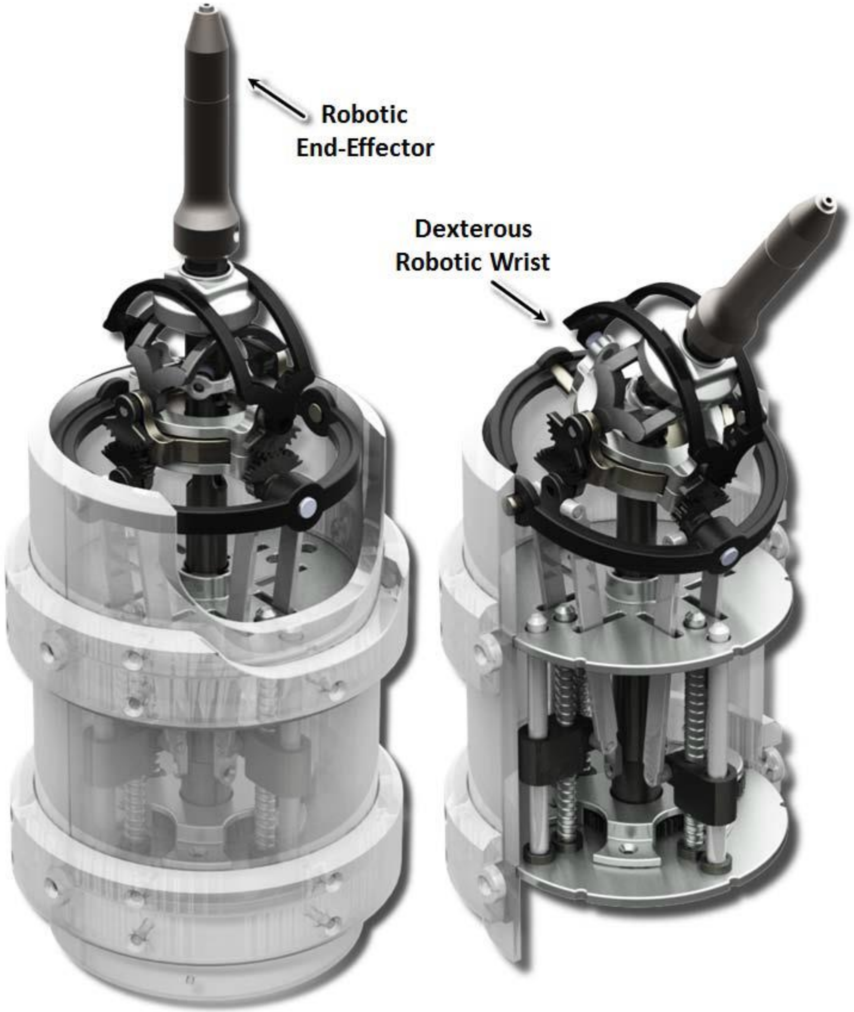
Prismatic joint:

- Larger
- More complex construction
- Uneven reaction forces
- More difficult to lubricate
- More sensitive to dirt
- Produces linear motion

https://medias.schaeffler.com/

Spherical Wrist Joint





Hammond et al. 2013

- Three axes of rotation: roll, yaw pitch
- All three axes intersect at a single point: wrist center point

Robot Grippers

Robot Grippers

https://thinkbotsolutions.com

Two-fingered parallel jaw gripper

Parallelogram gripper

Adaptive gripper

Bajaj et al. 2019

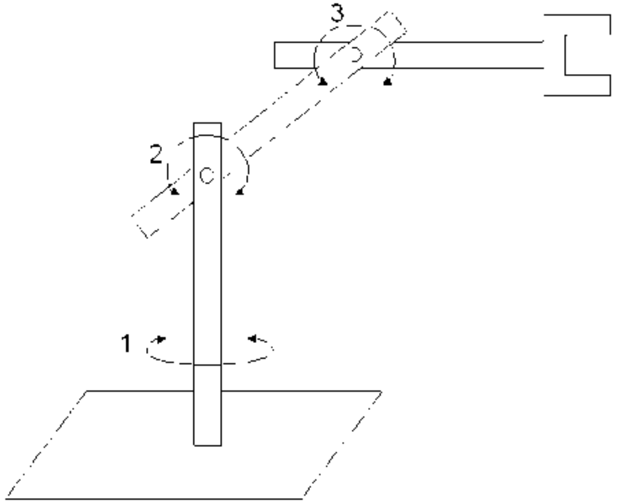
Scissor gripper

Three-fingered Anthropomorphic Hand

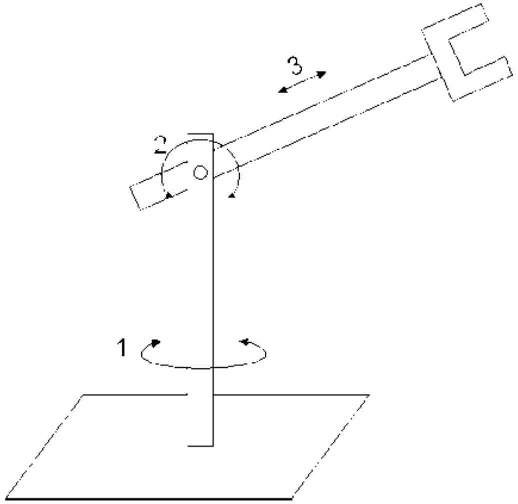
- More dexterity
- Better ability to manipulate objects of various sizes and geometries

Robot Configurations

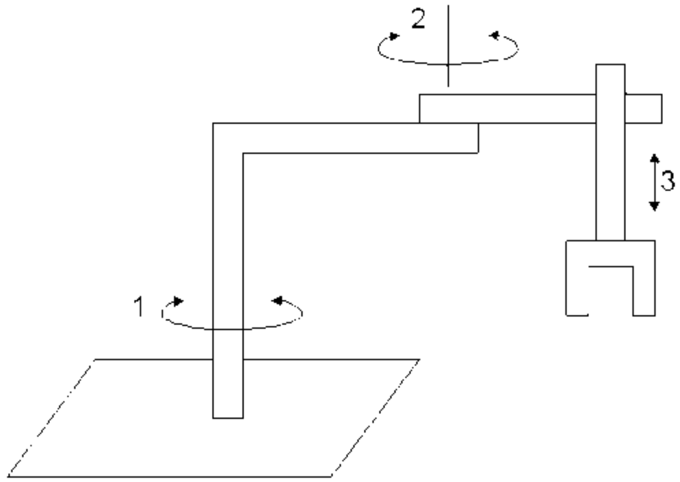
Robot Arm Examples



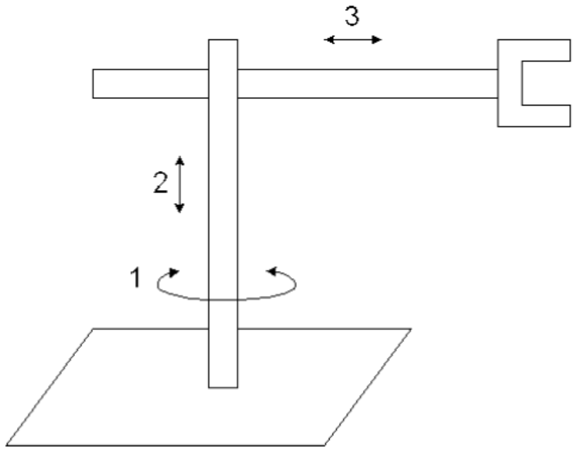
Articulated: RRR



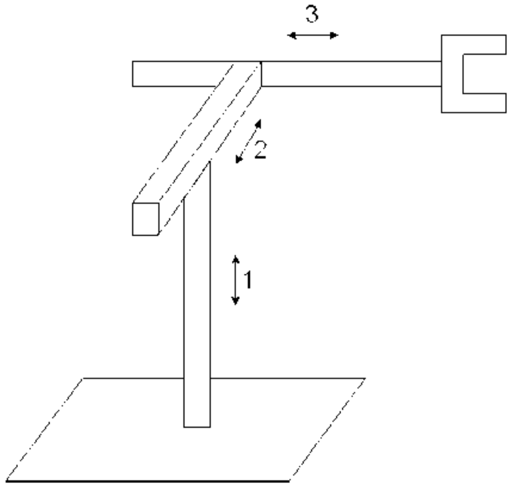
Spherical: RRP



SCARA: RRP



Cylindrical: RPP



Cartesian: PPP

Articulated Manipulator (RRR)

- A.k.a. revolute/elbow/anthropomorphic manipulator
- $z_2 \parallel z_1$, $z_1 \perp z_0$, $z_2 \perp z_0$

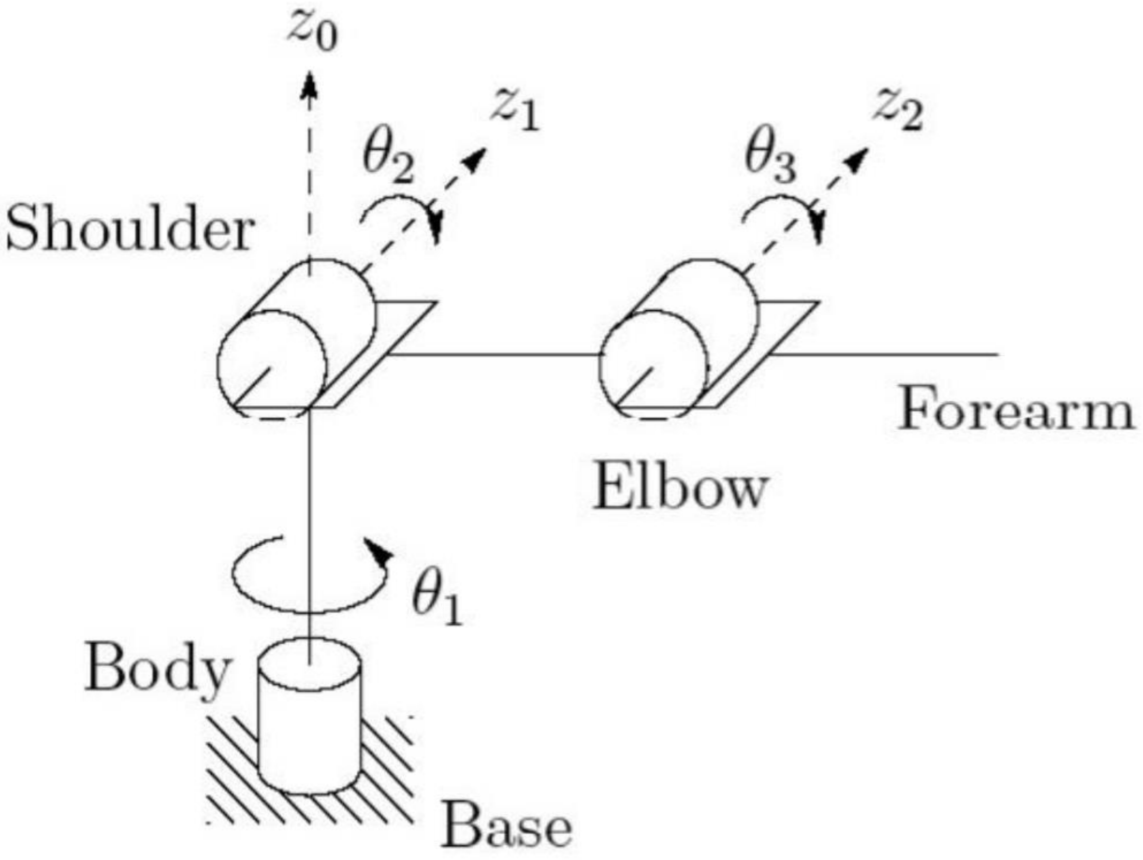
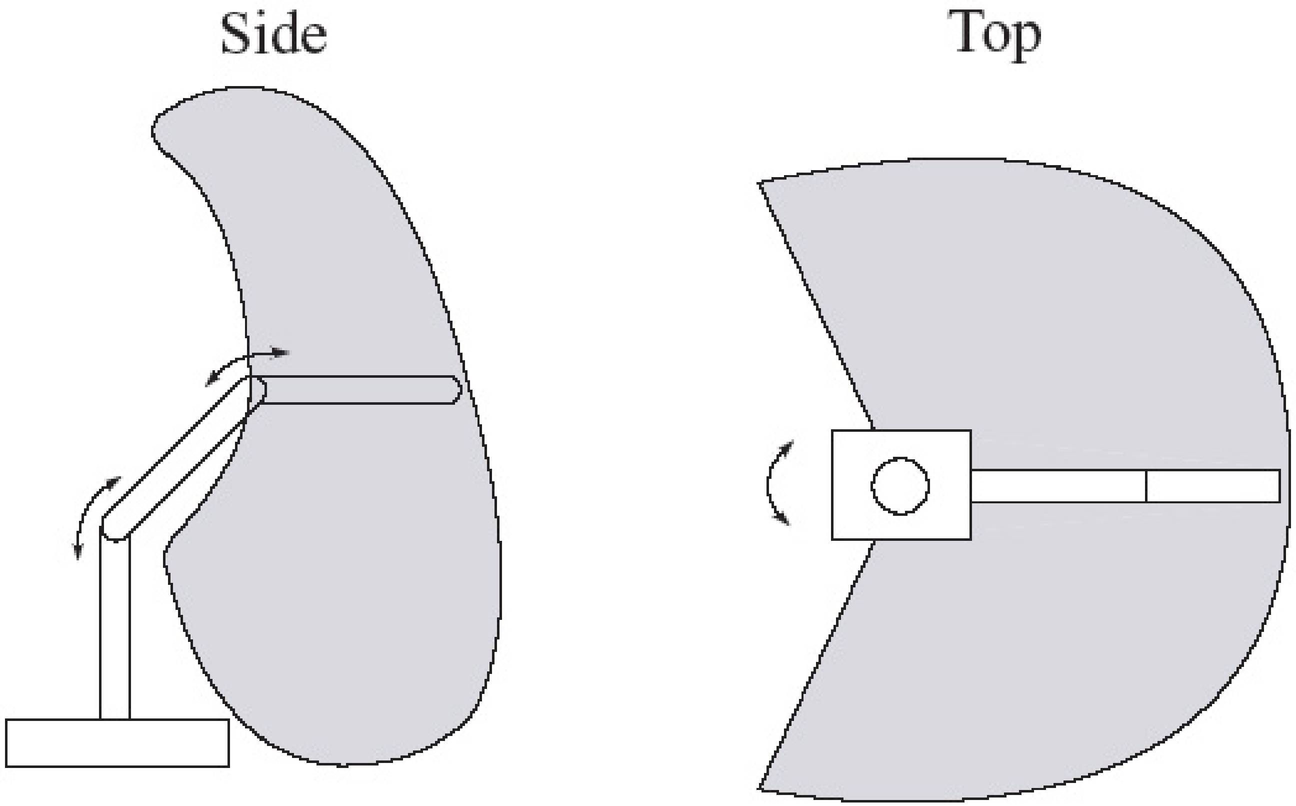


ABB IRB1400 Robot

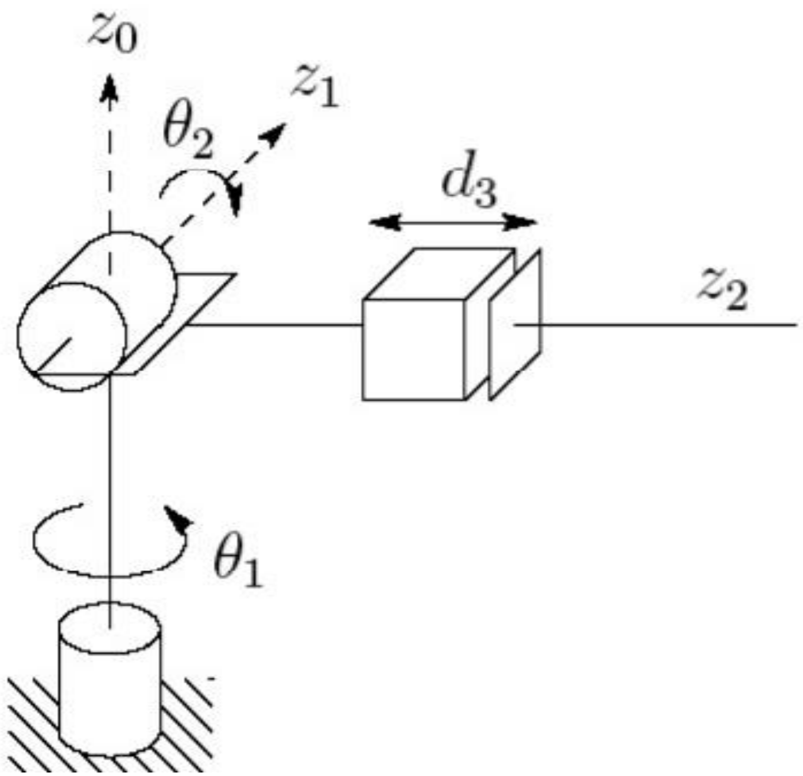
Workspace of the Articulated Manipulator (RRR)

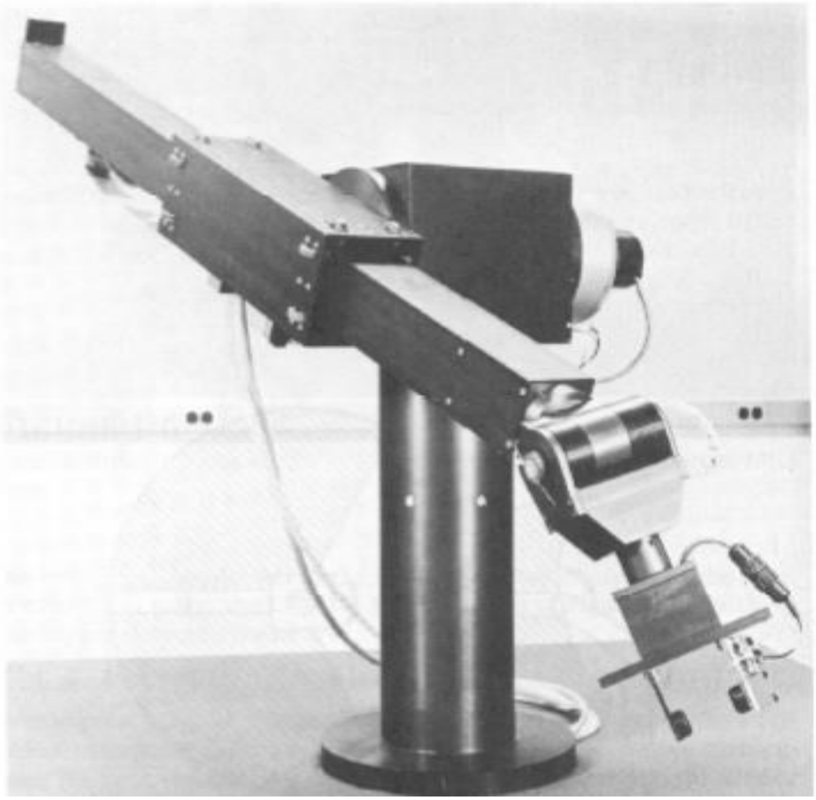


Provides a larger workspace than other kinematic designs

Spherical Manipulator (RRP)

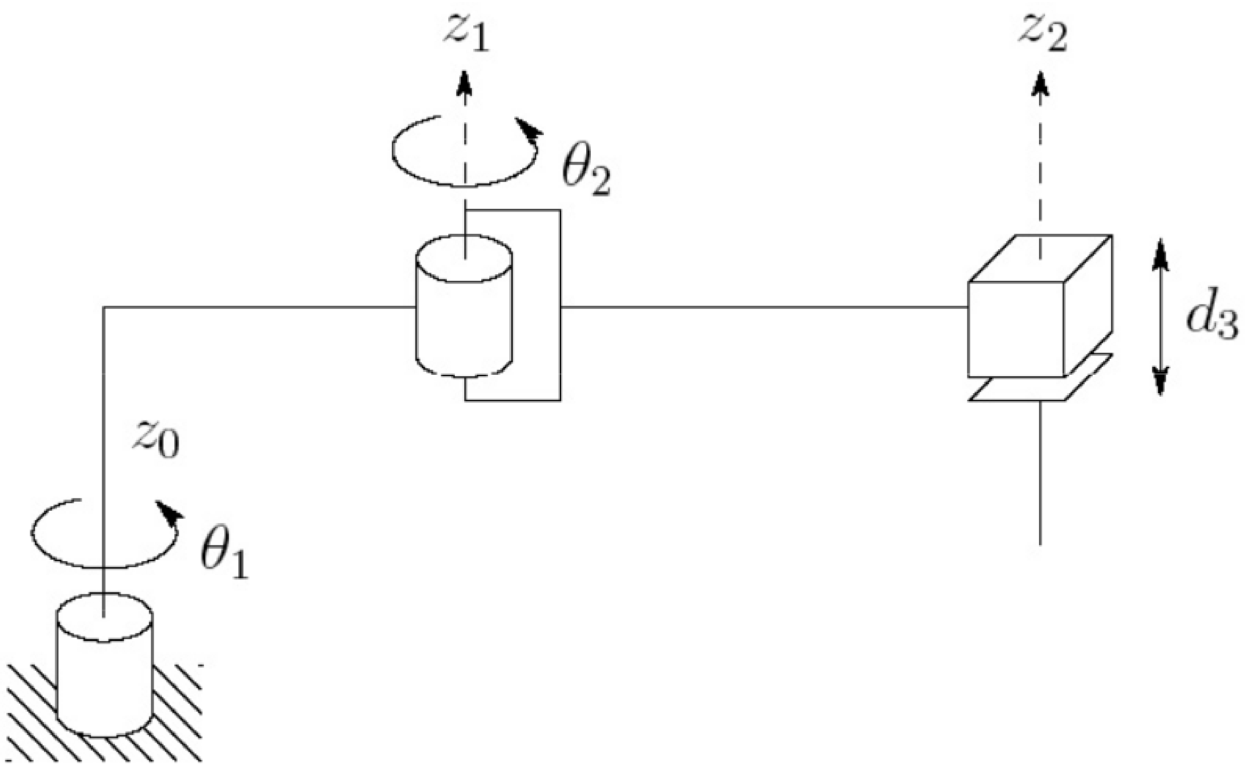
- ‘Spherical’ because the joint coordinates coincide with the spherical coordinates of the end-effector
- All three joint axes cross through a single point





SCARA Manipulator (RRP)

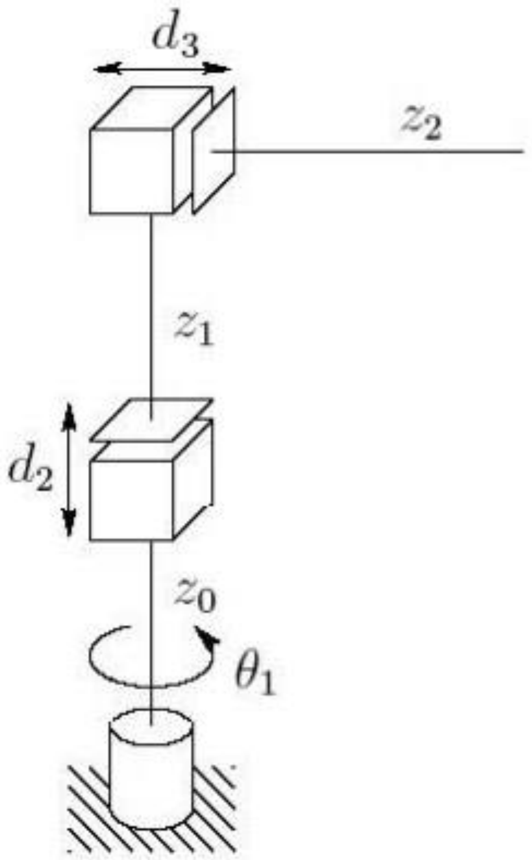
- SCARA: Selective Compliant Articulated Robot for Assembly
- $z_0 \parallel z_1 \parallel z_2$
- Ideal for table-top assembly, pick-and-place, packaging

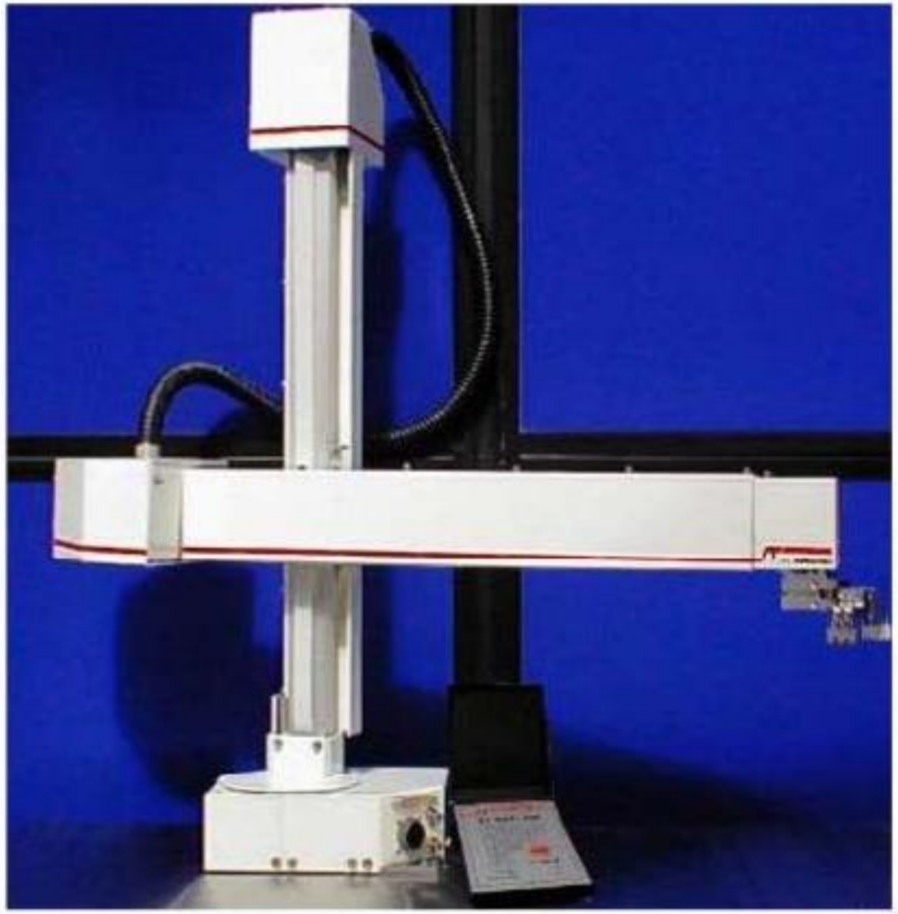


The Adept Cobra
Smart 600
SCARA arm

Cylindrical Manipulator (RPP)

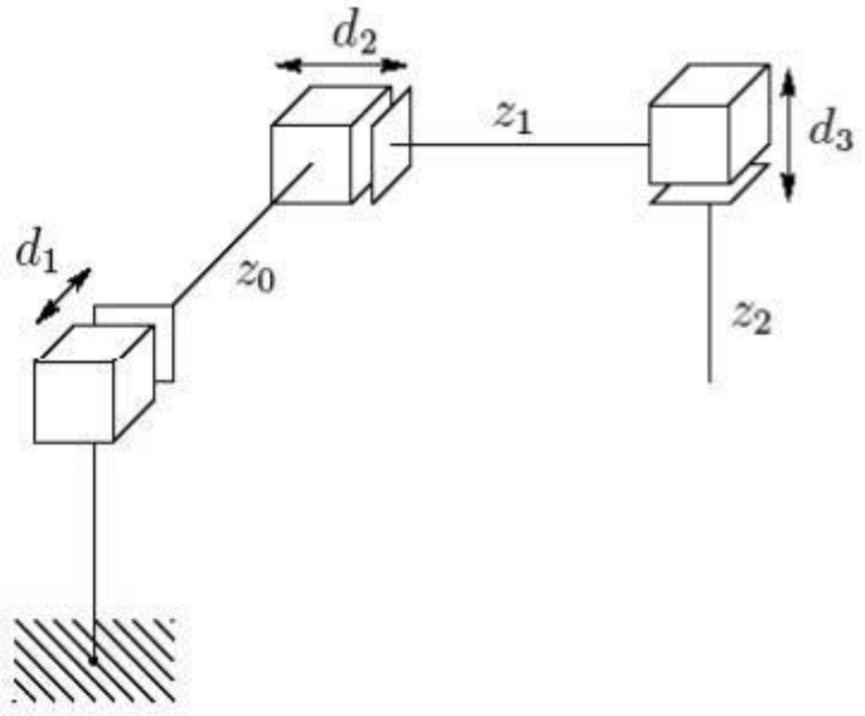
- Typically used in material transfer applications





Cartesian Manipulator (RPP)

- Simplest kinematic description
- Suitable for table-top assembly application, transfer of material or cargo



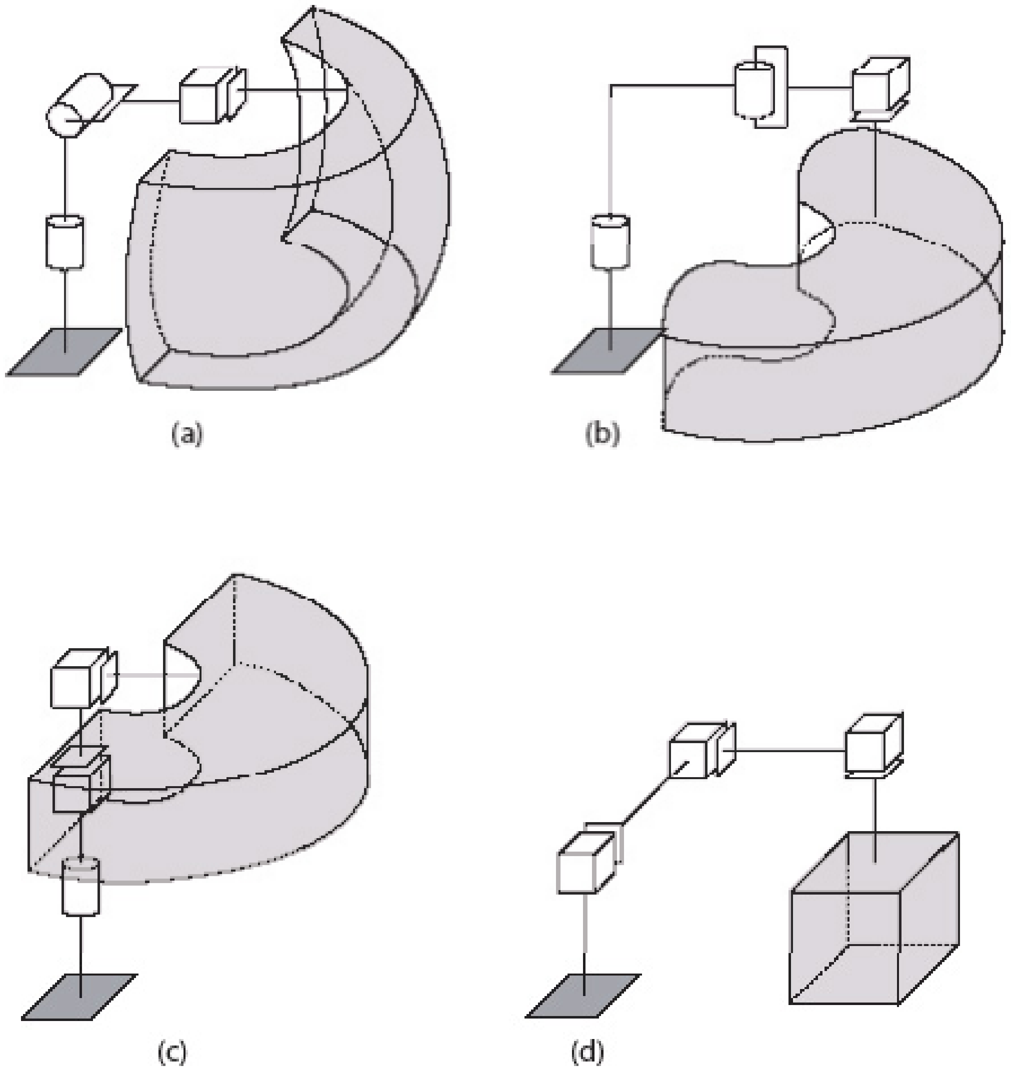
Workspace of Common Manipulators

Spherical robot

SCARA robot

Cylindrical robot

Cartesian robot



Parallel Manipulator

- A subset of the links form a closed chain
- Two or more kinematic chains connecting the base with the end-effector
- Greater rigidity and accuracy
- Fundamentally different to serial link robots

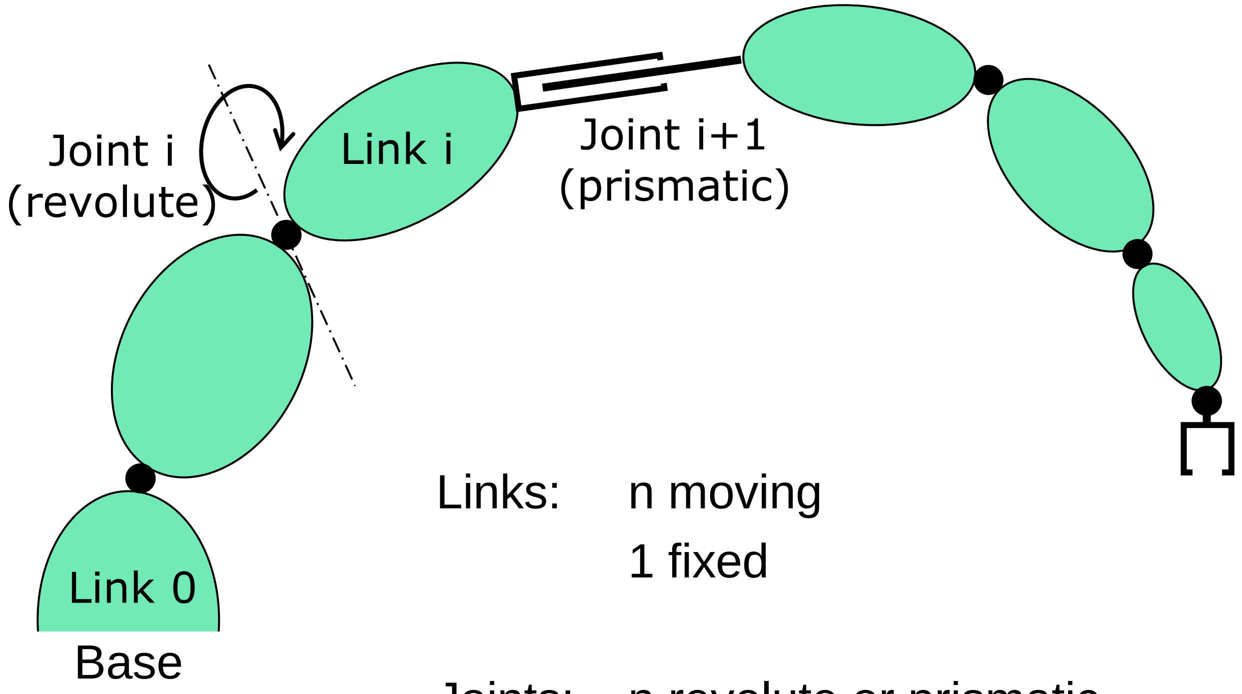
ABB IRB940 Tricept parallel robot

Kinematics

Kinematic Model of Robot Arm

- Serial link manipulator (a.k.a. robot arm, industrial robot)
  - An open chain of rigid bodies (links) connected by joints (revolute or prismatic)

- Manipulator specification
  - Degrees of freedom: n
  - Joint space
  - Work space
  - Redundancy: n > m



Links: n moving
1 fixed

Joints: n revolute or prismatic
1 dof each

Kinematics Notation and Terminology

- Definition: studying position and motion **without** considering **forces** (purely geometrical)
- Synthetic vs <u>analytic</u> approach
- Notation for representing position $p \equiv p^0 \equiv p^1$
- Notation for representing coordinate frame position o_1^0 versus o_0^1
- Notation for representing a free vector $v_a \equiv v_a^0 \equiv v_a^1$
- Rotation/Orientation in 2D/3D
- Position + Orientation = Placement

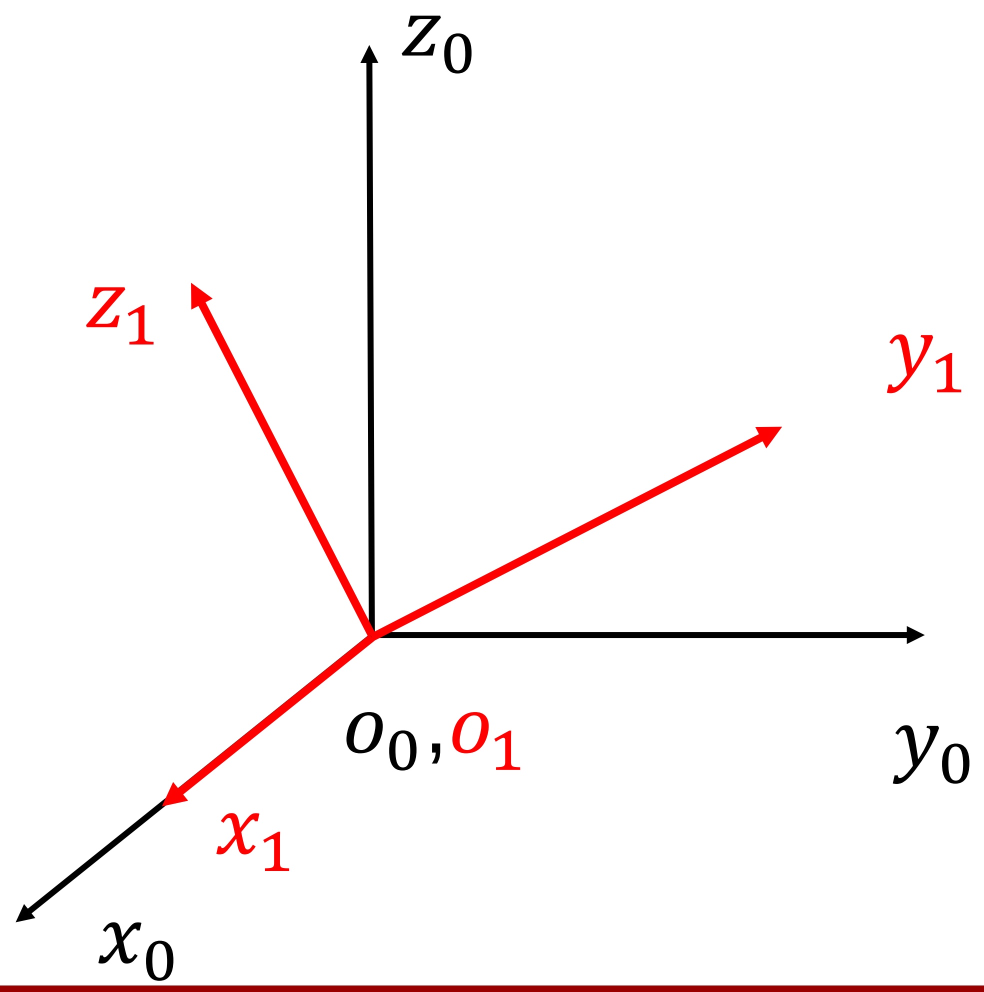
Rotation Matrix for Frame Orientation

- Basic rotation matrix between frame $o_0x_0y_0z_0$ and frame $o_1x_1y_1z_1$

$$R_1^0 = \begin{bmatrix} x_0 \cdot x_1 & x_0 \cdot y_1 & x_0 \cdot z_1 \\ y_0 \cdot x_1 & y_0 \cdot y_1 & y_0 \cdot z_1 \\ z_0 \cdot x_1 & z_0 \cdot y_1 & z_0 \cdot z_1 \end{bmatrix}$$

$$= \left[x_1^0 \,\middle|\, y_1^0 \,\middle|\, z_1^0 \right]$$

$$R_0^1 = (R_1^0)^{-1}$$



Basic Rotation Matrices

– About x-axis with θ

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix}$$

– About y-axis with θ

$$R_{y,\theta} = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}$$

– About z-axis with θ

$$R_{z,\theta} = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties:

$$R_{z,\theta} R_{z,\phi} = R_{z,\theta+\phi}$$

$$\left(R_{z,\theta}\right)^{-1} = R_{z,-\theta}$$

Rotation Matrix for Frame Transformation

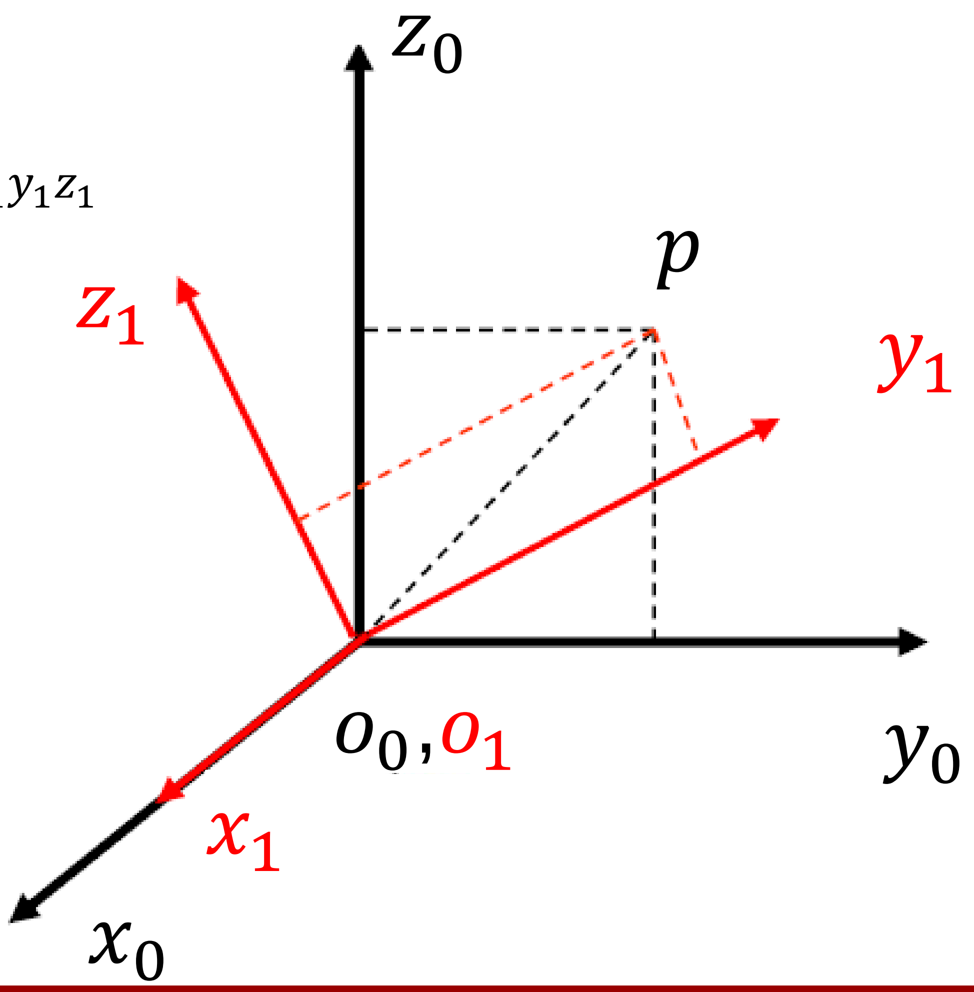
- Position vector

$$p \equiv p^0 \equiv p^1$$

- Basic rotation matrix between frame $o_0x_0y_0z_0$ and frame $o_1x_1y_1z_1$

$$p^0 = R_1^0 \, p^1$$

$$p^1 = (R_1^0)^{-1} \, p^0$$



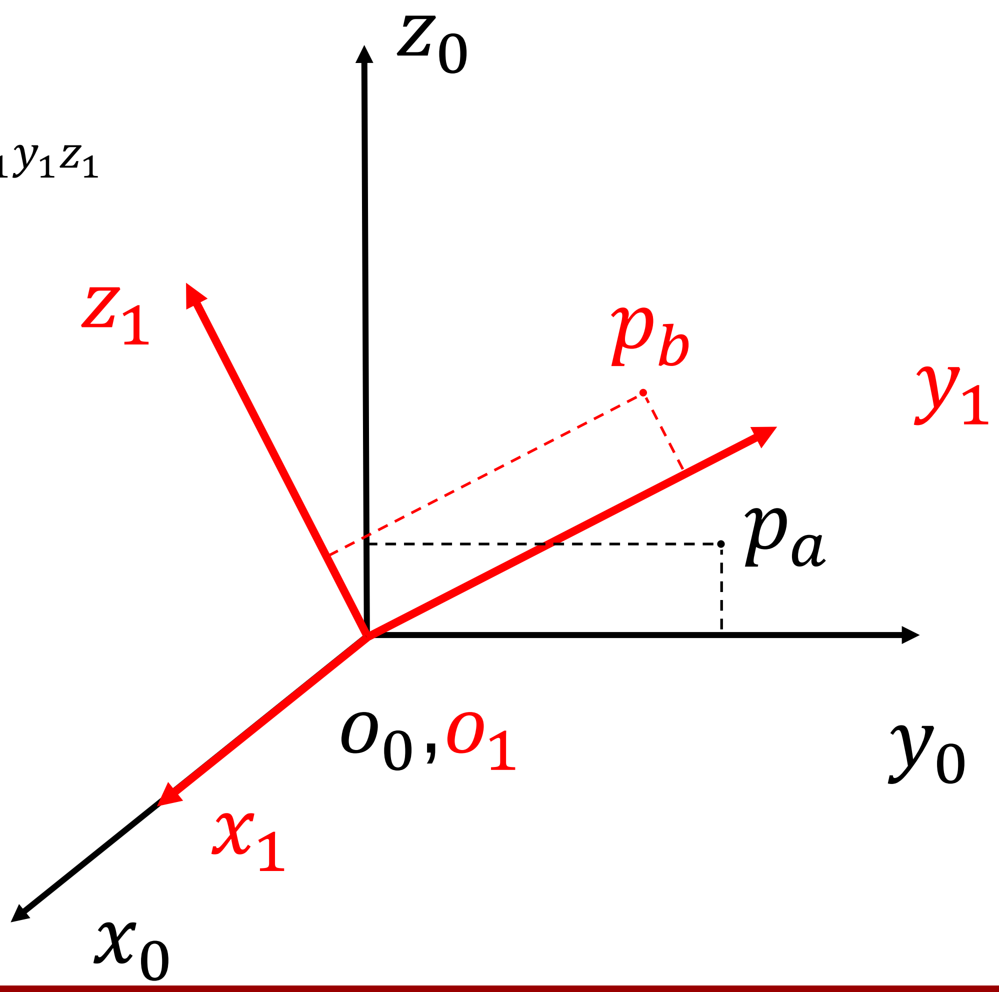
Rotation Matrix as Operator

- Position vector

$p_a^0 \rightarrow p_b^0$ $\qquad\qquad$ $p_a \not\equiv p_b$

- Basic rotation matrix between frame $o_0x_0y_0z_0$ and frame $o_1x_1y_1z_1$

$$p_b^0 = R_1^0 p_a^0$$



The Three Interpretations of a Rotation Matrix

- Describe the orientation of a frame 1 w.r.t. another frame 0

$$R_1^0 = \left[x_1^0 \,\middle|\, y_1^0 \,\middle|\, z_1^0\right]$$

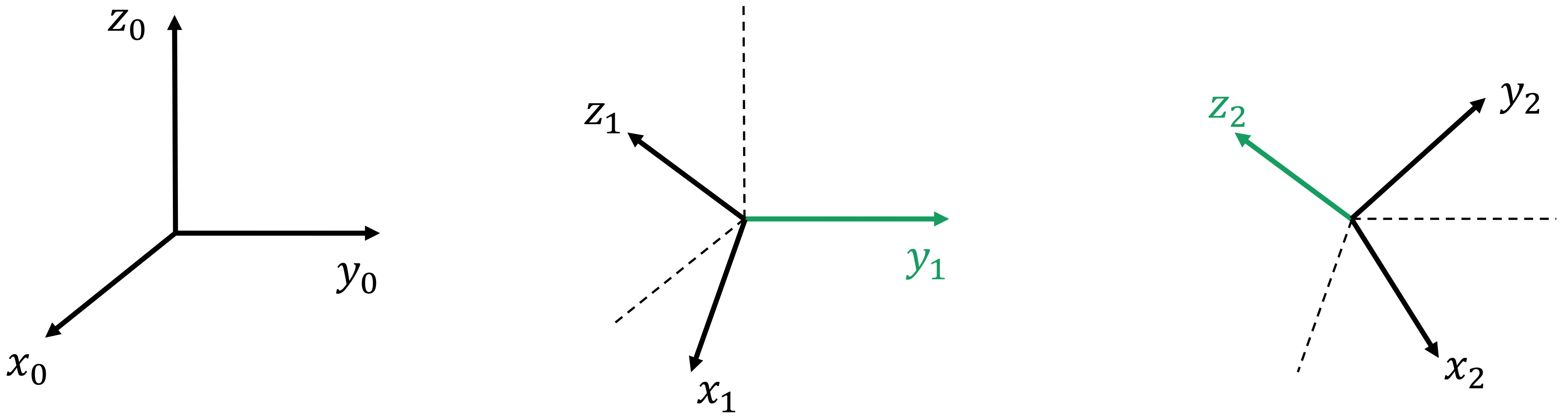
- Coordinates transformation for a fixed point from reference frame 1 to reference frame 0

$$p^0 = R_1^0 \, p^1$$

- Operator for rotating a point with the same rotation as the rotation between two frames

$$p_b^0 = R_1^0 p_a^0$$

Composition of Rotations (3D)



