Stochastic simulation exercises

ex123

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1 Exercises june 7th

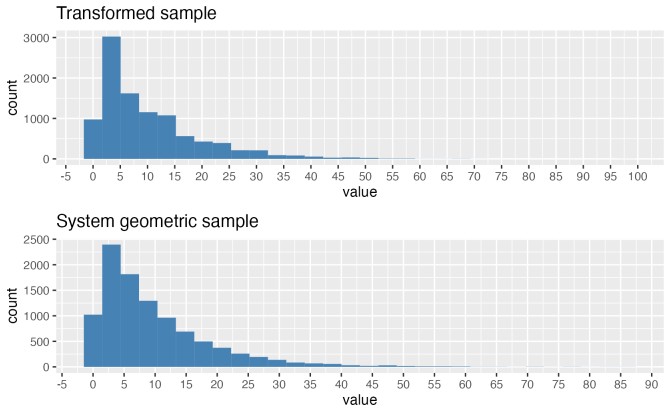
1.1 Exercise 1 (Sampling discrete random variables)

1.1.1 Simulate a geometric distribution

To simulate the geometric distribution we generated 10000 continuous uniform samples and then used the inverse distribution function technique to transform the uniform samples to samples following a geometric distribution. The transformation used was:

$$X = floor((\frac{\log(U)}{\log(1-p)}) + 1 \tag{1}$$

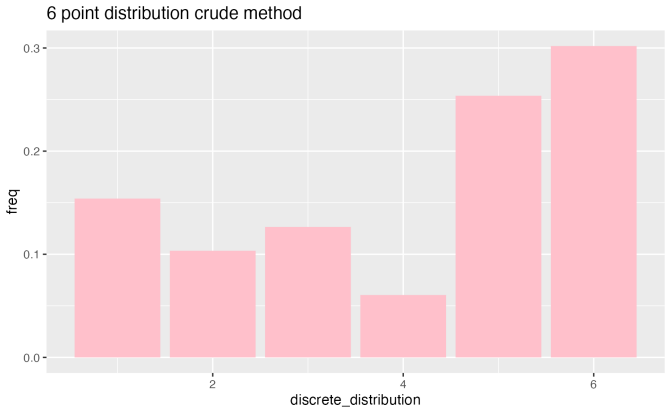
To compare the result we compared the resulting histograms of our transformed sample and a system available sampling from a geometric distribution.



The results are not quite the same as hoped since there was a smaller tail on our transformed sample compared to the geometric sample. However the compared frequencies did not seem too far off.
A good way to test sample would be too perform a chi squared test up until the highest observed value that has over 5 observation. Time did not allow this unfortunately.

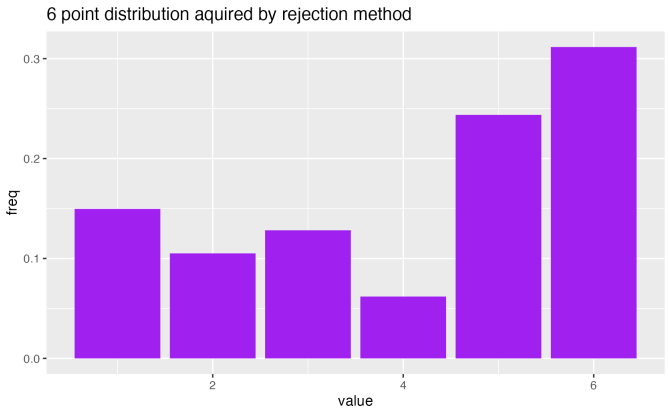
1.1.2 Simulate the 6 point distribution

- **The direct crude method:** This method revealed the frequencies show in the histogram (with frequencies on y) below:



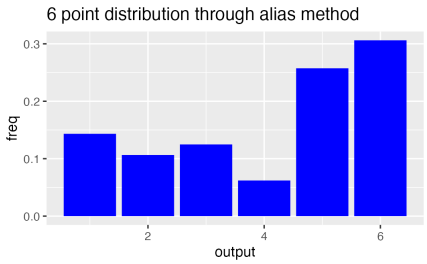
The resulting p value from a chi squared test revealed a p value of 0.997 which means our observation is extremely likely to come from the given 6 point distribution

- **The rejection method:** this method revealed the following histogram (with frequencies on the y axis)



A chi squared test gave the p value of 0.61 which is lower than the crude method but still concludes our sample to be likely from the given 6 point distribution

- **Alias method:** Using the Alias method the following frequencies were found using 10000 samples.



