Exercise 5 - Variance redudiction Methods

The tasks centers around simlation of the integral $\int_0^1 e^x \, \mathrm{d}x$

In [ ]:
```
import numpy as np
import numpy.random as rnd
import scipy.stats as stats
import math as math
```

UNCHANGED

1) Simulation of the integral using a Crude Monte Carlo Method

Using from the slides that the integral can be simplified to the expectation of e^U where $U \sim Uniform(0,1)$. Simulate $n = 100$ instances of e^x, and find the mean. To get a 95% CI, use quantiles from t distribution with $n-1$ dof and $\alpha = 0.05$.

$$CI = \left[\bar{\theta} + \frac{S_\theta}{\sqrt{n}} t_{\frac{\alpha}{2}}(n-1); \bar{\theta} + \frac{S_\theta}{\sqrt{n}} t_{1-\frac{\alpha}{2}}(n-1)\right]$$

In [ ]:
```
def crudeMC(n):
    U = rnd.uniform(size = n)
    x = np.exp(U)
    return x

def meanVar(x):
    mean = np.mean(x)
    var = np.var(x)
    return mean, var

def tConf(x, alpha, string):
    mean, var = meanVar(x)
    dof = len(x) - 1
    s = np.sqrt(var / len(x))
    a = stats.t.ppf(alpha/2, dof)
    b = stats.t.ppf(1 - alpha/2, dof)
    print("For", string, ":")
    print(f"Mean is {round(mean,3)}, with 95% confidence interval [{round(mean +

alpha = 0.05
n = 100
Xs_crude = crudeMC(n)
tConf(Xs_crude, alpha, "Crude Monte Carlo" )
print(f"The true value of the integral is: {round(np.exp(1) - np.exp(0),4)}")
```

```
For Crude Monte Carlo :
Mean is 1.763, with 95% confidence interval [1.67,1.857]
The true value of the integral is: 1.7183
```

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2) Antithetic variables

We can use Anithetic variables to exploit the fact that the integral $\int_{-\infty}^{\infty} e^x \, \mathrm{d}x$ is monotonely increasing. In practice this means we can use a single uniformly distributed value $U \sim Uniform(0,1)$, as in the Crude Monte Carlo Method, an use it to create a second uniformly distributed value $U-1$, almost for free.

Using these we can create 2 estimates of the integral by e^U and e^{U-1}, and take the average of these two to get a much more robust estimate of the integral

$$Y_i = \frac{e^{U_i - e^{U_i - 1}}}{2}.$$

The expectation of the integral is now

$$E\left(Y_i\right).$$

Y_i which on the slides is proved to have variance $\frac{1}{4}Var\left(e^{U_i}\right) + \frac{1}{4}Var\left(e^{1-U_i}\right) + \frac{1}{2}Cov\left(e^{U_i}, e^{1-U_i}\right)$. As e^{U_i} and e^{1-U_i} are obviosly negatively correlated, this variance is much lower than before.

By rewriting Y_i computing cost can be lowered, to only calculate a single exponential, resulting in an only marginally more expensive computation for n Y's compared to n X 's.

$$Y_i = \frac{e^{U_i + \frac{e}{e^{U_i}}}}{2}.$$

Note: Had the integral not been monotonely increasing, say we had attemted to estimate some other function f, we could have had a situation where $f\left(U_i\right)$ and $f\left(U_i - 1\right)$ had been positively correlated, which could lead to a higher variance on Y_i. For the exponential function one of e^{U_i} and $e^{U_i - 1}$ is always large, and one is always small.