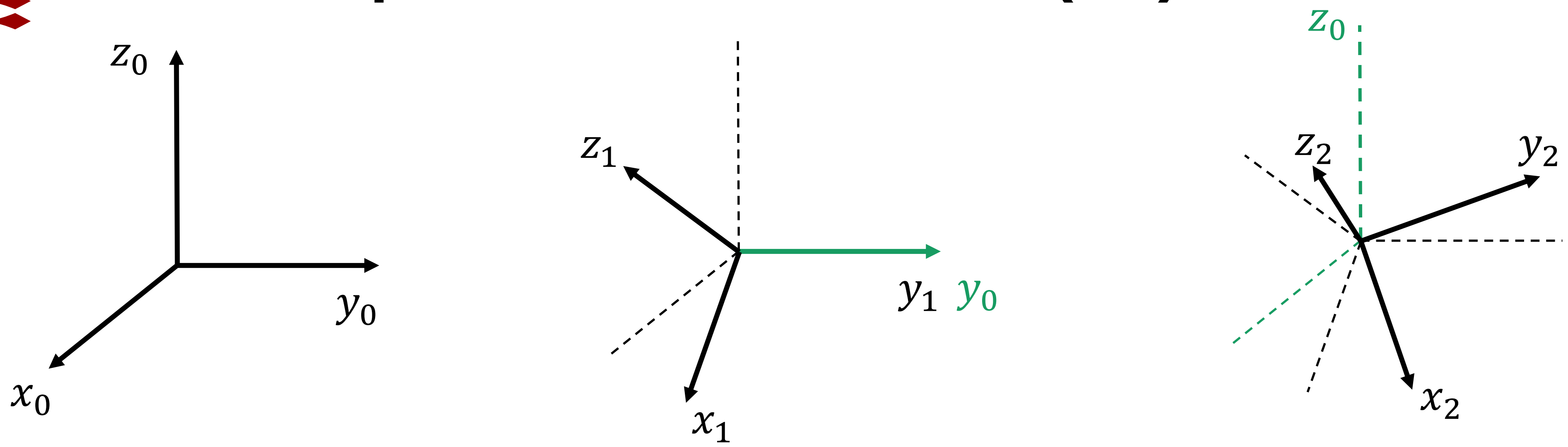
$$R_{y,45^\circ} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \qquad R_{z,45^\circ} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = R_2^0 = \left[x_2^0 \,\middle|\, y_2^0 \,\middle|\, z_2^0\right] = \left[R_1^0 x_2^1 \,\middle|\, R_1^0 y_2^1 \,\middle|\, R_1^0 z_2^1\right] = R_1^0\left[x_2^1 \,\middle|\, y_2^1 \,\middle|\, z_2^1\right] = R_1^0 R_2^1 = R_{y,45^\circ} R_{z,45^\circ}$$

$$\Rightarrow \; R = \begin{bmatrix} 1/2 & -1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/2 & 1/2 & 1/\sqrt{2} \end{bmatrix}$$

current axis → post-multiplication

Composition of Rotations (3D)



$$R_{y,45^\circ} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$R_{z,45^\circ} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R' = R_{z,45^\circ} R_{y,45^\circ}$$

$$\Rightarrow R' = \begin{bmatrix} 1/2 & -1/\sqrt{2} & 1/2 \\ 1/2 & 1/\sqrt{2} & 1/2 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

fixed axis → pre-multiplication

Parametrization of Rotations

- Express an arbitrary rotation matrix as a composition of 3 ”standard” rotations

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad \text{with} \quad \begin{array}{l} {r_{11}}^2 + {r_{21}}^2 + {r_{31}}^2 = 1 \\ {r_{12}}^2 + {r_{22}}^2 + {r_{32}}^2 = 1 \\ {r_{13}}^2 + {r_{23}}^2 + {r_{33}}^2 = 1 \\ r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} = 0 \\ r_{11}r_{13} + r_{21}r_{23} + r_{31}r_{33} = 0 \\ r_{13}r_{12} + r_{23}r_{22} + r_{33}r_{32} = 0 \end{array}$$

6 constraints $\rightarrow$ 3 independent parameters

- Euler angles (ZYZ current)

$$R = R_{z,\phi} R_{y,\theta} R_{z,\psi} \quad \longrightarrow \quad \phi, \theta, \psi$$

  - Use of atan2
  - Gimbal lock: infinitely many solutions for $\theta = 0$ or $\theta = \pi$

- Axis/angle

$$R = R_{k,\theta} \quad \longrightarrow \quad k, \theta$$

- Roll(ϕ)/Pitch(θ)/Yaw(ψ) (XYZ fixed)

$$R = R_{z,\phi} R_{y,\theta} R_{x,\psi} \quad \longrightarrow \quad \psi, \theta, \phi$$

Parametrization of Rotations

- Euler angles (ZYZ current)
  - Use of Atan2
  - Gimbal lock: infinitely many solutions for $\theta = 0$ or $\theta = \pi$

$$R = R_{z,\phi} R_{y,\theta} R_{z,\psi} \quad \longrightarrow \quad \phi, \theta, \psi$$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

$$\theta = Atan2(c_\theta, s_\theta) \qquad \theta = Atan2\left(r_{33}, \sqrt{1 - r_{33}^2}\right) \quad \text{or} \quad \theta = Atan2\left(r_{33}, -\sqrt{1 - r_{33}^2}\right)$$

$$\phi = \mathrm{Atan2}(c_\phi, s_\phi) \qquad \phi = \mathrm{Atan2}(r_{13}, r_{23}) \qquad \phi = \mathrm{Atan2}(-r_{13}, -r_{23})$$

$$\psi = \mathrm{Atan2}(c_\psi, s_\psi) \qquad \psi = \mathrm{Atan2}(-r_{31}, r_{32}) \qquad \psi = \mathrm{Atan2}(r_{31}, -r_{32})$$

In the textbook:

$$\mathrm{Atan2}(x, y) = \mathrm{Atan}(y/x)$$

In the Matlab:

atan2

Four-quadrant inverse tangent

Syntax

```
P = atan2(Y,X)
```

Homogeneous Transformations

Homogeneous Transformation Matrix

- Coordinate transformation from {1} to {0}

$$p^0 = R_1^0\, p^1 + o_1^0$$

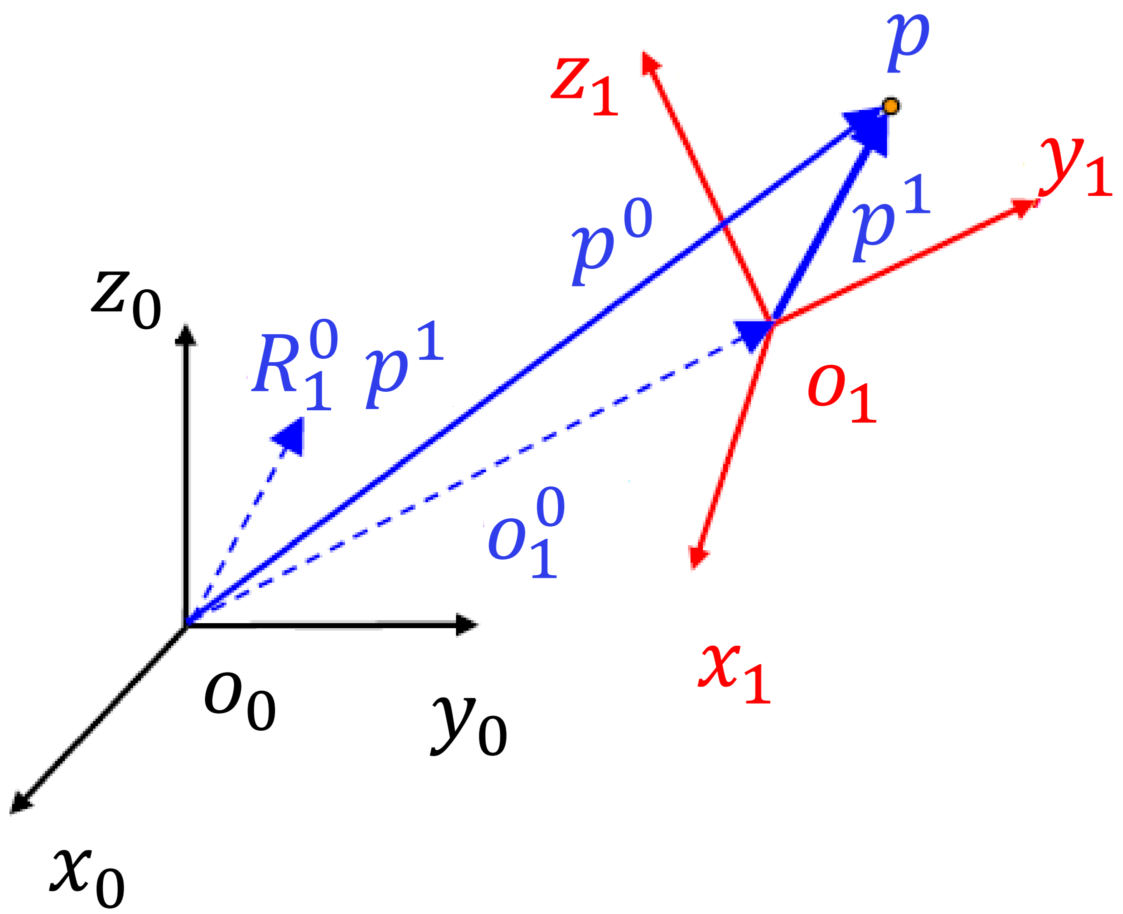
$$\begin{bmatrix} p^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 & o_1^0 \\ 0_{1x3} & 1 \end{bmatrix} \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$

- Homogeneous transformation matrix

rotation matrix

position vector

$$H_1^0 = \begin{bmatrix} R_1^0 & o_1^0 \\ 0_{1x3} & 1 \end{bmatrix}$$



Homogeneous Transformation Matrix

- Special cases
  - Translation

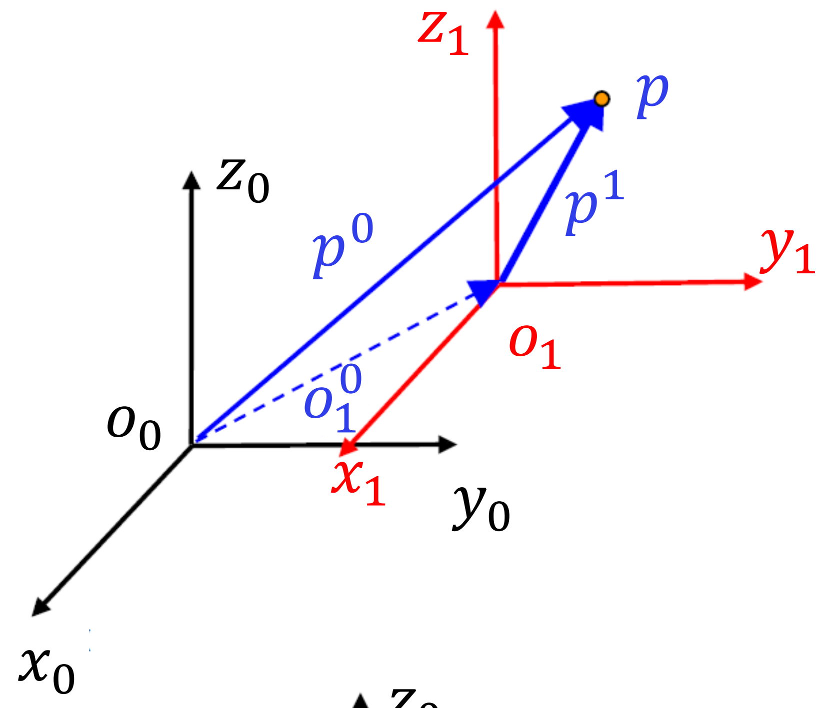
$$H_1^0 = \begin{bmatrix} I_{3\times3} & o_1^0 \\ 0_{1\times3} & 1 \end{bmatrix}$$

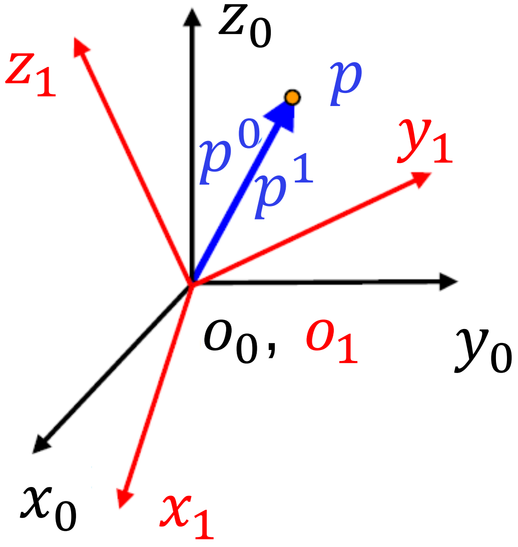
  - Rotation

$$H_1^0 = \begin{bmatrix} R_1^0 & 0_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix}$$

- General case

$$H_1^0 = \begin{bmatrix} x_1^0 & y_1^0 & z_1^0 & o_1^0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



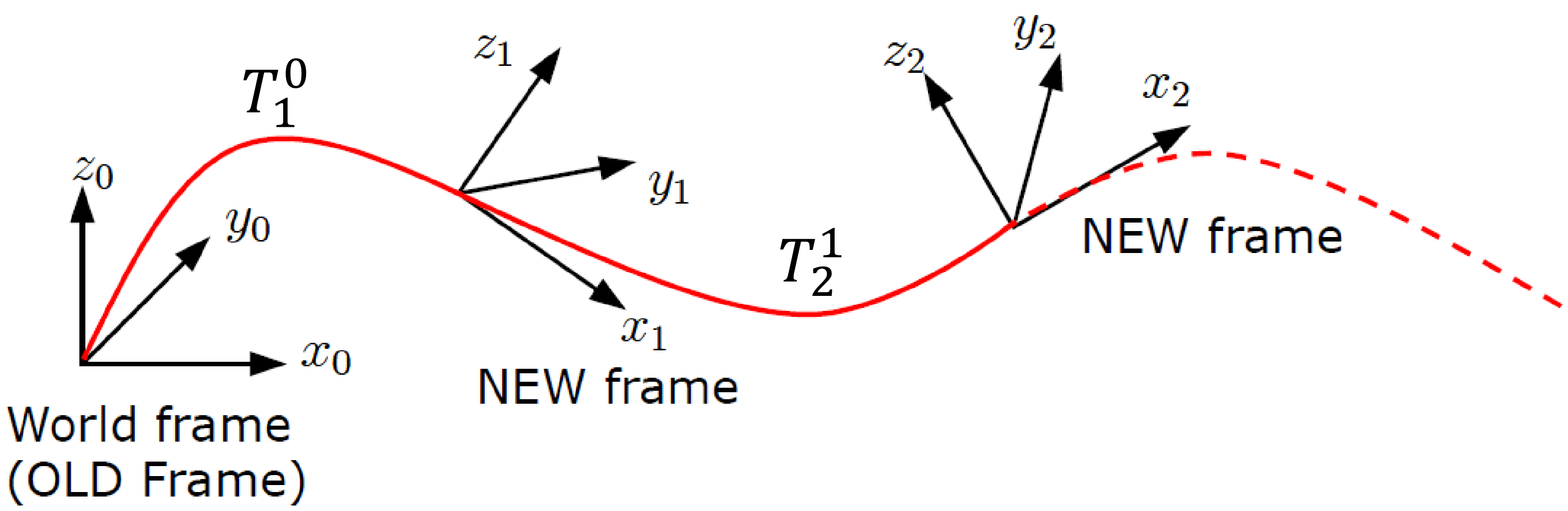


Composition of Transformation Matrices

- Composite Homogeneous Transformation Matrix

  - Rotation/Translation wrt. original/fixed/global/world frame → **Pre-Multiplication**

  - Rotation/Translation wrt. current frame → **Post-Multiplication**



Basic Homogeneous Transformation Matrices

$$\mathrm{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{z,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercises

Problem 1

What is the rotation matrix for a rotation of 30° about the world z-axis, followed by a rotation of 60° about the world x-axis, followed by a rotation of 90° about the world y-axis?

Problem 2

What is the rotation matrix for a rotation ϕ about the world x-axis, followed by a rotation ψ about the current z-axis, followed by a rotation θ about the world y-axis?

Problem 3

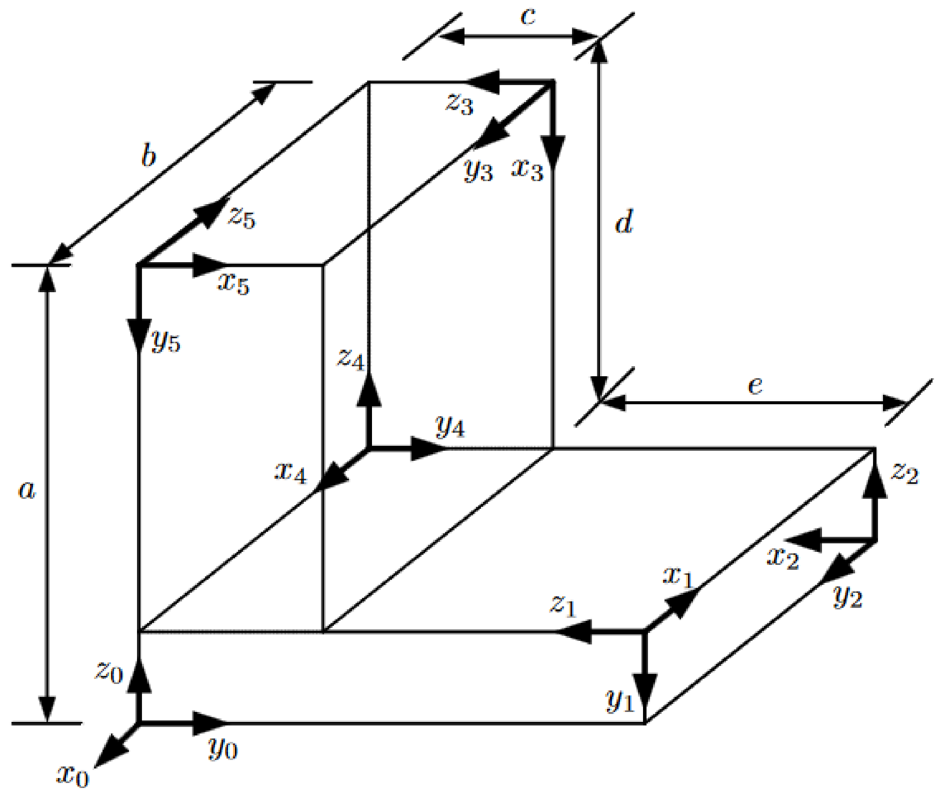
Find another sequence of rotations that is different from Prob. 2, but which results in the same rotation matrix.

Problem 4

Determine a homogeneous transformation matrix H that represents a rotation with an angle α about the world x-axis, followed by a translation with a length b along the world z-axis, followed by a rotation ϕ about the current y-axis.

Problem 5

For the figure shown below, find the 4×4 homogeneous transformation matrices A_1^0, A_2^1, A_3^2, A_4^3, and A_5^4, as well as A_5^0.

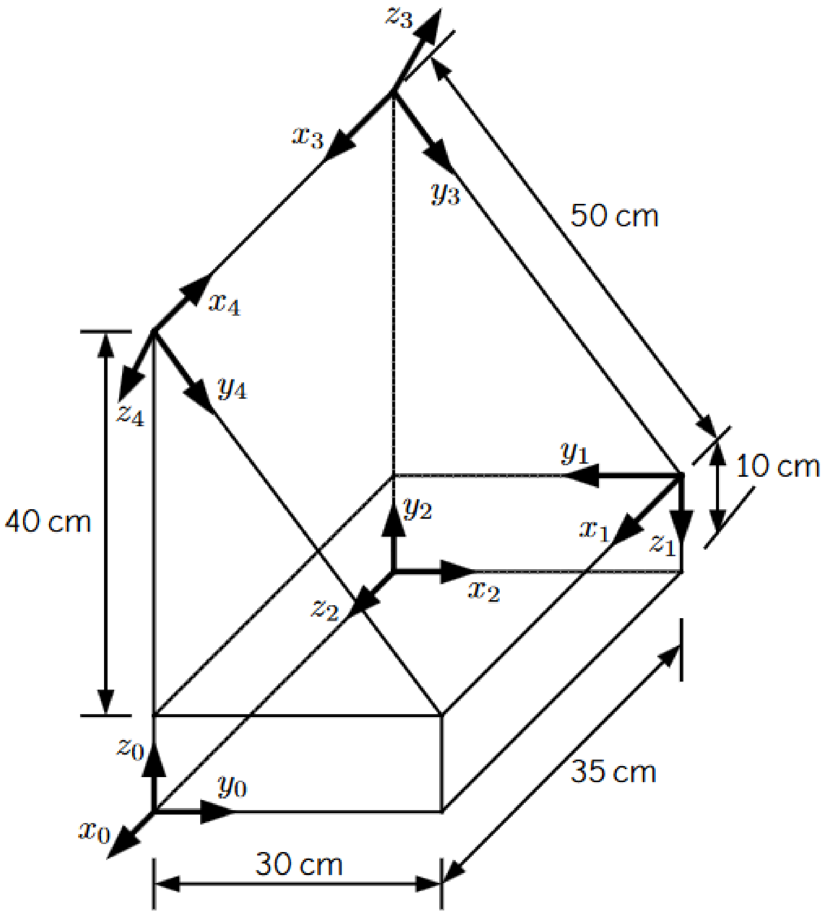


Hint: you can find the answers directly by observation, based on the geometric interpretation of each column in the homogeneous transformation matrix.

Exercises

Problem 6

For the figure shown below, find the 4×4 homogeneous transformation matrices A_1^0, A_2^1, A_3^2, and A_4^3, as well as A_4^0.



Hint: you can find the answers directly by observation, based on the geometric interpretation of each column in the homogeneous transformation matrix.

Robotics – 34753

Robot Kinematics II & Inverse Kinematics

Konstantinos Poulios
Associate Professor
Department of Civil and Mechanical Engineering
DTU Lyngby, building 404 / room 124

Robot Kinematics II & Inverse Kinematics – Lecture Overview

1. **Repetition**

2. **Kinematic Chains**

    - **Denavit-Hartenberg Convention (D-H)**
    - Coordinate Frame Assignment
    - End Effector Frame
    - Examples

3. **Inverse Kinematics**

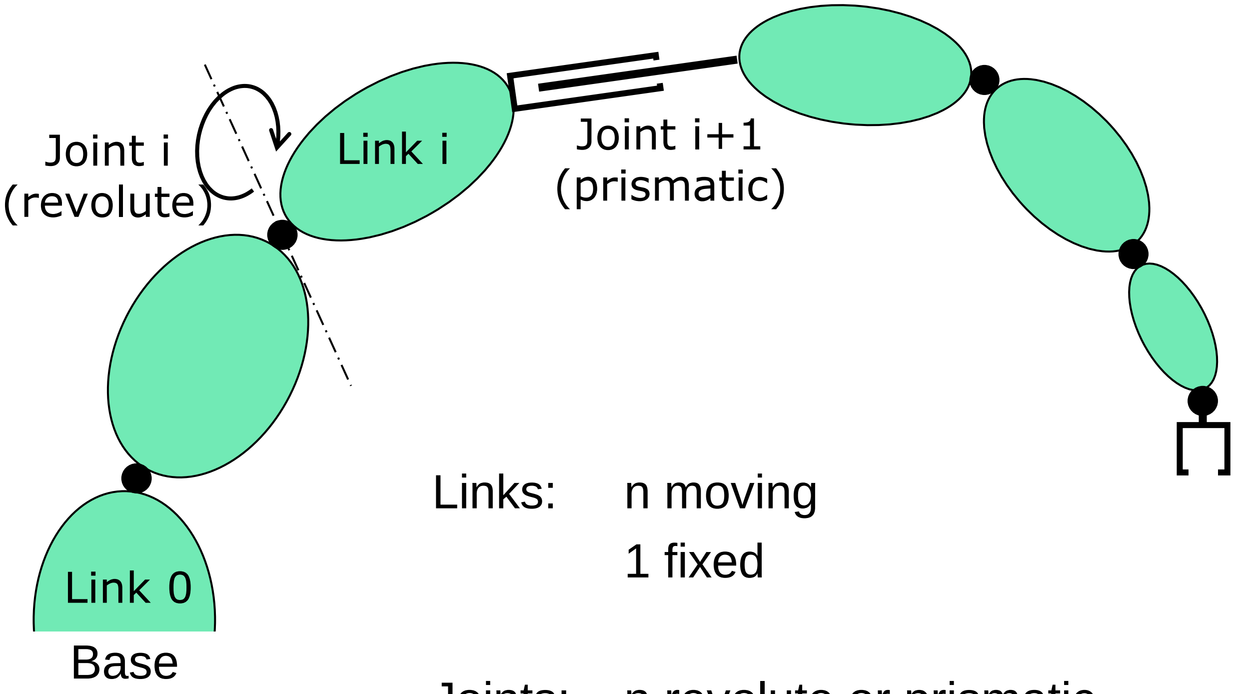
    - Inverse Position
    - Inverse Orientation

Repetition

Repetition

- Serial link manipulator (a.k.a. robot arm, industrial robot)
  - An open chain of rigid bodies (links) connected by joints (revolute or prismatic)

- Manipulator specification
  - Degrees of freedom: n
  - Joint space
  - Work space
  - Redundancy: n > m



Links: n moving
1 fixed

Joints: n revolute or prismatic
1 dof each

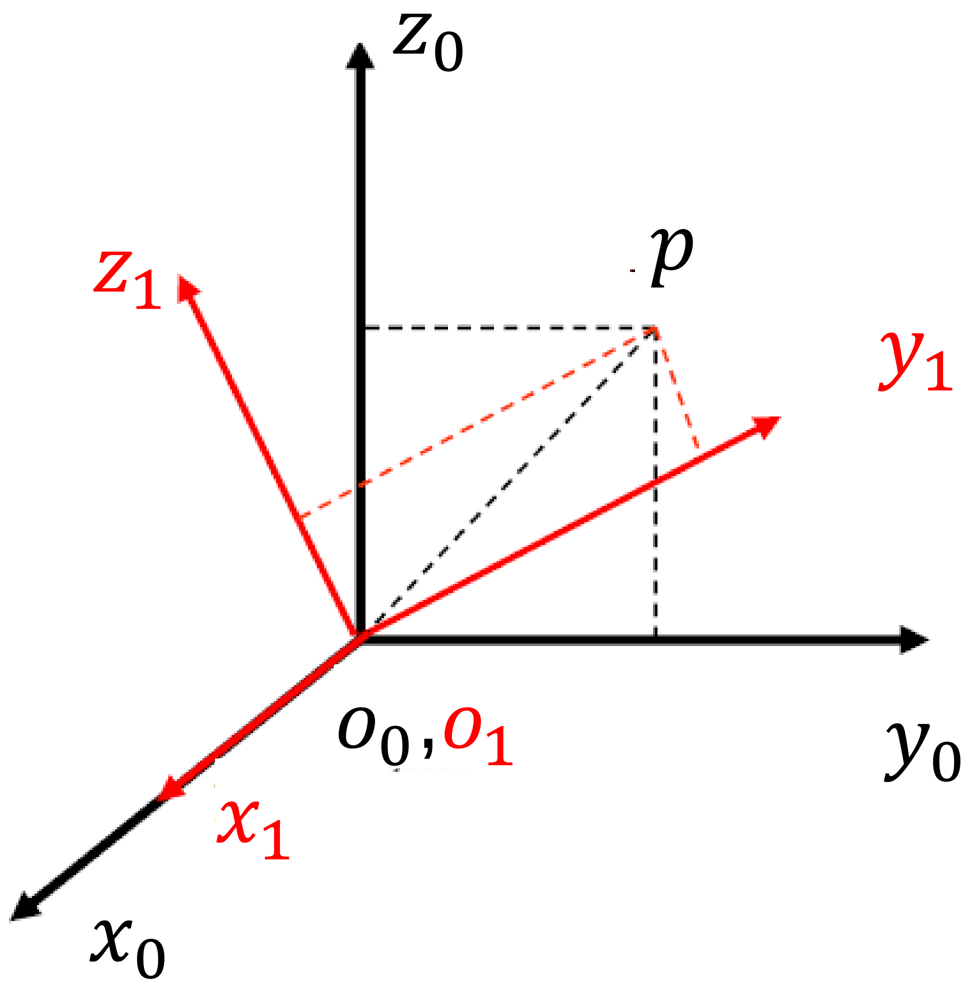
Repetition

- Basic rotation matrix between frame $o_0x_0y_0z_0$ and frame $o_1x_1y_1z_1$

$$p^0 = \begin{bmatrix} x_0 \cdot x_1 & x_0 \cdot y_1 & x_0 \cdot z_1 \\ y_0 \cdot x_1 & y \cdot y_1 & y_0 \cdot z_1 \\ z_0 \cdot x_1 & z_0 \cdot y_1 & z_0 \cdot z_1 \end{bmatrix} p^1$$

$$p^0 = R_1^0\, p^1$$

$$p^1 = (R_1^0)^{-1}\, p^0$$



Repetition

- Basic rotation matrices
  - About x-axis with θ

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix}$$

  - About y-axis with θ

$$R_{y,\theta} = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}$$

  - About z-axis with θ

$$R_{z,\theta} = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Repetition

- Coordinate transformation from {1} to {0}

$$p^0 = R_1^0\, p^1 + o_1^0$$

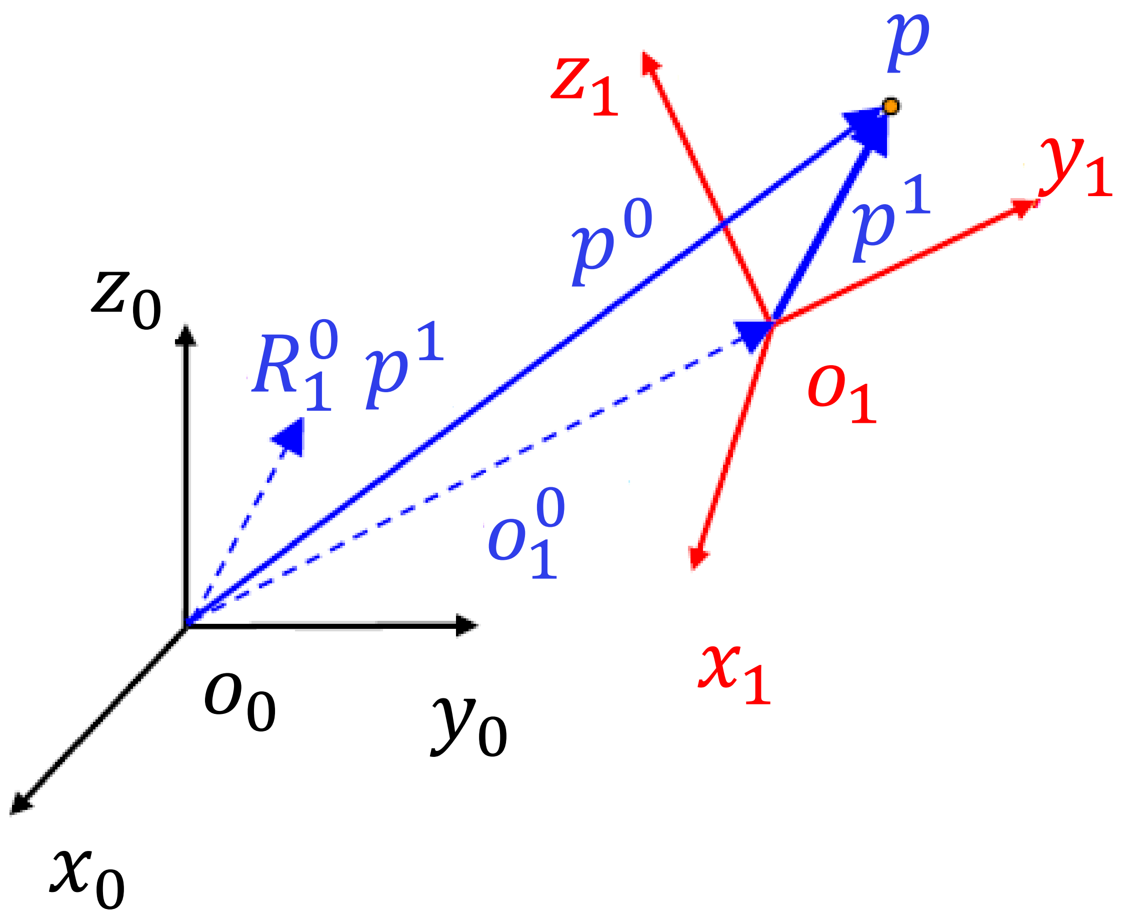
$$\begin{bmatrix} p^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 & o_1^0 \\ 0_{1x3} & 1 \end{bmatrix} \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$

- Homogeneous transformation matrix

rotation matrix

position vector

$$T_1^0 = \begin{bmatrix} R_1^0 & o_1^0 \\ 0_{1x3} & 1 \end{bmatrix}$$



Repetition

- Special cases

  – Translation

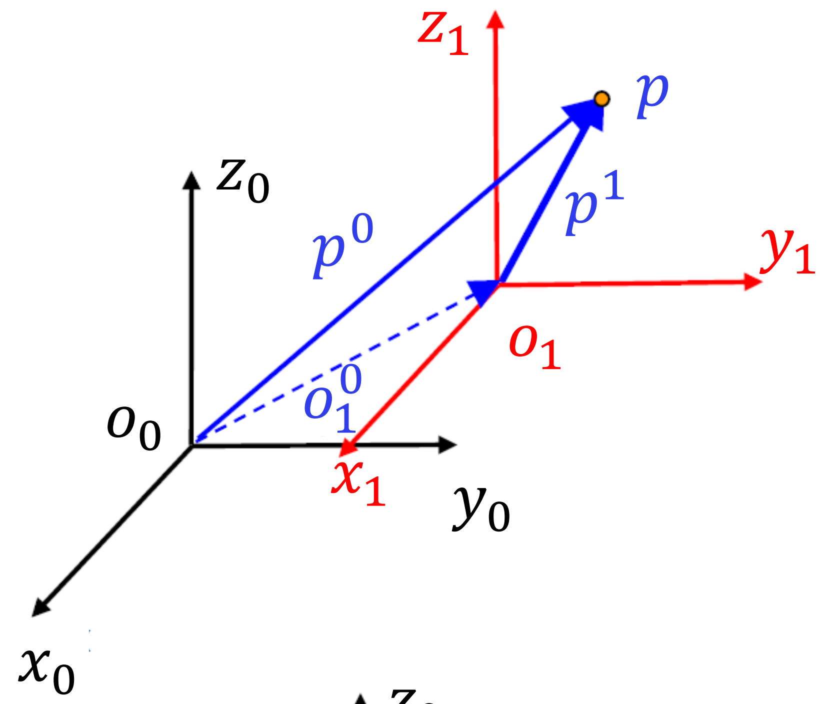
$$H_1^0 = \begin{bmatrix} I_{3\times3} & o_1^0 \\ 0_{1\times3} & 1 \end{bmatrix}$$

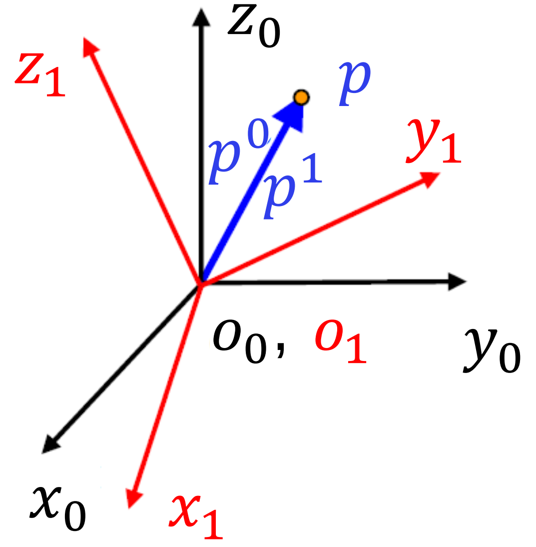
  – Rotation

$$H_1^0 = \begin{bmatrix} R_1^0 & 0_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix}$$

- General case

$$H_1^0 = \begin{bmatrix} x_1^0 & y_1^0 & z_1^0 & o_1^0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





Repetition

- Basic homogeneous transformation matrices

$$\mathrm{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{z,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Repetition

- Composition of Transformations

  - Rotation/Translation wrt. original/fixed/global/world frame → **Pre-Multiplication**

  - Rotation/Translation wrt. current frame → **Post-Multiplication**

- Example with rotations

  1. θ-rotation about **current** x
  2. ϕ-rotation about **current** z
  3. α-rotation about **fixed** z
  4. β-rotation about **current** y
  5. δ-rotation about **fixed** x

$$R_5^0 = \underset{5}{R_{x,\delta}}\,\underset{3}{R_{z,\alpha}}\,\underset{1}{R_{x,\theta}}\,\underset{2}{R_{z,\phi}}\,\underset{4}{R_{y,\beta}}$$

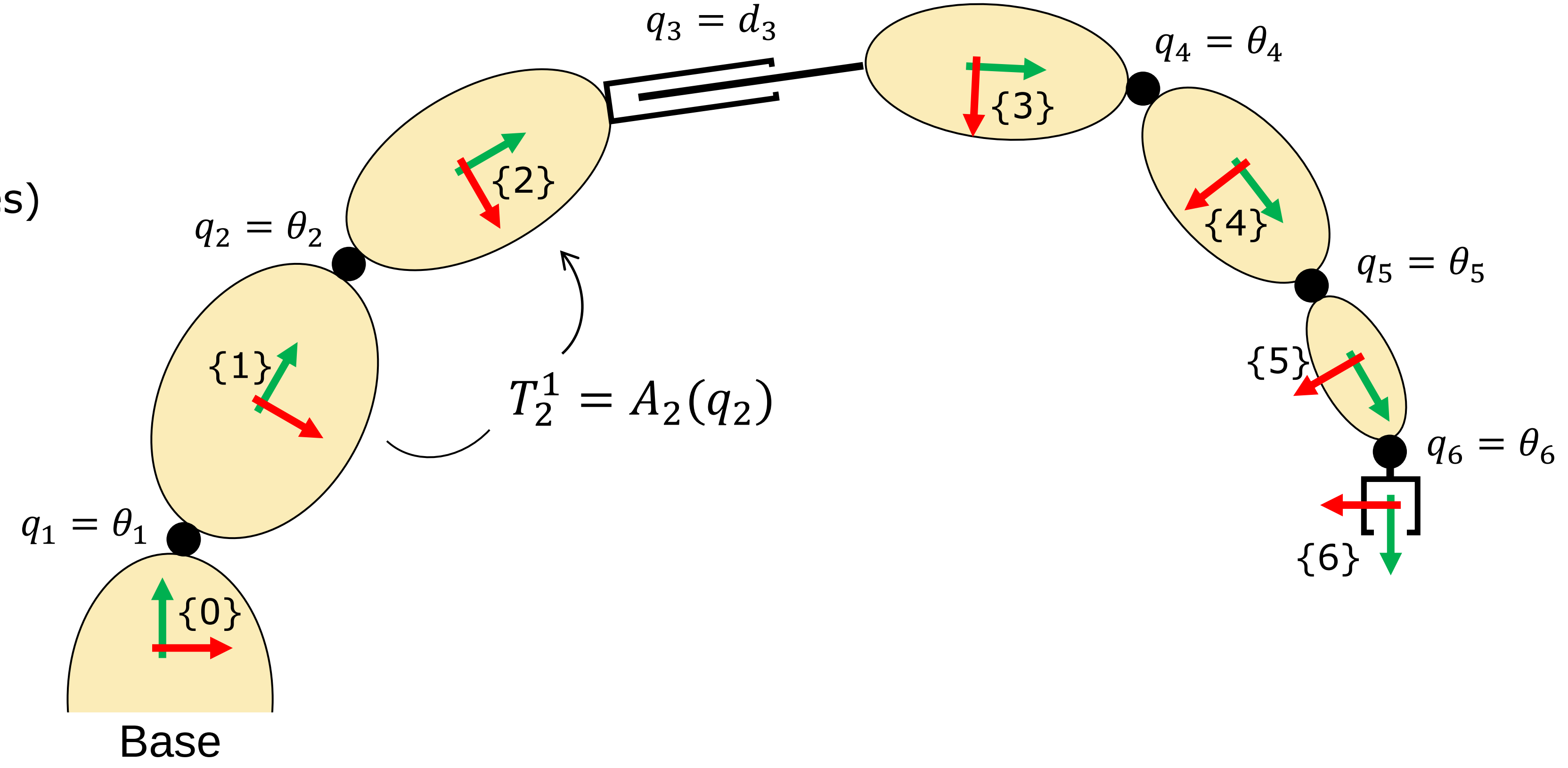
Kinematic Chains

Kinematic Chains

$n = 6$ joints/dofs

$n + 1 = 7$ links (bodies)

$n + 1 = 7$ frames



Note:
$A_i = T_i^{i-1}$

$$T_6^0 = A_1(q_1)\ A_2(q_2)\ A_3(q_3)\ A_4(q_4)\ A_5(q_5)\ A_6(q_6)$$

… but each homogeneous transformation matrix $A_j(q_j)$ is a rather complex function

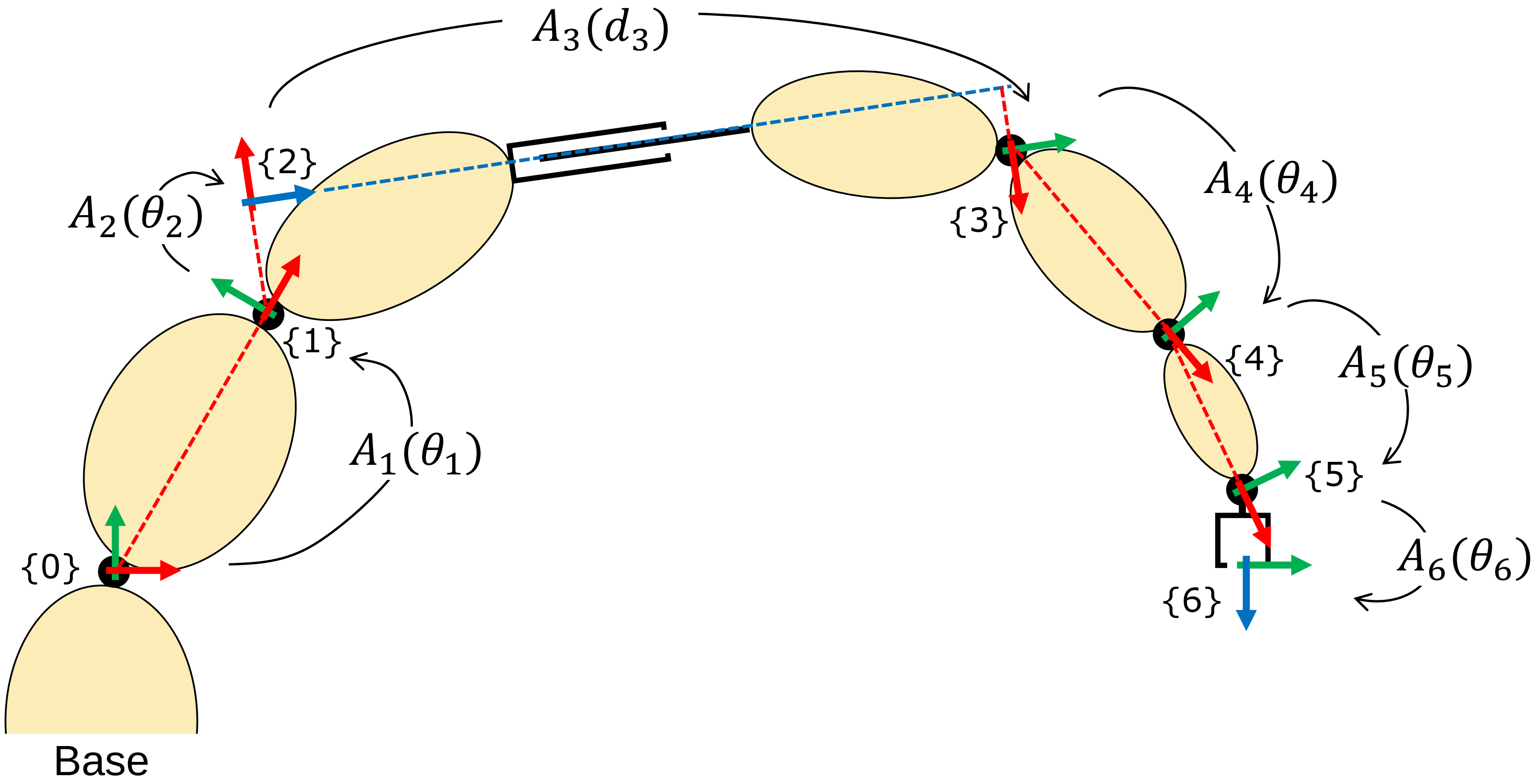
Kinematic Chains

$n = 6$ joints/dofs

$n + 1 = 7$ links (bodies)

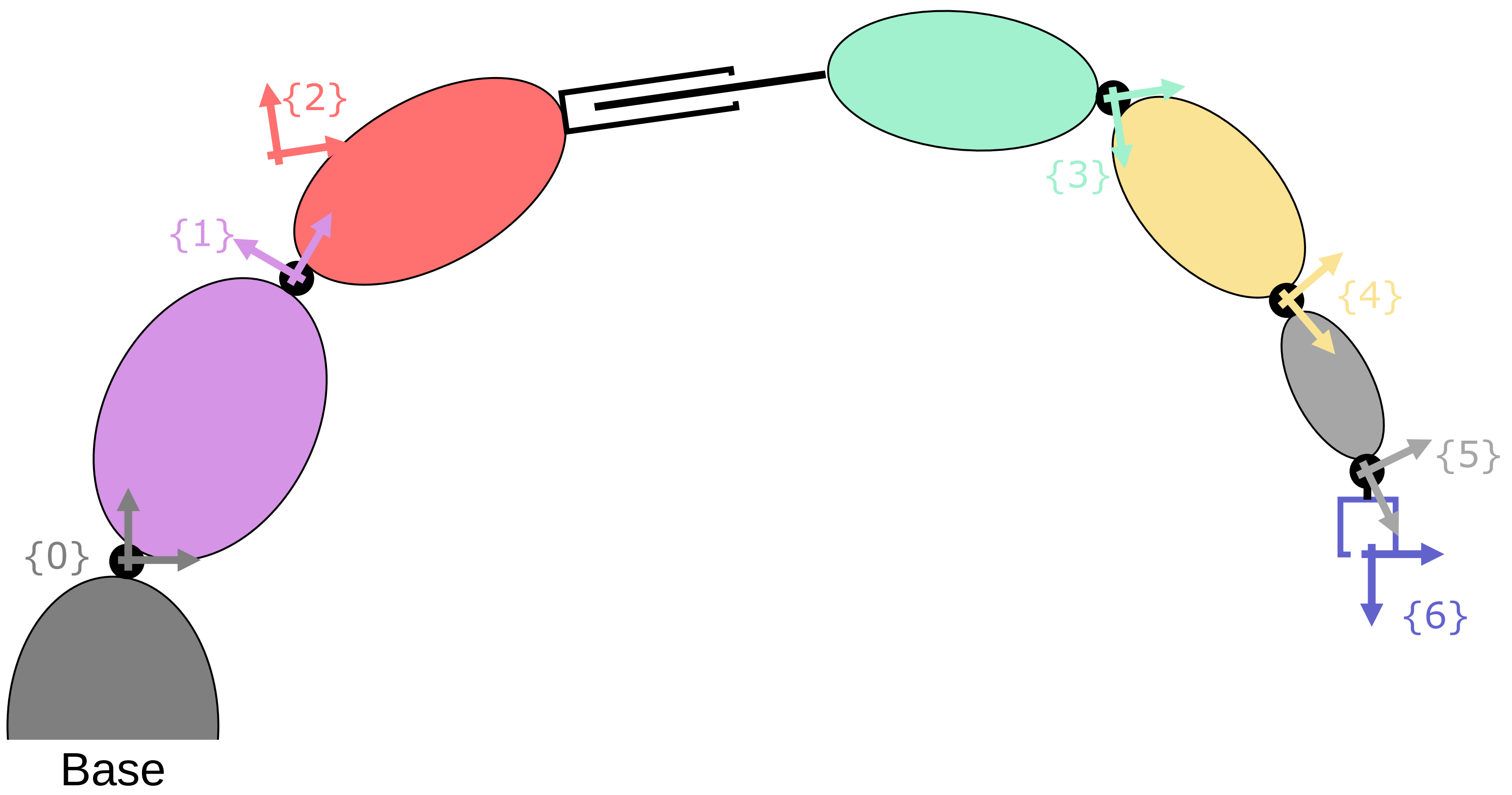
$n + 1 = 6 + 1$ frames

Last frame is special



$$T_6^0 = A_1(\theta_1)\ A_2(\theta_2)\ A_3(d_3)\ A_4(\theta_4)\ A_5(\theta_5)\ A_6(\theta_6)$$

Kinematic Chains



Denavit–Hartenberg Convention

- The coordinate frame {i} is determined from the frame {i-1} through a homogeneous transformation in the form:

$$A_i = Rot_{z,\theta_i}\, Trans_{z,d_i}\, Trans_{x,a_i}\, Rot_{x,\alpha_i}$$

$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Where:
  - θ_i : joint angle
  - d_i : link offset
  - a_i : link length
  - α_i : link twist

Only 4 parameters instead of 6.