Verifying with a chi squared test revealed a p value of 0.32 "verifying" that our observation or something more extreme is likely

1.1.3 Compare the methods using adeqaute criteria and discuss results

To compare the methods we use running time to estimate which are best.

- **Direct crude method:** This method had a running time of 0.64 seconds which is nicely fast considering $n = 10000$
- **The rejection method:** The rejection method had a runtime of 0.26 seconds which is even faster than the crude method.
- **The Alias method:** The alias method revealed a runtime of 0.19 seconds. Thus the alias method was the overall fastest algorithm.

Using computational time, the alias method woulp appear to be the best method overall for constructing and abitrary discrete distribution. However, there could very well be a possibility that the code could be written more effectively (in terms of runtime) for each method.

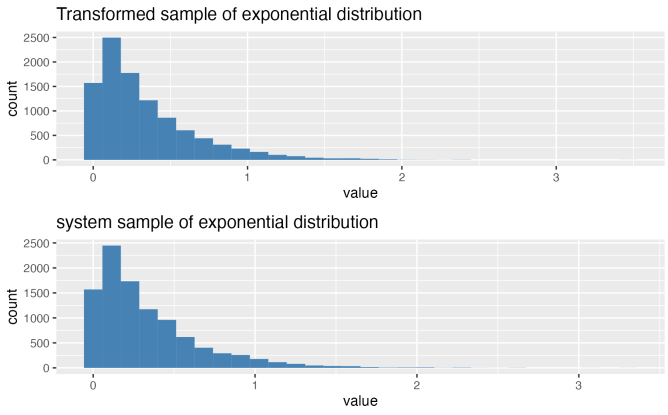
1.1.4 Give recommendations of how to choose the best suited method in different settings

- **The alias method:** This is great if we don't have too many classes since the setup process takes $\mathcal{O}(n)$ time, but the lookup process is ofcourse $\mathcal{O}(1)$ so in general if the number of classes are not too high, the alias method is good for a high speed generation.
- **The rejection method:** This method can also be fast and it doesn't have a "long" setup time like the alias method. However, for the sample to converge quckly it need to have a high acceptance rate. If the acceptance rate is too low (as a consequence of having a high c value) then the algorithm will take longer to finish.
- **The crude method:** If there are many classes or some classes have probability mass very close to each other and maybe some class with a very high probability you might need a high sample size to get the real probabilities.

1.2 Exercise 2 (Sampling continuous distributions)

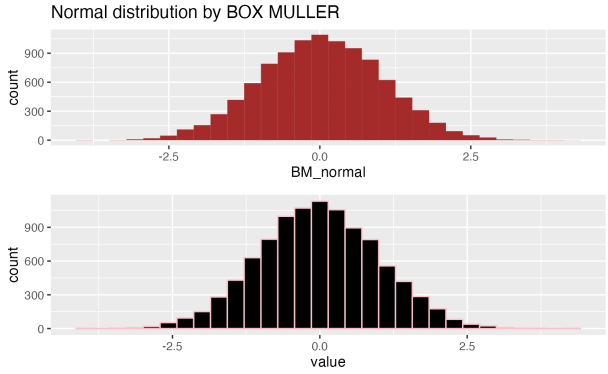
1.2.1 Generate samples from different continuous distributions

- **Exponential distribution:** Here we use the inverse transformation technique. Setting $\lambda = 3$ we compared again the transformed results with the system available generator.



