0.1 Lecture 4 - Master theorem

0.1.1 intro

$T(n) = aT(\frac{n}{b}) + f(n)$

1. $f(n) = n^{c+\epsilon}$ then $T(n) = \Theta(n^c)$
2. $f(n) = n^c$ then $T(n) = O(n^c)$
3. $f(n) = n^{c-\epsilon}$, then $T(n) = \Omega(f(n)))$

0.1.2 inequality

$T(n) = aT(\frac{n}{b}) + f(n)$

1. $f(n) = O(n^{c+\epsilon})$ then $T(n) = O(n^c)$
2. $f(n) = O(n^c)$ then $T(n) = O(n^c)$
3. $f(n) = O(n^{c-\epsilon})$, then $T(n) = O(f(n))$

> **Remark 0.1 —** when $f(n) = \Theta(n)$, we will split the cases by checking c > 1, c = 1 and c < 1

Proof when $f(n) = O(n)$

1.

$$\begin{aligned}
T(n) &\leq aT(\frac{n}{b}) + f(n) \\
&= a \cdot (T(\frac{n}{b^2}) + \cdot(\frac{n}{b})) + f(n) \\
&= \ldots \\
&= a^i T(1) + a^{i-1}(\frac{n}{b^i}) + \ldots + kn \\
&= n^c + kn \sum_{i=0}^{\log_b(n)-1} (\frac{a}{b})^i && \texttt{case 1:} \ c > 1 \\
&= n^c + kn \cdot \frac{(\frac{a}{b})^{\log_b n} - 1}{\frac{a}{b} - 1} \\
&\leq n^c + kn \cdot \frac{(\frac{a}{b})^{\log_b n}}{\frac{a}{b} - 1} \\
&= n^c + k \cdot \frac{n^{\log_b a}}{\frac{a}{b} - 1} \\
&= O(n^c)
\end{aligned}$$

.

2.

$$
\begin{aligned}
T(n) &\le aT(\frac{n}{b}) + f(n) \\
&= n + kn \sum_{i=0}^{\log_b(n)-1} (\frac{a}{b})^i & \texttt{case 2:}\ \ c = 1 \\
&= n + kn \cdot (\log_b n - 1) \\
&= O(n \log n)
\end{aligned}
$$

.

3.

$$
\begin{aligned}
T(n) &\le aT(\frac{n}{b}) + f(n) \\
&= n^c + kn \sum_{i=0}^{\log_b(n)-1} (\frac{a}{b})^i & \texttt{case 3:}\ \ c < 1 \\
&= n^c + kn \cdot O(1) & \texttt{decreasing geometric series} \\
&= O(n)
\end{aligned}
$$

.