In [ ]:
```
def antithetic(n):
    U = rnd.uniform(size = n)
    t = np.exp(U)
    Ys = 0.5 * (t + np.exp(1) / t )
    return Ys
Xs_antithetic = antithetic(100)
tConf(Xs_antithetic, alpha, "Antithetics Variables")
```

```
For Antithetics Variables :
Mean is 1.717, with 95% confidence interval [1.7,1.729]
```

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3) Control Variates

We can use the variable

$$Z = X + c\left(Y - \mu_i\right),$$

instead of X_i as the estimate of the integral. It can then be shown that $E\left(X\right) = E\left(Z\right)$, since

$$E\left(Z\right) = E\left(X\right) + E\left(c\left(Y - \mu_i\right)\right)$$
$$= E\left(X\right) + c\left(E\left(Y\right) - E\left(\mu_i\right)\right) = E\left(X\right) + c\left(\mu_i - \mu_i\right) = E\left(X_i\right).$$

From the slides we also know that choosing the optimal $c = -\frac{Cov(X,Y)}{Var(Y)}$, results in a variance of Z

$$Var(Z) = Var(X) - \frac{Cov(X,Y)^2}{Var(Y)}.$$

For this specific problem we are given $X_i = e^{U_i}$, and it is natural to choose $Y_i = U_i$. This again exploits the covariance between U_i and X_i, though this time negative correlation is not a requirement, as the covariance is squared. It should just be non-zero. As $E\left(U_i\right) = \frac{1}{2}$ we get

$$Z_i = e^{U_i} + c\left(U_i - \frac{1}{2}\right),$$

with $c \approx 0.14086$.

In [ ]:
```
def controlVariates(n, alpha):
    U = rnd.uniform(size = n)
    Xs = np.exp(U)
    dof = len(Xs) - 1
    meanZ = np.mean(Xs)
    cov = np.mean(U * Xs) - np.mean(U) * np.mean(Xs)
    varZ = np.var(Xs) - cov**2 / np.var(U)
    s = np.sqrt(varZ / len(Xs))
    a = stats.t.ppf(alpha/2, dof)
    b = stats.t.ppf(1 - alpha/2, dof)
    print("For Control Variates")
    print(f"Mean is {round(meanZ,2)}, with 95% confidence interval [{round(meanZ
controlVariates(n, alpha)
```

```
For Control Variates
Mean is 1.71, with 95% confidence interval [1.7,1.72]
```

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4) Statified Sampling

Stratified sampling attemts to lower variance by (almost) gaurantying samples from the entire sample space, by cutting the sample space into m pieces, and then sampling a number of variables from each. Here we´ll use $m = 10$ even intervals, to ensure even computational cost to the other methods, 10 U_i's are generated in each interval.

The method then takes one U_i from each interval and creates a single variable W_i

$$W_i = \frac{\sum_{k=1}^{m} X_{i,k}}{m},$$

where k now denotes which interval the X_i comes from. Still $X_i = e^{U_i}$, or $X_{i,k} = e^{U_{i,k}}$.

In [ ]:
```
def stratSamples(n):
    num_ints = 10
    k = 1 / num_ints
    Xs = np.zeros(n)
    Us = np.zeros(num_ints)
    for j in range(n):
        for i in range(num_ints):
            a = k * i
            b = a + k
            U = rnd.uniform(a, b, size = 1)
            Us[i] = U
        Xs[j] = np.mean(np.exp(Us))
    return Xs
n = 10
Xs_stratified = stratSamples(n)
tConf(Xs_stratified, alpha, "Stratified Sampling")
```

```
For Stratified Sampling :
Mean is 1.72, with 95% confidence interval [1.71,1.733]
```

1-4) Observations

The Crude MC method gives a very wide CI All the variance reduction methods have succesfully narrowed the CI. We had the largest success with Control Variates.

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5) Use control variates to reduce the variance of the estimator in exercise 4

We'll use the arrival intervals as control variate, i.e. control variate $C_i \sim Exp(\lambda)$, note the difference between arrival intervals, which are just sampled, and arrival times which are the actual times customers arrive at, which is the cummulative arrival intervals.

Using the formula

$$Z_i = X_i + c\,(C_i - 1)\,,$$

X_i is a list of bools of wether the customer with arrival interval C_i. Correlation is expected between X_i and C_i as a customer arriving quickly after another (small C_i) would presumably be blocked more often, as other customers have not had time to be served yet. We ofcourse know the mean of C_i, and can find the optimal $c = \frac{Cov(X,C)}{Var(C)}$ in each iteration.

We're concerned with the mean of X_i, which is the blocking probability, and Z_i is constructed to have the same mean.