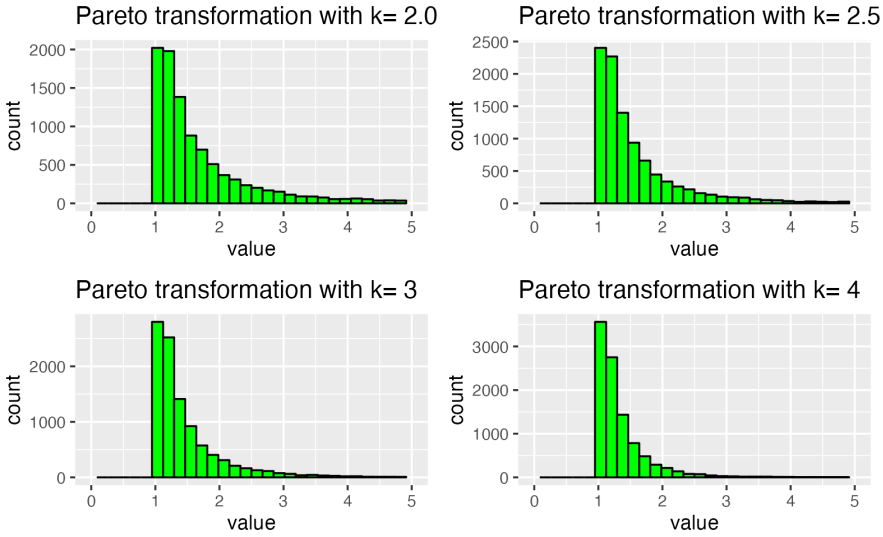
To test these results we used kolmogorov smirnov test. we arrived at a p value of 0.14 which is on the lower side, but still we cannot reject that the transformed observations are from the actual exponential distribution

- **Normal distribution:** Using the box muller method to transform our uniform samples to normal revealed the following results again compared to the system available normal distribution. We only compared one of the resulting transformed parameters.



A kolmogorov smirnov test revealed a p value of 0.71 meaning our transformed result was very likely to come from an actual standard normal distribution

- **Pareto:** Setting $\beta = 1$ and experimenting with the different values for k gave us the following histograms



We again tested these with the kolmogorov smirnov test. for $k = 2.05$ we got p value 0.45.
for $k = 2.5$ we got a p value of 0.07
for $k = 3$ we got p value of 0.18
for $k = 4$ we got p value of 0.17
A thing to not here is that the variance of the p values were very high! Each time we tested we frequently got results in either the very low range or the very high range.

1.2.2 Compare sample means with theoretical means for the pareto distribution

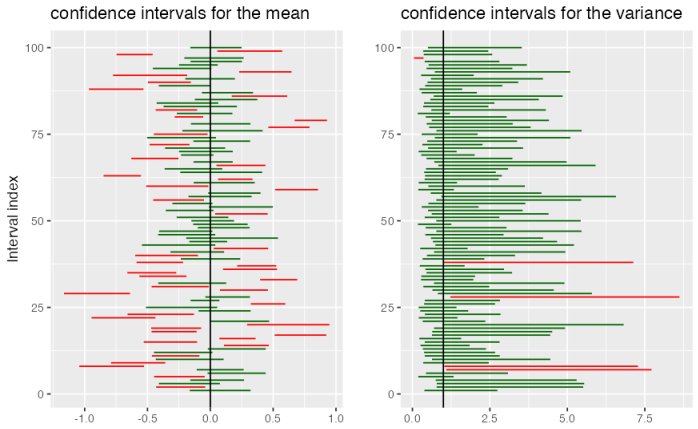
The means for every value of k in the first exercise were precise op to 3 digits. One could maybe expect a lower precision here since the tails are so large on the pareto distribution.

The variance however was very distant from the theoretical. This must be because of the have tails as a high value very far from the mean would would explode the variance as it is the average squared distance from the mean. The results can be seen on the table below. From the observed one could theorise that if we bootstrapped the variance parameter the variance of this would indeed also have a high variance and maybe also that the variance would increase for smaller values of k.

value <dbl>	sample_var <dbl>	theory_var <dbl>
2.05	5.483863	39.0476190
2.50	1.509980	3.3333333
3.00	0.613147	1.5000000
4.00	0.200078	0.6666667

1.2.3 Generate 100 95% confidence intervals for the mean based on 10 observations

Generating 100 confidence intervals for the mean and variance revealed the results below. Green line contain the actual parameter value and red lines don't.