Why?

Denavit–Hartenberg Convention

- The coordinate frame {i} is determined from the frame {i-1} through a homogeneous transformation in the form:

$$A_i = Rot_{z,\theta_i}\, Trans_{z,d_i}\, Trans_{x,a_i}\, Rot_{x,\alpha_i}$$

- Where: θ_i : joint angle
  d_i : link offset
  a_i : link length
  α_i : link twist

Only 4 parameters instead of 6.

Why?

- New coordinate frame with z-axis aligned with the **next** joint axis
- Position along and rotation around the axis **not free** to choose anymore

Denavit–Hartenberg Convention

- Where:

θ_i	: joint angle
d_i	: link offset
a_i	: link length
α_i	: link twist

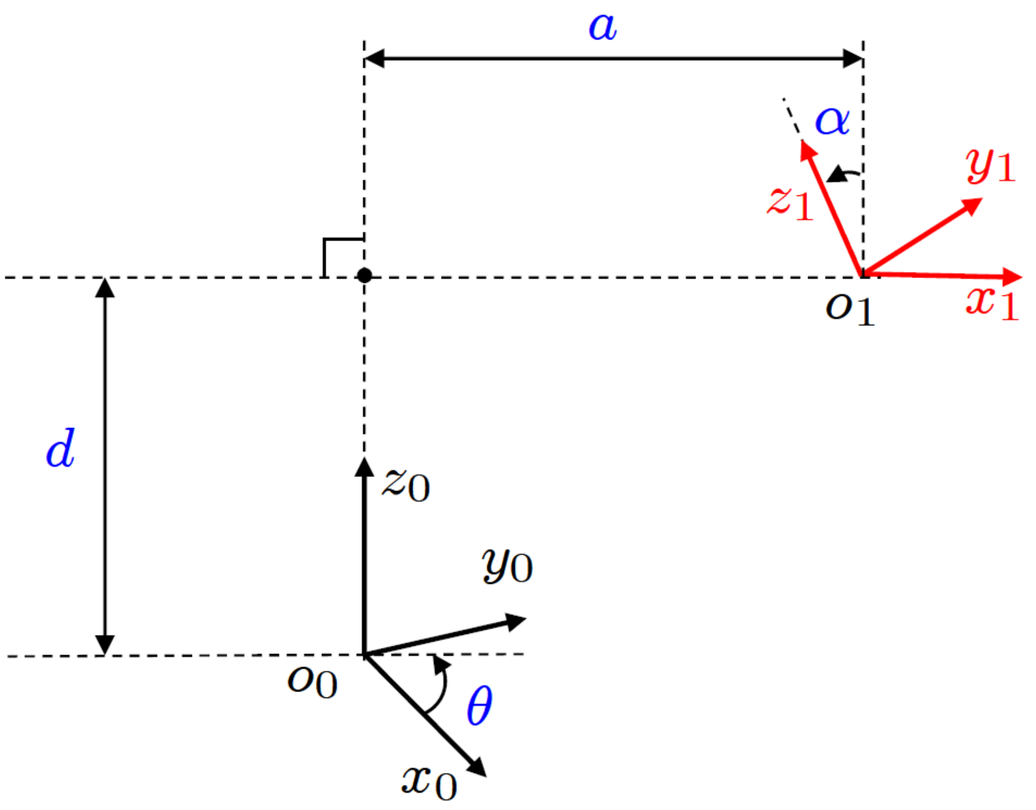
Only 4 parameters instead of 6

Why?

- The missing two parameters are due to the two Denavit–Hartenberg conditions:

DH1: The axis x_i is perpendicular to the axis z_{i-1}

DH2: The axis x_i intersects the axis z_{i-1}



Physical Interpretation of θ, d, a, α

θ_1 : angle from x_0 to x_1 about z_0

d_1 : distance from o_0 to x_1 (along z_0)

a_1 : distance from z_0 to o_1 (along x_1)

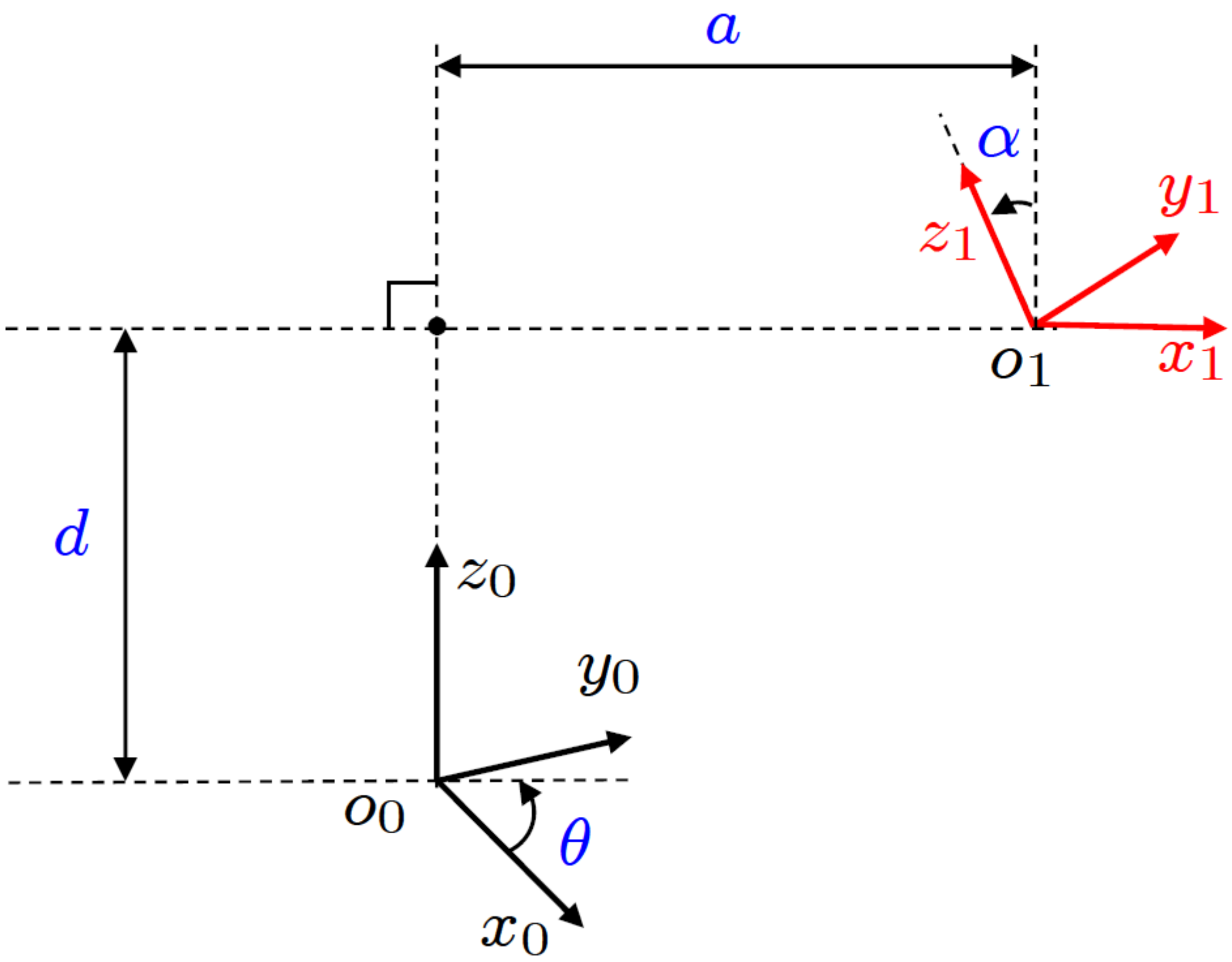
α_1 : angle from z_0 to z_1 about x_1

Mnemonic:

θ - d - a - α

about-along - along-about

z_{i-1} x_i



Physical Interpretation of θ, d, a, α

θ_1 : angle from x_0 to x_1 about z_0

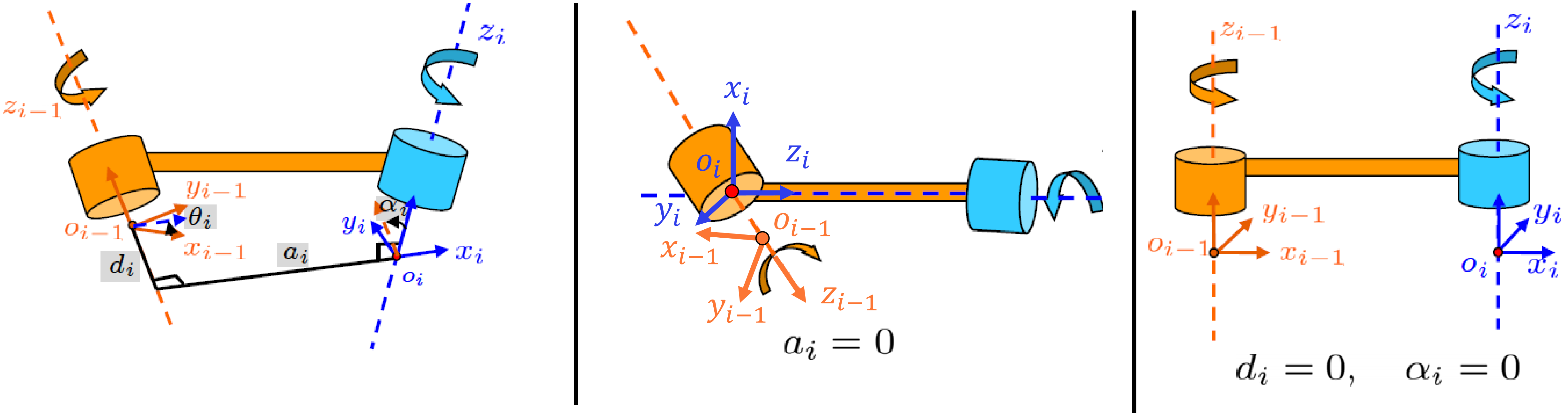
d_1 : distance from o_0 to x_1 (along z_0)

one of both is a variable: θ_1 for revolute, d_1 for prismatic

a_1 : distance from z_0 to o_1 (along x_1)

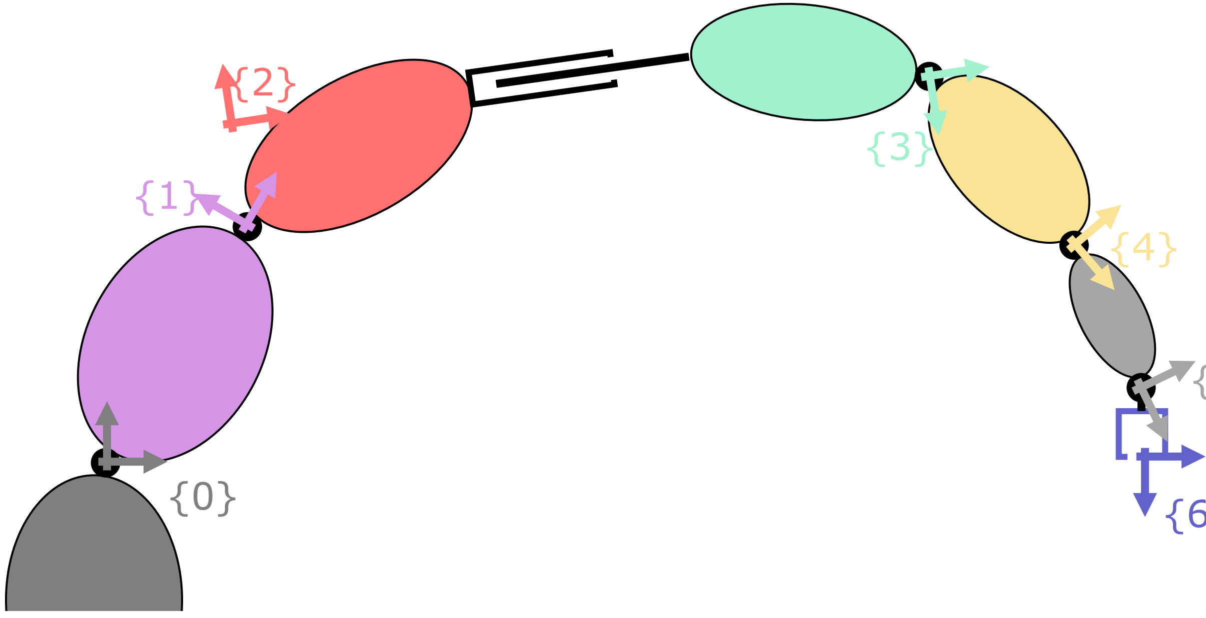
α_1 : angle from z_0 to z_1 about x_1

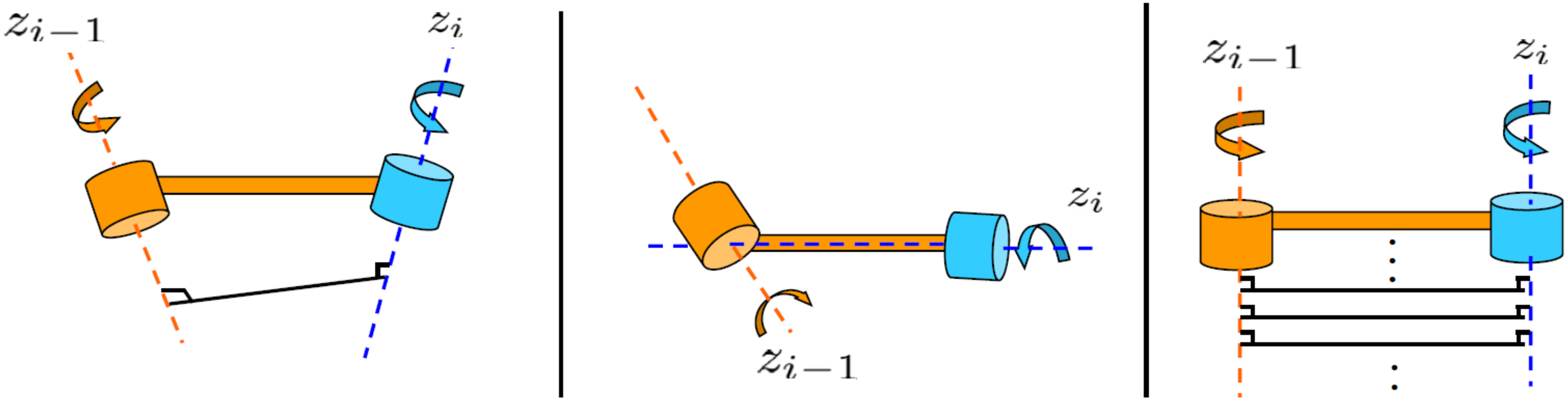
always constant characteristic of the manipulator



Assignment of Coordinate Frames

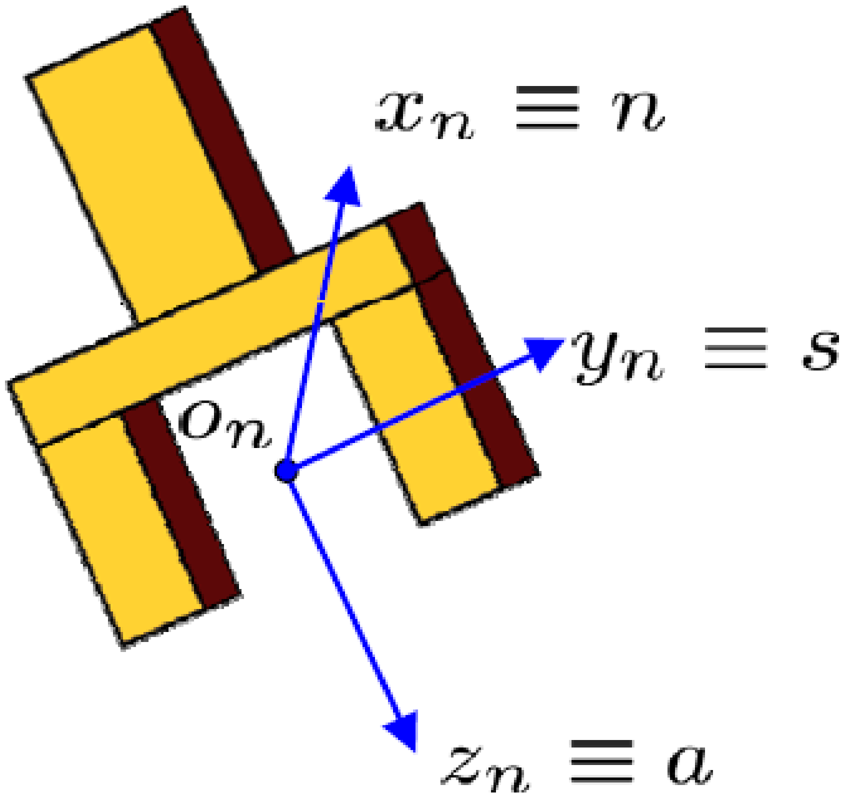
- z_i axis along the $i + 1$ joint axis
- x_i axis parallel to $z_i \times z_{i-1}$
- y_i axis parallel to $z_i \times x_i$
- Origin o_i along z_i at the point of shortest distance to z_{i-1}





End Effector Coordinate Frame

- Origin o_n in the middle between the fingers
- z_n axis parallel to the fingers, also denoted as a (approach)
- y_n axis parallel to the closing direction of the fingers, also denoted as s (sliding)
- x_n axis parallel to $y_n \times z_n$, also denoted as n (normal)



Examples

Three-Link Cylindrical Robot

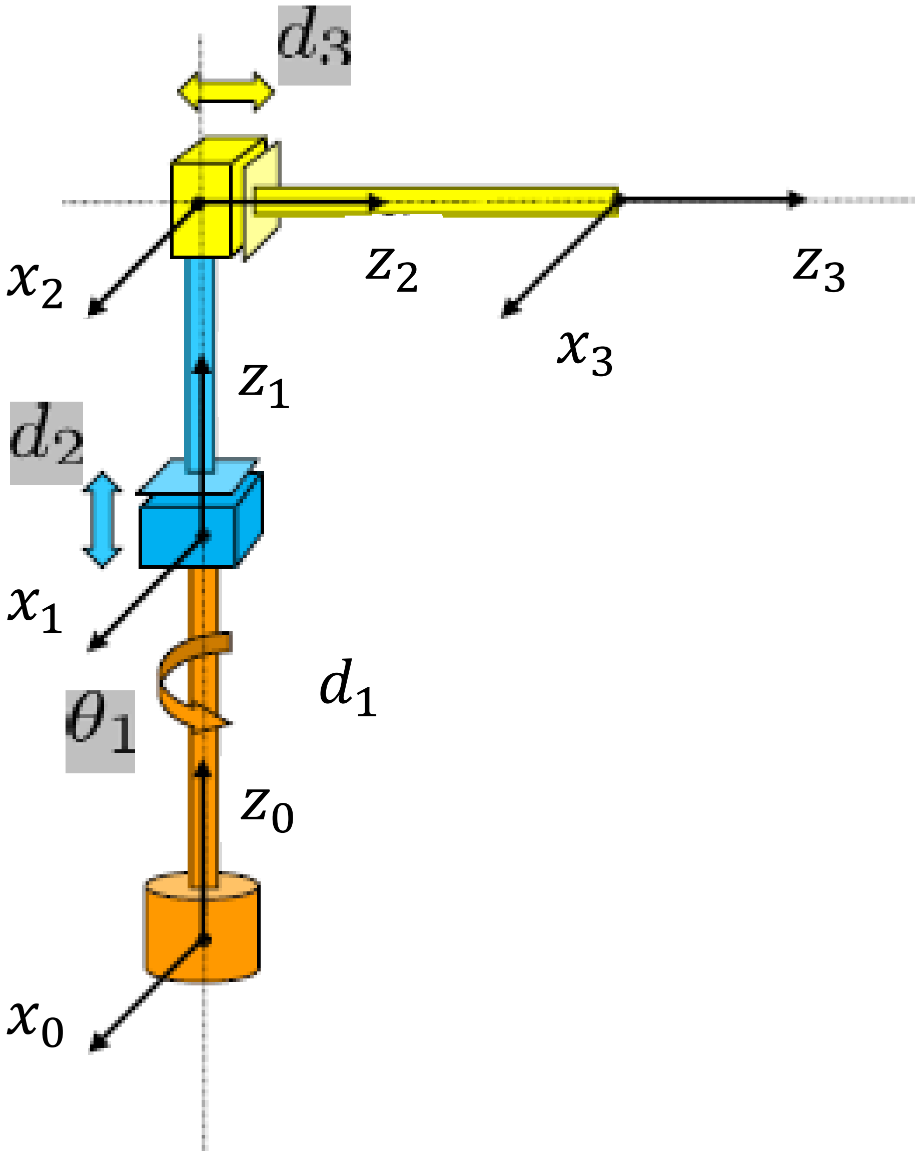
θ_i : angle from x_{i-1} to x_i about z_{i-1}

d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Joint i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1	θ_1^*	d_1	0	0
2	0	d_2^*	0	-90
3	0	d_3^*	0	0



SCARA Manipulator

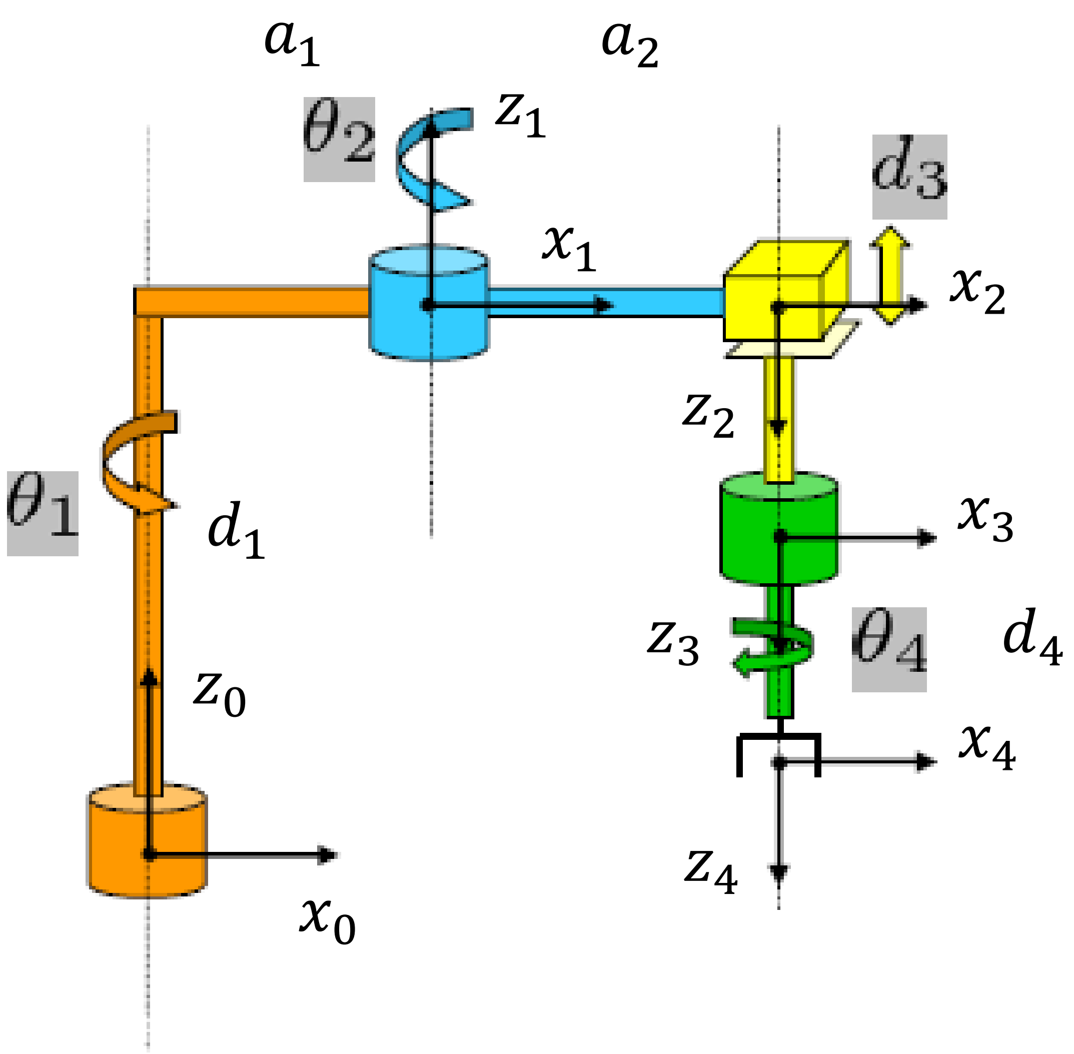
θ_i : angle from x_{i-1} to x_i about z_{i-1}

d_i : distance from o_{i-1} to x_i (along z_{i-1})

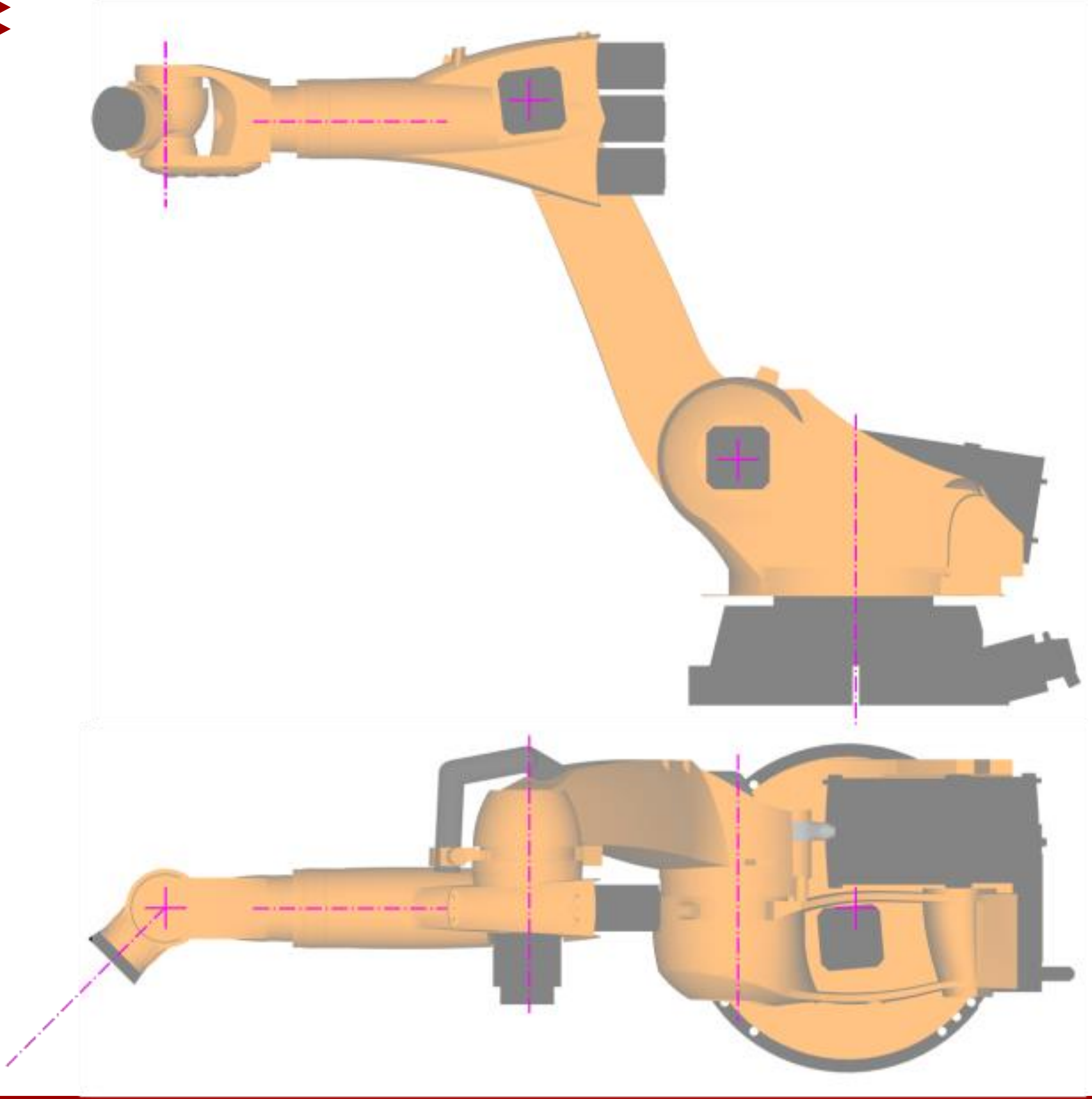
a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Joint i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1	θ_1^*	d_1	a_1	0
2	θ_2^*	0	a_2	180
3	0	d_3^*	0	0
4	θ_4^*	d_4	0	0

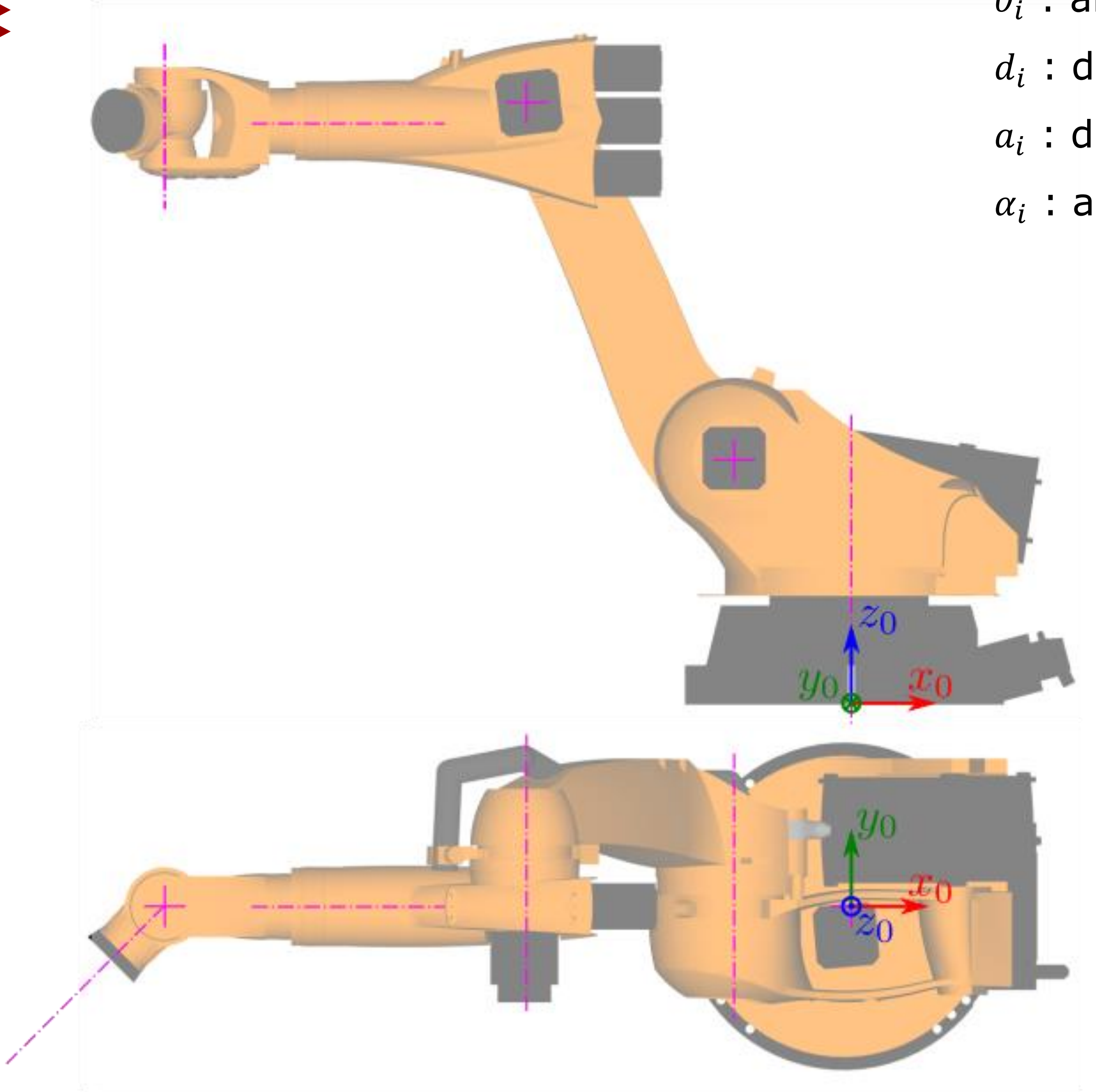


KUKA KR 210



- Number the joints
- Establish base frame
- Establish joint axes z_i
- Locate origin o_i
- Establish x_i and y_i

KUKA KR 210



θ_i : angle from x_{i-1} to x_i about z_{i-1}

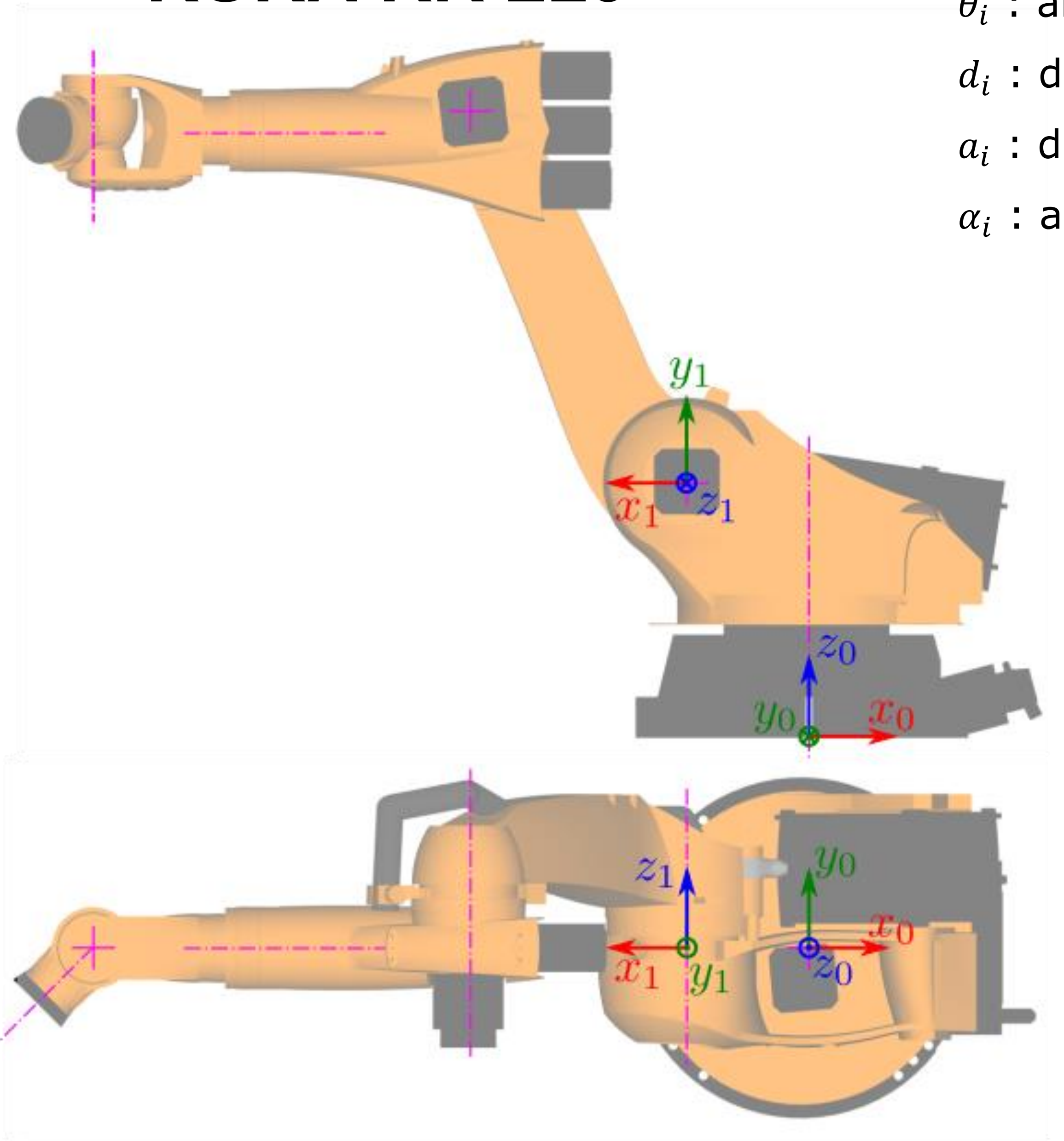
d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Frame i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1				
2				
3				
4				
5				
6				

KUKA KR 210



θ_i : angle from x_{i-1} to x_i about z_{i-1}

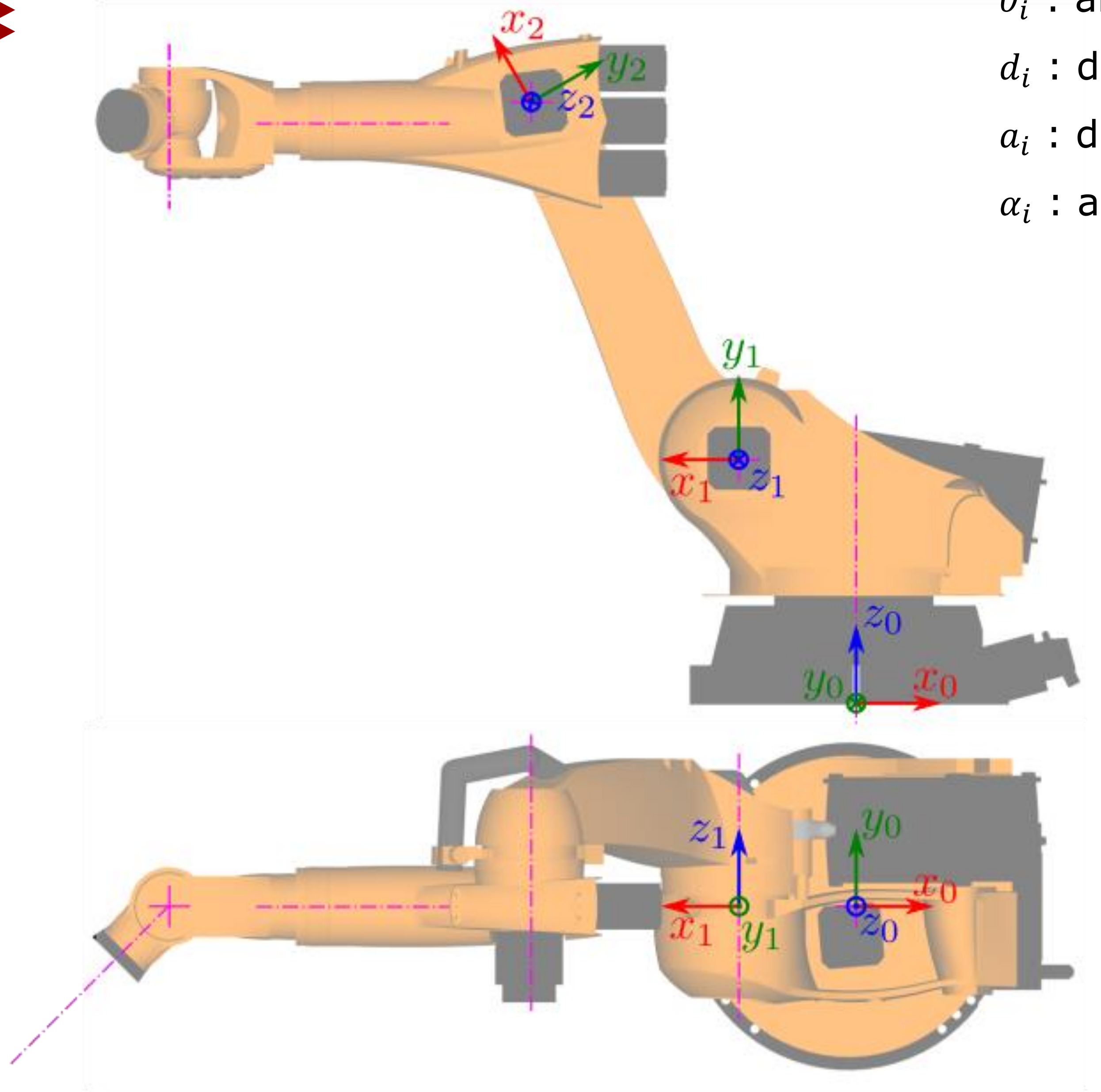
d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Frame i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1	$\theta_1^* = 180$	200	100	90
2				
3				
4				
5				
6				

KUKA KR 210



θ_i : angle from x_{i-1} to x_i about z_{i-1}

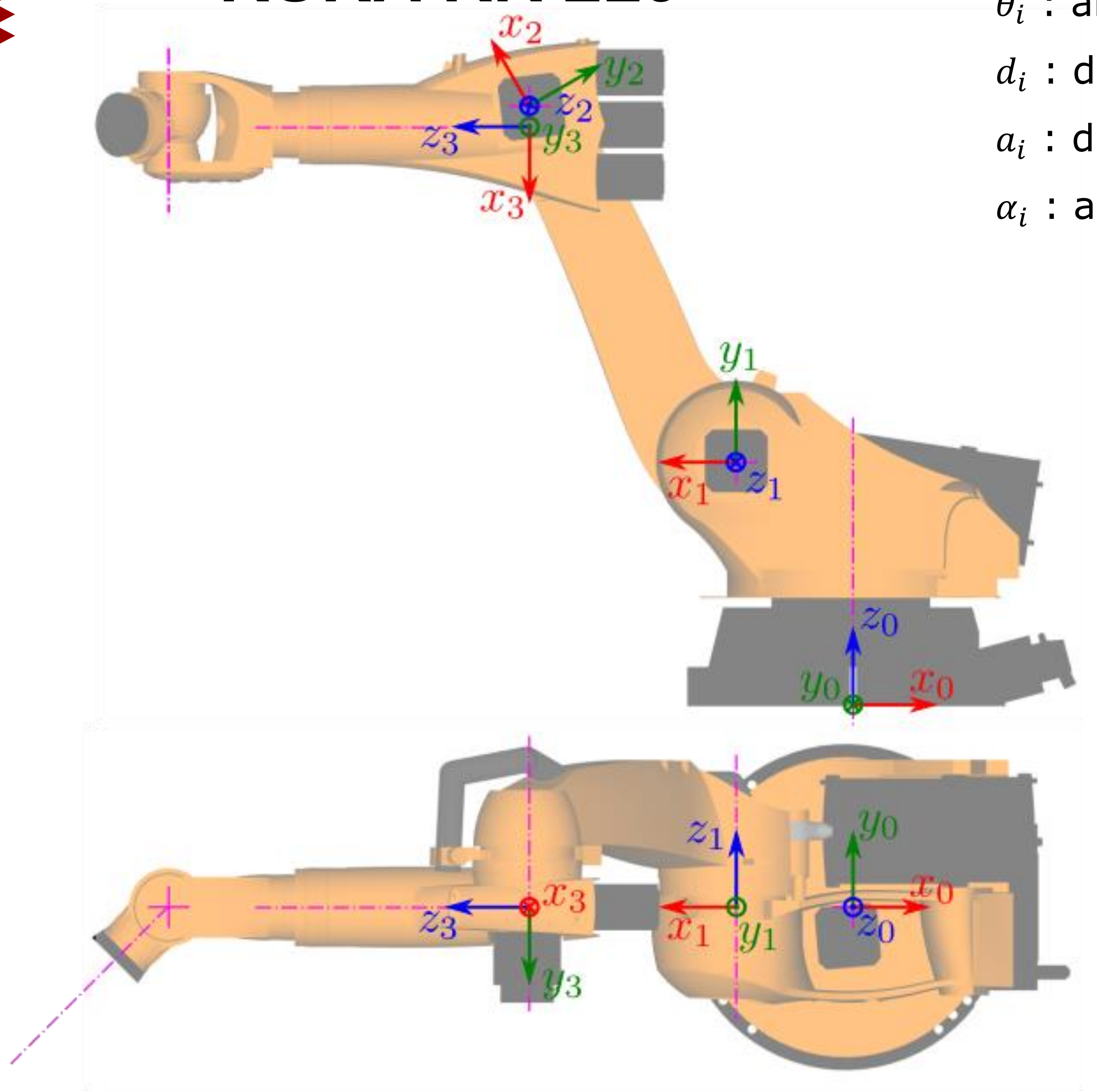
d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Frame i	θ_i	d_i	a_i	α_i
1	$\theta_1^* = 180$	200	100	90
2	$\theta_2^* = 60$	0	300	0
3				
4				
5				
6				

KUKA KR 210



θ_i : angle from x_{i-1} to x_i about z_{i-1}

d_i : distance from o_{i-1} to x_i (along z_{i-1})

