Week 4 - LP and Correlated Equilibrium

Very Brief Intro to Linear Programming

Linear programming is about maximizing a linear function over a polytope

- We describe the polytope P as a set of **linear** (in)equalities
- We optimize a **linear** function on that set

$$\begin{aligned} \max c^T x \\ Ax \leq b \\ x \geq 0 \end{aligned}$$

- or equivalently using equalities (and slack variables z)

$$\begin{aligned} Ax + z = b \\ x, z \geq 0 \end{aligned}$$

- We say that x is **feasible** for P if it satisfies the (in)equalities

Duality

- Motivation/main idea.
- The Primal Linear Program (and with introduced slack variables)

maximize $c^T x$

$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \end{aligned} \qquad\qquad \begin{aligned} Ax + z &= b \\ x, z &\geq 0 \end{aligned}$$

- The Dual Linear Program (and with introduced slack variables)

min $y^T b$

$$\begin{aligned} A^T y &\geq c \\ y &\geq 0 \end{aligned} \qquad\qquad \begin{aligned} A^T y - w &= c \\ y, w &\geq 0 \end{aligned}$$

Duality

P is the primal linear program, D is the dual linear program.

Lemma

If x is feasible for P, y for D, then $c^T x \leq y^T b$

Theorem - Weak Duality

If x is feasible for P, y for D and $c^T x = y^T b$, than x is an optimal solution to P, y is an optimal solution for D

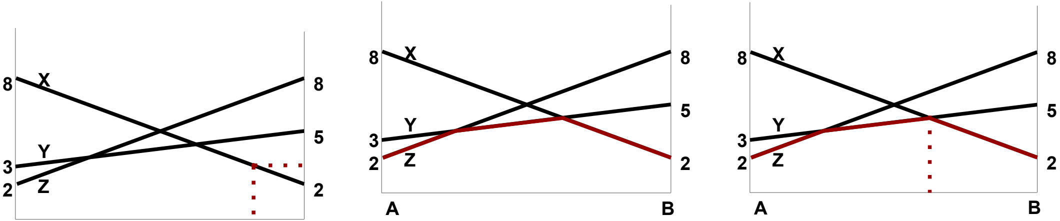
Theorem - Strong Duality

If P and D are both feasible, then there exist feasible x, y such that $c^T x = y^T b$

Nash and Linear Programming

- Let's see how is the linear programming related to Nash equilibrium
- Let's consider only the two-players zero-sum games case
- We will try to write a linear program that finds the Nash

Maximin Mixed Strategies



(a) Visualization of Game

(b) Worst case function we try to maximize.

(c) Strategy that maximizes the worst case function.

Nash and Linear Programming

The plan: let's write a LP that finds Nash equilibrium

Player 1's point of view

- Given any strategy x that I play in NE, player 2 plays best response against me
- $\min_y x^T Ay$
- I want to maximize my value
- $\max_x \min_y x^T Ay$

Player 2's point of view

- Given any strategy y that I play in NE, player 1 plays best response
- $\max_x x^T Ay$
- I want to maximize my value = minimize the negative value
- $\min_y \max_x x^T Ay$

Nash and Linear Programming

- $\max_x \min_y x^T Ay$
- The player 2 plays best response, but player 1 might not
- $\min_y \max_x x^T Ay$
- The player 1 plays best response, but player 2 might not
- But we need both players to play best response to get NE

Theorem - von Neumann MiniMax Principle

$\min_y \max_x x^T Ay = \min_y \max_x x^T Ay = x^{*T} Ay^*$

- How does this relate to NE?
- See the weak duality theorem!
- The optimal solutions x, y correspond to the optimal solutions

MinMax/MaxMin

- Write a LP that solves $\max_x \min_y x^T Ay$
- Write a LP that solves $\min_y \max_x x^T Ay$
- See that these are dual to each other!
- Thanks to the duality principle, the theorem is proven
- Thanks to the fact that we can solve the LP, we also have a way to compute the optimal strategies.

Interestingly, it works the other way around

- Given any two-player zero-sum normal-form game, we can construct an LP that finds the optimal solution
- Given any linear program, we can construct a game, where the optimal strategies in that game correspond to the optimal solution to the linear program

LP Construction

$$\max_{x \in \Pi_i} \min_{y \in \Pi_{-i}} x^\top A y \tag{1}$$

- Since (1) is bi-linear, we need to decouple the x and y by introducing new variable.
- Given a strategy x for the row player, the best-responding opponent simply chooses the column with the smallest utility.

$$\min_{y \in \Pi_{-i}} x^\top A y \tag{2}$$

- This can be re-formulated as

$$\begin{gathered} \max_{u \in \mathbb{R}} u \\ x^\top A \geq u\vec{1} \end{gathered} \tag{3}$$

