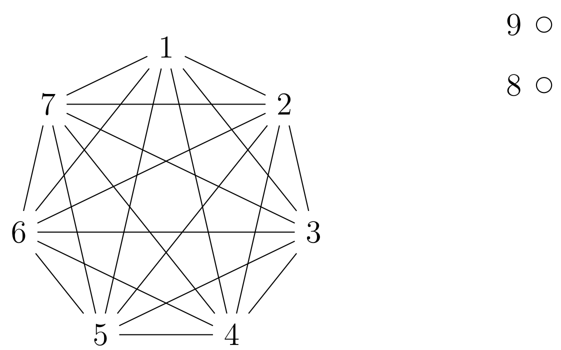
Marcus Bluestone April 3, 2023

6.1200 Problem Set 6

Problem 1 *(Collaborators: Avi Balsam)*

Part 1(a)



In the above graph, there are n = 9 nodes and with 21 edges, which is $\leq$ 3n - 6 = 21. However, it can NOT be 6-colored because K_7 is a subgraph of it, which requires 7 colors.

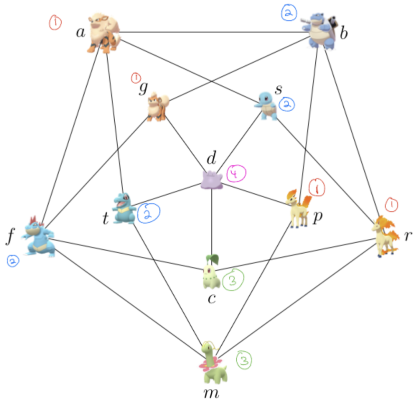
Part 1(b)

The problem is the line "We know that the induced subgraph G' formed in this way has n nodes, so by our inductive hypothesis G' is 6-colorable." This was true when we were discussing a planar graph (subgraph of a planar graph is planar), but removing a node from a graph that has at most $3n - 6$ edges may result in a graph that has more than $3n - 6$ edges, which means we can't necessarily apply our inductive hypothesis.

Problem 2 *(Collaborators: Avi Balsam)*

Part 2(a)

If we have 4 colors: c_1, c_2, c_3, c_4, we can color the graph with them in the following scheme:
Because we can color the graph with at worst 4 colors, $\chi(G) \leq 4$
c_1: a, g, p, r [fire]
c_2: b, f, t, s [water]
c_3: c, m [grass]
c_d: d [purple]



Part 2(b)

Let us assume, for the sake of contradiction, that G can be colored with 3 colors. Let us also define I = $\{g, t, s, p, c\}$ as the set of pokemon in the inner ring on the graph and the set O = $\{a, b, r, m, f\}$ as the set of pokemon on the outer ring of the graph. Also note that many steps require a "without loss of generality" assumption, which should be clear due to the symmetry of the problem (i.e. no pokemon in I is any different from another pokemon in I in terms of its relative relationship to the center and pokemon in the O).

Let us assume without loss of generality (wlog) that d (innermost pokemon) will be colored c_1. Because every pokemon in set I is adjacent to d, there are only 3 possible situations (all of which we will prove to be impossible):

1) All 5 members of the set I will be colored with the same color, say wlog c_2: In this case, because every $p \in O$ is adjacent to some $q \in I$, then O must be colored only with c_1 and c_3. However, O is a odd length cycle (C_5) which we already know can't be colored with only 2 colors. Therefore, this situation can not occur.

2) Four members of the set I will be colored with same color, say wlog c_2, except for one (say wlog g) which is colored with c_3. Because every pokemon $\in O$ is adjacent to TWO pokemon in I and only pokemon in I is colored with c_3, then all pokemon $\in O$ is still adjacent to c_2. By similar logic to part a, it would therefore be impossible to color O with just c_1, c_3 because O is an odd-length cycle.

3) Two members of I will be colored with (wlog) c_3 and three will be colored with c_2.
a)If the two members colored w/ c_3 are adjacent, then by inspection of the graph $\forall q \in O, \exists p \in I$ such that p is colored w/ c_2 and q is adjacent to p. This is evidenced in the graph since every pokemon $\in O$ is adjacent to two non-adjacent members of I. Thus, because every pokemon in O is adjacent the color c_2 we would have to color the members of O with just c_1 and c_3, which is impossible because O is of the type C_5.
b) If the two members colored w/ c_3 are not adjacent, then we arrive at a different issue. Let us assume wlog that c and s are colored with c_3 and thus g,p, and t are colored w/ c_2. Therefore, because a is adjacent to both t and s, a must be colored w/ c_1. In addition, because f is adjacent to both g and c, f must also be colored w/ c_1. However, this is impossible since f is adjacent to a.

Therefore, there is no possible situation in which the graph can be colored with just 3 colors.

Problem 3 *(Collaborators: Avi Balsam)*

Part 3(a)

Let T = the set of all trainers. Since there are no lucky trainers, we know that $\forall t \in T$ t is not lucky, which means that t did not get any pokemon from his top $\lceil n/2 \rceil$ choices. This means that t did NOT get any applications from pokemon in his top half, b/c if he did, he would've chosen it instead and been "lucky." Therefore, the maximum number of applications t could've received is all pokemon in his lower half, i.e. $\lfloor n/2 \rfloor$. Since since t accepted one, he must have rejected at most $\lfloor n/2 \rfloor - 1 < n/2$ applicants.

