# Stochastic Simulation The Bootstrap method

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#### The Bootstrap method



- A technique for estimating the variance (etc) of an estimator.
- Based on sampling from the empirical distribution.
- Non-parametric technique

#### Recall the simple situation



- We have n observations  $x_i$ ,  $i = 1, \ldots, n$ .
- If we want to estimate the mean value of the underlying distribution, we (typically) just use the estimator  $\bar{x} = \sum x_i/n$ .
- This estimator has the variance  $\frac{1}{n}Var(X)$ . To estimate this, we (typically) just use the sample variance.

#### A not-so-simple-situation



- Assume we want to estimate the median, rather than the mean.
- (This makes much sense w.r.t. robustness)
- The natural estimator for the median is the sample median.
- But what is the variance of the estimator?

#### The variance of the sample median



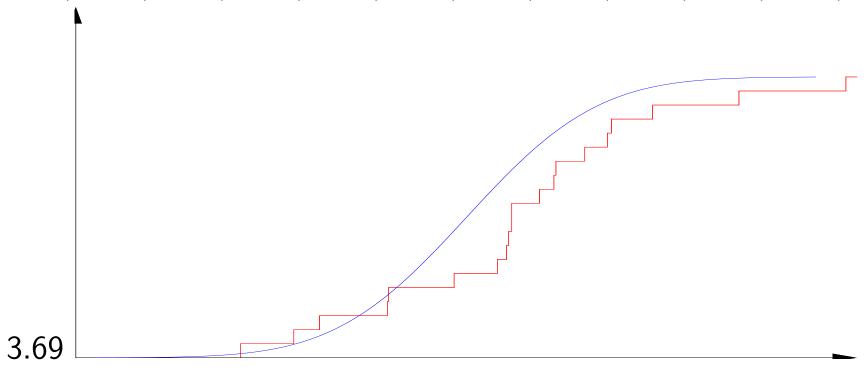
If we had access to the "true" underlying distribution, we could

- 1. Simulate a number of data sets like the one we had.
- 2. For each simulated data set, compute the median.
- 3. Finally report the variance among these medians.

We don't have the true distribution. But we have the **empirical** distribution!

### Empirical distribution

20 N(0,1) variates (sorted): -2.20, -1.68, -1.43, -0.77, -0.76, -0.12, 0.30, 0.39, 0.41, 0.44, 0.44, 0.71, 0.85, 0.87, 1.15, 1.37, 1.41, 1.81, 2.65,



#### Empirical distribution



 $X_i$  iid random variables with  $F(x) = P(X \le x)$ 

Each leads to a (simple) random function  $F_{e,i}(x) = \mathbf{1}_{\{\mathbf{X_i} \leq \mathbf{x}\}}$ 

leading to 
$$F_e(x) = \frac{1}{n} \sum_{i=1}^n F_{e,i}(x) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{\mathbf{X_i} \leq \mathbf{x}\}}$$

$$\mathsf{E}\left(F_e(x)\right) = \mathsf{E}\left(\frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{\mathbf{X_i} \leq \mathbf{x}\}}\right) = \frac{1}{n}\sum_{i=1}^n \mathsf{E}\left(\mathbf{1}_{\{\mathbf{X_i} \leq \mathbf{x}\}}\right) = F(x)$$

Once we have sample  $x_i$ , i = 1, 2, ..., n we have a realised version of the empirical distribution function

$$F_e(x) = \frac{1}{n} \sum_{i=1}^n F_{e,i}(x) = \frac{1}{n} \sum_{i=1}^n \delta_{\{x_i \le x\}}$$

where  $\delta$  is Kroneckers delta-function

## The Bootstrap Algorithm for the variance of a

#### parameter estimator



- ullet Given a data set with n observations.
- Simulate *r*
- (e.g., r = 100)
- data sets,
- each with n "observations"
- ullet sampled form the empirical distribution  $F_e$ .
- (To simulate such one data set, simply take n samples from the original data set with replacement)
- For each simulated data set, estimate the parameter of interest (e.g., the median). This is a *bootstrap replicate* of the estimate.
- Finally report the variance among the bootstrap replicates.

#### Advantages of the Bootstrap method



- Does not require the distribution in parametric form.
- Easily implemented.
- Applies also to estimators which cannot easily be analysed.
- Generalizes e.g. to confidence intervals.

#### Exercise 8

1. Exercise 13 in Chapter 8 of Ross (P.152).



- 2. Exercise 15 in Chapter 8 of Ross (P.152).
- 3. Write a subroutine that takes as input a "data" vector of observed values, and which outputs the median as well as the bootstrap estimate of the variance of the median, based on r=100 bootstrap replicates. Simulate N=200 Pareto distributed random variates with  $\beta=1$  and k=1.05.
  - (a) Compute the mean and the median (of the sample)
  - (b) Make the bootstrap estimate of the variance of the sample mean.
  - (c) Make the bootstrap estimate of the variance of the sample median.
  - (d) Compare the precision of the estimated median with the precision of the estimated mean.