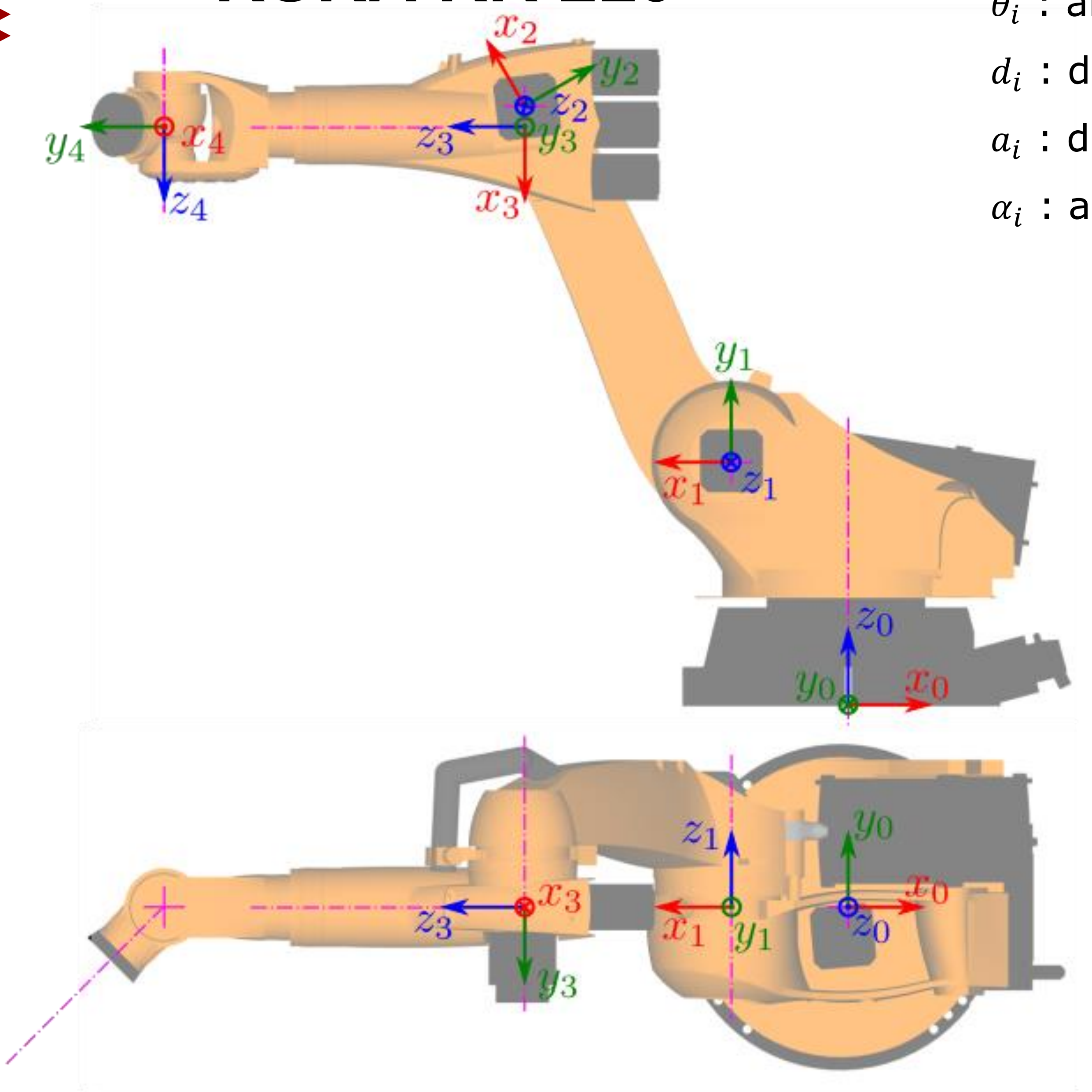
a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Frame i	θ_i	d_i	a_i	α_i
1	$\theta_1^* = 180$	200	100	90
2	$\theta_2^* = 60$	0	300	0
3	$\theta_3^* = -150$	0	20	-90
4				
5				
6				

DTU

KUKA KR 210



θ_i : angle from x_{i-1} to x_i about z_{i-1}

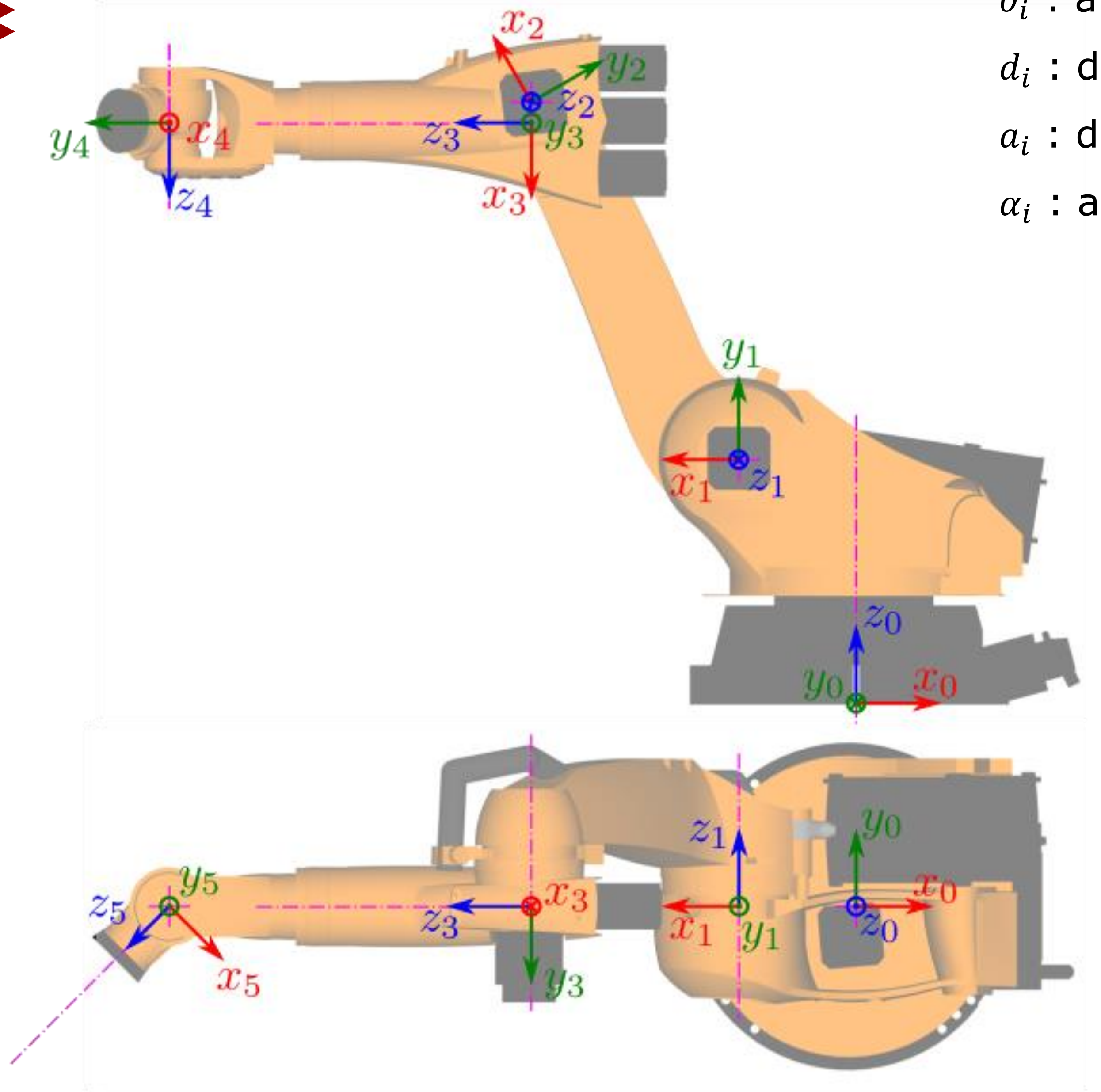
d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Frame i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1	$\theta_1^* = 180$	200	100	90
2	$\theta_2^* = 60$	0	300	0
3	$\theta_3^* = -150$	0	20	-90
4	90	$d_4^* = 300$	0	90
5				
6				

KUKA KR 210



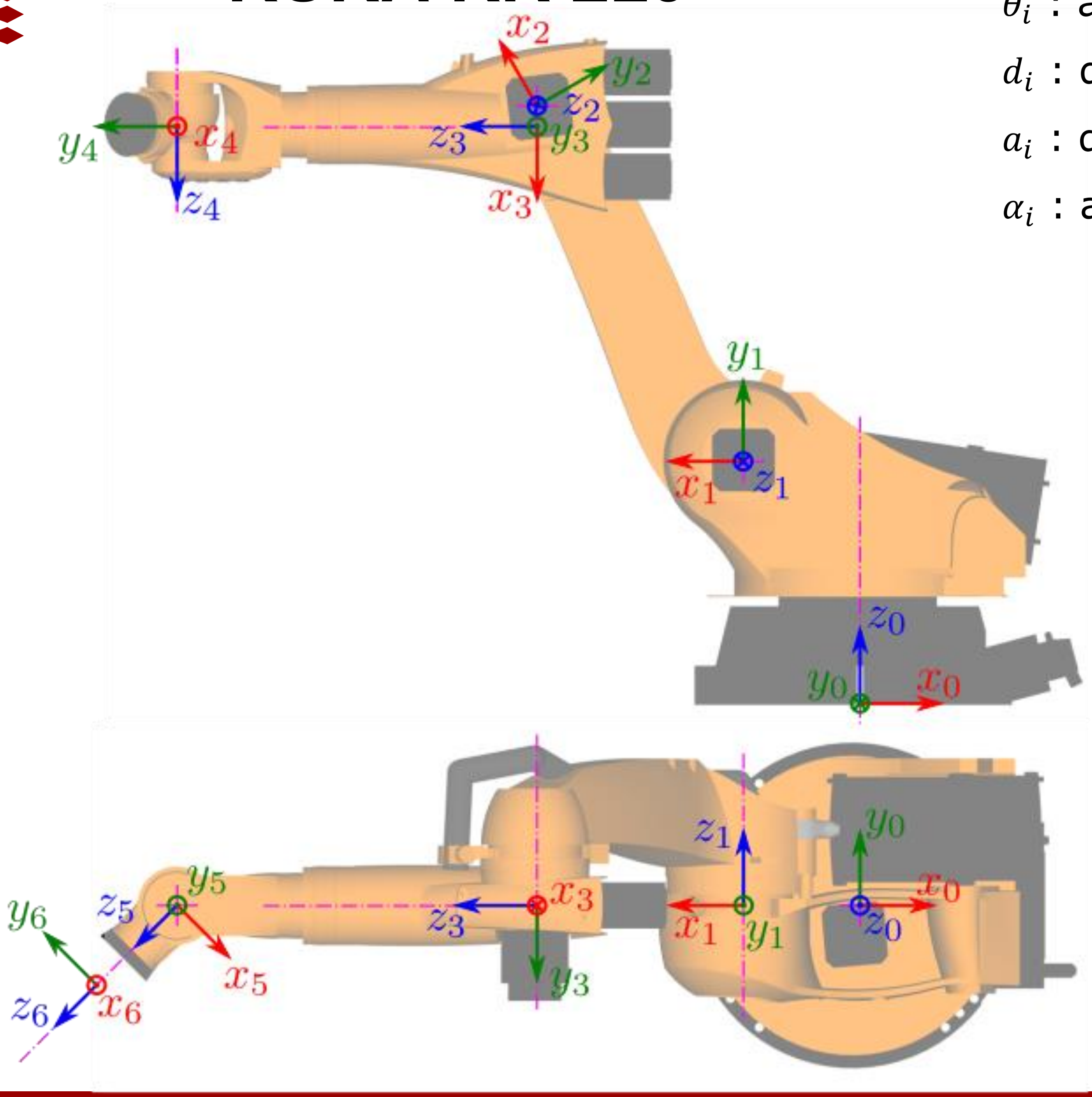
θ_i : angle from x_{i-1} to x_i about z_{i-1}

d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Frame i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1	$\theta_1^* = 180$	200	100	90
2	$\theta_2^* = 60$	0	300	0
3	$\theta_3^* = -150$	0	20	-90
4	90	$d_4^* = 300$	0	90
5	$\theta_5^* = -45$	0	0	-90
6				



KUKA KR 210

θ_i : angle from x_{i-1} to x_i about z_{i-1}

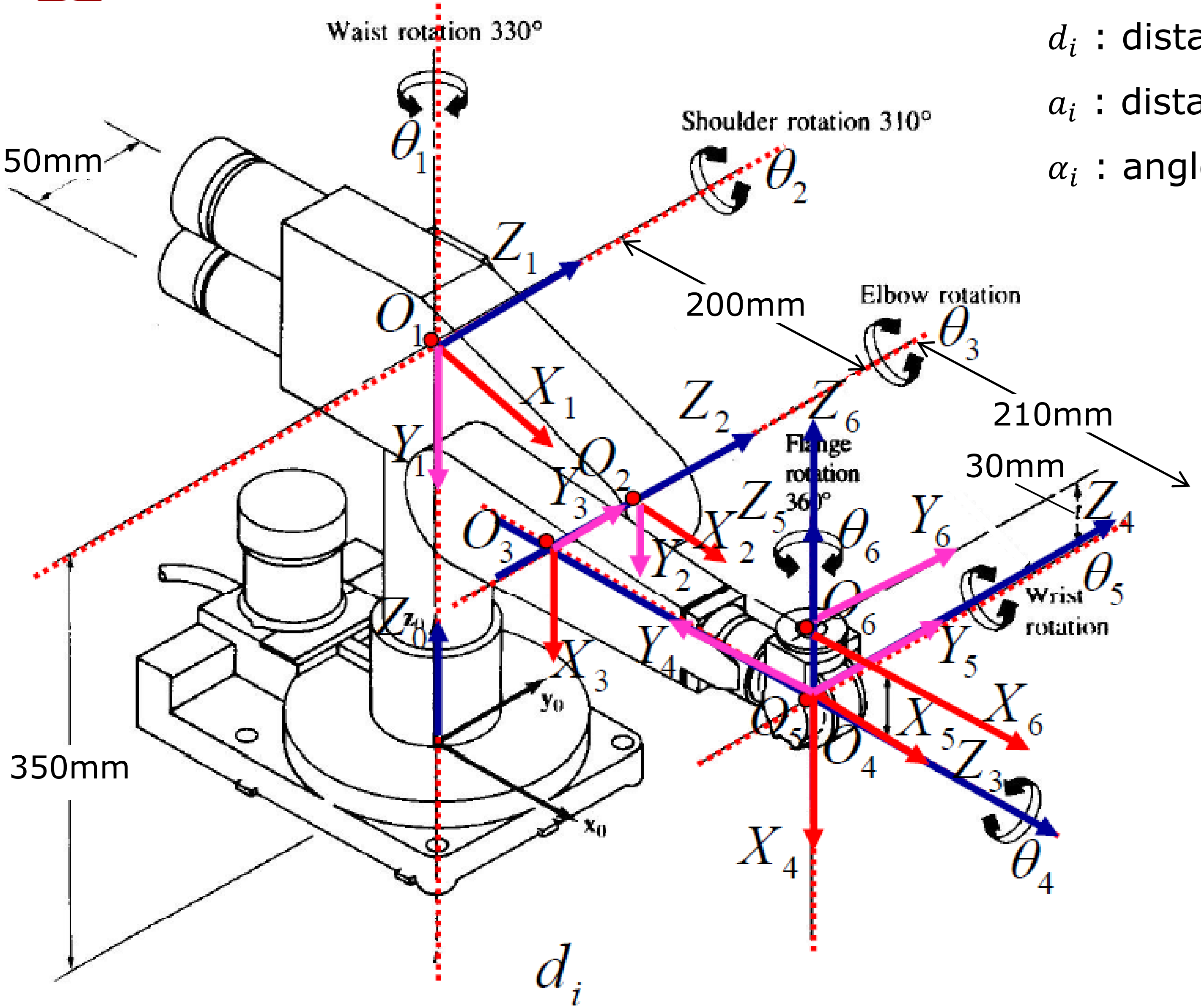
d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Frame i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1	$\theta_1^* = 180$	200	100	90
2	$\theta_2^* = 60$	0	300	0
3	$\theta_3^* = -150$	0	20	-90
4	90	$d_4^* = 300$	0	90
5	$\theta_5^* = -45$	0	0	-90
6	$\theta_6^* = 90$	50	0	0

PUMA 260



θ_i : angle from x_{i-1} to x_i about z_{i-1}

d_i : distance from o_{i-1} to x_i (along z_{i-1})

a_i : distance from z_{i-1} and o_i (along x_i)

α_i : angle from z_{i-1} to z_i about x_i

Frame i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1	θ_1^*	350	0	-90
2	θ_2^*	0	200	0
3	θ_3^*	-50	0	0
4	θ_4^*	210	0	90
5	θ_5^*	0	0	90
6	θ_6^*	30	0	0

Inverse Kinematics

Inverse Kinematics

- **Forward** kinematics:

Known joint degrees of freedom $q_1, q_2, \dots q_n$

→ find the pose of the end effector as:

$$H = T_n^0 = A_1(q_1)\ A_2(q_2)\ A_3(q_3)\ \dots\ A_n(q_n)$$

- **Inverse** kinematics:

Known homogeneous transformation H for end effector

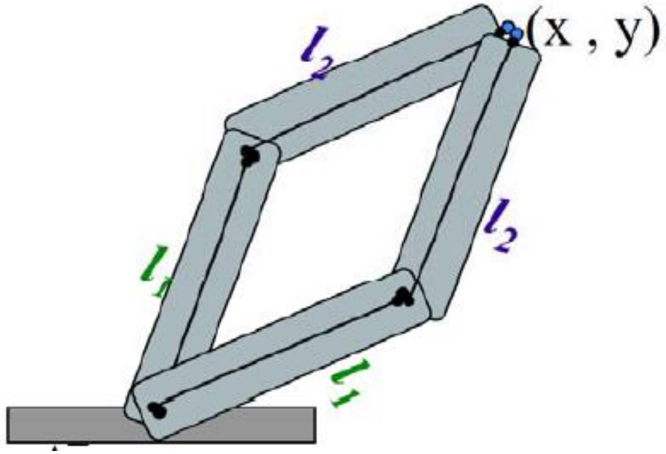
→ solve the nonlinear system of equations

$$A_1(q_1)\ A_2(q_2)\ A_3(q_3)\ \dots\ A_n(q_n) = H$$

for $q_1, q_2, \dots q_n$

→ 12 nonlinear equations → too difficult to find analytical solutions for

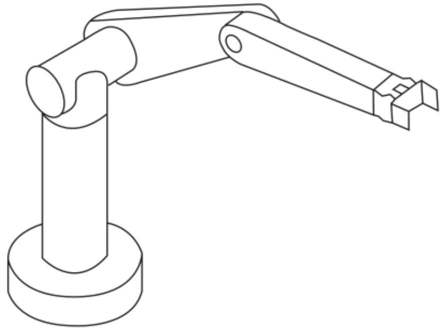
Inverse Kinematics



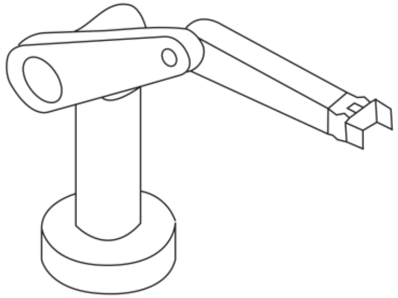
- 12 nonlinear equations
  → too difficult to find analytical solutions

- Not unique solutions
  – Redundant manipulator
  – Elbow-up/elbow-down solutions

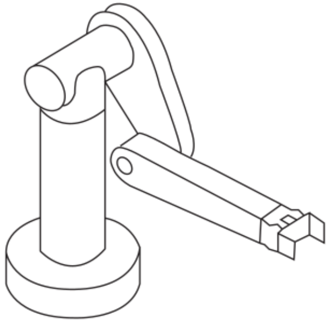
- Kinematic decoupling
  – Inverse position: geometric approach
  – Inverse orientation Euler angles



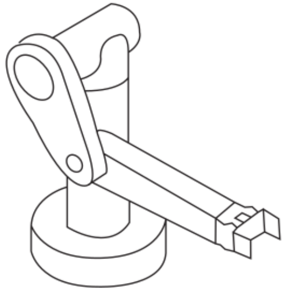
Left Arm Elbow Up



Right Arm Elbow Up



Left Arm Elbow Down



Right Arm Elbow Down

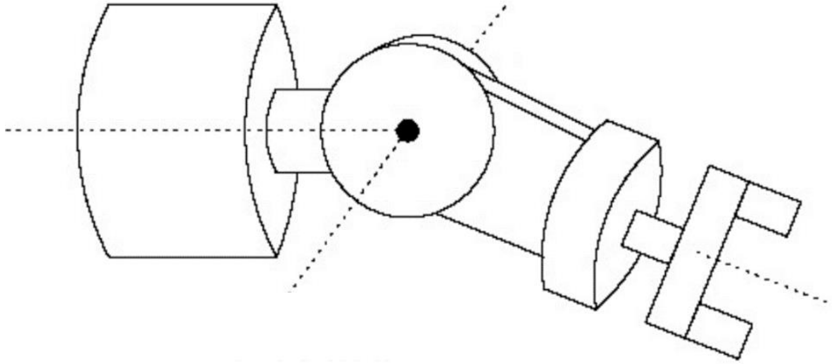
Inverse Kinematics – Kinematic Decoupling

- Decoupling of position and orientation

  – Assume a manipulator where the last 3 dofs correspond to a spherical wrist

  – First solve for the position of the wrist center

  – Then solve for the orientation of the end-effector



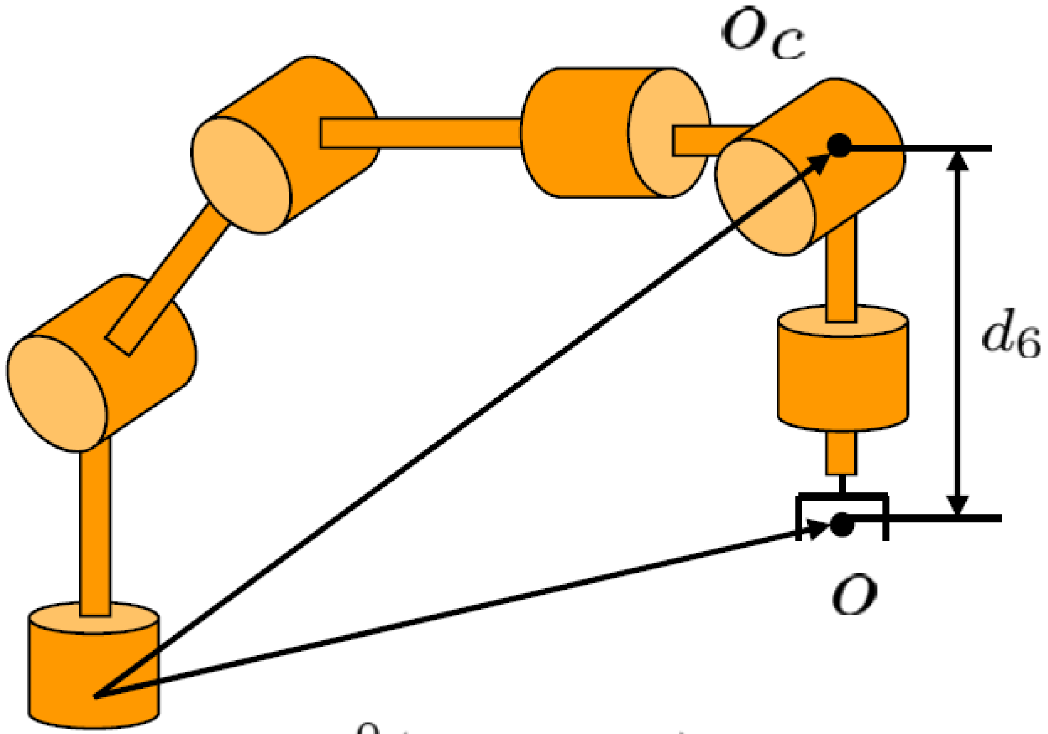
Inverse Kinematics – Kinematic Decoupling

Kinematic decoupling for 6-DoF manipulator with spherical wrist:

- Inverse position kinematics
  → wrist center

- Inverse orientation kinematics
  → wrist orientation

Axes z_3, z_4, z_5 intersect at o_c:
their rotations will not affect the position of o_c



$$\begin{cases} R_6^0(q_1, \ldots, q_6) = R \\ o_6^0(q_1, \ldots, q_6) = o \end{cases}$$

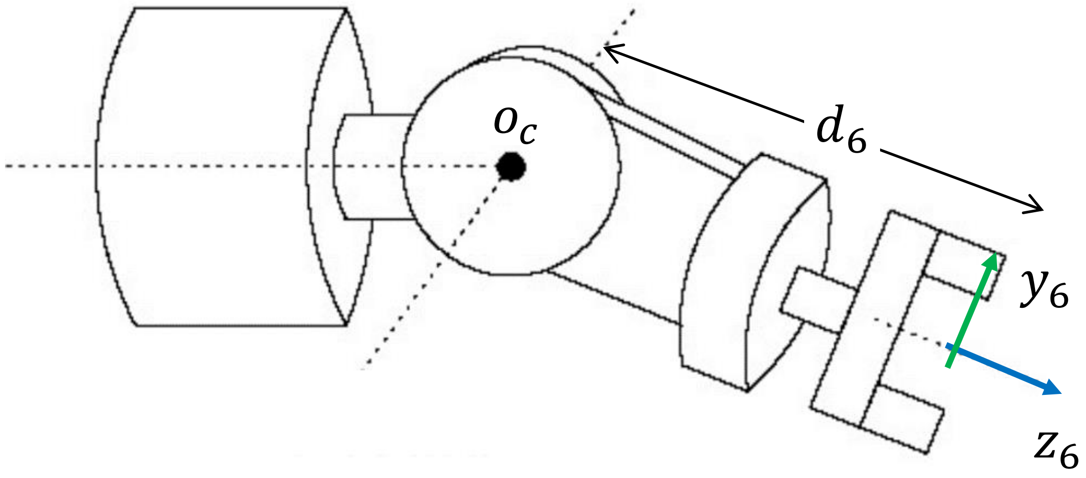
Kinematic Decoupling

- Intended homogeneous transformation matrix:

$$H = A_1A_2A_3A_4A_5A_6 = T_6^0 = \begin{bmatrix} R_6^0 & o_6^0 \\ 0_{1\times 3} & 1 \end{bmatrix}$$

Based on the wrist configuration we know:

$$o_c^0 = o_6^0 - d_6R_6^0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} o_x - d_6r_{13} \\ o_y - d_6r_{23} \\ o_z - d_6r_{33} \end{bmatrix}$$



- We find q_1, q_2, q_3 that satisfy o_c^0
- We apply forward kinematics to calculate

$$R_3^0 = R_1^0(q_1)R_2^1(q_2)R_3^2(q_3)$$

- We find q_4, q_5, q_6 from solving:

$$R_6^0 = R_3^0R_6^3(q_4, q_5, q_6) \quad \Rightarrow \quad R_6^3(q_4, q_5, q_6) = (R_3^0)^T R_6^0$$

Kinematic Decoupling

Kinematic decoupling for 6-DoF manipulator with spherical wrist:

Inverse position

$$o_c^0(q_1, q_2, q_3) = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

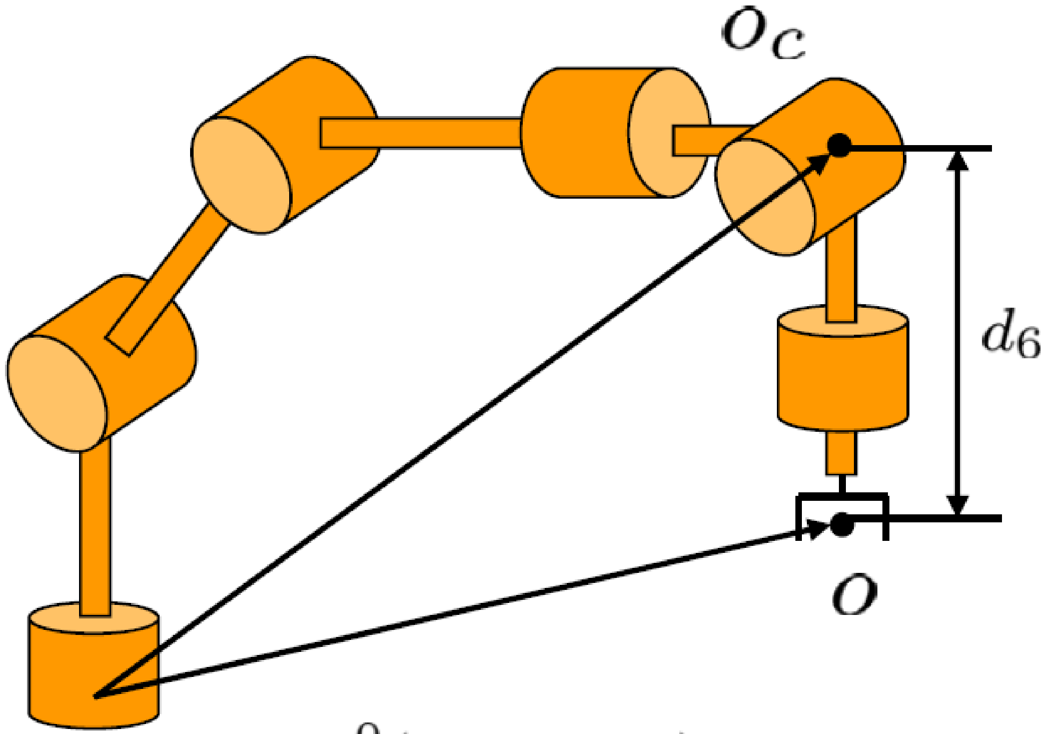
$\longrightarrow$ q_1, q_2, q_3

Inverse orientation

$$R_6^3(q_4, q_5, q_6) = (R_3^0(q_1, q_2, q_3))^T R_6^0$$

$\longrightarrow$ q_4, q_5, q_6

Axes z_3, z_4, z_5 intersect at o_c:
their rotations will not affect the position of o_c



$$\begin{cases} R_6^0(q_1, \ldots, q_6) = R \\ o_6^0(q_1, \ldots, q_6) = o \end{cases}$$

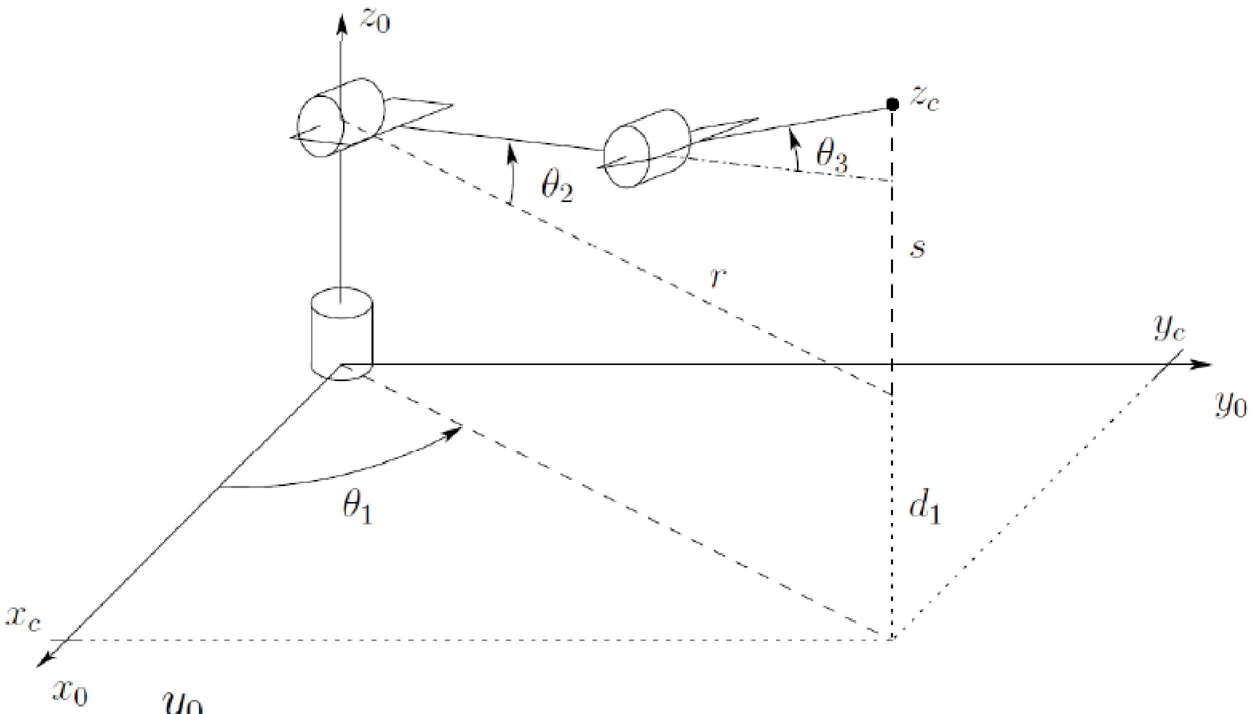
Inverse position

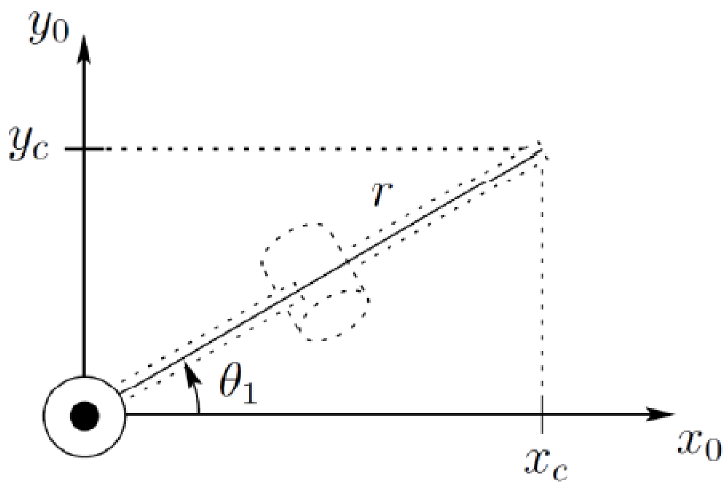
- We find q_1, q_2, q_3 that satisfy:

$$o_c^0(q_1, q_2, q_3) = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

- But How?
- By a graphical method:
  - Projecting the manipulator onto the xy-plane of a link frame
  - Applying trigonometry on the projected geometry

Inverse Kinematics of Articulated Manipulator



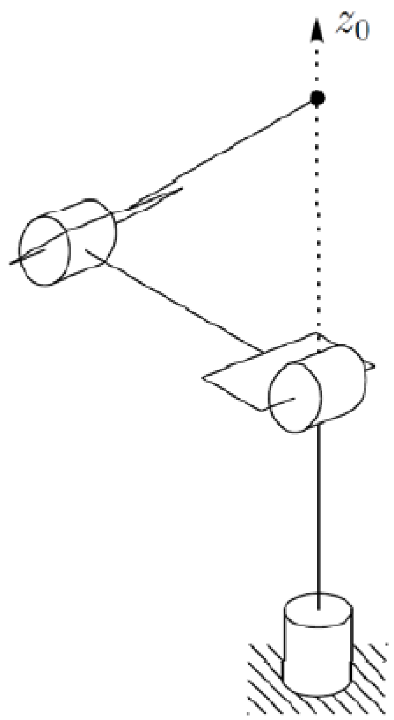


The angle θ_1 is easy to determine:

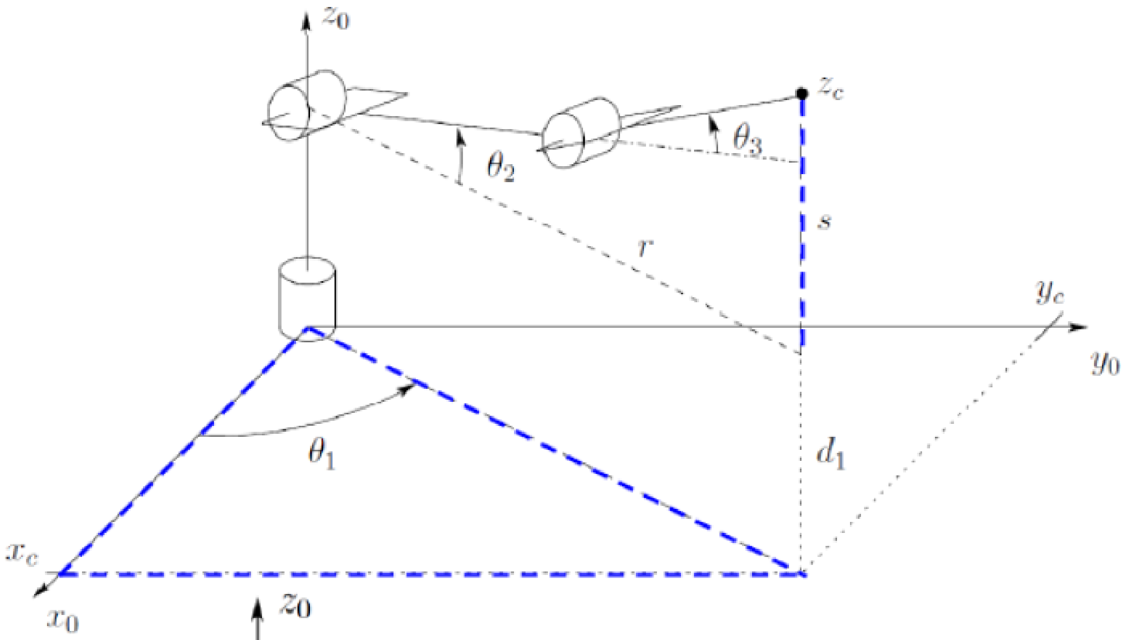
$$\theta_1 = \text{Atan2}(x_c, y_c)$$

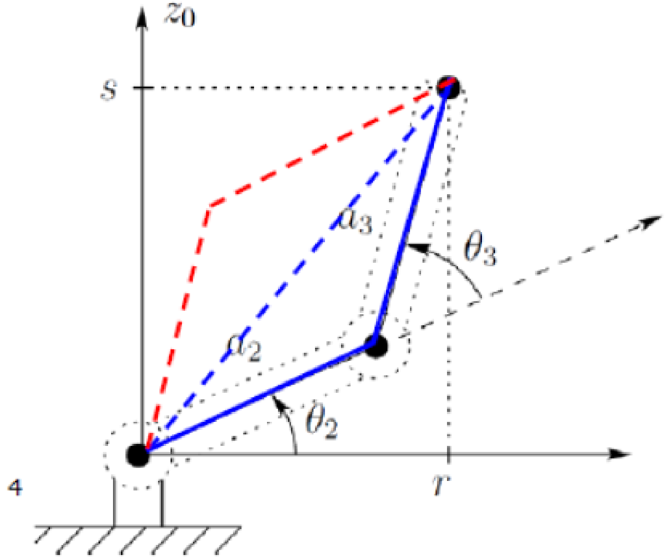
when $(x_c, y_c) \neq (0{,}0)$

Otherwise for $(x_c, y_c) = (0{,}0)$ there is a singularity w.r.t. determining θ_1:



Inverse Kinematics of Articulated Manipulator





From the law of cosines:

$$\cos(\theta_3) = \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3}$$

where:

$$r^2 = x_c^2 + y_c^2$$
$$s = z_c - d_1$$

There are two solutions:

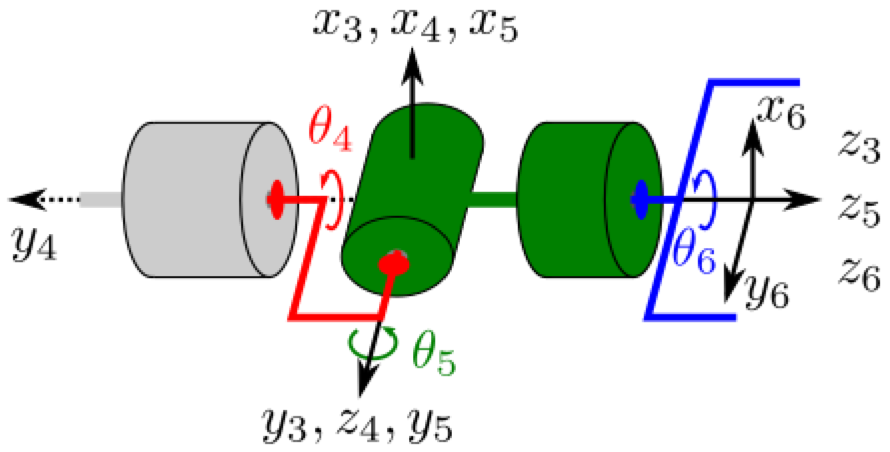
$$\theta_3 = \text{Atan2}\left(c_3, \pm\sqrt{1 - c_3^2}\right)$$

(elbow-down or elbow-up)

Then one can also find:

$$\theta_2 = \text{Atan2}(r, s) - \text{Atan2}(a_2 + a_3c_3, a_3s_3)$$

Inverse Orientation



- Find θ_4, θ_5, θ_6 that satisfy

$$R_6^3(\theta_4, \theta_5, \theta_6) = (R_3^0)^T R_6^0$$

$$\begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix} = (R_3^0)^T R_6^0$$

→Same problem as finding the Euler angles in Lecture 2:

$$\theta_5 = \theta = \text{Atan2}\left(r_{33}, \sqrt{1 - r_{33}^2}\right)$$

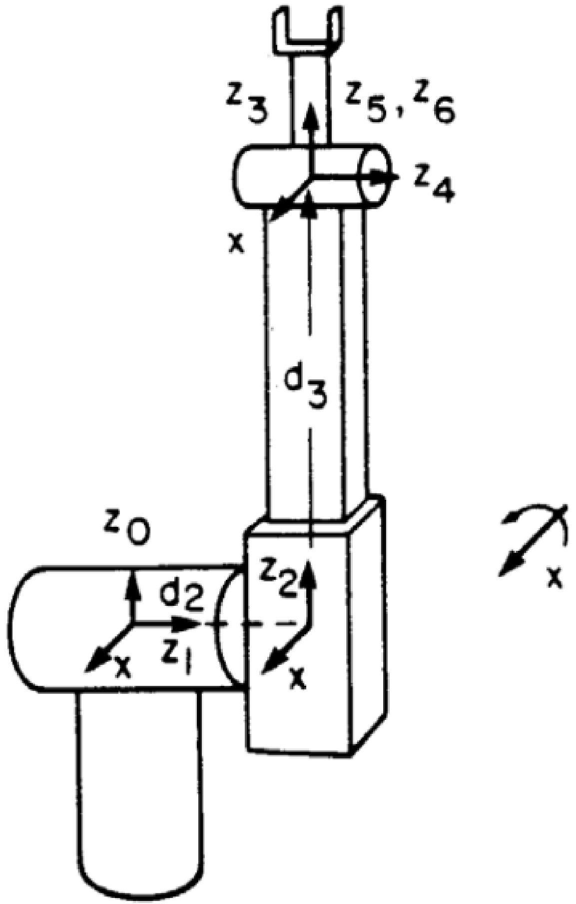
$$\theta_4 = \phi = \text{Atan2}(r_{13}, r_{23})$$

$$\theta_6 = \psi = \text{Atan2}(-r_{31}, r_{32})$$

Exercises

Problem 1

Given the Stanford arm in the figure below, with $d_2 = 0.1\,\text{m}$, answer the following questions.



Question 1

Find the link parameters for the robotic arm (d_3 is a prismatic joint variable, other joints are rotational joints, the link coordinate frames have been established as shown in the figure).

Joint i	θ_i	d_i	a_i	α_i
1				
2				
3				
4				
5				
6				

Question 2

Find the forward kinematic model for the arm and represent it in homogeneous matrix form.

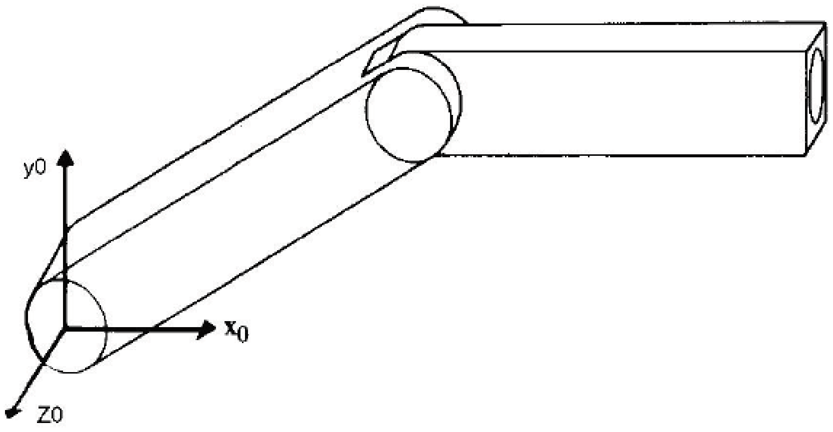
Question 3

Represent the orientation of the end-effector with Yaw-Pitch-Roll angles.

Exercises

Problem 2

A two degree-of-freedom manipulator is shown in the figure below. Given that the length of each link is $1\,\mathrm{m}$, establish its link coordinate frames and find T_1^0, T_2^1 and the kinematics matrix. For coordinate frame 2, assume a revolute joint at the tip of the robotic arm with its axis parallel to z_0.



Question 1

Find the forward kinematics solution for this manipulator, i.e. the homogeneous transformation matrix for the end-effector as a function of the joint angles.