To calculate the confidence intervals for the mean, we've used the fact that the sample variance given iid conditions will be normally distributed and since we have a low sample size, the t distribution is used to accommodate in terms of the quantiles used in the interval. For the confidence intervals for the variance, we use the fact that the $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$ and thus can extract a confidence interval from this. So in the end we've used the intervals:

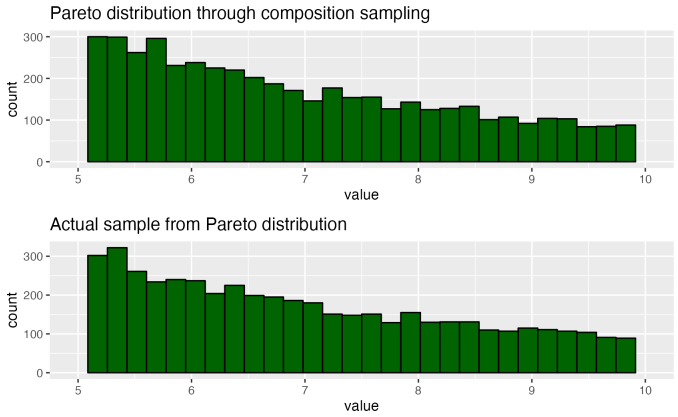
$$\bar{X} \pm t_{\alpha/2}(9)\frac{s}{\sqrt{n}} \tag{2}$$

$$\frac{(10-1)s^2}{\chi^2_{1-(\alpha/2)}(9)} \leq \sigma^2 \leq \frac{(10-1)s^2}{\chi^2_{(\alpha/2)}(9)} \tag{3}$$

To conclude on the results we see that small samples sizes impact the precision of the CI for the mean. However, the intervals are also much very long.

1.2.4 Simulate from the pareto distribution through composition

From slides we know that if we choose some β and let $Y \sim exp(\beta)$. Then the density $f(X|Y = y) = ye^{-yx}$ (I.e $X_i \sim exp(y_i)$ will describe a Pareto distribution. Generating this with $n = 10000$ and $\beta = 5$ and comparing with a system available pareto distribution, gave the following histograms.



Testing the distributions against eachother with a kolmogorov smirnov test gave us a p-value of 0.75 which makes our composition sample very likely for a pareto distribution with $\beta = 5$.

2 Code Appendix

2.1 Exercises 2 june 7th

2.1.1 Simulate a geometric distribution

```
    ```{r}
library(gridExtra)

## Get uniform continuous

n <- 10*1000

rand_nums <- runif(n,0,1)

p <- 0.1

geo_nums <- floor(log(rand_nums)/(log(1-p)))+1

p1 <- geo_nums %>%
  as.tibble() %>%
  ggplot(aes(value)) +
  geom_histogram(fill="steelblue",bins = 30) +
  scale_x_continuous(n.breaks = 25) +
  ggtitle("Transformed sample")

geo_nums2 <- (rgeom(n,p)+1) %>% as.tibble()

p2 <- geo_nums2 %>%
```
```

```
  ggplot(aes(value)) +
  geom_histogram(fill="steelblue",bins = 30) +
  scale_x_continuous(n.breaks = 25) +
  ggtitle("System geometric sample")

grid.arrange(p1,p2)

## Compare counts
geo_nums2 %>%
  filter(value<=5) %>%
  group_by(value) %>%
  summarise(
    n())
  )
```

2.1.2 Simulate a 6 point distribution

```
    ```{r}
## ---------------Direct crude method
start <- Sys.time()
## Define classes
classes <- 1:6

prob_vec <- c(7/48,5/48,1/8,1/16,1/4,5/16)

prob_vec %>%
  as.tibble() %>%
  rename("probs"=value) %>%
  mutate(cums = cumsum(probs))

