Quadratic Envelopes; Read Me

Sparse recovery

In order to promote the use of Quadratic Envelopes for applications in compressed sensing, sparse recovery, low-rank recovery and similar problem, this site contains free versions of some key proximal operators. All code is written in Matlab, and is decorated with explanations of what is going on. At the moment, there are 4 proximal operators. For each, γ is the tuning parameter of the quadratic envelope and ρ is the "step-size" parameter in the proximal operator. We need $\rho > \gamma$ for these to work out, see [2] for the general idea behind quadratic envelopes.

ProxQmucard This script solves

$$\operatorname*{arg\,min}_{x \in \mathbb{C}^n} \mathcal{Q}_\gamma(\mu \mathrm{card}) + \frac{\rho}{2}\|x-y\|^2$$

where $\gamma > 0$ and $\mu > 0$ are parameters, and $\mathrm{card}(x) = \|x\|_0$. For documentation, see e.g. Example ... [1] or [4], which is devoted to the study of this. Note that in the limit $\rho = \gamma$ this becomes hard thresholding with threshold $\sqrt{2\mu/\gamma}$. The proximal operator is identical to the one used for the Minimax Concave Penalty [6].

ProxQgammaiota This script solves

$$\operatorname*{arg\,min}_{x \in \mathbb{C}^n} \mathcal{Q}_\gamma(\iota_K) + \frac{\rho}{2}\|x-y\|^2$$

where $\iota_K(x) = \infty$ if $\mathrm{card}(x) > K$ and 0 else. For more info, see e.g. Example ... [1] or [4], which is devoted to the study of this.

The first proximal operator can be used when the sparsity degree of the sought solution is completely unknown, whereas the second can be used when it is exactly known. In between these there are many more advanced proximal operators which can be used when upper and lower limits of degree of sparsity is known, but we do not have online versions of them yet. Instructions for how to make them yourself is found in [5].

The coming two proximal operators come from quadratic envelopes on $\mathbb{R}^n$, where all vectors with negative values are banned. Let ι_+ be the indicator functional of the non-negative quadrant, i.e. $\iota_+(x) = 0$ if and only if $x_j \geq 0$ for all j, and ∞ else.

ProxQmucardplus This script solves

$$\arg\min_{x\in\mathbb{R}^n} \mathcal{Q}_\gamma(\mu\text{card} + \iota_+) + \frac{\rho}{2}\|x-y\|^2.$$

For documentation, see e.g. Example ... [1].

ProxQgammaiota This script solves

$$\arg\min_{x\in\mathbb{C}^n} \mathcal{Q}_\gamma(\iota_K + \iota_+) + \frac{\rho}{2}\|x-y\|^2.$$

See [3] for documentation.

Low-rank recovery

Each of the above proximal operators has a counterpart for low rank matrix problems. Given a matrix X we let $\sigma(X)$ be its singular values and $\lambda(X)$ its eigenvalues (in the self-adjoint case). Note that e.g. $\text{card}(\sigma(X)) = \text{rank}(X)$, so each of the above functionals is low rank inducing when combined with either σ or λ. Thus, to compute e.g.

$$\arg\min_X \mathcal{Q}_\gamma(\mu \text{ rank}) + \frac{\rho}{2}\|X-Y\|^2,$$

one proceeds by letting $U\text{diag}yV^*$ be the SVD of Y, then one computes $x = ProxQmucard(y,\mu,\gamma,\rho)$ and the solution is $X = U\text{diag}_x V^*$. This is further explained in [**?**].

For problems involving Hermitian matrices and Positive SemiDefinite conditions, one does the same but using the eigendecomposition $Y = U\text{diag}_y U^*$, see [3].

References

[1] Marcus Carlsson. On convexification/optimization of functionals including an l2-misfit term. *arXiv preprint arXiv:1609.09378*, 2016.

[2] Marcus Carlsson. On convex envelopes and regularization of non-convex functionals without moving global minima. *Journal of Optimization Theory and Applications, to appear*, 2019.

[3] Marcus Carlsson and Daniele Gerosa. On phase retrieval via matrix completion and the estimation of low rank psd matrices. *Inverse Problems*, 2019.

[4] Marcus Carlsson, Daniele Gerosa, and Carl Olsson. An unbiased approach to compressed sensing. *arXiv preprint*, arXiv:1806.05283, 2018.

[5] Viktor Larsson and Carl Olsson. Convex low rank approximation. *International Journal of Computer Vision*, 120(2):194–214, 2016.

[6] Cun-Hui Zhang et al. Nearly unbiased variable selection under minimax concave penalty. *The Annals of Statistics*, 38(2):894–942, 2010.