LP Construction II

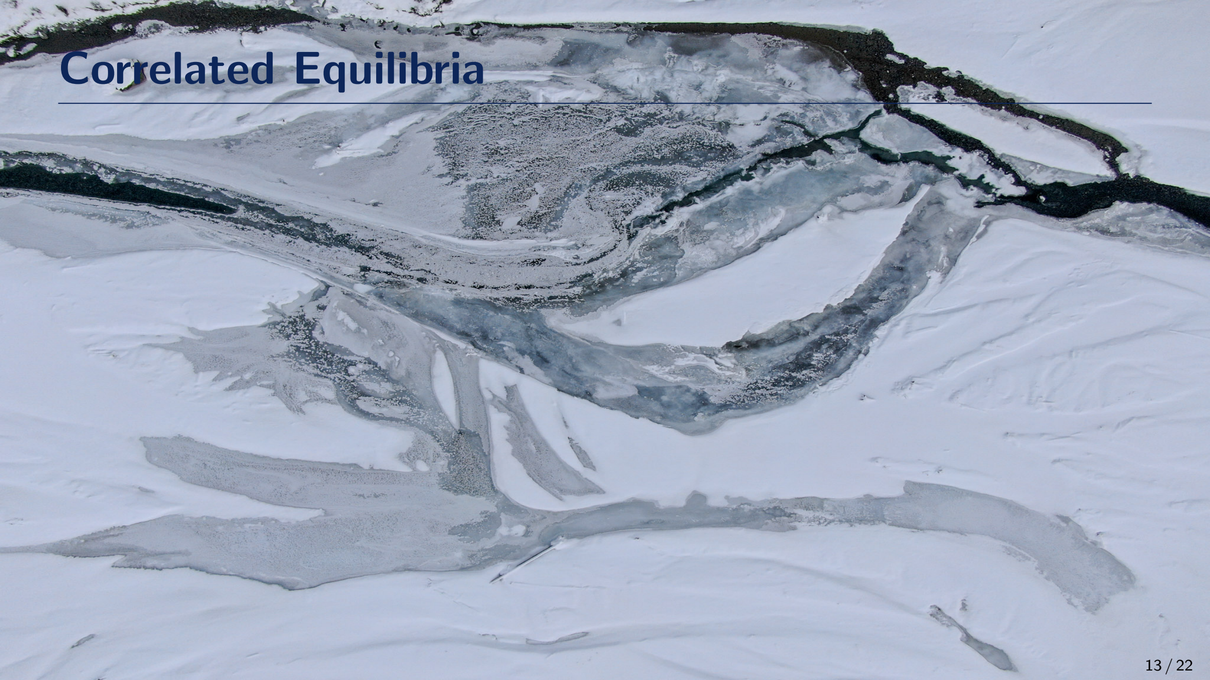
- Putting the reformulated best response back to the maxmin (1), we end up with

$$
\begin{aligned}
&\max_{u \in \mathbb{R}, x \in \Pi_i} \; u \\
&x^\top A \geq u\vec{1}
\end{aligned}
\tag{4}
$$

Summary

- Linear programming
- Two-players zero-sum games as a linear program
- Thanks to duality, we know the optimum exists
- Constructive

Correlated Equilibria



Nash equilibrium relaxation

- Coordinating can lead to better rewards
- Let's consider the crossroad game again:

	Stop	**Go**
Stop	(0, 0)	(0, 1)
Go	(1, 0)	(-10, -10)

Table: Chicken's game

- The game has three Nash equilibria.

Nash equilibrium relaxation

- Let's see all these equlibria and resulting probability distributions over strategy profiles:
- $\sigma_1 = (1, 0)$, $\sigma_2 = (0, 1)$ and probability distribution is then:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- $\sigma_1 = (0, 1)$, $\sigma_2 = (1, 0)$ and probability distribution is then:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- $\sigma_1 = (0.0099, 0.9900)$, $\sigma_2 = (0.0099, 0.9900)$ and probability distribution is then:

$$\begin{pmatrix} 0.000098 & 0.0098 \\ 0.0098 & 0.98 \end{pmatrix}$$

Nash equilibrium relaxation

- But what about following probability distribution:

$$\begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}$$

- Can we obtain it as a product of players' strategies?
- We can add an external coordinator - the traffic light.

Correlated Equilibrium

- In a Nash equilibrium, players choose their strategies independently.
- In a correlated equilibrium a coordinator can choose strategies for both players
- Chosen strategies have to be stable - it is in each player's interest to follow coordinator advice.

Formal definition

- A correlated equilibrium is a probability distribution over strategy profiles a.
- Let $p(a)$ denote the probability of strategy profile a.
- The distribution is a correlated equilibrium if for all players i and all strategies a_i, a_i' following inequality holds:

$$\sum_{a_{-i}} p(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i}} p(a_i, a_{-i}) u_i(a_i', a_{-i})$$

Relation to the Nash Equilibrium

- For both zero and non-zero sum games, think about

Examples

Does correlated equilibrium always represent some Nash equilibrium?

Examples

Are Nash equilibria a subset of correlated equilibria?

Examples

Is the set of all correlated equilibria convex?

Finding Correlated Equilibria

- Recall the definition of correlated equilibrium:
- The distribution $p(a)$ is a correlated equilibrium if for all players i and all strategies a_i, a_i' following inequality holds:

$$\sum_{a_{-i}} p(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i}} p(a_i, a_{-i}) u_i(a_i', a_{-i})$$

- Let's take definition as set of inequalities.
- The only non-constant part is $p(a_i, a_{-i})$ - these are our variables
- The resulting inequalities are linear.
- We need just guarantee that $p(a)$ forms a probability distribution:

$$\sum_{a} p(a) = 1$$

$$p(a) \geq 0$$

Correlated Equilibriuam as LP

- We have now a LP describing all correlated equilibria in the game!!
- We can even optimize any linear function of the $p(a)$
- For example, we can find an correlated equilibrium with maximal sum of players' utilities

Week 4 Homework

1. Find a Nash Equilibrium in a given zero-sum game using the LP formulation
2. Find a Correlated Equilibrium in a given zero-sum game using the LP formulation
3. Find a Correlated Equilibrium in a given non-zero-sum game using the LP formulation