## Setup if between cumulative probability

rand_nums %>%
  as.tibble() %>%
  mutate(
    discrete_distribution  = case_when(between(value,0,0.1458333) ~ 1,
              between(value,0.1458333,0.25) ~ 2,
              between(value,0.25,0.3750000) ~ 3,
              between(value,0.3750000 ,0.4375000 ) ~ 4,
              between(value,0.4375000 ,0.6875000 ) ~ 5,
              between(value,0.6875000 ,1) ~ 6,
    ) ) %>%
  group_by(discrete_distribution) %>%
  summarise(freq = n()/length(rand_nums)) %>%
  mutate(real_p = prob_vec) %>%
  ggplot(aes(discrete_distribution,freq)) +
  geom_bar(fill="pink",stat = "identity") +
  ggtitle("6 point distribution crude method")

## Calculate chi sqrd test

crude_table <- rand_nums %>%
  as.tibble() %>%
  mutate(
    discrete_distribution  = case_when(between(value,0,0.1458333) ~ 1,
              between(value,0.1458333,0.25) ~ 2,
              between(value,0.25,0.3750000) ~ 3,
              between(value,0.3750000 ,0.4375000 ) ~ 4,
              between(value,0.4375000 ,0.6875000 ) ~ 5,
              between(value,0.6875000 ,1) ~ 6,
    ) ) %>%
  group_by(discrete_distribution) %>%
  summarise(freq = n()/length(rand_nums)) %>%
  mutate(real_p = prob_vec)

stop_time <- Sys.time()
```
```

```
Time_comp <- stop_time-start
obs <- crude_table$freq*(10000)

exp <- crude_table$real_p*10000

stat <- sum(((obs-exp)^2)/exp)

pchisq(stat,df=6-1,lower.tail = F)

cat("computation time",Time_comp)

Time_comp
```

    ```{r}
## ------------------ Simple rejection method
n <- 10*1000
classes <- 1:6

prob_vec <- c(7/48,5/48,1/8,1/16,1/4,5/16)

c <- max(prob_vec)
samples <- c()
num_samples <- 0
start <- Sys.time()
while(num_samples<=n){
  rand_samp <- runif(2,0,1)

  I <- floor(6*rand_samp[1])+1
  if(rand_samp[2]<=prob_vec[I]/c){
    samples <- append(samples,I)
    num_samples <- length(samples)
  }
}

stop <- Sys.time()

time <- stop-start
rej_table <- samples %>%
  as.tibble() %>%
  group_by(value) %>%
  summarise(
    freq = n()/(10*1000)
  ) %>%
  mutate(real_p = prob_vec)

## Plot observed frequencies
rej_table %>%
  as.tibble() %>%
  ggplot(aes(value,freq)) +
```

```
  geom_bar(fill="purple",stat="identity") +
  ggtitle("6 point distribution aquired by rejection method")

## Perform chisq test

obs <- rej_table$freq*(10000)

exp <- rej_table$real_p*10000

stat <- sum(((obs-exp)^2)/exp)

pchisq(stat,df=6-1,lower.tail = F)
time
```

```{r}
### ----------------- ALIAS METHOD
## Probabilities

prob_vec

## Generate uniform sample

U <- runif(n)

## Generate alias table and F values
start <- Sys.time()
F_vals <- 6*prob_vec

F_table <- F_vals %>%
  as.tibble() %>%
  mutate(class = 1:6) %>%
  mutate(less_than1 = F_vals<=1)

G <- F_table %>%
  filter(less_than1==F) %>%
  pull(class)

S <- F_table %>%
  filter(less_than1==T) %>%
  pull(class)

L <- 1:6

while(length(S)!=0){
  i = G[1] ; j = S[1]
  L[j]=i ;  F_vals[i] = F_vals[i]-(1-F_vals[j])
  if(F_vals[i]<1){
    G=G[-1] ; S <- append(S,i)
  }
  S = S[-1]
```

```
}

## Perform algorithm

U %>%
  as.tibble() %>%
  mutate(U_2 = runif(n)) %>%
  mutate(I = floor(6*value)+1) %>%
  mutate(output = ifelse(U_2<=F_vals[I],I,L[I])) %>%
  group_by(output) %>%
  summarise(
    freq = n()/n
  ) %>%
  ggplot(aes(output,freq)) +
  geom_bar(stat="identity",fill="blue") +
  ggtitle("6 point distribution through alias method")

stop <- Sys.time()