Question 2

Find the inverse kinematics solution for this manipulator assuming the position of the robot tip is known, i.e. elements r_{14} and r_{24} in the homogeneous transformation matrix. (Hint: use trigonometry and the law of cosines)

Robotics – 34753

Velocity Kinematics & The Jacobian Matrix

Konstantinos Poulios
Associate Professor
Department of Civil and Mechanical Engineering
DTU Lyngby, building 404 / room 124

Velocity Kinematics & The Jacobian Matrix – Lecture Overview

1. **Repetition**

2. **Angular Velocities**
   - Skew Symmetric Matrices
   - Addition of Angular Velocities

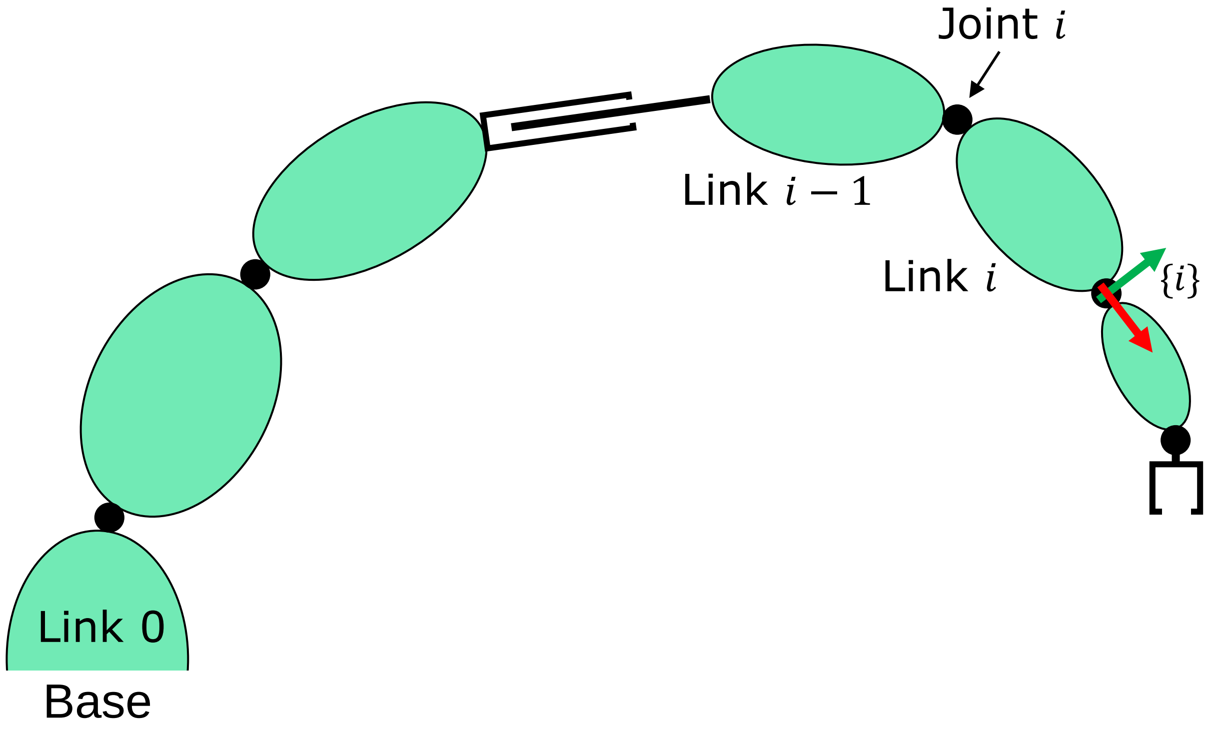
3. **Linear Velocities**
   - Fixed Point on a Link

4. **The Manipulator Jacobian**
   - Singularities
   - Inverse Jacobian

Repetition

Repetition

- Kinematic Model
  - Joint i is fixed to link $i-1$
  - Joint i actuation → motion of link i and frame $o_i x_i y_i z_i$



Repetition

- D-H convention: 4 basic transformations → Homogeneous transformation

$$A_i = Rot_{z,\theta_i}\, Trans_{z,d_i}\, Trans_{x,a_i}\, Rot_{x,\alpha_i}$$

$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

θ_i	: joint angle
d_i	: link offset
a_i	: link length
α_i	: link twist

versus

Six parameters

$\phi, \theta, \psi,$
d_x, d_y, d_z

Repetition

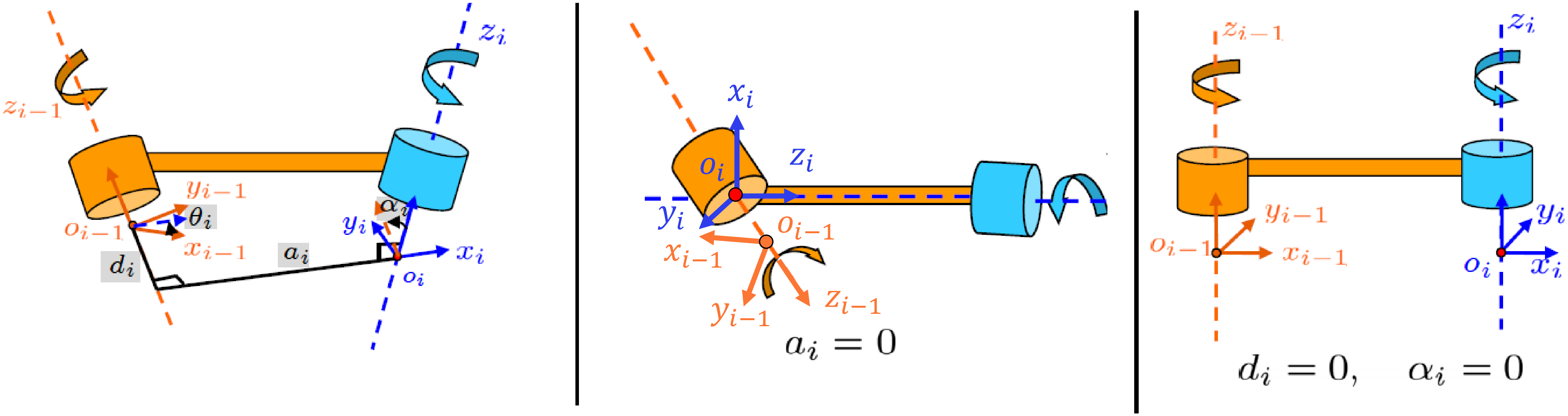
Physical interpretation of θ, d, a, α:

θ_1 : angle from x_0 to x_1 about z_0
d_1 : distance from o_0 to x_1 (along z_0)

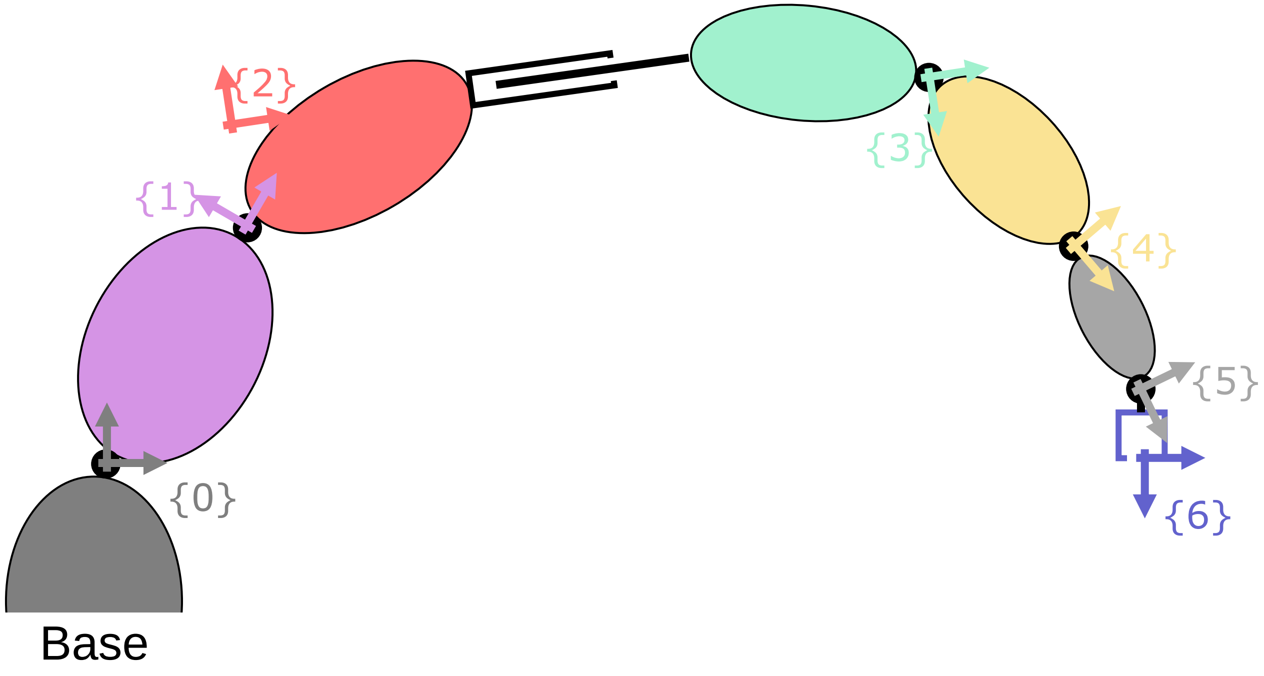
one of both is a variable:
θ_1 for revolute, d_1 for prismatic

a_1 : distance from z_0 to o_1 (along x_1)
α_1 : angle from z_0 to z_1 about x_1

always constant
characteristic of the manipulator

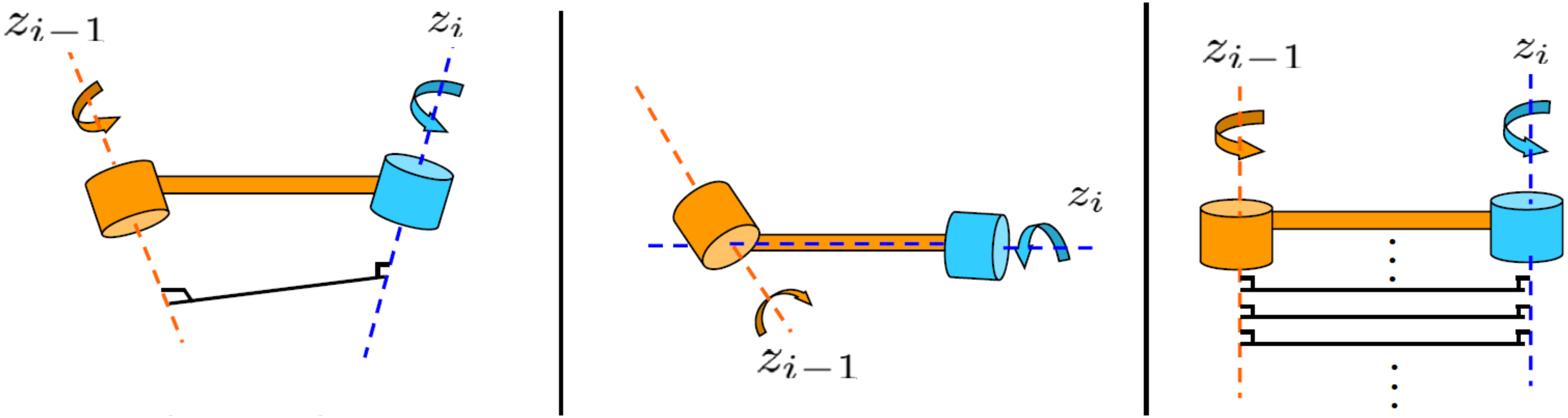


Repetition



Assignment of Coordinate Frames:

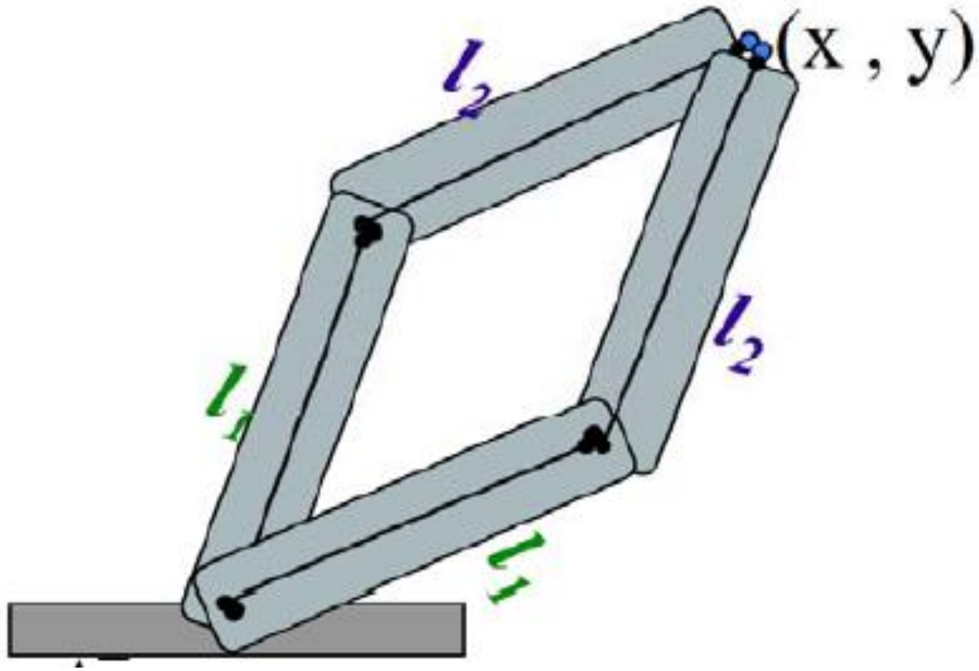
- z_i axis along the $i+1$ joint axis
- x_i axis parallel to $z_i \times z_{i-1}$
- y_i axis parallel to $z_i \times x_i$
- Origin o_i along z_i at the point of shortest distance to z_{i-1}



Repetition

Inverse kinematics:

- 12 nonlinear equations
  → too difficult to find analytical solutions
- Not unique solutions
  - Redundant manipulator
  - Elbow-up/elbow-down solutions
- Kinematic decoupling
  - Inverse position: geometric approach
  - Inverse orientation Euler angles

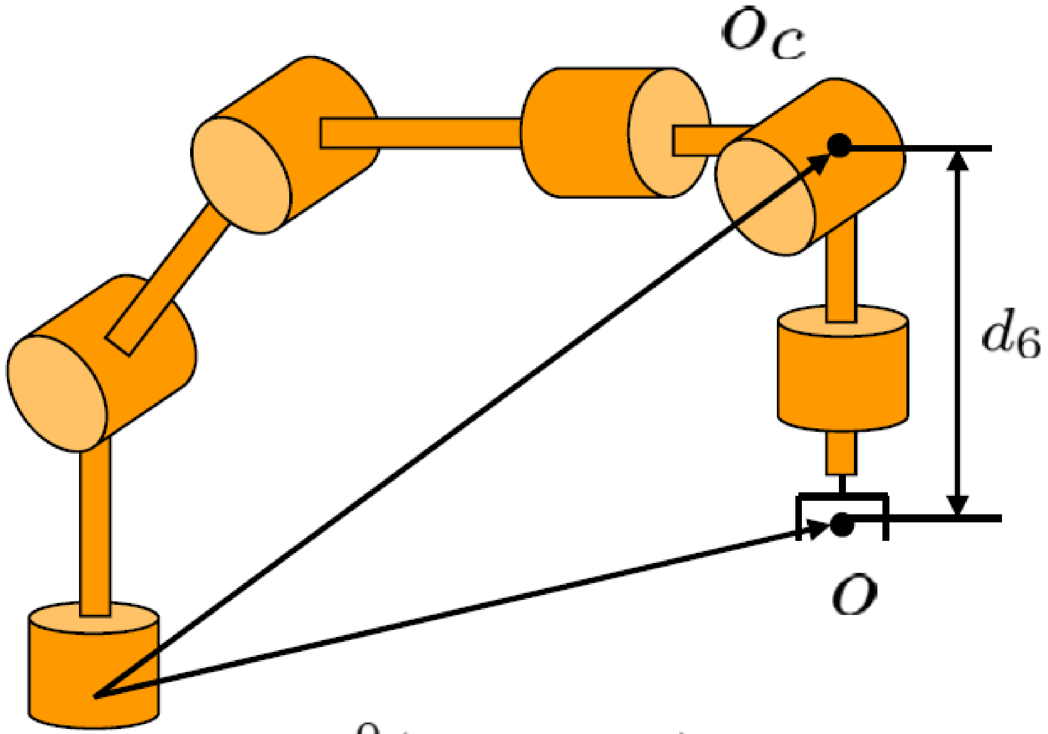


Repetition

Kinematic decoupling for 6-DoF manipulator with spherical wrist:

- Inverse position kinematics
  → wrist center
- Inverse orientation kinematics
  → wrist orientation

Axes z_3, z_4, z_5 intersect at o_c: their rotations will not affect the position of o_c



$$\begin{cases} R_6^0(q_1, \ldots, q_6) = R \\ o_6^0(q_1, \ldots, q_6) = o \end{cases}$$

Repetition

Kinematic decoupling for 6-DoF manipulator with spherical wrist:

Inverse position

$$o_c^0(q_1, q_2, q_3) = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

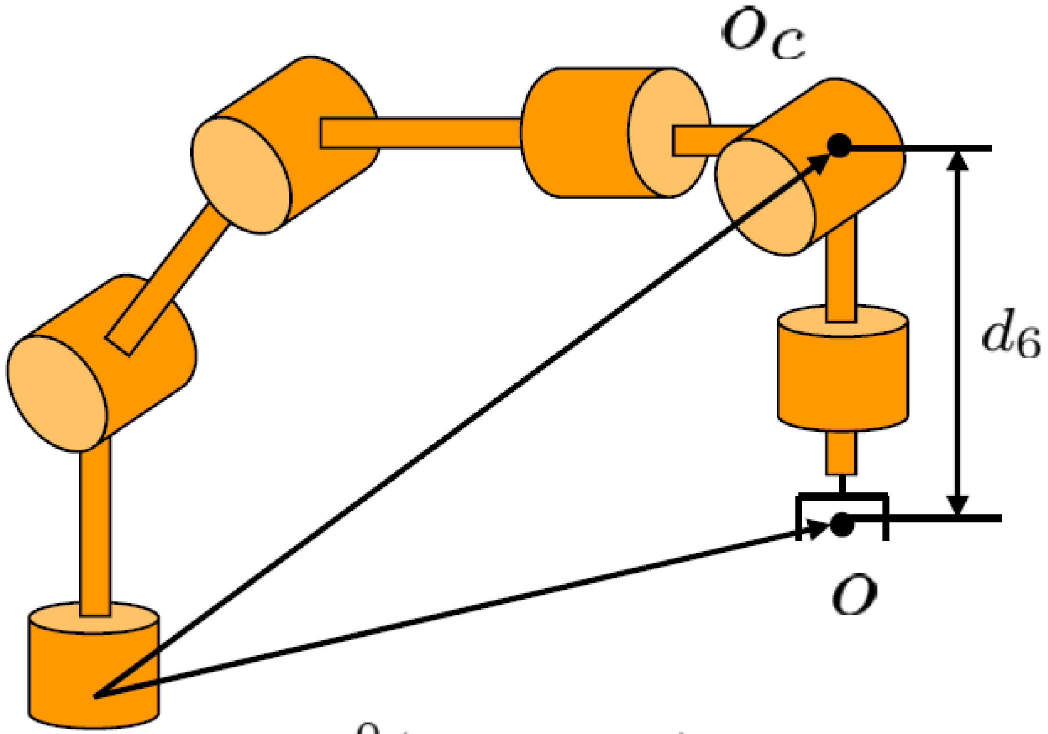
$\longrightarrow$ $q_1,\ q_2,\ q_3$

Inverse orientation

$$R_6^3(q_4, q_5, q_6) = (R_3^0(q_1, q_2, q_3))^T R_6^0$$

$\longrightarrow$ $q_4,\ q_5,\ q_6$

Axes z_3, z_4, z_5 intersect at o_c: their rotations will not affect the position of o_c

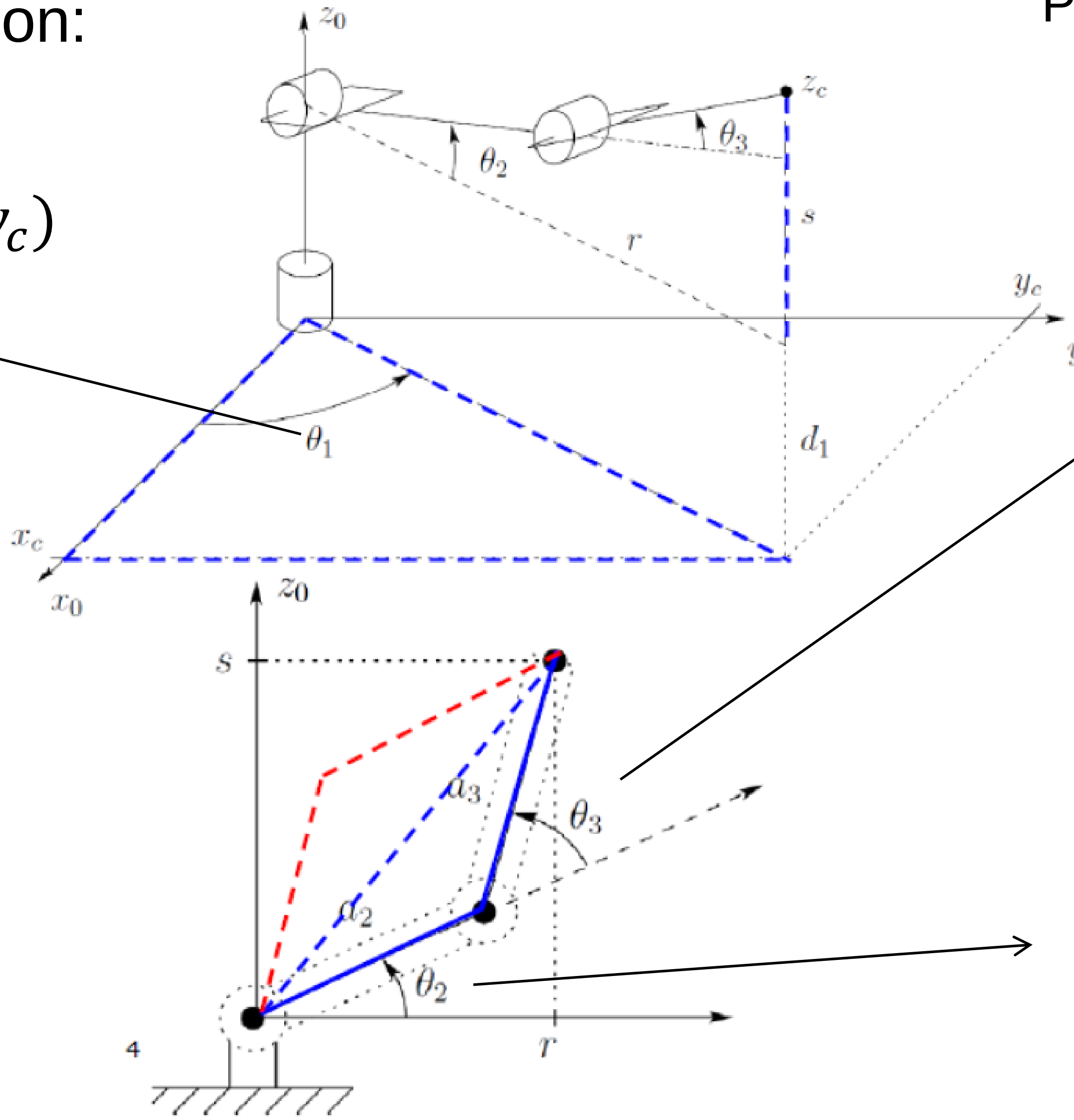


$$\begin{cases} R_6^0(q_1, \ldots, q_6) = R \\ o_6^0(q_1, \ldots, q_6) = o \end{cases}$$

Repetition

Inverse position:

$$\theta_1 = \text{Atan2}(x_c, y_c)$$



Projecting the manipulator onto an $x_i y_i$ plane

$$\theta_3 = \text{Atan2}\left(c_3, \pm\sqrt{1 - c_3^2}\right)$$

$$c_3 = \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2 a_3}$$

$$r^2 = x_c^2 + y_c^2$$

$$s = z_c - d_1$$

$$\theta_2 = \begin{array}{c} \text{Atan2}(r, s) \\ -\text{Atan2}(a_2 + a_3 c_3, a_3 s_3) \end{array}$$

Repetition

Inverse orientation:

- Find θ_4, θ_5, θ_6 that satisfy

$$\begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix} = (R_3^0)^T R_6^0$$

$$\theta_5 = \theta = \mathrm{Atan2}\left(r_{33}, \sqrt{1 - r_{33}^2}\right)$$

$$\theta_4 = \phi = \mathrm{Atan2}(r_{13}, r_{23})$$

$$\theta_6 = \psi = \mathrm{Atan2}(-r_{31}, r_{32})$$

Angular Velocities

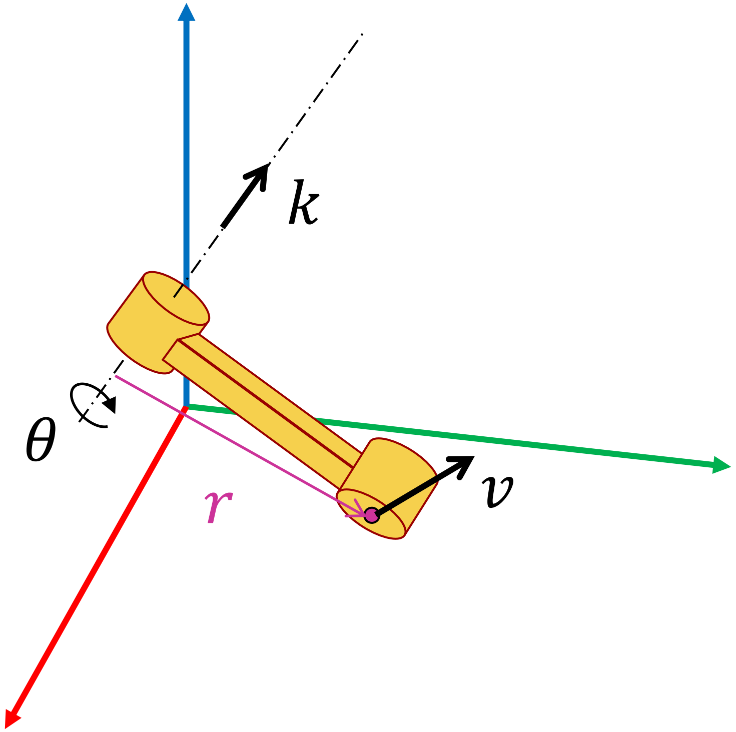
Angular Velocity as a Vector

- Axis of rotation k (unit vector)
- Rotation angle θ → Rotation matrix $R_{k,\theta}$
- Time derivative $\dot{\theta}$
- Angular velocity (vector)

$$\omega = \dot{\theta} k$$

- Linear velocity of a point at a distance r

$$v = \omega \times r$$



Cross Product through a Skew Symmetric Matrix

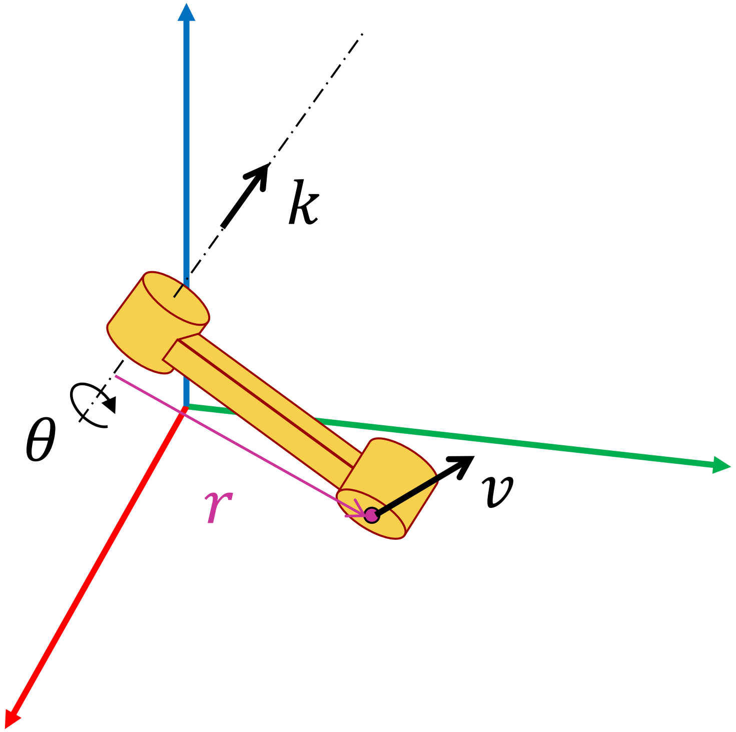
- Angular velocity (vector)

$$\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \dot{\theta} k$$

- Linear velocity of a point p

$$v = \omega \times r = \underbrace{\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}}_{S(\omega)} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

skew symmetric matrix



Properties of Skew Symmetric Matrices

$$\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \rightarrow \quad S(\omega) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Properties:

- $S + S^T = 0$
- $S(\alpha a + \beta b) = \alpha S(a) + \beta S(b)$
- $S(R\,a) = RS(a)R^T$, for any orthonormal matrix R
- $x^T S(a) x = 0,$ for any vector x

Example:

- $S(\omega) = \begin{bmatrix} 0 & 3 & -1 \\ -3 & 0 & -4 \\ 1 & 4 & 0 \end{bmatrix}$
- $\omega = ?$
- $\dot{\theta} = ?$
- $k = ?$

Derivative of a Rotation Matrix

$$RR^T = I \quad \Rightarrow \quad \frac{dR}{d\theta}R^T + R\frac{dR^T}{d\theta} = 0$$

Which means

- $\frac{dR}{d\theta}R^T = S \quad \Rightarrow \quad \frac{dR}{d\theta} = S\,R(\theta)$

Example:

- $R_{z,\theta} = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $\frac{dR_{z,\theta}}{d\theta}R_{z,\theta}^T = \begin{bmatrix} -s_\theta & -c_\theta & 0 \\ c_\theta & -s_\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Derivative of a Rotation Matrix

$$RR^T = I \quad \Rightarrow \quad \frac{dR}{d\theta}R^T + R\frac{dR^T}{d\theta} = 0$$

Which means

- $\frac{dR}{d\theta}R^T = S \quad \Rightarrow \quad \frac{dR}{d\theta} = S\,R(\theta)$

Time derivative

- $\dot{R}_{k,\theta} = \frac{dR}{d\theta}\frac{d\theta}{dt} = S(k)R_{k,\theta}\,\dot{\theta} = \dot{\theta}\,S(k)\ R_{k,\theta} \quad \Rightarrow \quad \dot{R}_{k,\theta} = S\big(\underbrace{\dot{\theta}k}_{\omega}\big)\,R_{k,\theta}$

Addition of Angular Velocities

General case:

$$\dot{R}(t) = S\big(\omega(t)\big)R(t)$$

$\omega(t)$: instantaneous angular velocity vector

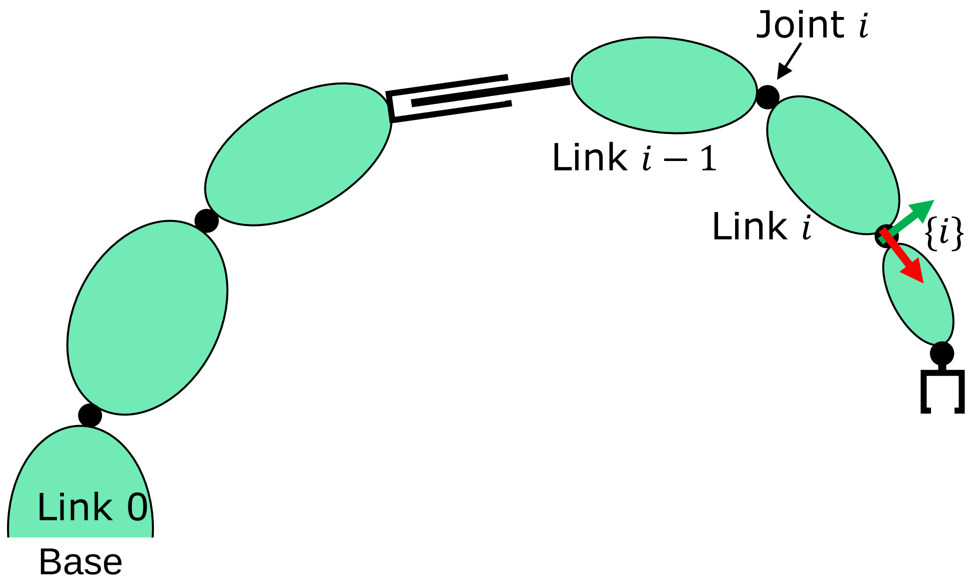
Angular velocity vector of link i w.r.t link $i-1$ expressed in frame $i-1$:

$$\omega_{i-1,i}^{i-1}$$

With this notation the angular velocity of the end-effector is

$$\omega_{0,n}^{0} = \omega_{0,1}^{0} + R_1^0\omega_{1,2}^{1} + \cdots + R_{n-1}^{0}\omega_{n-1,n}^{n-1}$$

$$\omega_{0,n}^{0} = \omega_{0,1}^{0} + \omega_{1,2}^{0} + \cdots + \omega_{n-1,n}^{0}$$

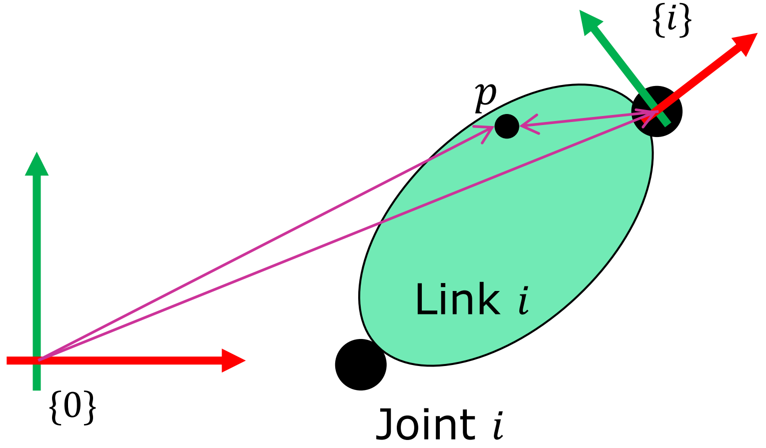


Linear Velocities

Linear Velocity of a Fixed Point on a Link

Frame i attached to link i with

$$H_i^0 = \begin{bmatrix} R_i^0 & o_i^0 \\ 0 & 1 \end{bmatrix}$$



Consider a point $p = p^i$ on link i fixed in frame i. The point coordinates wrt. frame 0:

$$p^0 = R_i^0 p^i + o_i^0$$

Its linear velocity:

$$\dot{p}^0 = \dot{R}_i^0 p^i + \dot{o}_i^0 = \underbrace{S(\omega_i^0)}_{\omega} \times \underbrace{R_i^0 p^i}_{r} + \underbrace{\dot{o}_i^0}_{v}$$

where $r = p^0 - o_i^0$

Based on the fact that p^i fixed, i.e. $\dot{p}^i = 0$

The Manipulator Jacobian

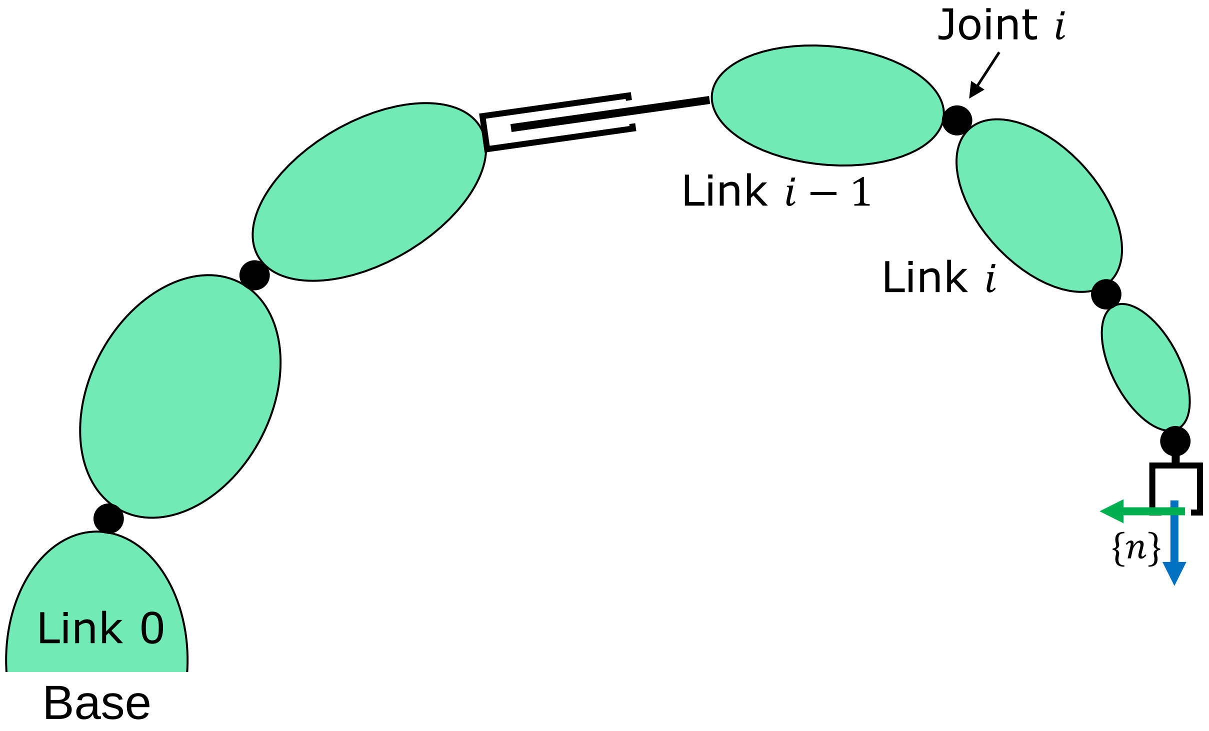
The Manipulator Jacobian

- Joint variables $q_i(t)$ → joint velocities $\dot{q}_i$
- End effector position $o_n^0(t)$ → linear velocity $v_n^0 = \dot{o}_n^0$
- End effector orientation $R_n^0(t)$ → angular velocity ω_n^0 such that $S(\omega_n^0) = \dot{R}_n^0 (R_n^0)^T$

For a given configuration q
we seek a relationship in the form:

$$\xi = \begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = \begin{bmatrix} J_{11} & \cdots & J_{1n} \\ \vdots & \ddots & \vdots \\ J_{61} & \cdots & J_{6n} \end{bmatrix} \dot{q} = J\, \dot{q}_i$$

$$\begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{q}$$



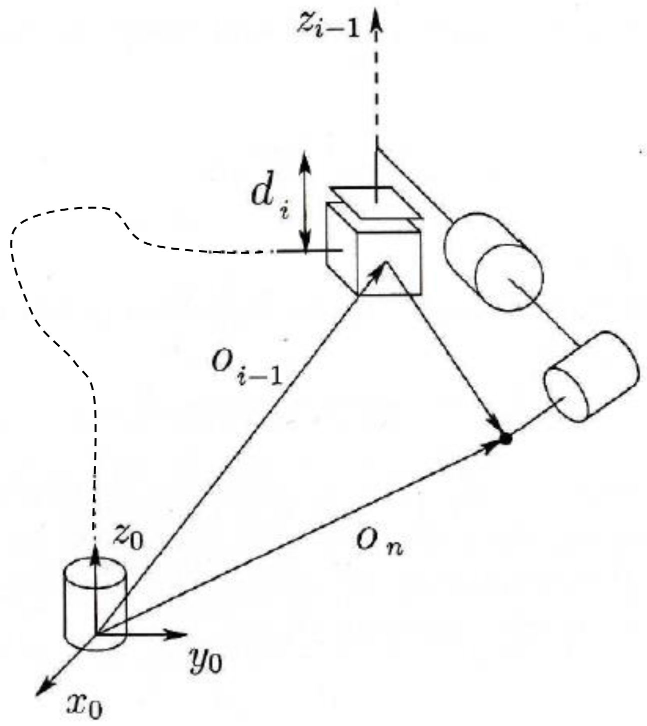
Derivation of the Jacobian

$$\begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{q}$$

Prismatic Joint

$$v_n^0 = \dot{o}_n^0 = \dot{d}_i z_{i-1}$$

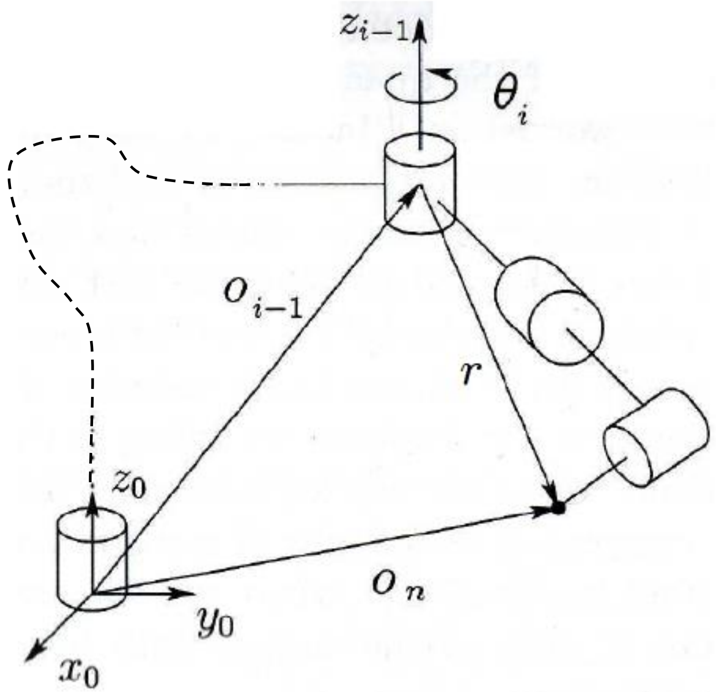
i.e. $J_{v_i} = z_{i-1}$



Revolute Joint

$$v_n^0 = \dot{o}_n^0 = \omega \times r = \dot{\theta}_i \, z_{i-1} \times (o_n - o_{i-1})$$

i.e. $J_{v_i} = z_{i-1} \times (o_n - o_{i-1})$



Derivation of the Jacobian

$$\begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{q}$$

Prismatic Joint

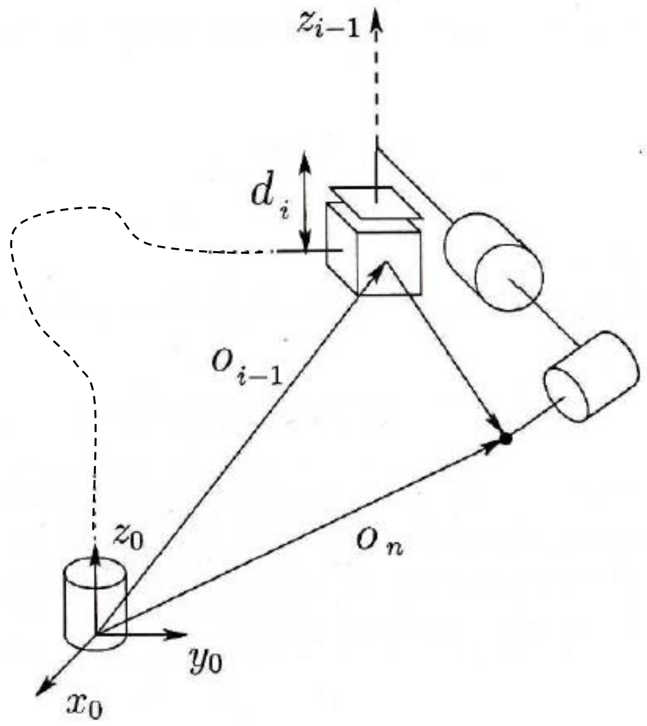
$$\omega_n^0 = 0$$

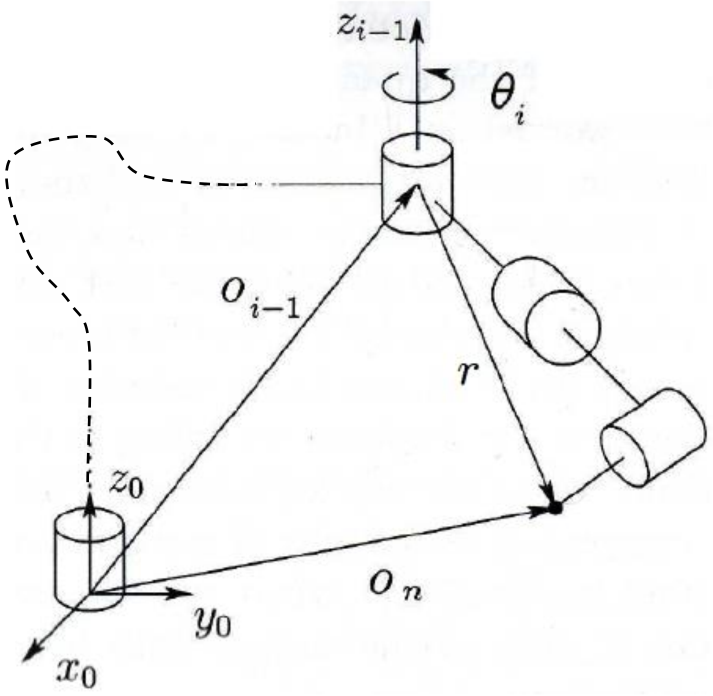
i.e. $J_{\omega_i} = 0$

Revolute Joint

$$\omega_n^0 = \dot{\theta}_i \, z_{i-1}^0$$

i.e. $J_{\omega_i} = z_{i-1}$





Derivation of the Jacobian

$$\begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{q}$$

Prismatic Joint	Revolute Joint
$J_{v_i} = z_{i-1}$	$J_{v_i} = z_{i-1} \times (o_n - o_{i-1})$
$J_{\omega_i} = 0$	$J_{\omega_i} = z_{i-1}$

$$J = [J_1 \quad \cdots \quad J_i \quad \cdots \quad J_n]$$

where

$$J_i = \begin{cases} \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix}, & i \text{ prismatic} \\ \begin{bmatrix} z_{i-1} \times (o_n - o_{i-1}) \\ z_{i-1} \end{bmatrix}, & i \text{ revolute} \end{cases}$$

Where do we get z_{i-1} and o_{i-1} from:

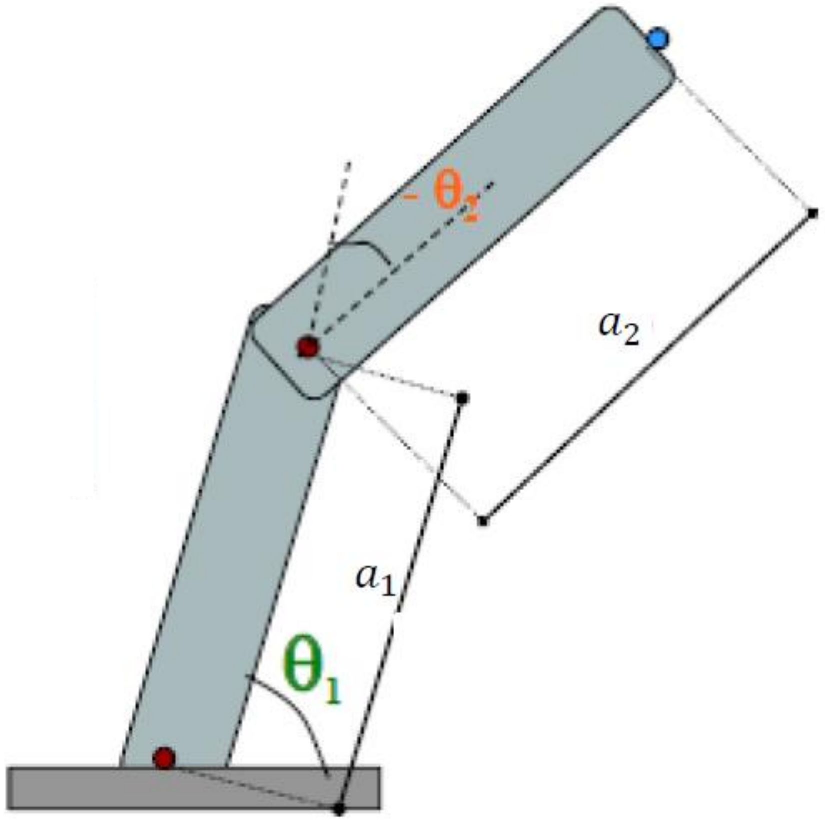
$$\mathrm{T}_{i-1} = \begin{array}{c} \begin{array}{cccc} & & z_{i-1} & o_{i-1} \\ & & \searrow & \swarrow \end{array} \\ \begin{bmatrix} r_{11} & r_{12} & \boxed{r_{13}} & \boxed{o_x} \\ r_{21} & r_{22} & \boxed{r_{23}} & \boxed{o_y} \\ r_{31} & r_{32} & \boxed{r_{33}} & \boxed{o_y} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Example of a Jacobian

Jacobian matrix of a two-link planar manipulator:

- $J = \begin{bmatrix} z_0 \times (o_n - o_0) & z_1 \times (o_n - o_1) \\ z_0 & z_1 \end{bmatrix}$
- $o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad o_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \quad o_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix}$
- $z_0 = z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Prismatic Joint	Revolute Joint
$J_{v_i} = z_{i-1}$	$J_{v_i} = z_{i-1} \times (o_n - o_{i-1})$
$J_{\omega_i} = 0$	$J_{\omega_i} = z_{i-1}$



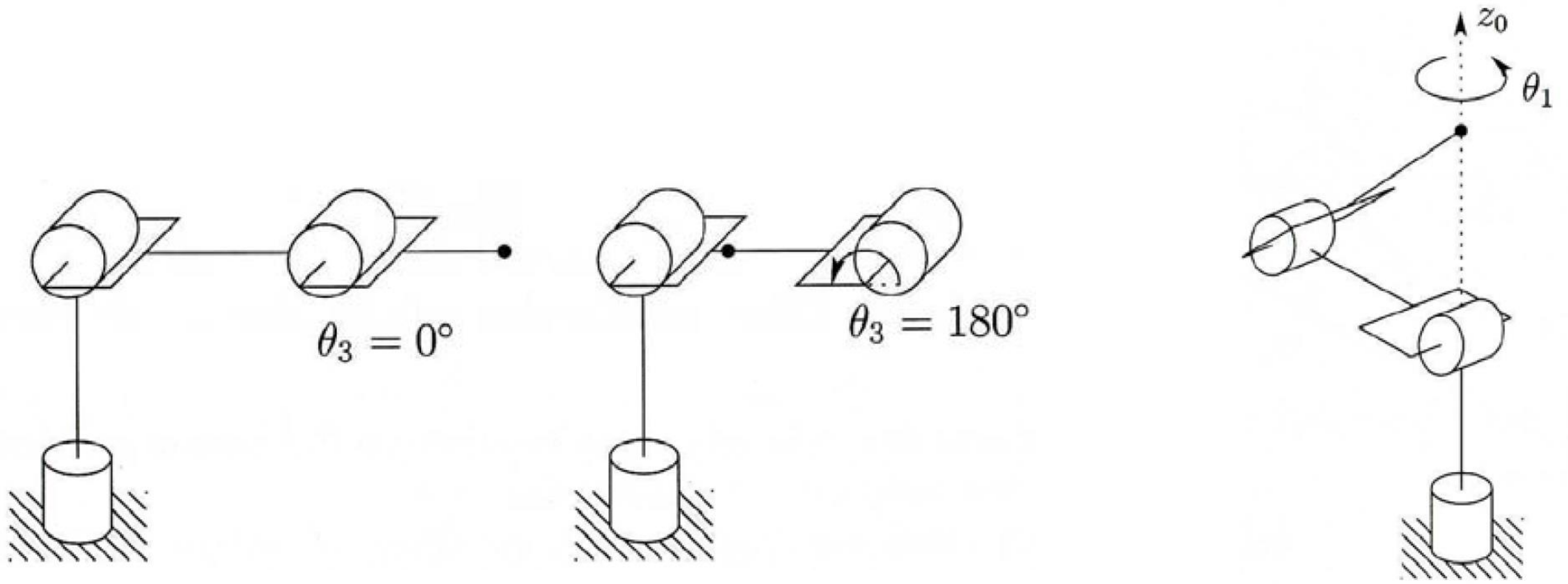
$$\Rightarrow \qquad J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Singularities

Singularities

- Singular configuration

$$q^s = [q_1^s, q_2^s, \dots, q_n^s] \quad \text{for} \qquad \operatorname{rank}[J(q^s)] < \max_q \{\operatorname{rank}[J(q)]\}$$



Singularity Implications

- Certain directions of motion may be unatainable
- Infinite joint velocity may correspond to bounded end-effector velocities
- Bounded joint torques may correspond to infinite end-effector torques/forces
- Often at the boundaries of the workspace
- Positions that may become unreachable by small perturbations of the robot link geometry

Static force/torque relationships

- Joint torques/forces vector τ can be found as

$$\tau = J^T(q)F$$

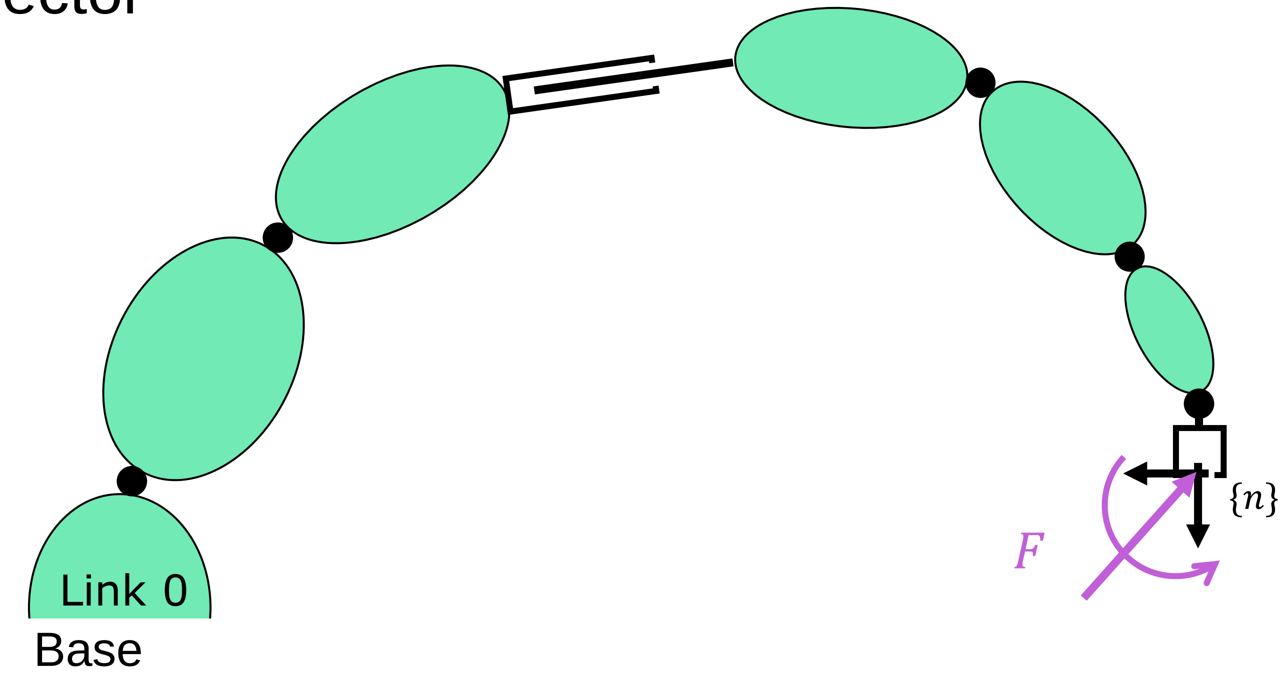
where F is a 6-component vector with forces/torques applied to the end-effector

- Remember that

$$\xi = J(q)\dot{q}$$

- Work conjugate pairs

$$(\xi, F)$$
$$(\dot{q}, \tau)$$



Inverse Velocity

Inverse Velocity

- Inverse Jacobian
  – Pseudoinverse

$$A^+ = \begin{cases} A^T[AA^T]^{-1}, & m < n \\ A^{-1}, & m = n \\ [A^TA]^{-1}A^T, & m > n \end{cases}$$

→ overactuated arm

→ fully actuated arm

→ underactuated arm

- Inverse velocity

$$\dot{q} = J^+ \xi$$

in general

$$\dot{q} = J^+ \xi + (I_{n\times n} - J^+J)b_{n\times 1}$$

Exercises

Exercises

Problem 1

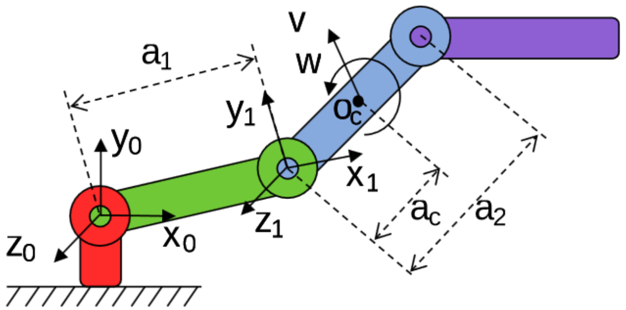
Two frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ are related by the time-independent homogeneous transformation

$$H_1^0 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A particle has velocity $v^1 = \{3, 1, 0\}^T$ relative to frame $o_1x_1y_1z_1$. What is the velocity of the particle in frame $o_0x_0y_0z_0$?