In [ ]:
```
import numpy as np
import numpy.random as rnd
import scipy.stats as stats
import math as math
from scipy.stats import expon
import numpy as np
from discrete_event import Customer, main_loop, confidence_intervals, erlang_b,

#arrival time differences are exponentially distributed
lam = 1
total_customers = 10000
m = 10
s = 8
repititions = 10
#arrival time differences are exponentially distributed
arrival_interval = lambda : np.random.exponential(1/lam, size = total_customers)
service_time =lambda : expon.rvs(scale = s, size = total_customers)
```

In [ ]:
```
blocked = main_loop(arrival_interval, service_time, m, repititions = repititions
#print("Blocking probability", blocked/total_customers * 100)
#confidence interval for the mean
theta = np.mean(blocked)
confint = confidence_intervals(blocked)
print(f"Estimated blocking probability {round(theta,3)*100}%\nTrue blocking prob
print(f"95% CI for mean of blocking probability [{round(confint[0],4)}, {round(c
print("Interval width", round(confint[1]-confint[0],4))
```

```
Estimated blocking probability 12.3%
True blocking probability 12.2% (From Erlang formula)
95% CI for mean of blocking probability [0.1194, 0.1259]
Interval width 0.0065
```

In [ ]:
```
np.random.seed(2)
n = 10
Zs = []

for i in range(n):
    arrival_intervals = np.random.exponential(1/lam, size = total_customers)
    service_times = expon.rvs(scale = s, size = total_customers)
    blocked = main_loop_array(arrival_intervals, service_times, m)#/total_custom
    # Construct new variable
    c = -np.cov(arrival_intervals, blocked)[0,1] / np.var(arrival_intervals)
    Z = blocked + c * (arrival_intervals - 1)
    Zs.append(np.mean(Z))

tehta_Z = np.mean(Zs)
confint_Z = confidence_intervals(Zs)
print(f"Estimated blocking probability {round(theta,3)*100}%\nTrue blocking prob
print(f"95% CI for mean of blocking probability [{round(confint_Z[0],4)}, {round
print("Interval width", round(confint_Z[1]-confint_Z[0],4))
```

```
Estimated blocking probability 11.899999999999999%
True blocking probability 12.2% (From Erlang formula)
95% CI for mean of blocking probability [0.1208, 0.1251]
Interval width 0.0043
```

6) Reduce variance for the difference between solutions with

poisson arrivals and hyperexponential renewal process in exercise 4

Do this using Common Random Numbers (CRN). Using CRN we should be able to reduce the width of the CI og the difference in number of blocked customers between the two arrival processes.

Using non-CRN's it should require 5 differerent processes of generating arrivals and leaves to compare the two processes. 2 generates the serice times for the processes, one generates arrivals for the poisson process and 2 generates renewals for the hyperexponential arrivals.

When using CRN's this can be reduced to 3. The services times can be generated from the same U, since they should be the same here. We can use a single U as the exponential for the poisson and hyperexponential processes. We also need one more U to flip between the two exponentials in the hyperexponential distribution.

Both test runs predict more blocked customers when assuming a hyperexponential renewal process. The width of the CI when using CRN's can be reduced to $\sim 1/3$ compared to just simulating the two processes independently.