Time_comp <- stop-start
### Perform chi squared test

obs <- U %>%
  as.tibble() %>%
  mutate(U_2 = runif(n)) %>%
  mutate(I = floor(6*value)+1) %>%
  mutate(output = ifelse(U_2<=F_vals[I],I,L[I])) %>%
  group_by(output) %>%
  summarise(
    freq = n()/n
  ) %>%
  pull(freq)

obs <- obs*10000
exp <- prob_vec*10000
stat <- sum(((obs-exp)^2)/exp)

pchisq(stat,6-1,lower.tail = F)

‘‘‘
```

2.2 Exercises 3 June 7th

2.2.1 Generate samples from continuous distributions

```
    ‘‘‘{r}
## ----------- EXPONENTIAL DISTRIBUTION
### Generate for the following distributions.

lambda <- 3
```

```
exp_randnums <- -log(rand_nums)/3


p1 <- exp_randnums %>%
  as.tibble() %>%
  ggplot(aes(value)) +
  geom_histogram(fill="steelblue") +
  ggtitle("Transformed sample of exponential distribution")

real_exp <- rexp(10000,rate=lambda)

p2 <- real_exp %>%
  as.tibble() %>%
  ggplot(aes(value)) +
  geom_histogram(fill="steelblue") +
  ggtitle("system sample of exponential distribution")

grid.arrange(p1,p2)

## Test with Kolmogorov smirnov

ks.test(exp_randnums,"pexp",lambda)

```

    ```{r}
## ------------- BOX MULLER NORMAL DISTRIBUTION
n <- 10*1000
U_1 <- runif(n)
U_2 <- runif(n)

samp_data <- cbind(U_1,U_2) %>% as.tibble()

bm_norm <- samp_data %>%
  mutate(BM_normal = sqrt(-2*log(U_1))*cos(2*pi*U_2)) %>%
  pull(BM_normal)

p1 <- bm_norm %>%
  as.tibble() %>%
  ggplot(aes(value)) +
  geom_histogram(fill="brown") +
  ggtitle("Normal distribution by BOX MULLER")

real_norm <- rnorm(n)

p2 <- real_norm %>%
  as.tibble() %>%
  ggplot(aes(value)) +
  geom_histogram(fill="black",color="pink") +
  ggtitle("System standard normal distribution")
```

```
grid.arrange(p1,p2)

## Perform KS test

ks.test(bm_norm,"pnorm",0,1)

bm_norm
```

    ```{r}
beta <- 1

k_vals <- c(2.05,2.5,3,4)
plots <- list()
for(i in 1:length(k_vals)){

k <- k_vals[i]

X <- beta*(rand_nums^(-1/k))

X %>% as.tibble() %>%
  ggplot(aes(value)) +
  geom_histogram()

plots <- append(plots,p)
X %>% hist(xlim=c(0,18),breaks=1000)
}

## Construct table comparing values and creating plots.
paret_table <- k_vals %>%
  as.tibble() %>%
  group_by(value) %>%
  mutate(samp = map(value,~(rand_nums))) %>%
  mutate(pareto = map2(value,samp,~(
    (.y)^(-1/.x)
  ))) %>%
  mutate(
    plot = map2(value,pareto,~(
      .y %>%
        as.tibble() %>%
        ggplot(aes(value)) +
        geom_histogram(fill="green",color="black") +
        xlim(0,5) +
        ggtitle(paste("Pareto transformation with k=",.x))
    )
  )) %>%
  mutate(sampmean_paret = map_dbl(pareto,~(
    mean(.x)
  ))) %>%
  mutate(thrymean_paret = beta*(value/(value-1)))
```

```
plots <- paret_table$plot

grid.arrange(plots[[1]],
             plots[[2]],
             plots[[3]],
             plots[[4]])

## Test each transformed sample

for(i in 1:length(k_vals)){
  k_choice <- paret_table$value[i]

real_samp <- rpareto(10*1000,1,k_choice)

trans_samp <- paret_table$pareto[[i]]

p_val <- ks.test(trans_samp,real_samp)$p.value

cat("for k=",k_choice,"the p_value is = ",p_val,"    ")
}

### Comparing means

paret_table %>%
  select(sampmean_paret,thrymean_paret)