Problem 2

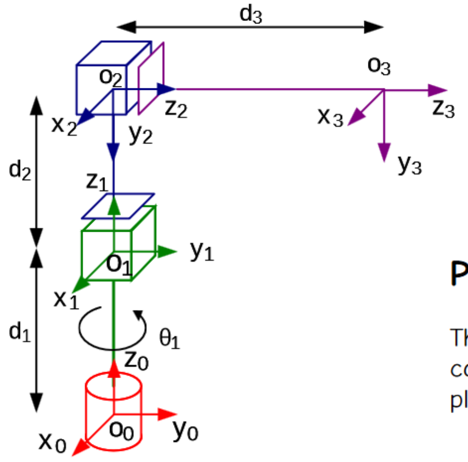
Consider the three-link planar manipulator shown below.



Compute the linear velocity v and angular velocity ω at the center o_c of link 2, where $a_c = a_2/2$, as functions of the joint variables $\{\theta_1, \theta_2\}$ and joint velocities $\{\dot\theta_1, \dot\theta_2\}$.

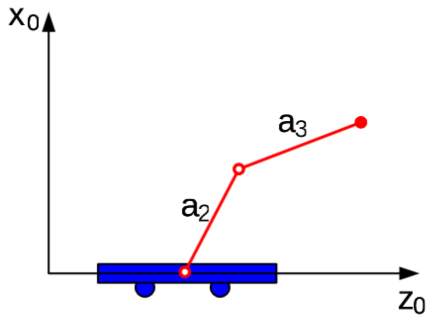
Problem 3

Find the 6×3 Jacobian matrix corresponding to the end-effector frame 3 of the cylindrical manipulator shown below.



Problem 4

The planar manipulator with a mobile platform moving in Z direction shown below can be interpreted as a PRR manipulator. The robot operates only in the XZ plane in the working space.



Define the coordinate frames 1 and 2 according to the Denavit-Hartenberg convention. Define frame 3 according to the usual convention for the end-effector, assuming that the grip opens and closes in the out of plane direction. Determine the transformation matrix T_3^0, and find the Jacobian matrix for this mobile robot.
Note: the end-effector frame does not follow the Denavit-Hartenberg convention, hence there is some ambiguity in how you define angle θ_3.

Robotics – 34753

Trajectory Planning & Manipulator Dynamics

Konstantinos Poulios
Associate Professor
Department of Civil and Mechanical Engineering
DTU Lyngby, building 404 / room 124

Trajectory Planning & Manipulator Dynamics – Lecture Overview

1. **Repetition**

2. **Trajectory Planning**

   - Point to Point Motion
   - Constant Jerk Curve

3. **Manipulator Dynamics**

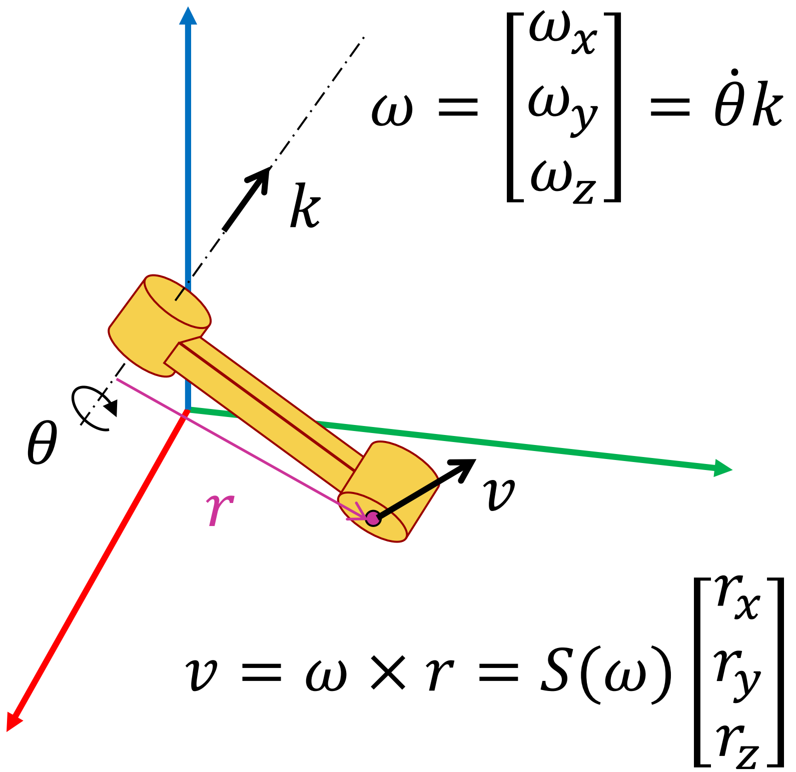
   - Euler-Lagrange Equations
   - Kinetic Energy
   - Potential Energy
   - Generalized Forces

Repetition

Repetition

Rotation by means of a skew symmetric matrix:

$$\omega = \begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} \rightarrow \quad S(\omega) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$



Properties:

- $S + S^T = 0$
- $S(\alpha a + \beta b) = \alpha S(a) + \beta S(b)$
- $S(R\, a) = R S(a) R^T$, for any orthonormal matrix R
- $x^T S(a) x = 0,$ for any vector x

Derivative of a rotation matrix:

$$\frac{dR}{d\theta} = S\, R(\theta) \qquad \text{and} \qquad \dot{R}_{k,\theta} = S(\dot{\theta} k)\, R_{k,\theta}$$

Repetition

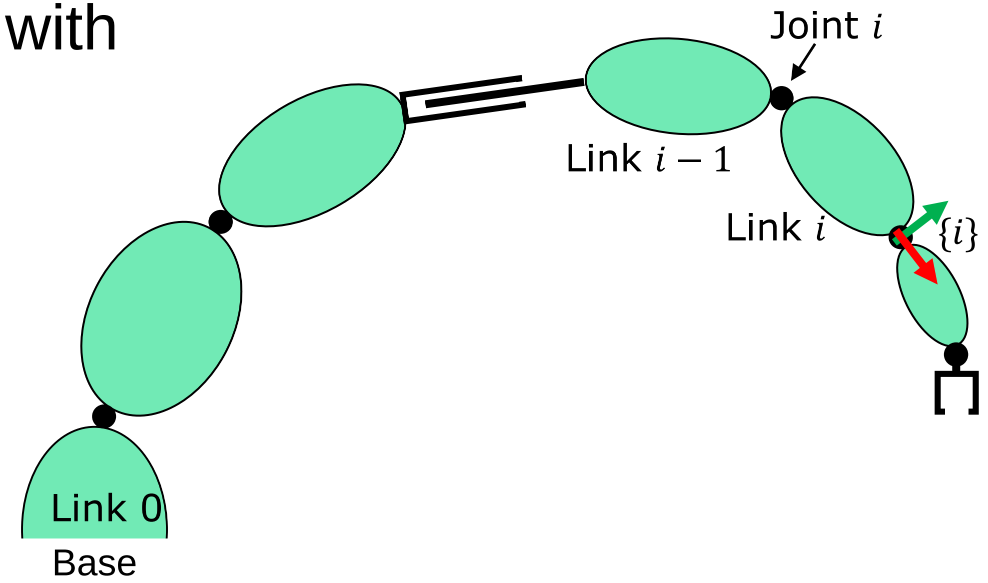
- Angular velocities (relative to the previous link and relative to the world frame)

$$\omega_{0,n}^{0} = \omega_{0,1}^{0} + R_1^0 \omega_{1,2}^{1} + \cdots + R_{n-1}^{0} \omega_{n-1,n}^{n-1}$$

$$\omega_{0,n}^{0} = \omega_{0,1}^{0} + \omega_{1,2}^{0} + \cdots + \omega_{n-1,n}^{0}$$

- Linear velocities (of arbitrary points on a link with respect to the world frame)

$$\dot{p}^0 = S(\omega_i^0) R_i^0 p^i + \dot{o}_i^0$$



Repetition

The **manipulator Jacobian** connects joint velocities to (end-effector) frame velocities:

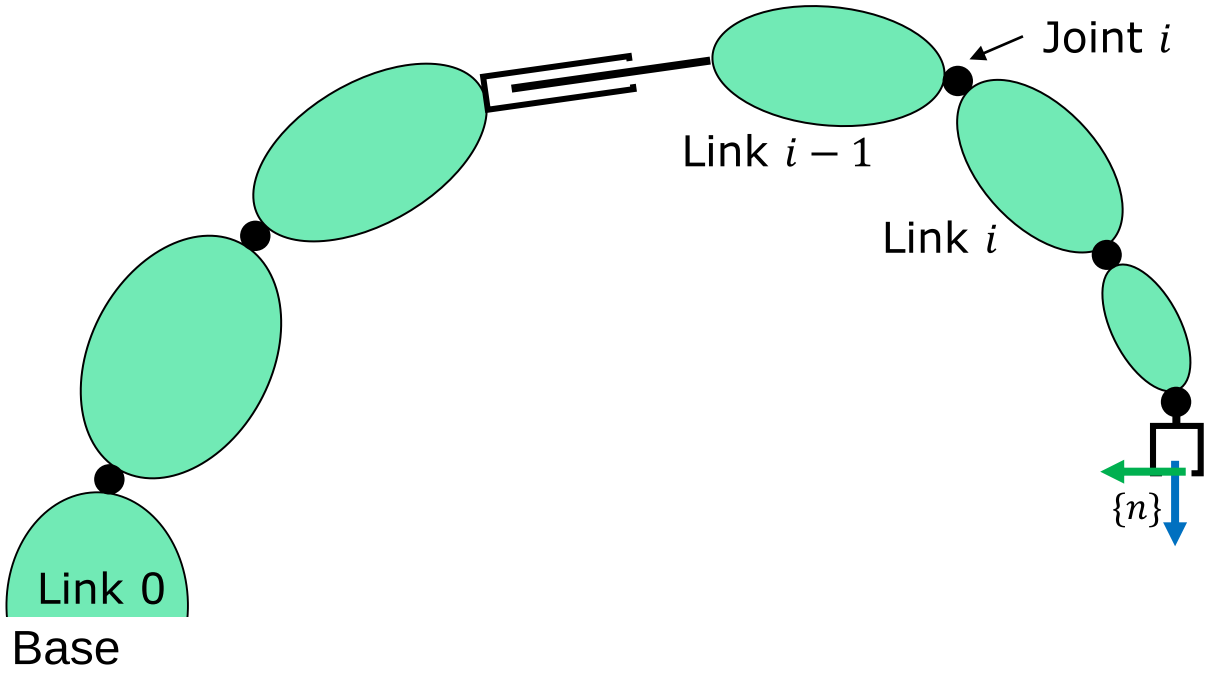
$$\xi = \begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = J\,\dot{q} = \begin{bmatrix} J_{11} & \cdots & J_{1n} \\ \vdots & \ddots & \vdots \\ J_{61} & \cdots & J_{6n} \end{bmatrix} \dot{q} = \begin{bmatrix} J_v(q(t)) \\ J_\omega(q(t)) \end{bmatrix} \dot{q}$$

$$\begin{array}{cccccc} & & & z_{i-1} & o_{i-1} & \\ \mathrm{T}_{i-1} = & \left[\begin{array}{c} r_{11} \\ r_{21} \\ r_{31} \\ 0 \end{array}\right. & \begin{array}{c} r_{12} \\ r_{22} \\ r_{32} \\ 0 \end{array} & \begin{array}{c} r_{13} \\ r_{23} \\ r_{33} \\ 0 \end{array} & \left.\begin{array}{c} o_x \\ o_y \\ o_y \\ 1 \end{array}\right] & \end{array}$$

Calculation of the manipulator Jacobian:

$$J = [J_1 \quad \cdots \quad J_i \quad \cdots \quad J_n]$$

where $$J_i = \begin{cases} \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix}, & i \text{ prismatic} \\ \begin{bmatrix} z_{i-1} \times (o_n - o_{i-1}) \\ z_{i-1} \end{bmatrix}, & i \text{ revolute} \end{cases}$$

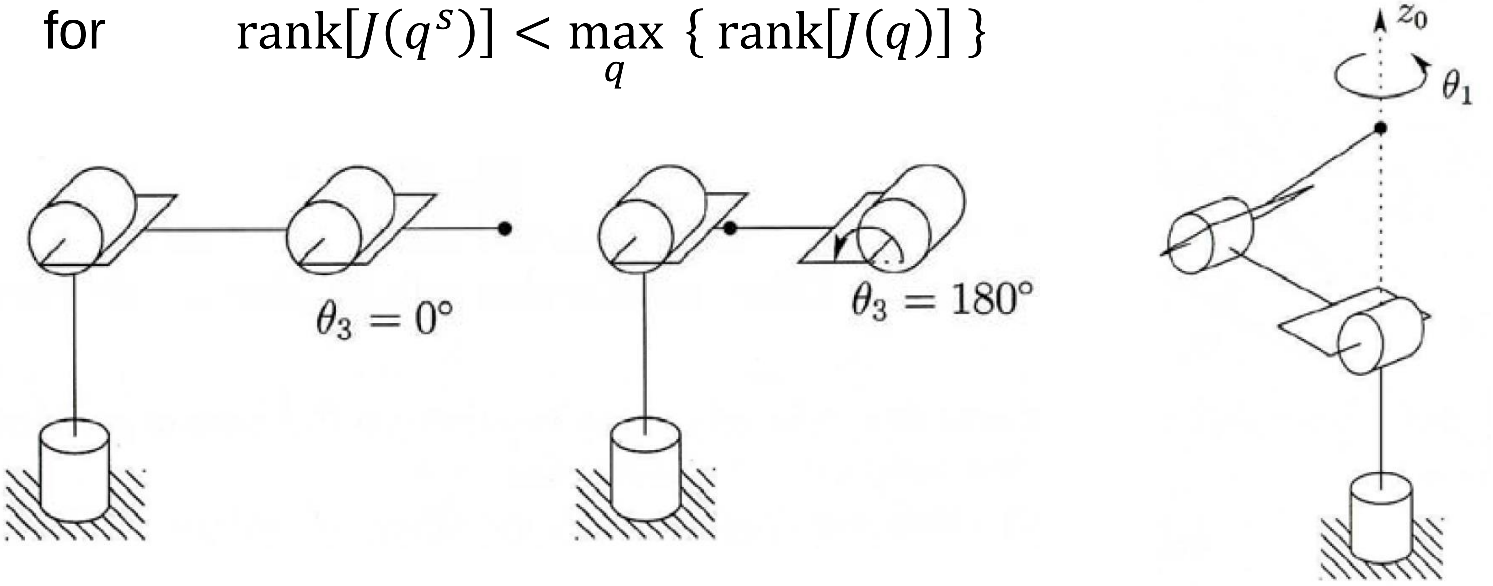


Repetition

Singularities

- Singular configuration

$$q^s = [q_1^s, q_2^s, \dots, q_n^s] \quad \text{for} \quad \operatorname{rank}[J(q^s)] < \max_q \{\operatorname{rank}[J(q)]\}$$



- Inverse Jacobian
  - Pseudoinverse

$$A^+ = \begin{cases} A^T[AA^T]^{-1}, & m < n \\ A^{-1}, & m = n \\ [A^TA]^{-1}A^T, & m > n \end{cases}$$

Repetition

Static force/torque relationships

- Joint torques/forces vector τ can be found as

$$\tau = J^T(q)F$$

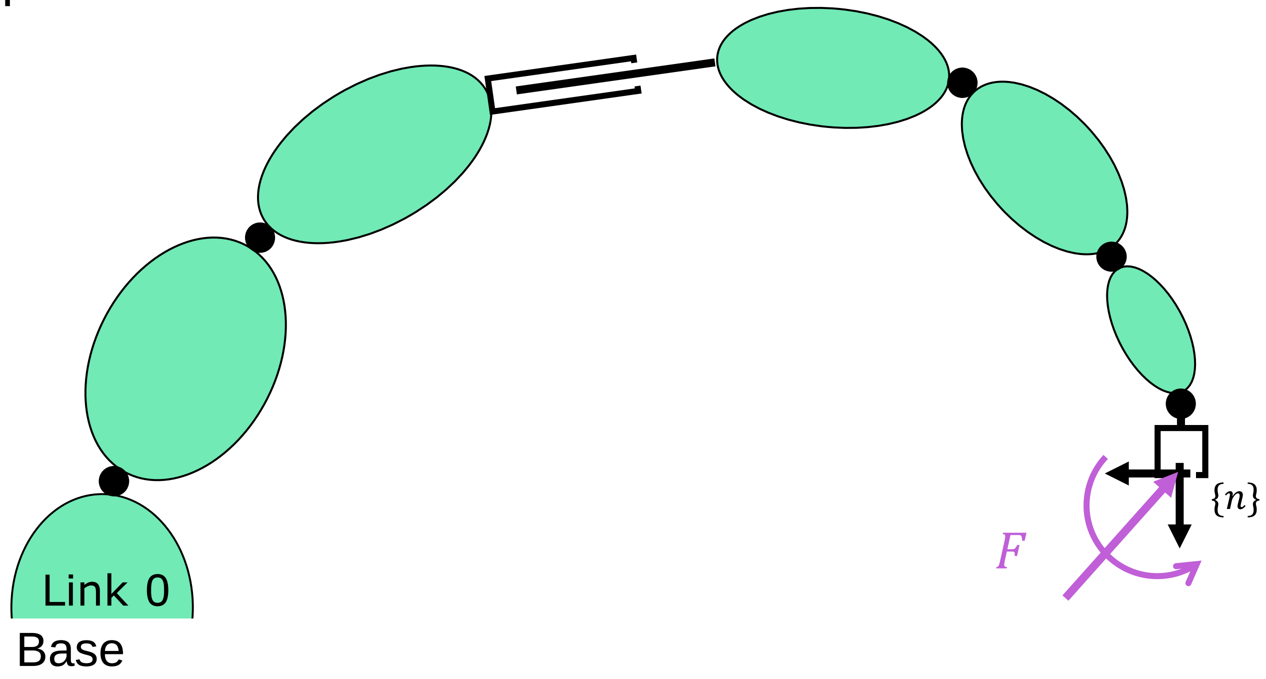
where F is a 6-component vector with forces/torques applied to the end-effector

- Remember that

$$\xi = J(q)\dot{q}$$

- Work conjugate pairs

$$(\xi, F)$$
$$(\dot{q}, \tau)$$



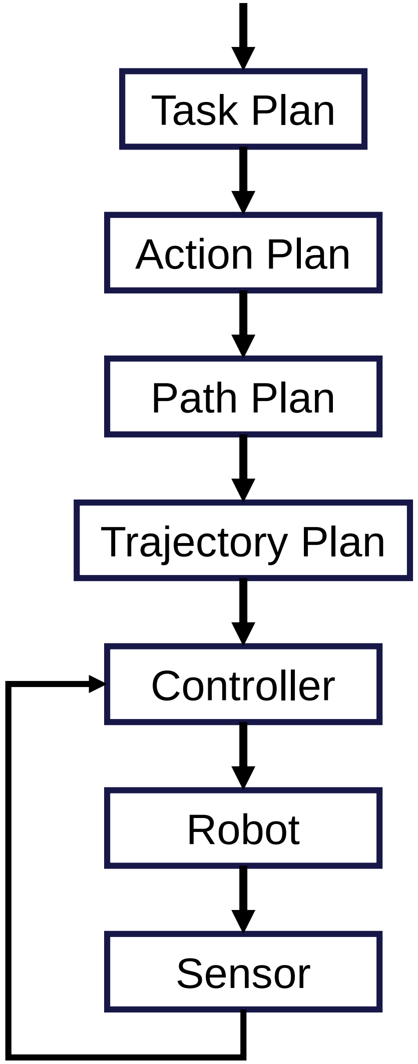
Trajectory Planning

Path versus Trajectory

- **Path**: Purely geometric description of the route that the robot manipulator will follow

- **Trajectory**: Time-aware description of a followed path, defined as $q(t)$ → velocities/accelerations

Robot Motion Planning



- **Path planning**:
  - Geometric path, e.g. point-wise in configuration space or in workspace
  - Avoidance of obstacles, shortest path …
- **Trajectory planning**:
  - Interpolate or approximate the desired path from point to point by a class of **polynomial functions of time**
  - Satisfy position, velocity and acceleration constraints
  - Define $q(t)$ piecewise to control the manipulator from the initial to the final configuration

Trajectory Planning

Path constraints
(e.g. continuity/smoothness)

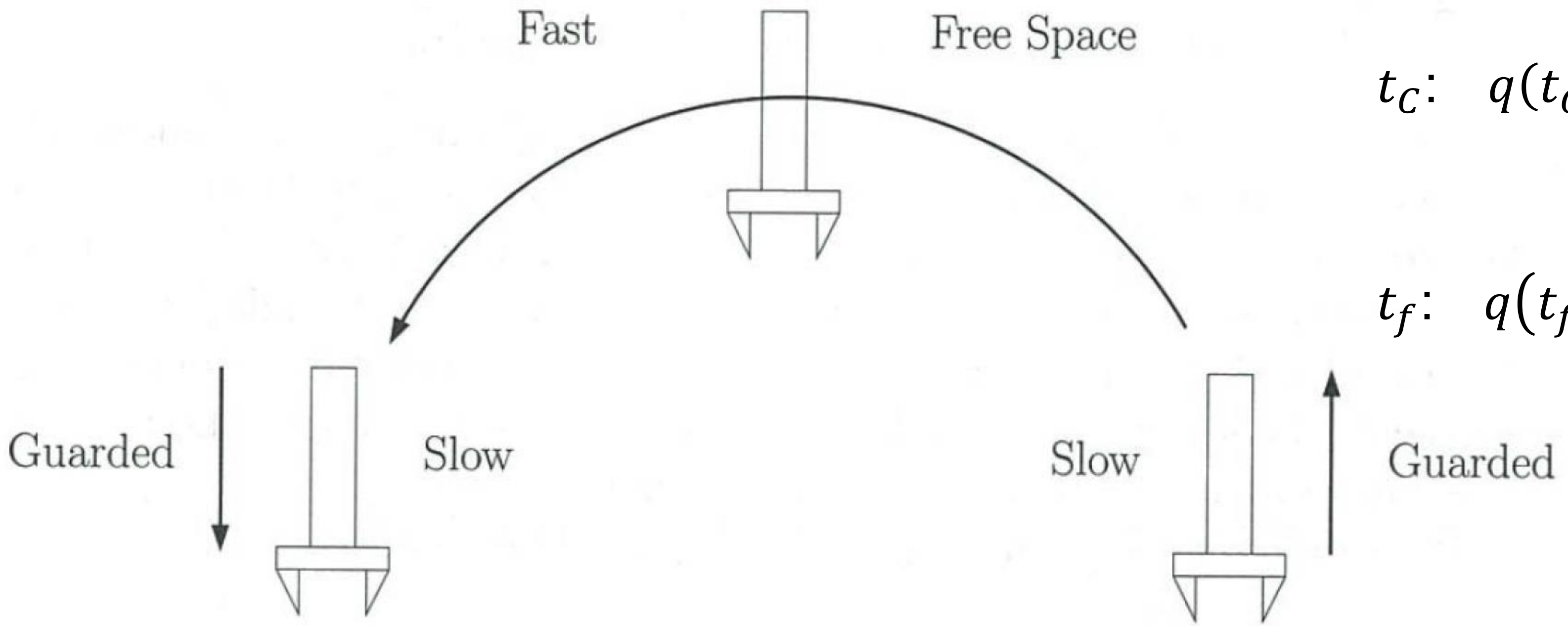
Path description
(geometry)

Trajectory planner

Joint/configuration space
$q(t), \dot{q}(t), \ddot{q}(t)$
or
Cartesian space/workspace
$p(t), v(t), a(t)$

Trajectory Planning

- Decomposition of a path into segments with fast and slow velocity profiles



$t_{in} \rightarrow t_f$

t_{in}: $q(t_{in}) = s_{in}$, $\dot{q}(t_0) = v_{in}$, $\ddot{q}(t_{in}) = a_{in}$

t_A: $q(t_A) = s_A$, $\dot{q}(t_A) = v_A$, $\ddot{q}(t_A) = a_A$

t_B: $q(t_B) = s_B$, $\dot{q}(t_B) = v_B$, $\ddot{q}(t_B) = a_B$

t_C: $q(t_C) = s_C$, $\dot{q}(t_C) = v_C$, $\ddot{q}(t_C) = a_C$

....

t_f: $q(t_f) = s_f$, $\dot{q}(t_f) = v_f$, $\ddot{q}(t_f) = a_f$

Trajectory Planning

- Interpolation between two points in configuration space

$$t_A \rightarrow t_B$$

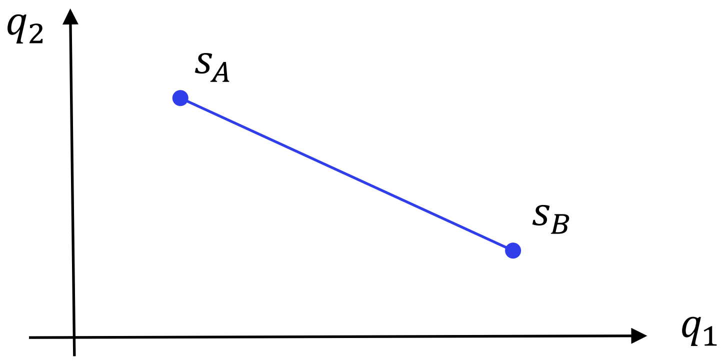
$$t_A: \quad q(t_A) = s_A, \qquad \dot{q}(t_A) = v_A, \qquad \ddot{q}(t_A) = a_A$$

$$t_B: \quad q(t_B) = s_B, \qquad \dot{q}(t_B) = v_B, \qquad \ddot{q}(t_B) = a_B$$

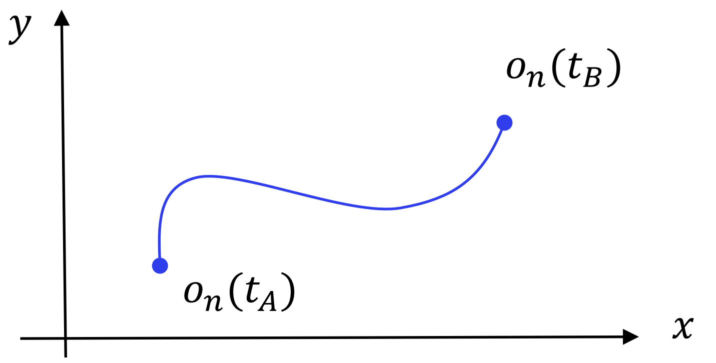
- E.g. linear interpolation

$$q(t) = \frac{t_B - t}{t_B - t_A} s_A + \frac{t - t_A}{t_B - t_A} s_B$$

Path in a 2D configuration space

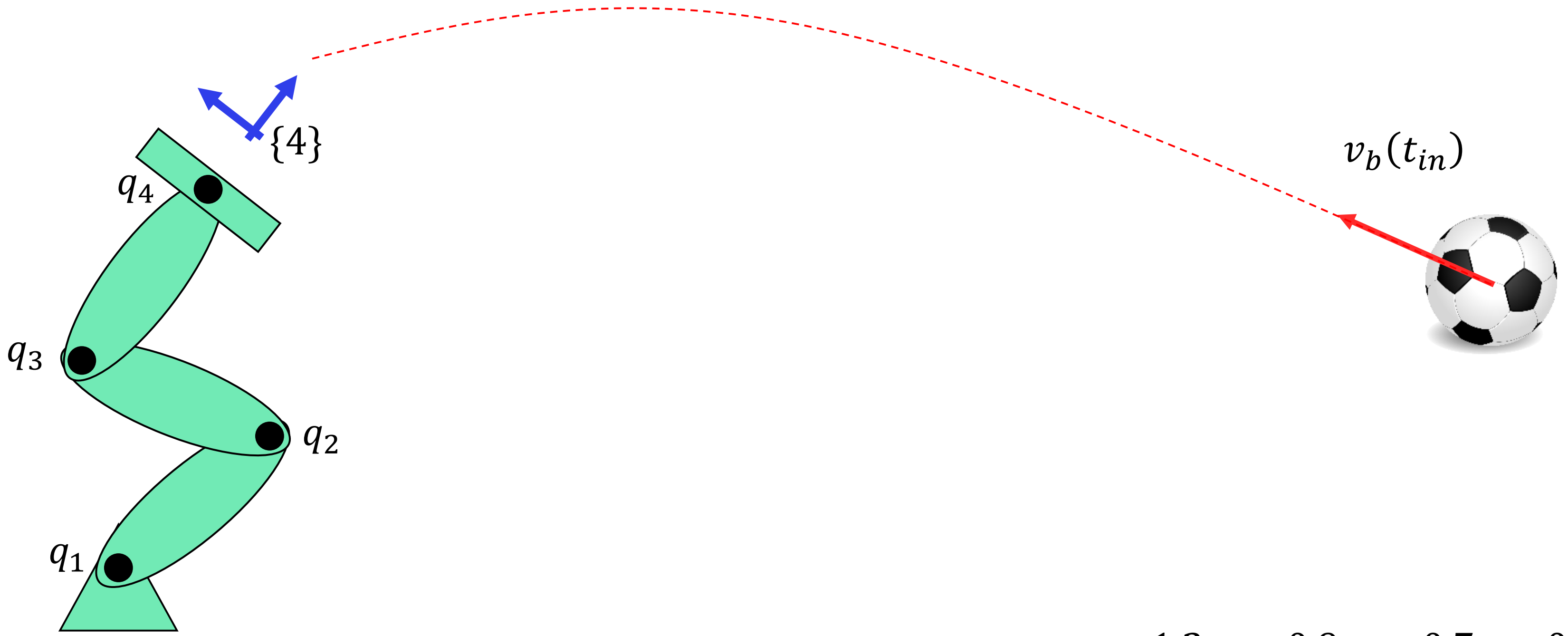


End effector path in workspace



Trajectory Planning Example

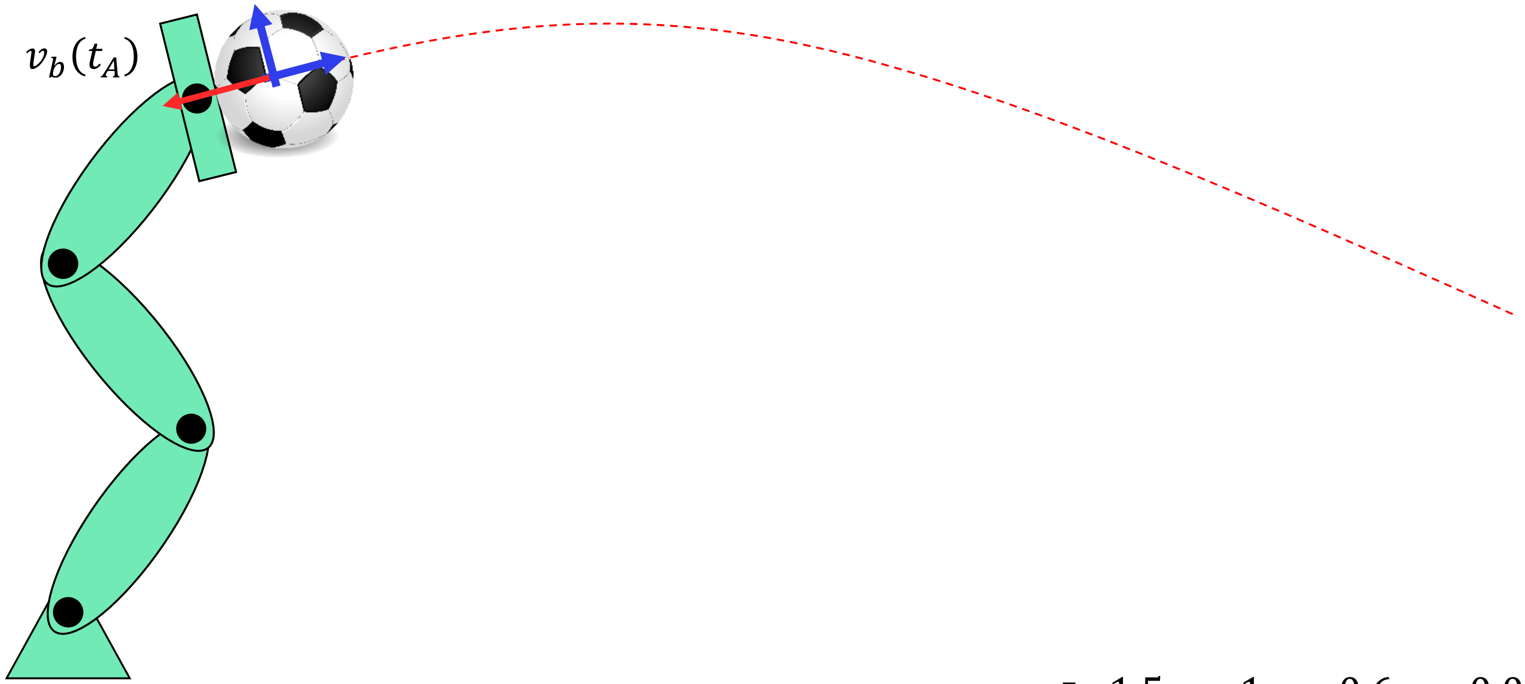
- Soccer ball catcher



$$q(t_{in}) = \begin{bmatrix} 0.25\pi \\ 0.6\pi \\ -0.52\pi \\ 0 \end{bmatrix} \quad T_4^0 = \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0.3 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 1.2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \dot{q} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad J_4 = \begin{bmatrix} -1.2 & -0.9 & -0.7 & -0.1 \\ 0.3 & -0.02 & 0.4 & 0.1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad v_4^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \omega_4^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Trajectory Planning Example

- Soccer ball catcher



$$q(t_A) = \begin{bmatrix} 0.33\pi \\ 0.4\pi \\ -0.4\pi \\ -0.2\pi \end{bmatrix} \quad T_4^0 = \begin{bmatrix} 0 & -0.15 & 0.989 & 0.4 \\ 0 & 0.989 & 0.15 & 1.5 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \dot{q} = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}$$

$$J_4 = \begin{bmatrix} -1.5 & -1 & -0.6 & -0.02 \\ 0.4 & 0.1 & 0.4 & 0.15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad v_4^0 = v_b(t_A) = \begin{bmatrix} -9.89 \\ -1.5 \\ 0 \end{bmatrix} \quad \omega_4^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Trajectory Planning Example

$$\dot{q} = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \qquad J_4 = \begin{bmatrix} -1.5 & -1 & -0.6 & -0.02 \\ 0.4 & 0.1 & 0.4 & 0.15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$v_4^0 = v_b(t_A) = \begin{bmatrix} -9.89 \\ -1.5 \\ 0 \end{bmatrix}$$

$$\omega_4^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{q} = J_4^T (J_4 J_4^T)^{-1} \begin{pmatrix} v_4^0 \\ \omega_4^0 \end{pmatrix} = \begin{bmatrix} -0.51 & 0.966 & 0 & 0 & 0 & -0.401 \\ -0.494 & -2.91 & 0 & 0 & 0 & 0.629 \\ 0.411 & 2.452 & 0 & 0 & 0 & -0.073 \\ 0.592 & -0.508 & 0 & 0 & 0 & 0.845 \end{bmatrix} \begin{bmatrix} -9.89 \\ -1.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.596 \\ 9.246 \\ -7.746 \\ -5.096 \end{bmatrix}$$

Check that indeed

$$J_4 \dot{q} = \begin{bmatrix} -1.5 & -1 & -0.6 & -0.02 \\ 0.4 & 0.1 & 0.4 & 0.15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3.596 \\ 9.246 \\ -7.746 \\ -5.096 \end{bmatrix} = \begin{bmatrix} -9.89 \\ -1.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Trajectory Planning – Interpolation Functions

In the previous example, we know the state at time t_{in}:

$$q(t_{in}) = \begin{bmatrix} 0.25\pi \\ 0.6\pi \\ -0.52\pi \\ 0 \end{bmatrix} \quad \text{and} \quad \dot{q}(t_{in}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and then, at the second time instant t_A we need to reach a state with

$$q(t_A) = \begin{bmatrix} 0.33\pi \\ 0.4\pi \\ -0.4\pi \\ -0.2\pi \end{bmatrix} \quad \text{and} \quad \dot{q}(t_A) = \begin{bmatrix} 3.596 \\ 9.246 \\ -7.746 \\ -5.096 \end{bmatrix}$$

How do we control the joint variables q_1, q_2, q_3 and q_4 from t_{in} to t_A?