For n trainers, this means that the number of total rejections $S < n \cdot (n/2) = n^2/2$

Part 3(b)

Assume, for the sake of contradiction, that both all pokemon and trainers were unlucky. Because all trainers are unlucky, we know from part a that $S < n^2/2$.

However, if all pokemon are unlucky $\rightarrow$ any given p $\in$ Pokemon was not accepted from an evaluator from its top $\lceil n/2 \rceil$ choices $\rightarrow$ p must have been accepted by someone in his lower half. Because p always applies to the highest-ranking remaining evaluator on its list, p must have been rejected at least $\lceil n/2 \rceil \geq n/2$ times. Thus, for n pokemon, there must be at least $n \cdot n/2$ rejections, which means that $S \geq n^2/2$.

This leads to a contradiction since S can't be both $<$ and $\geq n^2/2$, and so our initial assumption must be wrong. Therefore, there exists at least one pokemon or trainer who is lucky.

Problem 4 *(Collaborators: Avi Balsam)*

Part 4(a)

Because each hideout can only be raided once and there are four hideouts and four admins, every hideout must be raided.

If Giovanni was not captured at Viridian City Gym at 12:00 pm, then Viridian City Gym must have been raided at some other time [9:00 am, 10:00 am, 11:00 am are the only options]. B/c Giovanni's first inspection occurs at 12:00 pm, any raid in Viridian City Gym at an earlier time would cause Giovanni to flee at 12:00 pm, making Jiggleypuff's mission fail.

Therefore, it must be true that Giovanni was captured at the Viridian City Gym if Jiggleypuff's mission were to be successful.

Part 4(b)

We know that each hideout must be raided, so that means the Haunted Tower must be raided. There are 4 options for when to raid and who to capture:
1) 10:30 am, Sabrina
2) 12:30 am, Lt. Surge
3) 1:30 pm, Koga
4) 3:00 pm, Giovanni

Choice 4 will not work b/c Giovanni has already been captured from part a) and thus raiding Haunted Tower at 3:00 pm won't result in an additional capture, which we need in order for the mission to be successful.

If we choose either Choice 1 or Choice 2, then Koga will flee at 1:30 b/c there is no way to capture Koga earlier. Since each hideout can only be raided once, and from part a) we know that Viridian City Gym is raided at 12:00 pm, the possibility of raiding it at 10:00 am to capture Koga is impossible. Thus, raiding Haunted Tower at 10:30 or 12:30 will cause Koga to flee at 1:30.

Therefore, it must be Choice 3, i.e. raiding Haunted Tower at 1:30 pm to capture Koga is necessary for the mission to be successful.

Part 4(c)

1) 9:30 am, Lt. Surge, Pokemon Labratory
2) 11:30 am, Sabrina, Rocket Game Corner
3) 12:00 pm, Giovanni, Viridian City Gym
4) 1:30 pm, Koga, Haunted Tower

Part 4(d)

Let us assume the pair (A,H) is a rogue couple, where A-H' and A'-H are current matchings. By definition, this means:
1) A prefers H to H' $\rightarrow$ A is scheduled to inspect H before H'.
2) H prefers A to A' $\rightarrow$ H is scheduled to be inspected by A after A'.
This means we know at least a partial order of the scheduling:
Time 1: A'-H [raid occurs here]
Time 2: A-H
Time 3: A-H' [raid occurs here]

The issue is that when the raid occurs at time 1, this will cause A to flee at time 2 before A is caught at time 3 when h' is raided. Therefore, the mission will be unsuccessful. (This is, of course, assuming that no two inspections coincide, b/c otherwise in that case more than one inspector can be caught in a single raid).

Part 4(e)

To prove that M is stable $\rightarrow$ mission succeeds, it is sufficient to prove the contra-positive: if the mission fails, then there must be a rogue match.

The mission can only fail if an inspector flees before they are caught. To illustrate the fleeing of inspector A, the following situation must occur:
We plan to capture A by raiding location H' at a certain time (say Time 3). Before that can occur, 2 things happen:
Time 1: raid occurs to capture inspector A' at location H.
Time 2: Inspector A carries out normal inspection on location H, notices its already been raided, and then flees.
Time 3: our plan to capture A at H' fails, as A has already fled in time step 2.

In other words, we must have the following rules:
1) A raid exists at A'-H [A'-H is a match]

2) A raid exists at A-H' [A-H' is a match]
3) A is scheduled to inspect H before H' [A prefers H to H']
4) H is scheduled to be inspected by A after A' [H prefers A to A']

By definition, this means that there is a rogue couple (A,H), where A-H' and A'-H are the current matchings. Since we have proved that the mission failing implies the existence of a rogue pair, we also know the contrapositive: a stable matching (no rogue pairs) implies the success of the mission.

Because this situation can be modeled as a bipartite graph (4 hideouts, 4 inspectors), there must exist a stable solution and so Jiggleypuff's mission is not impossible.

Part 4(f)

If we use the Gale-Shapely, with admins as applications and hideouts as evaluators, then all applicants will get their optimal choice. x is optimal for y if x is the most preferred of y's feasible matches. Since a preferring h_1 to h_2 means that a is scheduled to visit h_1 BEFORE h_2, this means that every admin will be raided at the earliest possible time.

Part 4(g)

If we use the Gale-Shapely, with hideouts as applications and admins as evaluators, then all hideouts will get their optimal choice. x is optimal for y if x is the most preferred of y's feasible matches. Since h preferring a_1 to a_2 means that h is scheduled to visited by a_1 AFTER a_2, this means that every hideout will be raided at the latest possible time.