## Comparing variance

paret_table %>%
  select(value,pareto) %>%
  mutate(
    sample_var = map_dbl(pareto,~(
      var(.x)
    )
  )) %>%
  mutate(theory_var = value/(((value-1^2)*(value-2)))) %>%
  select(-pareto)
```

2.2.2 Generate 100 CI for mean and variance

```
    ```{r}
## Construct 100 confidence intervals for the mean

p1 <- 1:100 %>%
  as.tibble() %>%
  ## take sample of size 10 from box muller normal distribution 100 times
  mutate(
    sample = map(value,~(
      sample(bm_norm,size=10,replace=F)
```
```

```
    )
  )) %>%
  ## Construct lower bound for the mean
  mutate(
    lower_bound_mean = map_dbl(sample,~(
      mean(.x)-qt(0.025,df=9)*(sd(.x)/10)
    )
  )) %>%
  ## Construct upper bound for mean
  mutate(
    upper_bound_mean = map_dbl(sample,~(
      mean(.x)+qt(0.975,df=9)*(sd(.x)/10)
    )
  )) %>%
  ## Construct logical if real value is within the interval
  mutate(contains_value = between(rep(0,100),lower_bound_mean,upper_bound_mean)) %>%
  ## Create plot
  ggplot(aes(value,ymin = lower_bound_mean,ymax=upper_bound_mean)) +
  geom_linerange(aes(color=contains_value)) +
  scale_color_manual(values = c("TRUE" = "darkgreen", "FALSE" = "red")) +
  coord_flip() +
  geom_hline(yintercept = 0) +
  xlab("Interval index") +
  theme(legend.position = "none") +
  ggtitle(paste("confidence intervals for the mean"))


### Generate plot for variance
p2 <- 1:100 %>%
  as.tibble() %>%
  mutate(
    sample = map(value,~(
      sample(bm_norm,size=10,replace=T)
    )
  )) %>%
  mutate(
    lower_bound_mean = map_dbl(sample,~(
      ((10-1)*var(.x))/(qchisq(0.975,9))
    )
  )) %>%
  mutate(
    upper_bound_mean = map_dbl(sample,~(
      ((10-1)*var(.x))/(qchisq(0.025,9))
    )
  )) %>%
  mutate(contains_value = between(rep(1,100),lower_bound_mean,upper_bound_mean)) %>%
  ggplot(aes(value,ymin = lower_bound_mean,ymax=upper_bound_mean)) +
  geom_linerange(aes(color=contains_value)) +
  scale_color_manual(values = c("TRUE" = "darkgreen", "FALSE" = "red")) +
  coord_flip() +
  geom_hline(yintercept = 1) +
```

```
  xlab("") +
  theme(legend.position = "none") +
  ggtitle(paste("confidence intervals for the variance"))

  grid.arrange(p1,p2,ncol=2)
```

‘‘‘

2.2.3 Simulate through pareto distribution

```
    ‘‘‘{r,warning=F}
## -------- Pareto distribution through composition

n <- 10*1000
### Generate Y through exponential distribution

mu <- 5

Y <- rexp(n,rate=mu)

X <- rexp(n,rate=Y)

## Add mu to X so support is X>=beta

comp_paret <- X + mu

## Thus X should follow a Pareto distribution with location = 5

### Compare by histograms

p1 <- comp_paret %>%
  as.tibble() %>%
  ggplot(aes(value)) +
  geom_histogram(fill="darkgreen",color="black") +
  xlim(mu,10) +
  ggtitle("Pareto distribution through composition sampling")

paret_realsamp <-  rpareto(n,location = mu,shape = 1)

p2 <- paret_realsamp %>%
  as.tibble() %>%
  ggplot(aes(value)) +
  xlim(mu,10) +
  geom_histogram(fill="darkgreen",color="black") +
  ggtitle("Actual sample from Pareto distribution")

grid.arrange(p1,p2)

### Test the sample against actual sample

ks.test(comp_paret,paret_realsamp)
```

‘ ‘ ‘