Naive linear interpolation? $$q_i(t) = \frac{t_A - t}{t_A - t_{in}} q_i(t_{in}) + \frac{t - t_{in}}{t_A - t_{in}} q_i(t_A) = c_0 + c_1 t$$

Trajectory Planning – Interpolation Functions

Cubic interpolation:

$$q(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$$

$$\Rightarrow \dot{q}(t) = c_1 + 2c_2 t + 3c_3 t^2$$

can satisfy two initial and two final conditions:

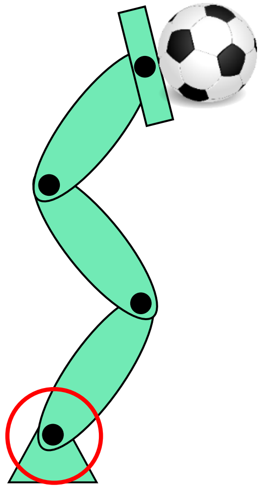
$$q(t_{in}) = q_{in} \quad \text{and} \quad \dot{q}(t_{in}) = v_{in}$$

$$q(t_A) = q_A \quad \text{and} \quad \dot{q}(t_A) = v_A$$

These four conditions can be solved for c_0, c_1, c_2 and c_3:

$$\begin{bmatrix} 1 & t_{in} & t_{in}^2 & t_{in}^3 \\ 0 & 1 & 2t_{in} & 3t_{in}^2 \\ 1 & t_A & t_A^2 & t_A^3 \\ 0 & 1 & 2t_A & 3t_A^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} q_{in} \\ v_{in} \\ q_A \\ v_A \end{bmatrix}$$

Trajectory Planning – Example



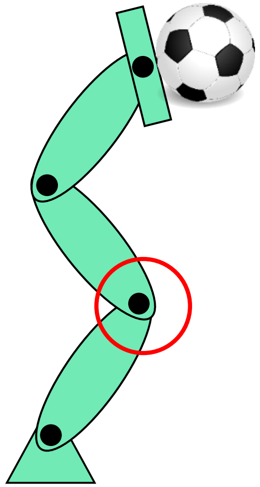
Assume $t_{in} = 0$ and $t_A = 1$, then:

$$q(0) = \begin{bmatrix} 0.25\pi \\ 0.6\pi \\ -0.52\pi \\ 0 \end{bmatrix}, \dot{q}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, q(1) = \begin{bmatrix} 0.33\pi \\ 0.4\pi \\ -0.4\pi \\ -0.2\pi \end{bmatrix}, \dot{q}(1) = \begin{bmatrix} 3.60 \\ 9.25 \\ -7.75 \\ -5.10 \end{bmatrix}$$

For $i = 1$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0.25\pi \\ 0 \\ 0.33\pi \\ 3.60 \end{bmatrix} \Rightarrow \begin{aligned} c_0 &= 0.785 \\ c_1 &= 0 \\ c_2 &= -2.842 \\ c_3 &= 3.093 \end{aligned}$$

Trajectory Planning – Example



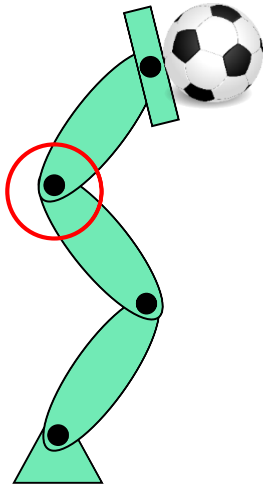
Assume $t_{in} = 0$ and $t_A = 1$, then:

$$q(0) = \begin{bmatrix} 0.25\pi \\ 0.6\pi \\ -0.52\pi \\ 0 \end{bmatrix}, \dot{q}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, q(1) = \begin{bmatrix} 0.33\pi \\ 0.4\pi \\ -0.4\pi \\ -0.2\pi \end{bmatrix}, \dot{q}(1) = \begin{bmatrix} 3.60 \\ 9.25 \\ -7.75 \\ -5.10 \end{bmatrix}$$

For $i = 2$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0.6\pi \\ 0 \\ 0.4\pi \\ 9.25 \end{bmatrix} \Rightarrow \begin{matrix} c_0 = 1.885 \\ c_1 = 0 \\ c_2 = -11.131 \\ c_3 = 10.503 \end{matrix}$$

Trajectory Planning – Example



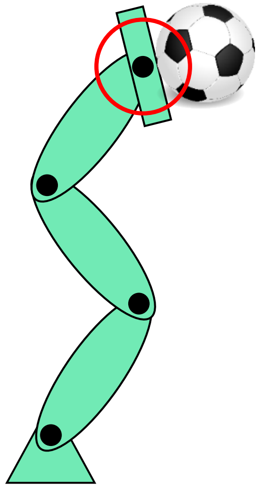
Assume $t_{in} = 0$ and $t_A = 1$, then:

$$q(0) = \begin{bmatrix} 0.25\pi \\ 0.6\pi \\ -0.52\pi \\ 0 \end{bmatrix}, \dot{q}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, q(1) = \begin{bmatrix} 0.33\pi \\ 0.4\pi \\ -0.4\pi \\ -0.2\pi \end{bmatrix}, \dot{q}(1) = \begin{bmatrix} 3.60 \\ 9.25 \\ -7.75 \\ -5.10 \end{bmatrix}$$

For $i = 3$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -0.52\pi \\ 0 \\ -0.4\pi \\ -7.75 \end{bmatrix} \Rightarrow \begin{aligned} c_0 &= -1.634 \\ c_1 &= 0 \\ c_2 &= 8.877 \\ c_3 &= -8.500 \end{aligned}$$

Trajectory Planning – Example



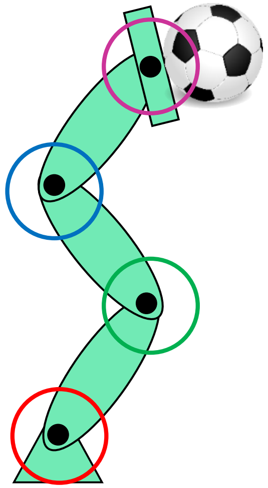
Assume $t_{in} = 0$ and $t_A = 1$, then:

$$q(0) = \begin{bmatrix} 0.25\pi \\ 0.6\pi \\ -0.52\pi \\ 0 \end{bmatrix}, \dot{q}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, q(1) = \begin{bmatrix} 0.33\pi \\ 0.4\pi \\ -0.4\pi \\ -0.2\pi \end{bmatrix}, \dot{q}(1) = \begin{bmatrix} 3.60 \\ 9.25 \\ -7.75 \\ -5.10 \end{bmatrix}$$

For $i = 4$:

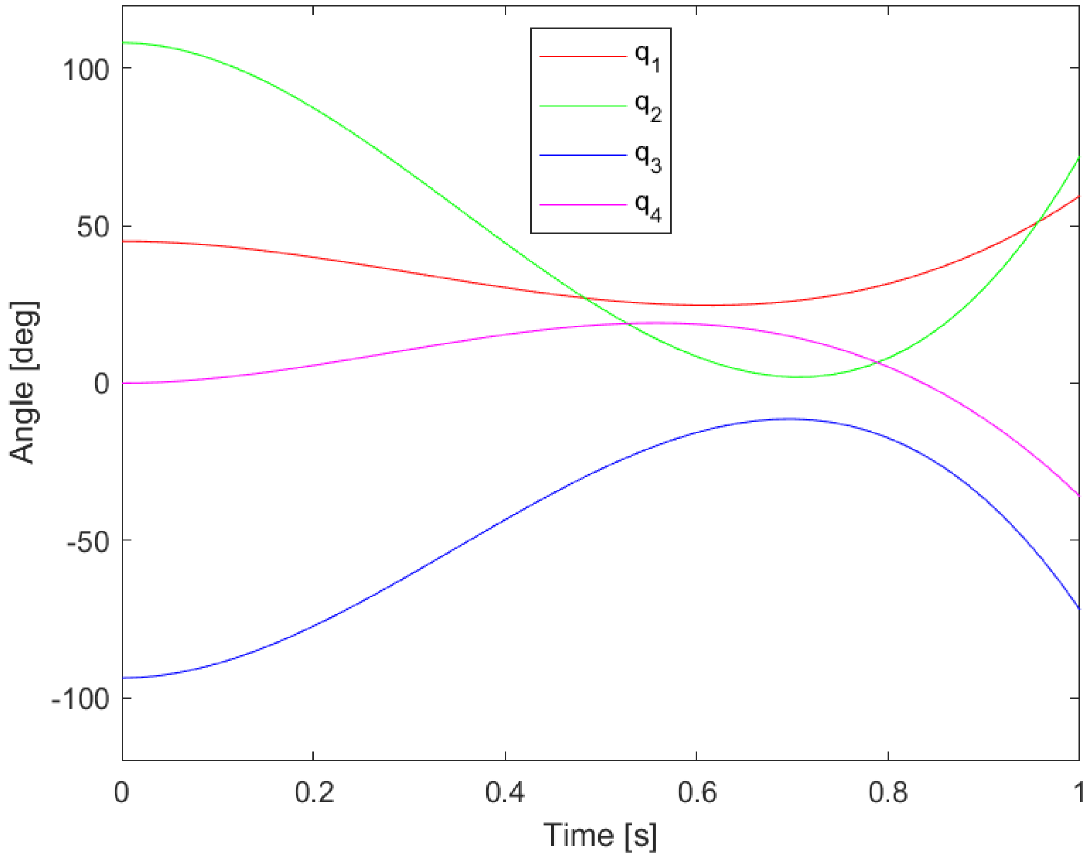
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.2\pi \\ -5.10 \end{bmatrix} \Rightarrow \begin{aligned} c_0 &= 0 \\ c_1 &= 0 \\ c_2 &= 3.211 \\ c_3 &= -3.839 \end{aligned}$$

Trajectory Planning – Example



Hence, the interpolation functions for the four joints between $t_{in} = 0$ and $t_A = 1$ sec are:

$$q_1 = 0.785 - 2.842\, t^2 + 3.093\, t^3$$
$$q_2 = 1.885 - 11.131\, t^2 + 10.503\, t^3$$
$$q_3 = -1.634 + 8.877\, t^2 - 8.5\, t^3$$
$$q_4 = 3.211\, t^2 - 3.839\, t^3$$



Trajectory Planning – Interpolation Functions

Quintic interpolation:

$$q(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5$$
$$\Rightarrow \dot{q}(t) = c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4$$
$$\Rightarrow \ddot{q}(t) = 2c_2 + 6c_3 t + 12c_4 t^2 + 20c_5 t^3$$

can satisfy three initial and three final conditions:

$$q(t_{in}) = q_{in} \quad \& \quad \dot{q}(t_{in}) = v_{in} \quad \& \quad \ddot{q}(t_{in}) = a_{in}$$
$$q(t_A) = q_A \quad \& \quad \dot{q}(t_A) = v_A \quad \& \quad \ddot{q}(t_A) = a_A$$

These six conditions can be solved for c_0, c_1, c_2, c_3, c_4 and c_5:

$$\begin{bmatrix} 1 & t_{in} & t_{in}^2 & t_{in}^3 & t_{in}^4 & t_{in}^5 \\ 0 & 1 & 2t_{in} & 3t_{in}^2 & 4t_{in}^3 & 5t_{in}^4 \\ 0 & 0 & 2 & 6t_{in} & 12t_{in}^2 & 20t_{in}^3 \\ 1 & t_A & t_A^2 & t_A^3 & t_A^4 & t_A^5 \\ 0 & 1 & 2t_A & 3t_A^2 & 4t_A^3 & 5t_A^4 \\ 0 & 0 & 2 & 6t_A & 12t_A^2 & 20t_A^3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} q_{in} \\ v_{in} \\ a_{in} \\ q_A \\ v_A \\ a_A \end{bmatrix}$$

Manipulator Dynamics

Kinetic and Potential Energy

- Kinematics < Statics < **Dynamics**
- **Rigid bodies** < Flexible couplings < Deformable bodies
- Kinetic energy of rigid bodies:

$$\mathcal{K} = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^T v_i + \frac{1}{2} \omega_i^T \mathcal{I}_i \omega_i$$

$$\mathcal{K} = \frac{1}{2} \dot{q}^T D(q) \dot{q}$$

- Potential energy of rigid bodies:

$$\mathcal{P} = -\sum_{i=1}^{n} m_i \boldsymbol{g}^T r_{c_i}$$

Inertia Tensor

- Kinetic energy of rigid bodies:

$$\mathcal{K} = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^T v_i + \frac{1}{2} \omega_i^T \mathcal{J}_i \omega_i$$

- Inertia tensor in global frame (inertial frame):

$$\mathcal{J}_i = R_i(q_1, q_2, \ldots)\, I_i\, R_i^T(q_1, q_2, \ldots)$$
$$\mathcal{J}_i = R_i\, I_i\, R_i^T$$

- Inertia tensor in local frame:

$$I_i = \begin{bmatrix} I_{xx} & I_{xy} & I_{zx} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{zx} & I_{yz} & I_{zz} \end{bmatrix}$$

e.g. for a $a \times b \times c$ box: $I_{xx} = \frac{m}{12}(b^2 + c^2),\ I_{yy} = \frac{m}{12}(a^2 + c^2),\ I_{zz} = \frac{m}{12}(a^2 + b^2)$

Kinetic and Potential Energy

- Location of the center of gravity of link i expressed in the world frame:

$$r_{c_i}$$

- Inertia tensor (3×3) of link i w.r.t its center of mass c_i and axes parallel to frame i:

$$I_i$$

- Inertia tensor (3×3) of link i w.r.t. its center of mass c_i and axes parallel to the world frame axes:

$$\mathcal{J}_i = R_i I_i R_i^T$$

- Jacobian of the center of mass c_i of link i:

$$J_{v,c_i} \quad \text{and} \quad J_{\omega_i}$$

- Translational and rotational velocities:

$$v_{c_i} = J_{v,c_i}\ \dot{q}$$
$$\omega_i = J_{\omega_i}\ \dot{q}$$

Kinetic and Potential Energy

- The inertia matrix of the robotic manipulator (n by n):

$$D(q) = \sum_{i=1}^{n} m_i\, J_{v,c_i}^T J_{v,c_i} + J_{\omega_i}^T R_i I_i R_i^T J_{\omega_i}$$

- The kinetic energy of the manipulator is then written in compact form:

$$\mathcal{K} = \frac{1}{2}\dot{q}^T D(q)\dot{q}$$

- The potential energy is also known as

$$\mathcal{P} = -\sum_{i=1}^{n} m_i \boldsymbol{g}^T r_{c_i}(q)$$

Euler-Lagrange Equations of Motion

- The Lagrangian function:

$$\mathcal{L}(q,\dot{q}) = \mathcal{K} - \mathcal{P} = \frac{1}{2}\dot{q}^T D(q)\dot{q} - \mathcal{P}(q) = \sum_{i,j}\frac{1}{2}d_{ij}(q)\dot{q}_i\dot{q}_j + \sum_i m_i \boldsymbol{g}^T r_{c_i}(q)$$

- Equations of motions for $k = 1, \dots, n$:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_k} - \frac{\partial\mathcal{L}}{\partial q_k} = \tau_k$$

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i,j} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k$$

where $$c_{ijk}(q) = \frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right)$$

- $\dot{q}_i^2$ terms express centrifugal effects
- $\dot{q}_i\dot{q}_j$ terms express Coriolis effects

Equations of Motion in Matrix Form

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

Where:

$$D(q) = \sum_{i=1}^{n} m_i \, J_{v,c_i}^T \, J_{v,c_i} + J_{\omega_i}^T \, R_i I_i R_i^T J_{\omega_i}$$

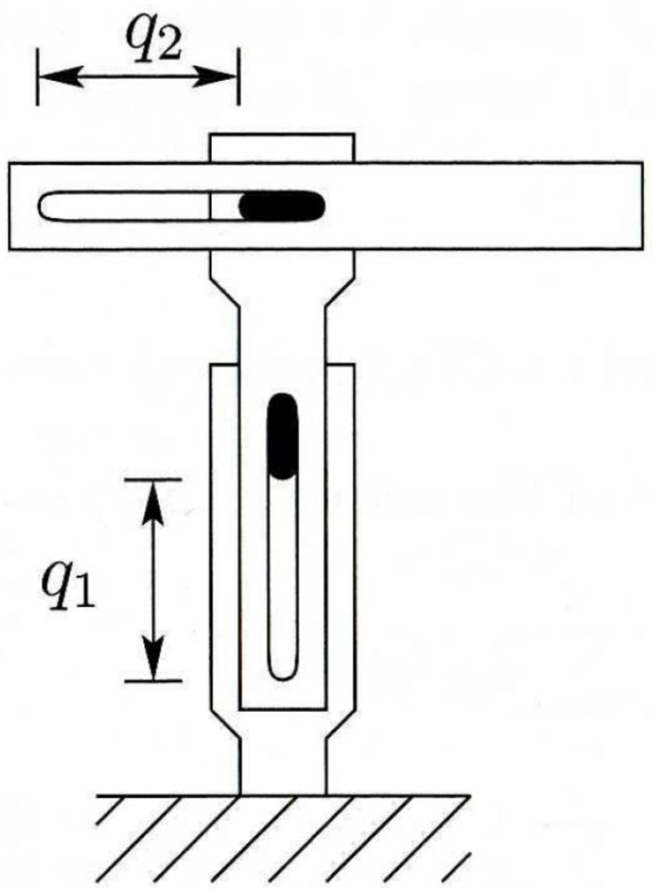
$$[C(q,\dot{q})]_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i = \sum_{i=1}^{n} \frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right)\dot{q}_i$$

$$g = \begin{bmatrix} g_1(q) \\ \dots \\ g_n(q) \end{bmatrix} \quad \text{where} \quad g_k = \frac{\partial \mathcal{P}}{\partial q_k}$$

Example

Two-link Cartesian manipulator:

- Forward kinematics?
- Manipulator Jacobian?
- Kinetic and potential energies?
- Euler-Lagrange equation?



Example

Two-link Cartesian manipulator:

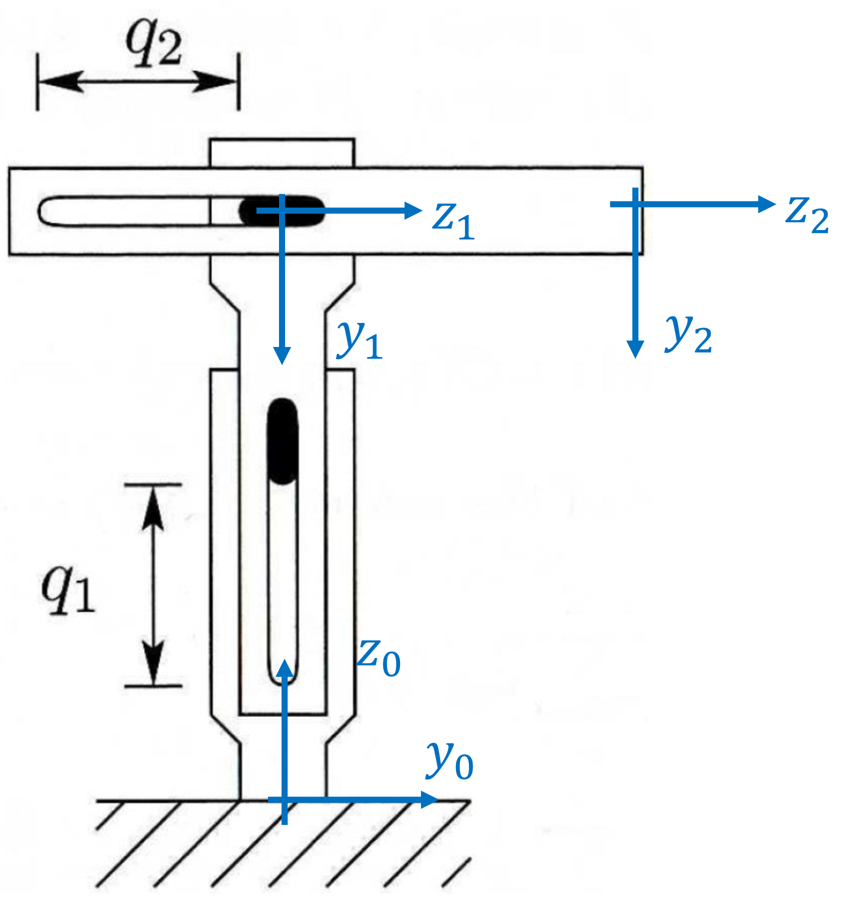
- Denavit-Hartenberg parameters

Link	θ	d	a	α
1	0	q_1	0	-90
2	0	q_2	0	0

- Jacobian matrices for centers of mass

$$J_{v,c_1} = [z_0^0, 0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$J_{v,c_2} = [z_0^0, z_1^0] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$



Example

Two-link Cartesian manipulator:

- Linear velocity of centers of mass

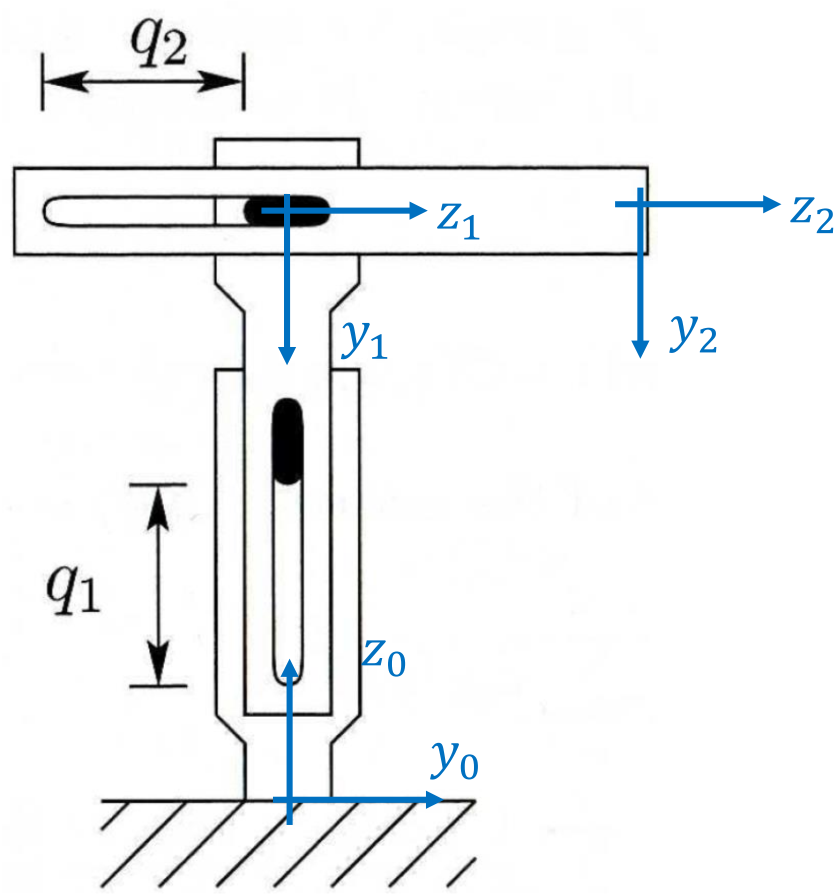
$$v_{c_1} = J_{v,c_1}\begin{bmatrix}\dot q_1\\ \dot q_2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}\dot q_1\\ \dot q_2\end{bmatrix}$$

$$v_{c_2} = J_{v,c_2}\begin{bmatrix}\dot q_1\\ \dot q_2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\dot q_1\\ \dot q_2\end{bmatrix}$$

- Kinetic energy

$$\mathcal{K} = \frac{1}{2}\dot q^T\left(m_1\, J_{v,c_1}^T\, J_{v,c_1} + m_2\, J_{v,c_2}^T\, J_{v,c_2}\right)\dot q$$

$$\mathcal{K} = \frac{1}{2}\begin{bmatrix}\dot q_1 & \dot q_2\end{bmatrix}\underbrace{\begin{bmatrix}m_1+m_2 & 0\\ 0 & m_2\end{bmatrix}}_{D}\begin{bmatrix}\dot q_1\\ \dot q_2\end{bmatrix}$$



Example

Two-link Cartesian manipulator:

- Potential energy

$$\mathcal{P} = g(m_1 + m_2)q_1$$

- Gravity vector

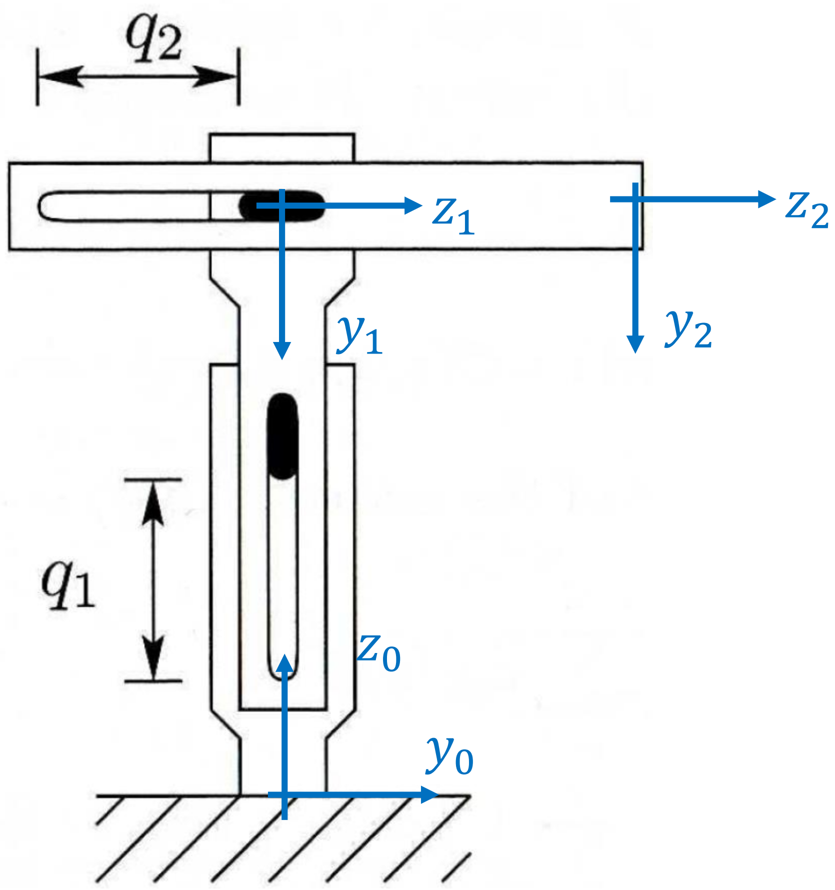
$$g_1 = \frac{\partial \mathcal{P}}{\partial q_1} = g(m_1 + m_2)$$

$$g_2 = \frac{\partial \mathcal{P}}{\partial q_2} = 0$$

- Christoffel symbols

$$c_{ijk} = 0$$

because inertia matrix D is constant and therefore independent of q.



Example

Two-link Cartesian manipulator:

- Euler-Lagrange equation

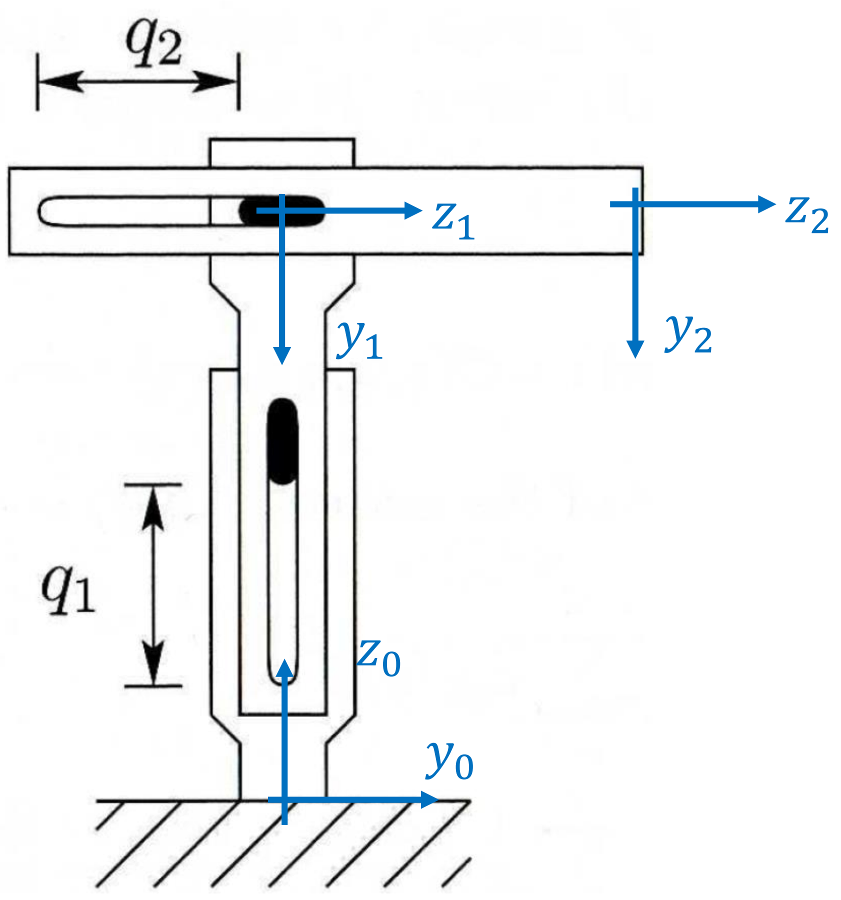
$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i,j} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k$$

with

$$d_{11} = m_1 + m_2 \quad d_{22} = m_2 \quad d_{12} = d_{21} = 0$$

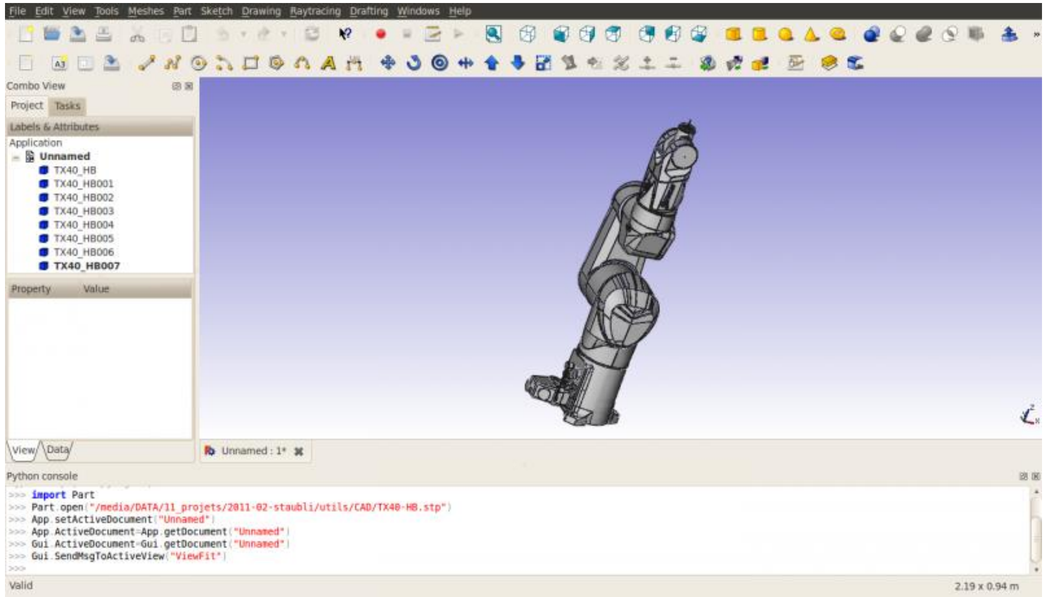
$$g_1 = g(m_1 + m_2) \quad g_2 = 0$$

$$(m_1 + m_2)\ddot{q}_1 + g(m_1 + m_2) = f_1$$
$$m_2\ddot{q}_2 = f_2$$



Student Projects

- Programming of a general serial link robotic manipulator module for FreeCAD (open source)

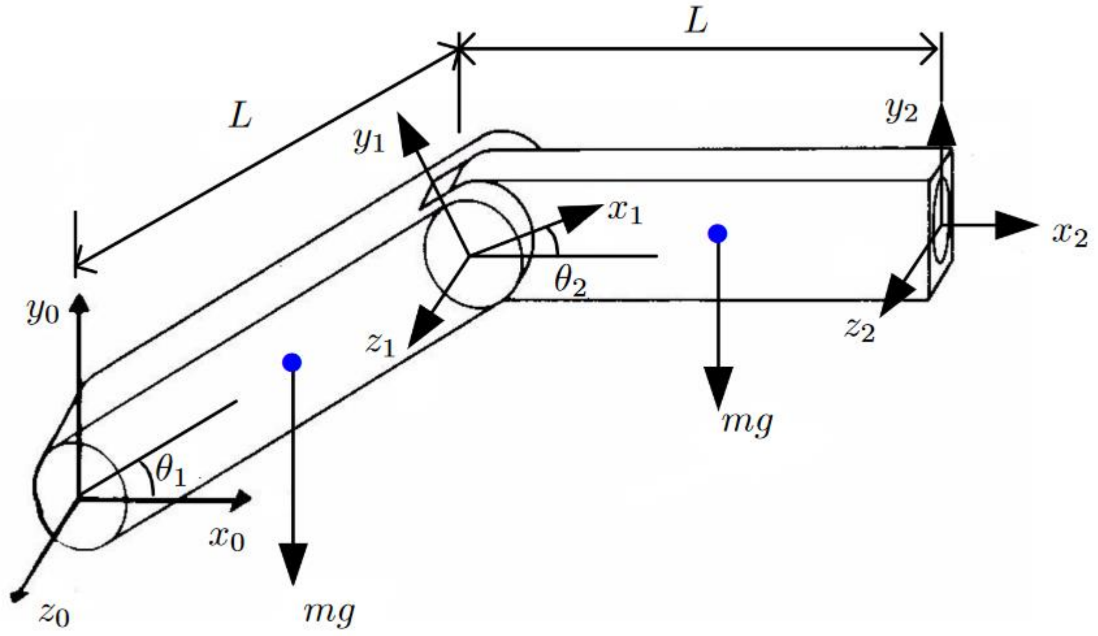


- Soft robotics design using computational tools (continuum mechanics + optimization)

Exercise

Problem 1

Consider a manipulator of two revolute joints, as shown in the figure below.



Given that the length of each link is L and the mass of each link is m, derive the dynamic model of this two-link robot arm using the Euler-Lagrange method. Approximate the geometry of the two arms as parallelepipeds of dimensions $L \times 0.1L \times 0.1L$ and assume that their mass is uniformly distributed.

DTU

Robotics - 34753

Recap on linear control

Matteo Fumagalli
Associate Professor
Automation and Control Group
Department of Electrical Engineering
DTU Lyngby, Building 326

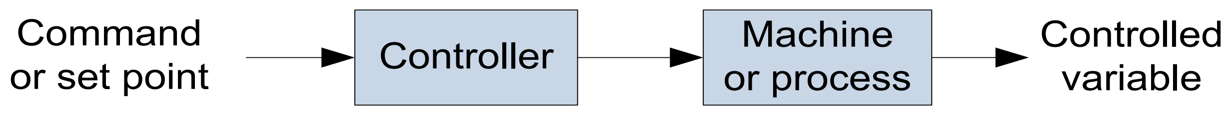
Contributors:
Niels Axel Andersen
Hayan Wu

Outline

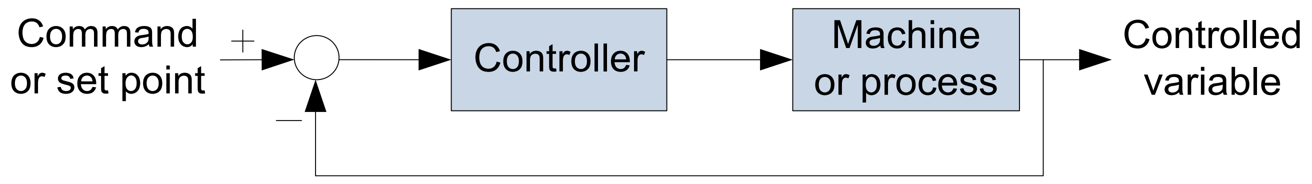
- Feedback control system?
- The Laplace transformation?
- Transfer function?
- System block diagram?
- First/second order systems?
- Basic controllers?

Open loop and closed loop control systems

- An open-loop control system utilizes an actuating device to control the process directly without using feedback.



- A closed-loop control system uses a measurement of the output and feedback of this signal to compare it with the desired output (reference or command).



Why do we need feedback?

+ Most systems are subject to disturbances that perturb the controlled variable. A major reason for the use of feedback is to reduce the effect of these disturbances.

+ The closed-loop system is more accurate than the open-loop system. It can be designed to provide extreme accuracy in the steady state.

+ Its response time can be adjust by appropriate design

- Stability problem, the controlled system may fluctuate continuously at some periodic rate and never reaches the desired steady-state condition

How to develop a feedback control system?

- Choose a way to adjust the variable to be controlled, e.g. the mechanical load will be positioned with an electric motor

- Select suitable sensors to complete the loop

- Determine what is required for the system to operate, e.g. accuracy in steady state and response time

- System stability analysis

- Modify the system regarding stability and other desired operating conditions

Tools for linear system analysis and design

- Modeling: mathematical description of the system
  – the differential equations of the system

- Laplace transformation: convert the differential equations to algebraic equations, and convert from the time domain to frequency domain

- System analysis based on e.g. Nyquist stability criterion or on the root locus method

- Analyse open loop system to predict the behavior of closed loop system

Laplace Transform

Definition: The Laplace Transform of a signal $f(t),\ t \geq 0$

is the function of the complex variable $s \in \mathbb{C}$

$$F(s) := \mathcal{L}[f(t)](s) = \int_0^{+\infty} f(t)e^{-st}dt$$

Laplace Transform

Typical Laplace Transform of canonical functions

$f(t)$	$\mathcal{L}(s)$
$imp(t)$	1
$step(t)$	$\frac{1}{s}$
$cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
e^{at}	$\frac{1}{s-a}$

Laplace Transform

Properties of the Laplace Transform:

LINEARITY

$$\mathcal{L}[\alpha_1 f_1(t) + \alpha_2 f_2(t)](s) = \alpha_1 \mathcal{L}[f_1(t)](s) + \alpha_2 \mathcal{L}[f_2(t)](s)$$

DERIVATION OVER TIME

$$\mathcal{L}[\dot{f}_1(t)](s) = s\mathcal{L}[f_1(t)](s) - f(0)$$

The properties allow us to compare differential and linear relations in time domain, to the equivalent in Laplace domain

Laplace Transform

Inverse Laplace transform

Consider F=N/D, with d>n

$$F(s) = \frac{N(s)}{D(s)} = \frac{n_0 + n_1 s + n_2 s^2 + \cdot + n_n s^n}{d_0 + d_1 s + d_2 s^2 + \cdot + d_d s^d}$$

it is possible to define the inverse transform function

$$f(t) = \mathcal{L}^{-1}[F(s)](t)\ ,\quad t \geq 0$$

it can be determined using the Heaviside method

Laplace Transform

Consider the Laplace transform of a function $f(t)$ with $d > n$ (Strictly proper)

It is possible to calculate the value of $f(t)$in $t = 0$ and the limit of $f(t)$, for $t \to \infty$, by using the laplace transform of $f(t)$, WITHOUT calculating $\mathcal{L}^{-1}[F(s)](t)$ explicitly

THEOREM OF THE INITIAL VALUE

$$f(0) = \lim_{s\to\infty} sF(s)$$

And if the roots of D(s) are equal to 0 or they have real part greater then 0:

THEOREM OF THE FINAL VALUE

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Transfer Function

A function $G(s)$ that describes the input-output relation of a linear system subject to an input $U(s)$:

$$Y(s) = G(s)U(s)$$

NOTE: a TF allows to describe differential equations as algebraic ones, and it allows to evaluate the motion of the output $y(t),\ t \geq 0$, due to an input $u(t),\ t \geq 0$ as:

$$u(t),\ t \geq 0 \overset{\mathcal{L}}{\rightarrow} U(s) \overset{\cdot G(s)}{\rightarrow} Y(s) \overset{\mathcal{L}^{-1}}{\rightarrow} y(t)$$

Transfer Function

$$G(s) = C(sI - A)^{-1}B + D = \frac{N(s)}{D(s)}$$

the roots of $D(s)$ are called POLES; the roots of $N(s)$ are called ZEROS

Poles and zeros are singularities of $G(s)$

Poles are the eigenvalues of the state matrix A . This means that the number of poles of a transfer function of a system of the n-th order is n

$$G(s) = \frac{\mu}{s^g} \frac{\prod_i (1 + T_i s) \prod_i \left(1 + 2\frac{\zeta_i}{\sigma_{ni}} s + \frac{s^2}{\sigma_{ni}^2}\right)}{\prod_i (1 + \tau_i s) \prod_i \left(1 + 2\frac{\xi_i}{\omega_{ni}} s + \frac{s^2}{\omega_{ni}^2}\right)}$$

Transfer Function

$$G(s) = \frac{\mu}{s^g} \frac{\prod_i (1 + T_i s) \prod_i \left(1 + 2\frac{\zeta_i}{\sigma_{ni}} s + \frac{s^2}{\sigma_{ni}^2}\right)}{\prod_i (1 + \tau_i s) \prod_i \left(1 + 2\frac{\xi_i}{\omega_{ni}} s + \frac{s^2}{\omega_{ni}^2}\right)}$$

Where:

$g \in \mathbb{Z}$ is the type of the transfer function. In general, it indicates the number of singularity ($s = 0$) of the transfer function. More precisely, if $g > 0$ it indicates the number of poles that are equal to zero. If $g < 0$, it indicates the number of zeros that are equal to zero.

$\mu \in \mathbb{R}$ is the gain of the transfer function.

Transfer Function

$$G(s) = \frac{\mu}{s^g} \frac{\prod_i (1 + T_i s) \prod_i \left(1 + 2\frac{\zeta_i}{\sigma_{ni}} s + \frac{s^2}{\sigma_{ni}^2}\right)}{\prod_i (1 + \tau_i s) \prod_i \left(1 + 2\frac{\xi_i}{\omega_{ni}} s + \frac{s^2}{\omega_{ni}^2}\right)}$$

Where:

$T_i \in \mathbb{R}$ and $\tau_i \in \mathbb{R}$ are the time constants of the zeros and poles that are real and not null ($s \neq 0$)

$\sigma_{n,i} \in \mathbb{R}^+$ and $\omega_{n,i} \in \mathbb{R}^+$ are the natural frequencies of the complex, conjugated poles

$\zeta_i \in (-1, 1)$ and $\xi_i \in (-1, 1)$ are the damping coefficients of the complex conjugated poles and zeroes that are

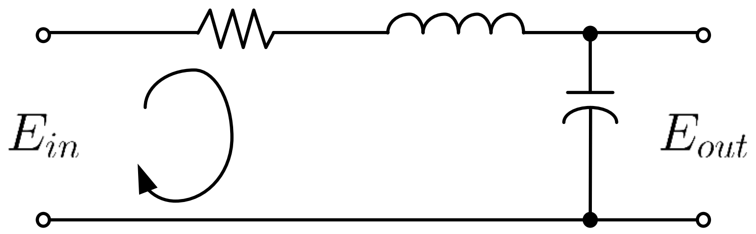
Transfer Function

- The transfer function describe the cause and effect relationship between input and output

The transfer function of a linear system is the Laplace transform of its response to a unit impulse input

---

Example: An R-L-C filter



i) applying Kirchhoff's law: $E_i = iR + L\frac{di}{dt} + \frac{1}{C}\int idt, \quad E_o = \frac{1}{C}\int idt$

ii) applying the Laplace transform: $E_i(s) = I(s)(R + sL + \frac{1}{sC}), \quad E_o(s) = I(s)\frac{1}{sC}$

iii) transfer function: $\frac{E_o(s)}{E_i(s)} = \frac{1}{s^2LC + sCR + 1}$

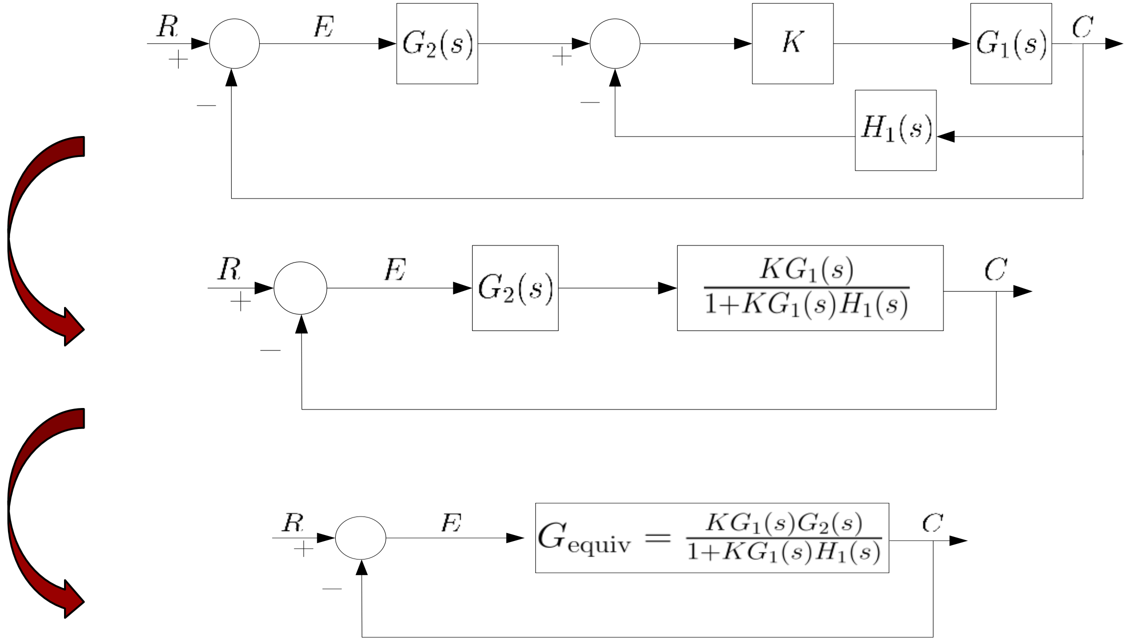
System block diagram

- The interconnection of components to form a system is conveniently shown by blocks arranged in some sort of diagram.

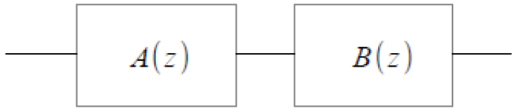
Example: manipulation of a system block diagram

Reduction of minor feedback loop to one equivalent cascade block

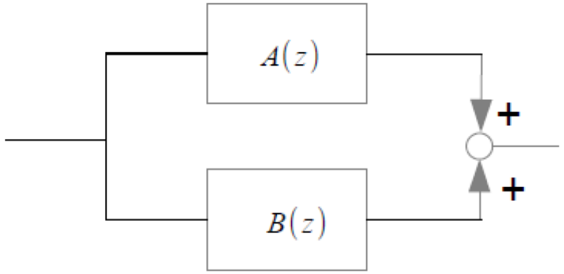
Cascaded blocks combined to give one equivalent block



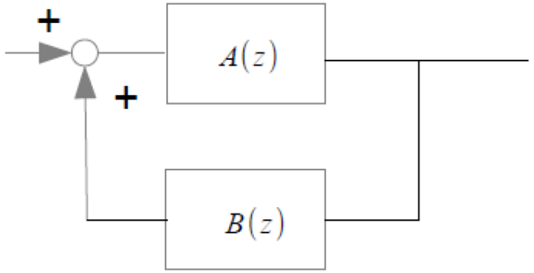
System block diagram



$$D(z) = A(z)\,B(Z)$$

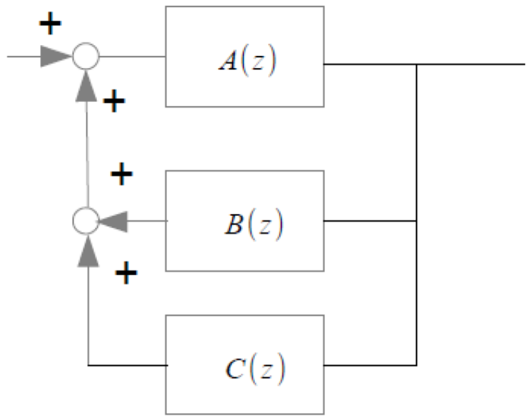


$$D(z) = A(z) + B(Z)$$

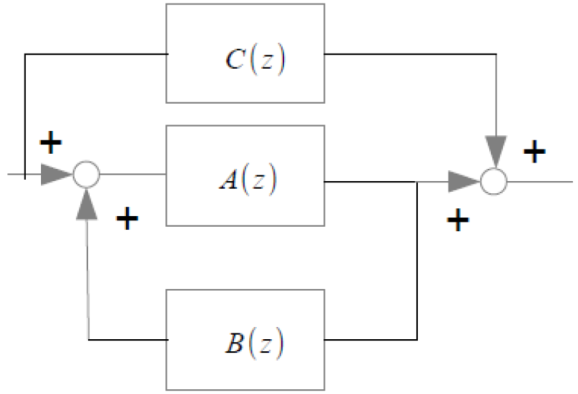


$$D(z) = \frac{A(z)}{1 - A(z)B(z)}$$

System block diagram

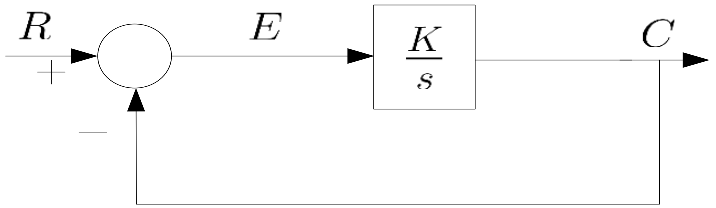


$$D(z) = \frac{A(z)}{1 - A(z)(B(z) + C(z))}$$



$$D(z) = C(z) + \frac{A(z)}{1 - A(z)B(z)}$$

First order system



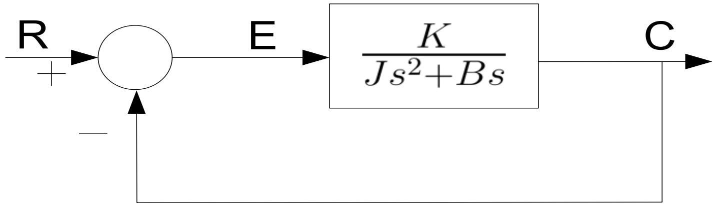
K : a positive real number serving as the gain of the closed-loop system

Closed loop system: $$\frac{C}{R} = \frac{1}{sT+1}$$

$T = \frac{1}{K}$ is defined as the time constant of the system

The closed-loop pole for this system is located at $s = -\frac{1}{T} = -K$.
Since $K > 0$ the closed loop system is guaranteed to be stable.

Second order system



Transfer function of a general second-order system is

$$\frac{C}{R} = \frac{K}{Js^2+Bs+K} = \frac{\omega_N^2}{s^2+2\xi\omega_N+\omega_N^2}$$

Note: This is second order because the highest power of s = 2

$\xi = \frac{B}{2\sqrt{KJ}}$ ➡ damping ratio, will determine how much the system oscillates as the response decays toward steady sate

$\omega_N = \sqrt{\frac{K}{J}}$ ➡ the undamped natural frequency, will determine how fast the system oscillates during any transient response

Basic controllers

- Proportional control
  - adjustable gain (amplifier)

$$u(t) = K_p e(t) \qquad \frac{U(s)}{E(s)} = K_p$$

$e(t)$: control error

- Integral control
  - eliminates bias (steady-state error)
  - may cause oscillations

$$u(t) = K_i \int_0^t e(t)dt \qquad \frac{U(s)}{E(s)} = \frac{K_i}{s}$$

- Differential control
  - effective in transient periods, provides faster response
  - never used alone

$$u(t) = K_d \frac{d}{dt} e(t) \qquad \frac{U(s)}{E(s)} = K_d s$$

- Integral and differential control are typically used in combination with at least proportional control

References

- “Automatic Control Systems”, G. J. Thaler, *West Publishing Company*, 1989.
- “An Introduction to Control Systems”, K. Warwick, *World Scientific Publishing Company*, 1996.
- “Control Engineering”, O. Jannerup og P. H. Sørensen, 3. edition (2004) or 4. edition (2006), Polyteknisk Forlag. (*In danish*)
- “The Control Handbook”, William S. Levine, editor, CRC press, 1996.
- “Feedback Control of Dynamical Systems”, G. F. Franklin, J. D. Powell, and A. Emami-Naeini, Prentice-Hall, Upper Saddle River, NJ, 4th edition, 2002.
- “PID Controllers: Theory, Design, and Tuning”, K. J. Åström and T. H¨agglund, International Society for Measurement and Control, Seattle, WA, 2nd edition,1995
- http://www.facstaff.bucknell.edu/mastascu/eControlHTML/CourseIndex.html
- ……

DTU

ROBOTICS 34753

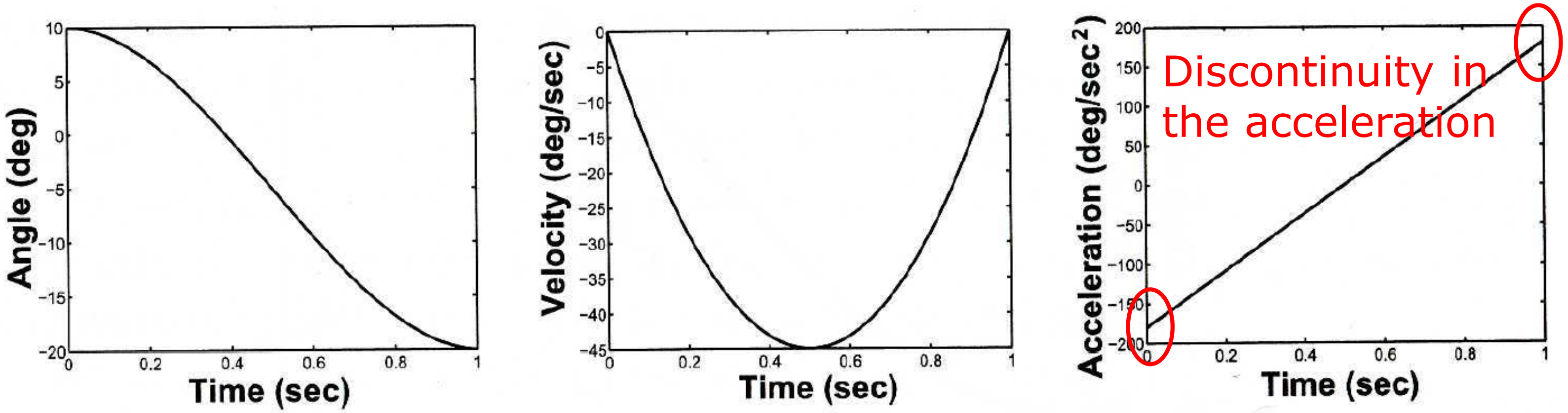
Independent Joint Control

Outline

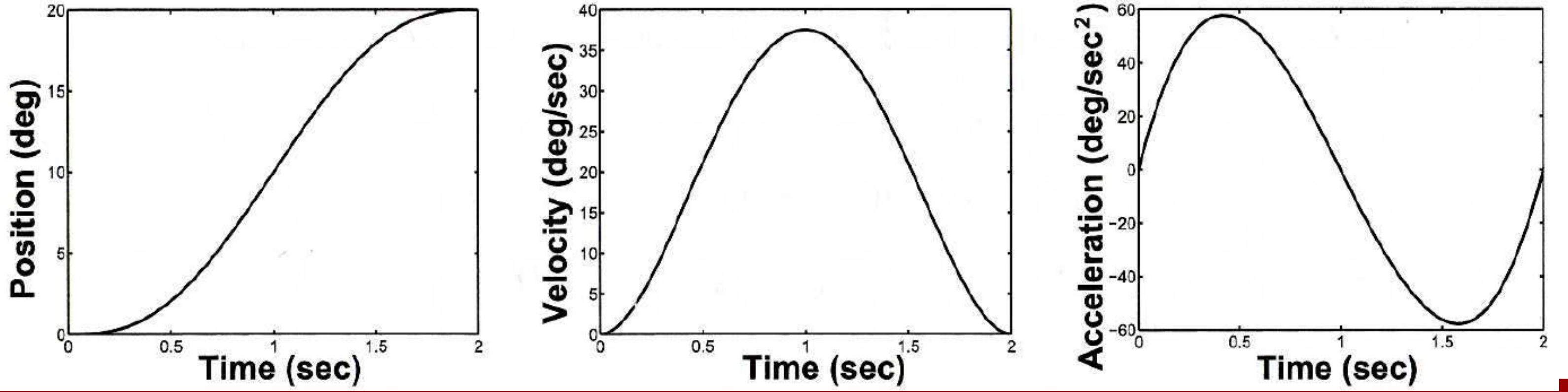
- Review
  - Trajectory Planning
  - Dynamics
- Actuator Dynamics
- Dynamical Model of a Robot with One Joint
- Controller Design

Review

- Trajectory planning
  - 3^{rd}rder polynomial



  - 5^{th}rder polynomial



Review

- Dynamics: Euler-Lagrange Equations
  - Euler-Lagrange formulation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i \qquad \mathcal{L} = \mathcal{K} - \mathcal{P}$$

  - Kinetic energy

$$\mathcal{K} = \frac{1}{2}\dot{q}^T \left[\sum_{i=1}^{n} m_i J_{v_i}^T(q) J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I R_i(q)^T J_{\omega_i}(q) \right] \dot{q}$$

  - Potential energy

$$\mathcal{P} = \sum_{i=1}^{n} \mathcal{P}_i = \sum_{i=1}^{n} m_i g^T r_{ci}$$

Review

- Equation of motion

$$\sum_{j=1}^{n} d_{kj}\ddot{q}_j + \sum_{i,j} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k$$

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

$$D(q) = \sum_{i=1}^{k}(m_i J_{v_i}^T(q)J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I R_i(q)^T J_{\omega_i}(q))$$

Element of $C(q,\dot{q})$: $c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i = \sum_{i=1}^{n} \frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right)\dot{q}_i$

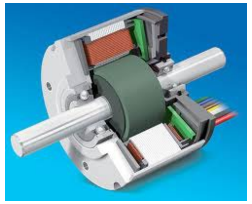
$$g(q) = [g_1, \ldots, g_n(q)]^T$$

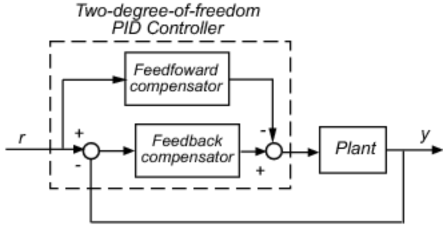
Development control systems for robots

- Modeling robot behaviors
  - Kinematics
  - Dynamics
  - …

- Choice of actuators, gears, sensors and their allocation

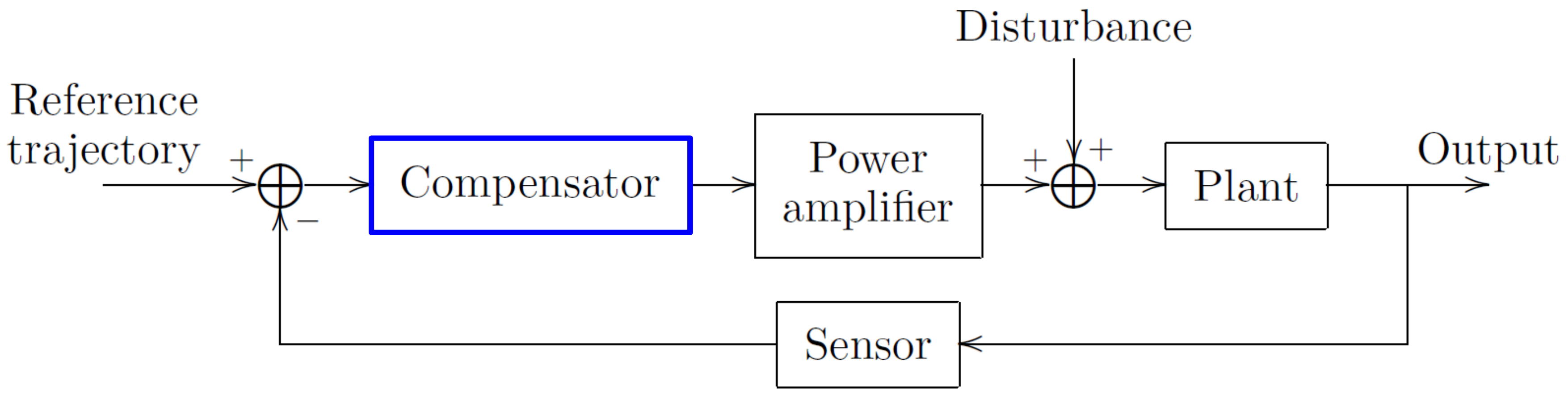
- Choice of control architecture
  - Linear vs. nonlinear
  - Delay compensation
  - Adaptive online parameter estimation
  - Robustness
  - …





⋮

Basic structure of a feedback system



Design objective: choose the **compensator** to drive the plant's output to **follow** a desired reference.

Actuator Dynamics

Permanent magnet DC motor

- A current-carrying conductor in magnetic field
  - Force

    $$F = i_a \times \phi$$

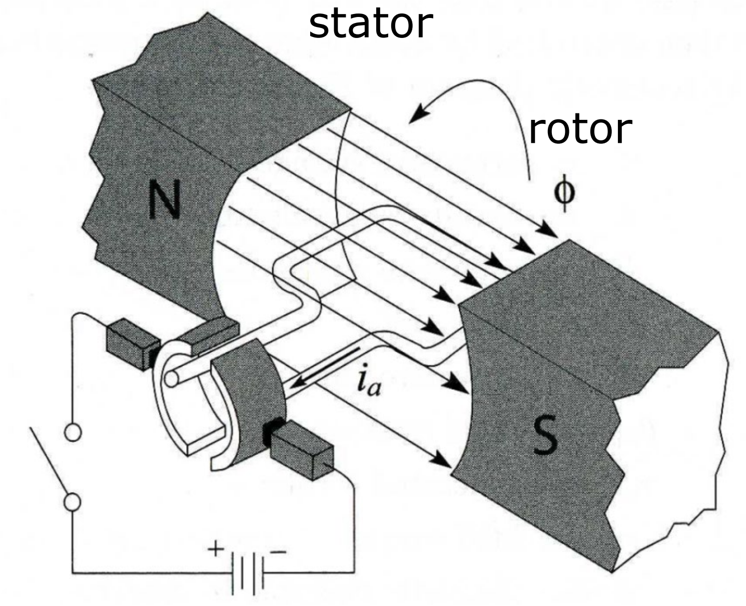
    i_a: current
    ϕ : magnetic flux
  - Torque

    $$\tau_m = K_1 \phi i_a$$

    K_1: physical constant
  - Back emf (electromotive force)

    $$V_b = K_2 \phi \omega_m$$

    K_2: proportionality constant
    ω_m: angular velocity



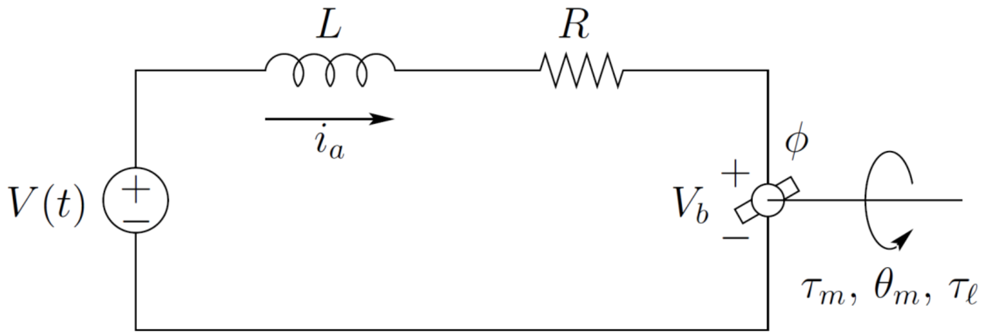
Permanent magnet DC motor

- Relations between i_a, V, ω_m, τ_m

$$L\frac{di_a}{dt} + Ri_a = V - V_b$$

$$\tau_m = K_1\phi i_a = K_m i_a$$

$$V_b = K_2\phi\omega_m = K_b\omega_m$$



i_a : armature current

V : armature voltage

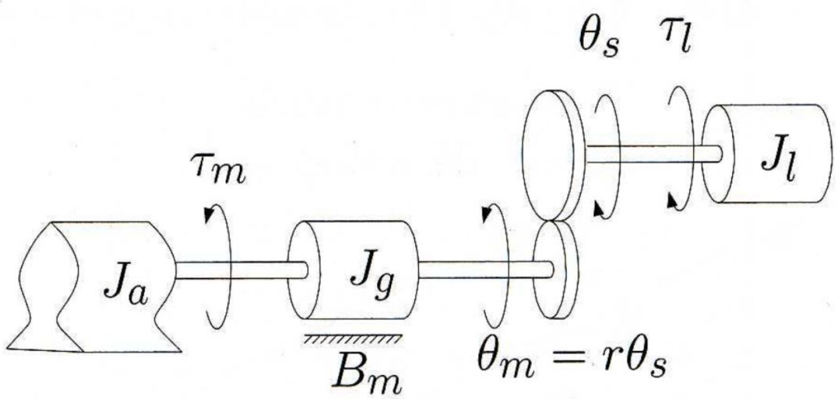
ω_m : angular velocity

τ_m : generated torque

Independent joint model

- Single link with actuator/gear train
  - Each link: independent **SISO** system
  - Large gear reduction: reduce the nonlinear coupling among the links
  - Equation of motion

$$J_m(\frac{d^2\theta_m}{dt^2}) + B_m\frac{\theta_m}{dt} = \tau_m - \tau_l/r$$
$$= K_m i_a - \tau_l/r$$



B_m: coefficient of motor friction
J_a: actuator inertia
J_g: gear inertia
J_l: load inertia
$J_m = J_a + J_g$

Independent joint model

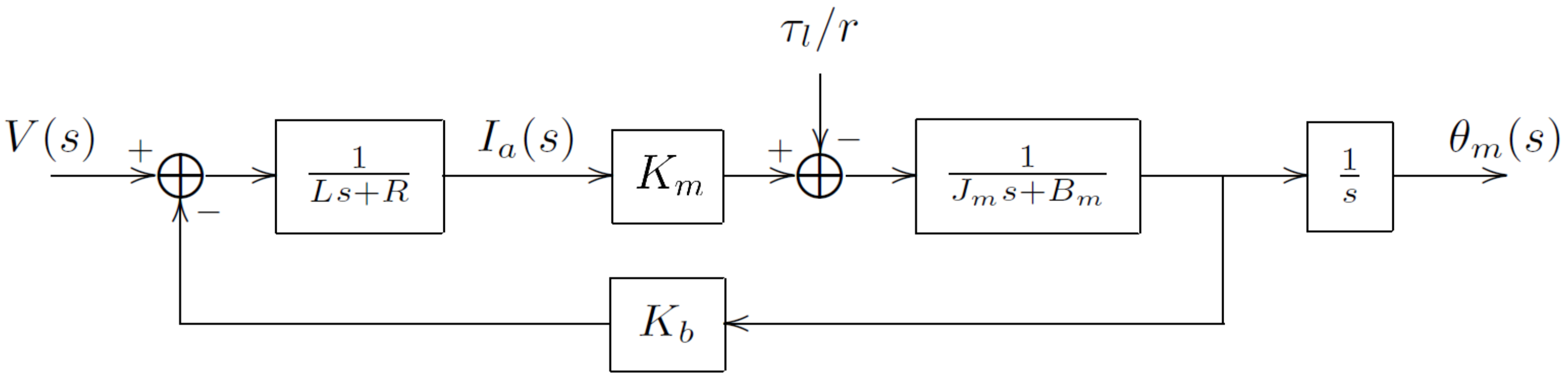
- Augmenting the mechanical and electrical models

$$J_m\left(\frac{d^2\theta_m}{dt^2}\right) + B_m\frac{\theta_m}{dt} = K_m i_a - \tau_l/r$$

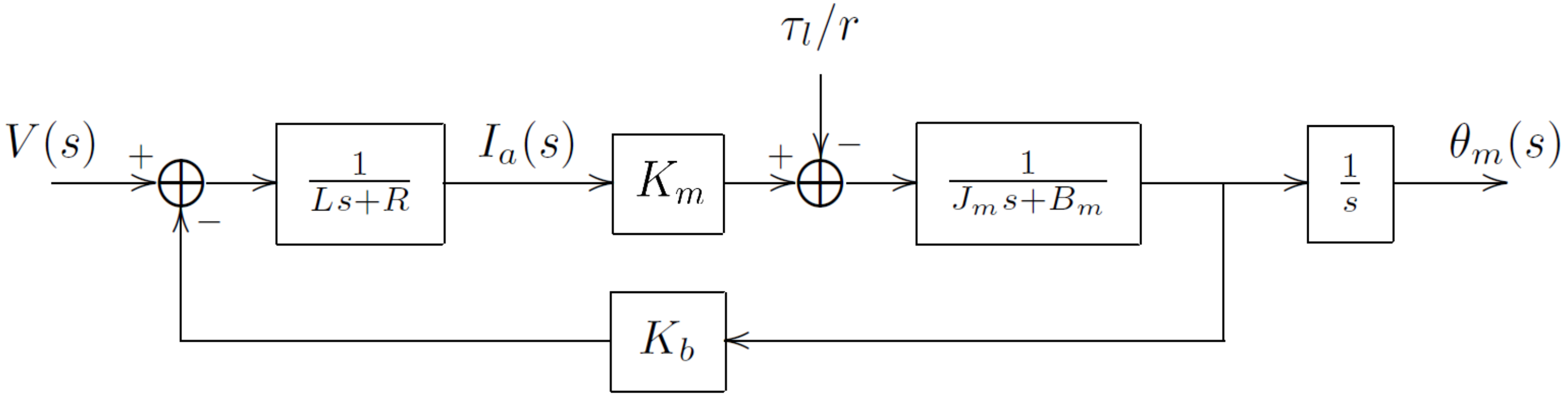
$$L\frac{di_a}{dt} + Ri_a = V - V_b$$

$$(J_m s^2 + B_m s)\theta_m(s) = K_m I_a(s) - \tau_l(s)/r$$

$$(Ls + R)I_a(s) = V(s) - K_b s\theta_m(s)$$



Independent joint model



- Transfer function $V(s) \to \Theta_m(s)$

$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m}{s[(LS+R)(J_m s+B_m)+K_b K_m]}$$

- Transfer function $\tau_l(s) \to \Theta_m(s)$

$$\frac{\Theta_m(s)}{\tau_l(s)} = \frac{-(Ls+R)/r}{s[(LS+R)(J_m s+B_m)+K_b K_m]}$$

- Reduction of the model based on $\dfrac{L}{R} \approx 0$

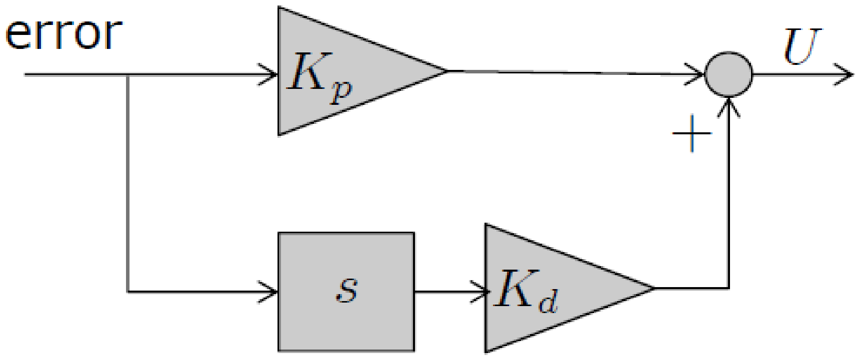
$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m/R}{s[J_m s+B_m+K_b K_m/R]}$$

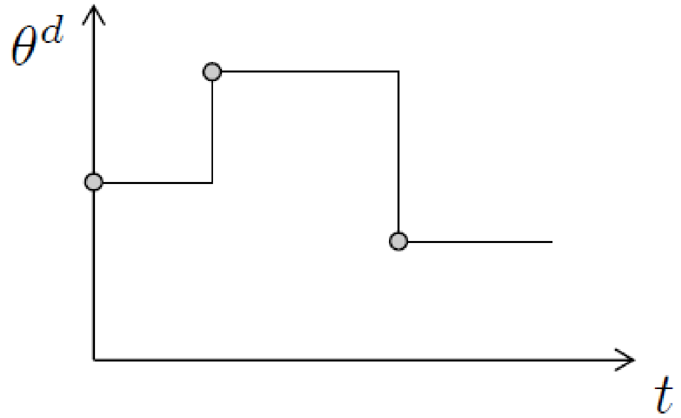
$$\frac{\Theta_m(s)}{\tau_l(s)} = \frac{-1/r}{s[J_m s+B_m+K_b K_m/R]}$$

Controller Design:
PD and PID controller

Set-Point tracking with PD controller

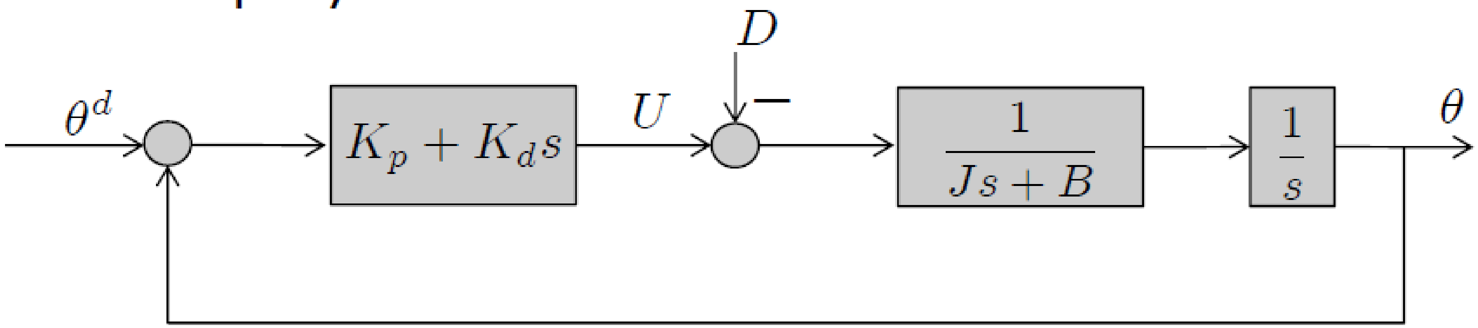
- PD controller





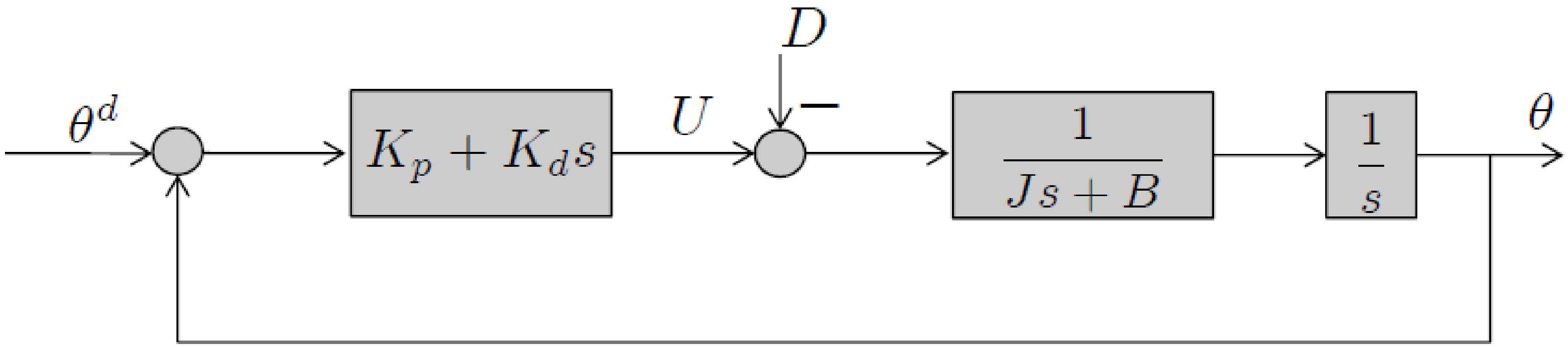
$$U(s) = (K_p + K_d s)(\theta^d(s) - \theta(s))$$

- Closed-loop system



Set-Point tracking with PD controller

- Relationship between output and input

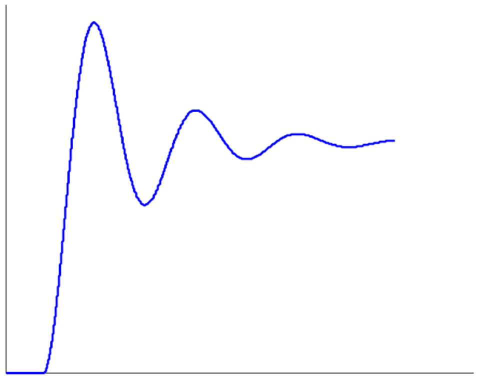


$$[(\theta^d - \theta)(K_p + K_D s) - D(s)]\frac{1}{Js+B}\frac{1}{s} = \theta$$

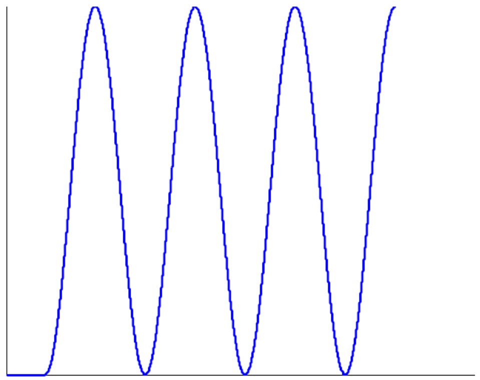
$$\theta = \frac{(K_p + K_D s)\theta^d(s) - D(s)}{Js^2 + (B + K_D)s + K_p}$$

- Type I system (the number of 1/s in the open-loop transfer function is 1)
- Stable for all positive values K_p, K_d

System stability



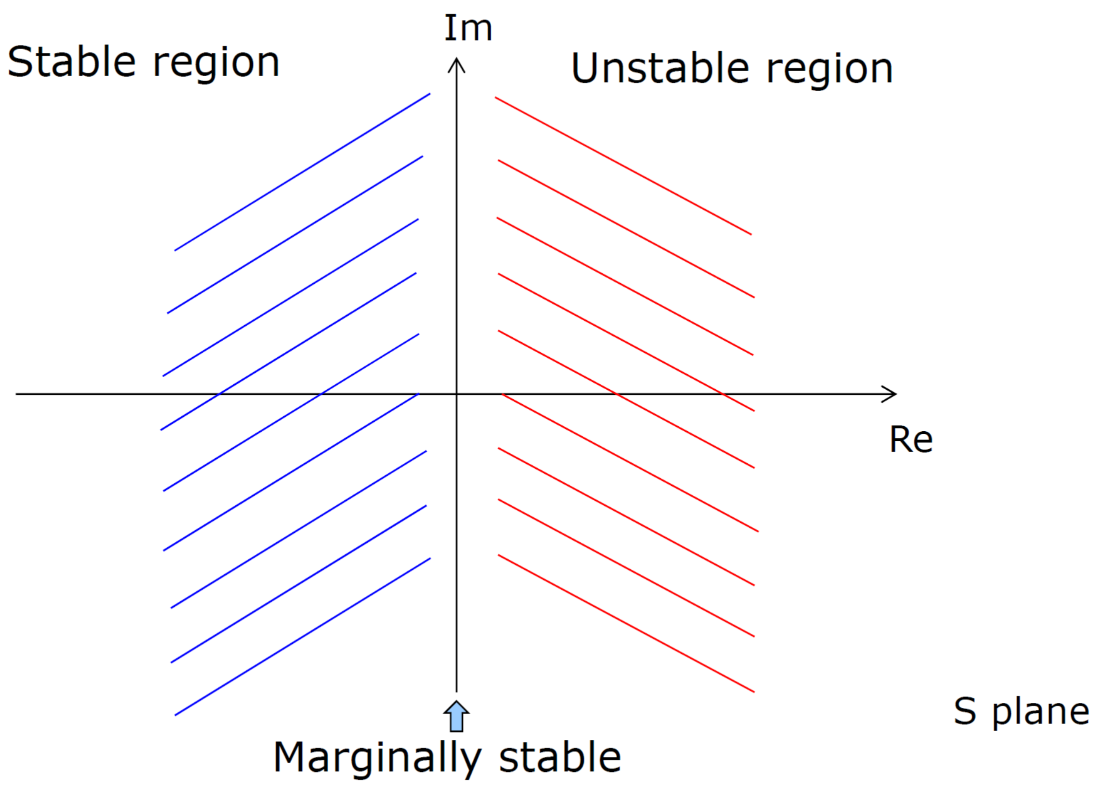
Asymptotically stable



Marginally stable

Unstable

S-plane for stability analysis



Root of characteristic polynomial (PD controller)

- Characteristic equation

$$Js^2 + (B + K_d)s + K_p = 0$$

- If roots are on the left half plane -> stable

$$s = \frac{-(B + K_d) \pm \sqrt{(B + K_d)^2 - 4JK_p}}{2J}$$

- For $(B + K_d)^2 - 4JK_p < 0$, complex conjugate $\rightarrow$ stable
- For $(B + K_d)^2 - 4JK_p = 0$ -> stable
- For $(B + K_d)^2 - 4JK_p > 0$:
  - $-(B + K_d) - \sqrt{(B + K_d)^2 - 4JK_p} < 0$ $\rightarrow$ stable
  - $-(B + K_d) + \sqrt{(B + K_d)^2 - 4JK_p} < 0$ $\rightarrow$ for $K_p > 0$ $\rightarrow$ stable

For all positive values Kp, Kd → the system with PD controller is stable

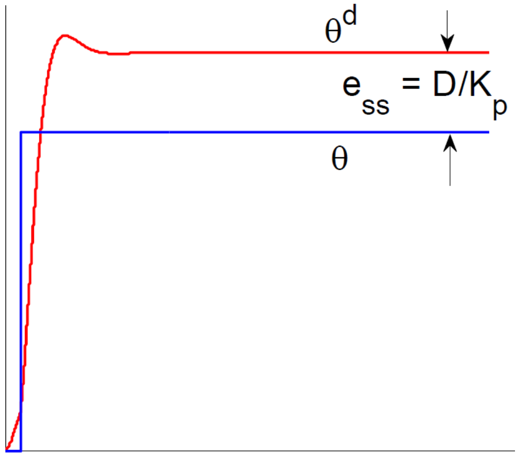
Steady-state error

- Tracking error

$$\theta(t) = \theta^d(t) - \theta(t)$$

$$E(s) = \theta^d(s) - \theta(s)$$

- Assume
  - A step reference input $\theta^d(s) = \dfrac{\Omega^d}{s}$
  - A constant disturbance $D(s) = \dfrac{D}{s}$



- Steady-state error
  - $\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$ ⇦ *Final value theorem*
  - $e_{ss} = \lim_{s\to 0} sE(s) = \dfrac{D}{K_p}$ ⇨ $\because D \propto \dfrac{1}{r}$ $\therefore r\uparrow, e_{ss}\downarrow$

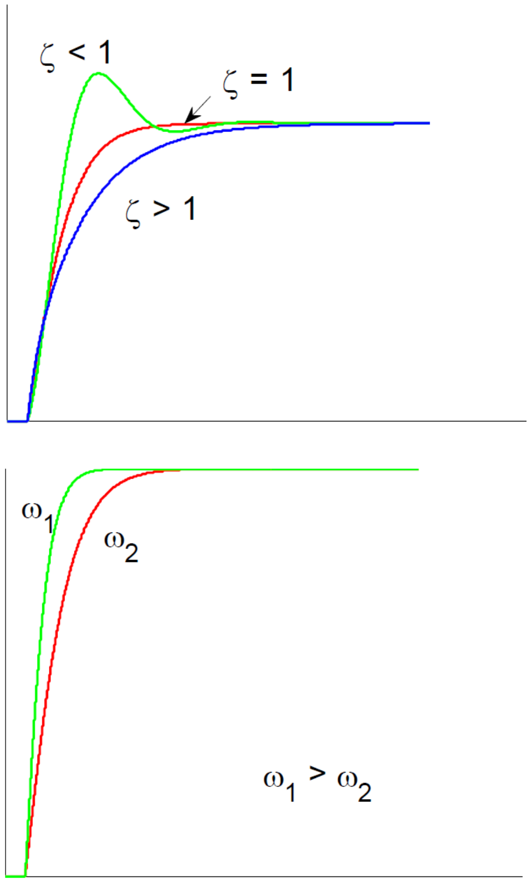
Natural frequency and damping ratio

- For a standard second order system, step response is determined by the closed loop natural frequency ω and damping ratio ζ

- $$s^2 + \frac{B + K_d}{J}s + \frac{K_p}{J} \quad \leftrightarrow \quad s^2 + 2\omega\zeta s + \omega^2$$

  $\Rightarrow$ $K_p = J\omega^2, \quad K_d = 2\omega J\zeta - B$

- Usually choose $\zeta = 1 \rightarrow$ critically damped response

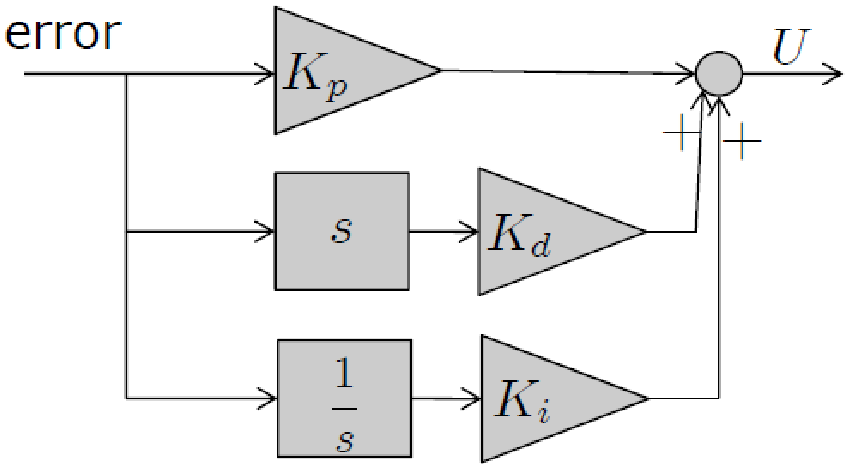


PID controller

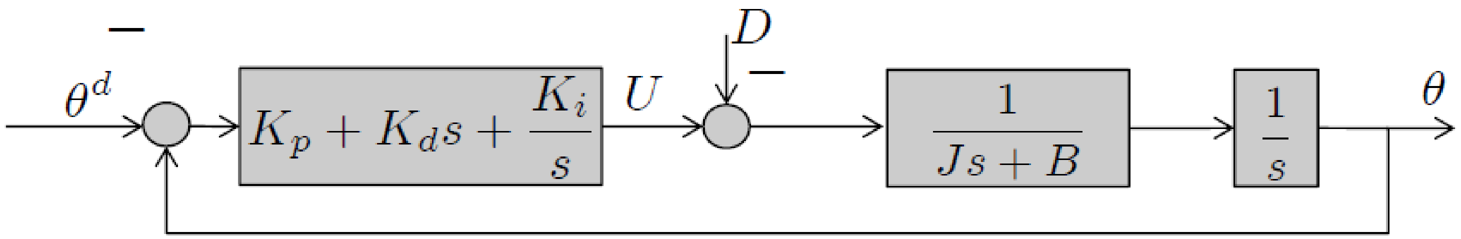
- Control output

$$U(s) = (K_p + K_d s + \frac{K_i}{s})(\theta^d(s) - \theta(s))$$

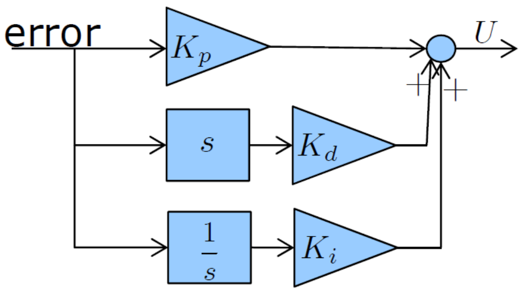
$$K_p, K_d, K_i > 0$$



- Closed-loop system with PDI controller



PID Controller

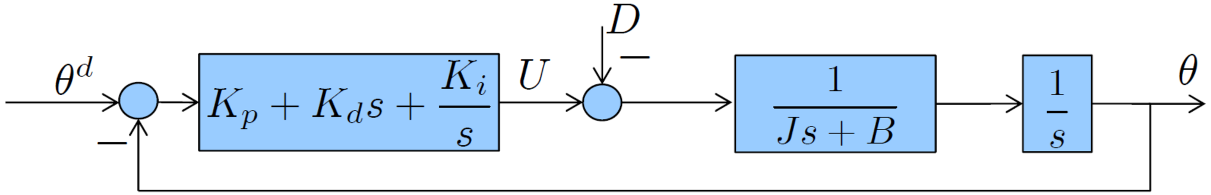


- Control output

$$U(s) = (K_p + K_d s + \frac{K_i}{s})(\theta^d(s) - \theta(s))$$

$$K_p, K_d, K_i > 0$$

- Closed-loop system with PID controller



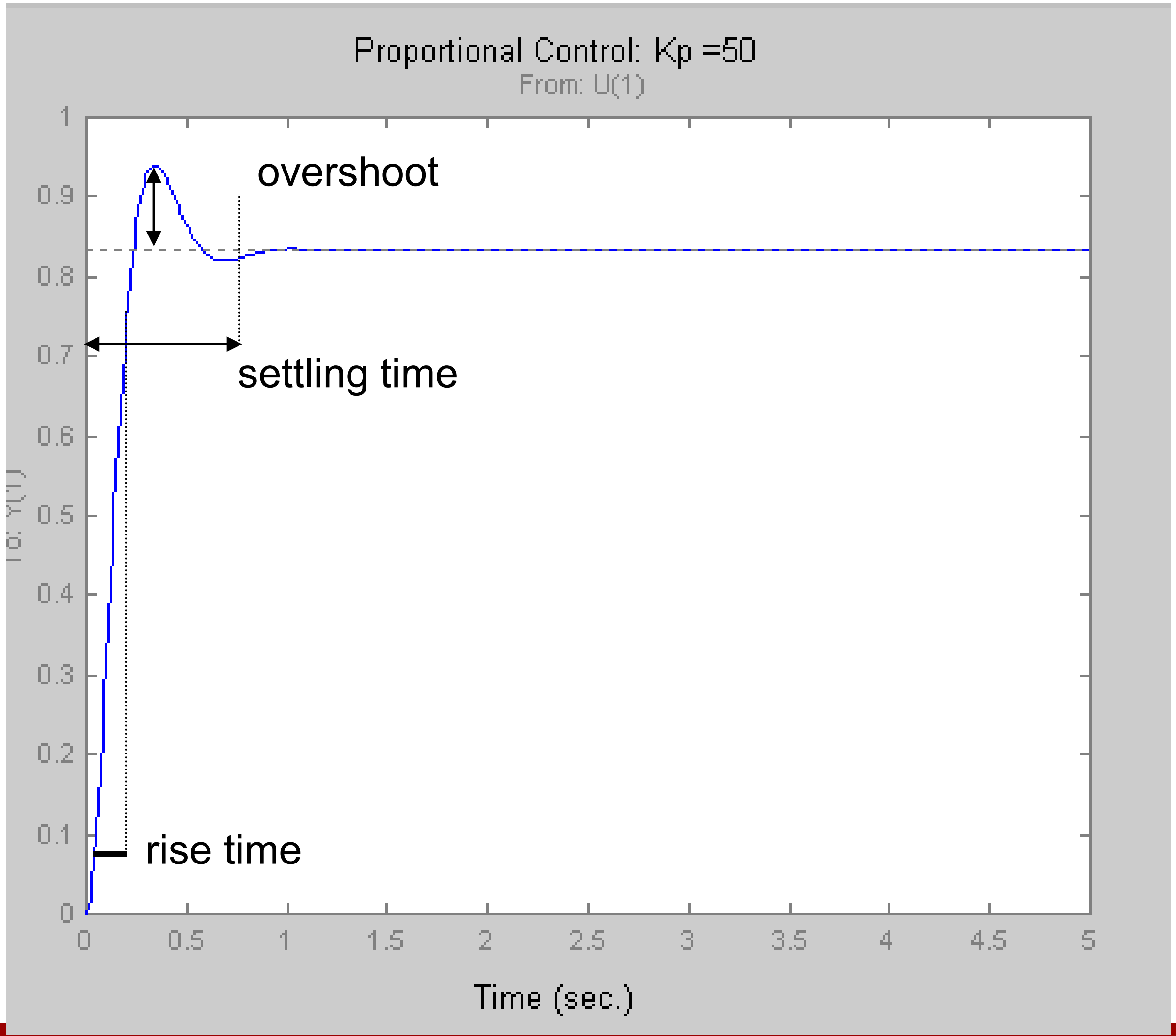
$$\theta(s) = \frac{(K_d s^2 + K_p s + \frac{K_i}{s})\theta^d(s) - sD}{Js^3 + (B + K_d)s^2 + K_p s + K_i}$$

⇨ third-order system

- Stable for $K_i < \frac{(B+K_d)K_p}{J}$ ⇦ *Routh-Hurwitz criterion*

- Steady-state error: $e_{ss} = 0$

Evaluating the response



steady-state error

ss error -- difference from the system's desired value

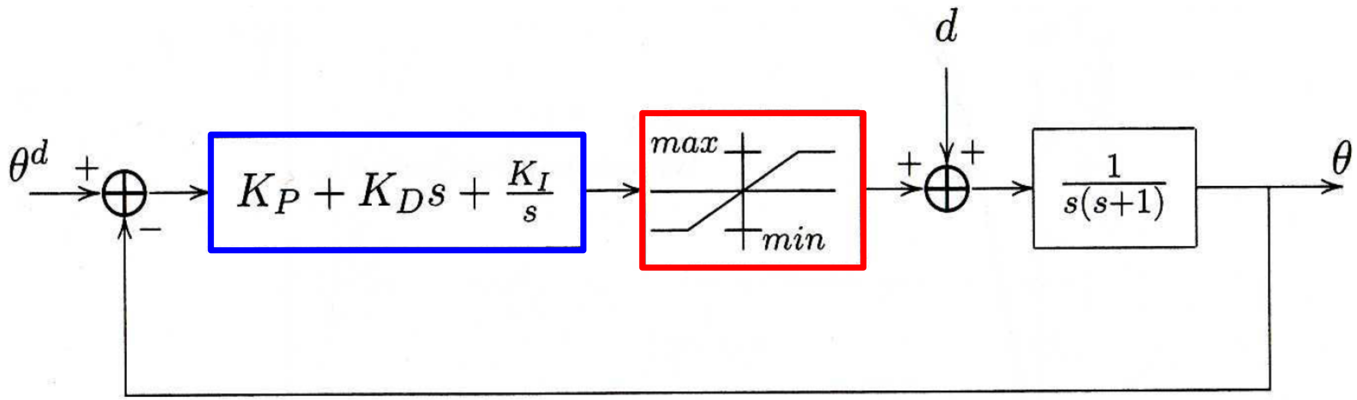
overshoot -- % of final value exceeded at first oscillation

rise time -- time to span from 10% to 90% of the final value

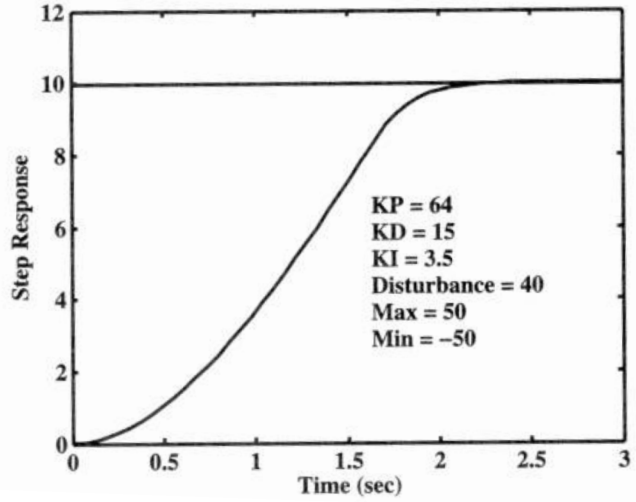
settling time -- time to reach within 2% of the final value

Input saturation

- Saturation: limits on the maximum torque (or current) input



Example:



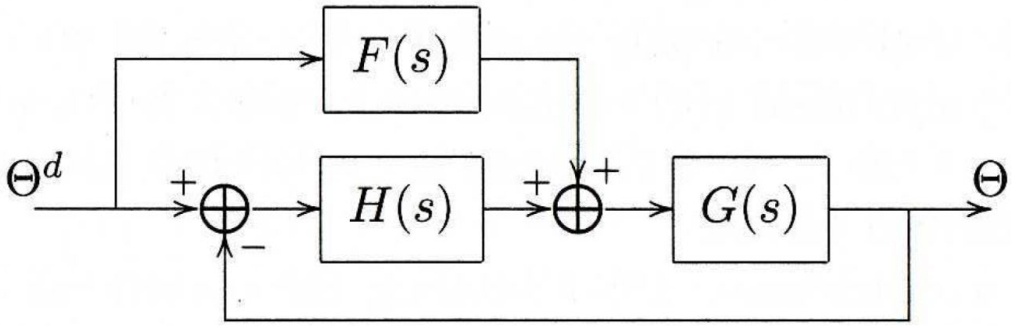
Effect of saturation: much slower rise time

Controller Design:
Feedforward control

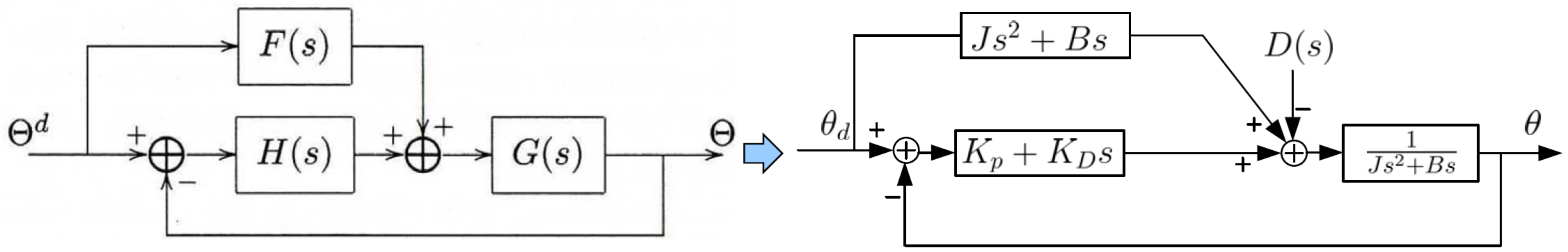
Feedforward control

- Track **time varying** trajectories
- Transfer functio $F(s)$: new parameter for design
- Choice of $F(s)$ stable, proper

Arbitrary joint space reference trajectory

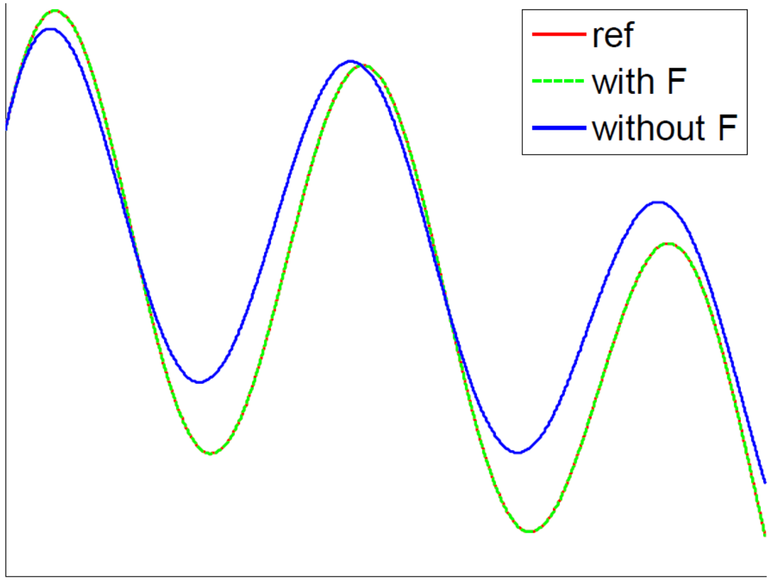


Feedforward control for tracking time varying trajectories

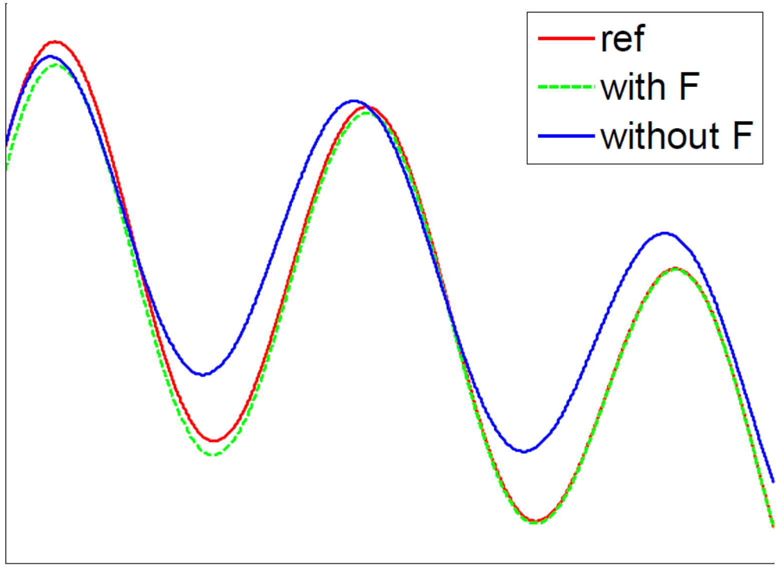


- Assume: $G(s) = \dfrac{q(s)}{p(s)}, H(s) = \dfrac{c(s)}{d(s)}, F(s) = \dfrac{a(s)}{b(s)}$
- Transfer function: $\dfrac{\theta}{\theta^d} = \dfrac{(bc + ad)q}{(pd + cq)b}$
- Stable if $pd + cq$ and b are Hurwitz $\rightarrow$ the feedforward transfer function itself must be stable
- Choose $F(s) = \dfrac{1}{G(s)}$: $\theta = \theta^d \rightarrow$ tracking error: $Err = \theta^d - \theta = 0$

With vs Without Feedforward transfer function



Without disturbance = 0



With disturbance

The closed-loop system will track any desired trajectory. The steady state error is only determined by the disturbance.