In [ ]:
```
import bisect as bisect
class Customer:
    def __init__(self, arrival_time, service_time):
        self.service_time = service_time
        self.blocked = False

        self.event = "arrival"
        self.event_time = arrival_time


    def arrive(self, servers, event_list):
        if servers < 1:
            self.blocked = True
            return servers
        else:
            servers -= 1
            servers = max(servers, 0)
            self.event = "departure"
            self.event_time += self.service_time
            bisect.insort(event_list, self, key=lambda x: x.event_time)
            return servers

    def depart(self, servers):
        servers += 1
        servers = min(servers, m)
        return servers

from distributions import getExponential, getUniform

def main_loop(event_list, m, repititions = 10):

    blocked = 0
    for i in range(repititions):
        event_list.sort(key=lambda x: x.event_time)
```

```
        open_servers = m
        while event_list:
            event = event_list.pop(0)
            if event.event == "arrival":
                open_servers = event.arrive(open_servers, event_list)
                blocked += event.blocked
            elif event.event == "departure":
                open_servers = event.depart(open_servers)
    return blocked

num_customers = 10000
lam = 1
s = 1/8
# Generate two eventlists
p1 = 0.8
lam1 = 0.8333
p2 = 0.2
lam2 = 5.0

def generateHexp(Us, Us2, p2, lam1, lam2):
    Xs = np.zeros(len(Us))
    for i in range(len(Us)):
        if Us2[i] < p2:
            Xs[i] = -np.log(Us[i]) / lam2
        else:
            Xs[i] = -np.log(Us[i]) / lam1
    return Xs

m = 10
rep = 5
blocked_hexp = np.zeros(rep)
blocked_poisson = np.zeros(rep)
blocked_poisson_common = np.zeros(rep)
blocked_hexp_common = np.zeros(rep)
for i in range(rep):

    Us1 = rnd.uniform(0,1,num_customers)
    Us2 = rnd.uniform(0,1,num_customers)
    Us3 = rnd.uniform(0,1,num_customers)
    service_intervals = -np.log(Us1) / s
    poisson_arrival_times = np.cumsum(-np.log(Us2) / lam)
    hexp_arrival_times = np.cumsum(generateHexp(Us2, Us3, p2, lam1, lam2))

    event_list_poisson = [Customer(poisson_arrival_times[i], service_intervals[i
    event_list_hexp = [Customer(hexp_arrival_times[i], service_intervals[i]) for

    blocked_poisson_common[i] = main_loop(event_list_poisson, m, repititions=1)
    blocked_hexp_common[i] = main_loop(event_list_hexp, m, repititions=1)

    Us1 = rnd.uniform(0,1,num_customers)
    Us2 = rnd.uniform(0,1,num_customers)
    Us3 = rnd.uniform(0,1,num_customers)
    Us4 = rnd.uniform(0,1,num_customers)
    Us5 = rnd.uniform(0,1,num_customers)
    service_intervals1 = -np.log(Us2) / s
    service_intervals2 = -np.log(Us3) / s

    poisson_arrival_times = np.cumsum(-np.log(Us1) / lam)
    hexp_arrival_times = np.cumsum(generateHexp(Us4, Us5, p2, lam1, lam2))
```

```
    event_list_poisson = [Customer(poisson_arrival_times[i], service_intervals1[
    event_list_hexp = [Customer(hexp_arrival_times[i], service_intervals2[i]) fo

    blocked_poisson[i] = main_loop(event_list_poisson, m, repititions=1)
    blocked_hexp[i] = main_loop(event_list_hexp,m, repititions=1)
```

In [ ]:

```
theta_crn = blocked_poisson_common - blocked_hexp_common
theta_irn = blocked_poisson - blocked_hexp

mean_theta_crn = round(np.mean(theta_crn),0)
var_theta_crn = np.var(theta_crn)
s = np.sqrt(var_theta_crn / len(theta_crn))
dof = len(theta_crn)-1
a = stats.t.ppf(alpha/2, dof)
b = stats.t.ppf(1 - alpha/2,dof)
print("For common random numbers")
print(f"Mean is {mean_theta_crn}, with confidence interval [{round(mean_theta_cr
print(f"Width of CI = {round(np.abs(mean_theta_crn + s * a - mean_theta_crn - s*


mean_theta_irn = int(np.mean(theta_irn))
var_theta_irn = np.var(theta_irn)
s = np.sqrt(var_theta_irn / len(theta_irn))
print("For independent random numbers")
print(f"Mean is {mean_theta_irn}, with confidence interval [{round(mean_theta_ir
print(f"Width of CI = {round(abs(mean_theta_irn + s * a - mean_theta_irn - s*b),
```

```
For common random numbers
Mean is -132.0, with confidence interval [-170.009,-93.991]
Width of CI = 76.017
For independent random numbers
Mean is -152, with confidence interval [-284.092,-19.908]
Width of CI = 264.184
```

7) Using a Crude Monte Carlo estimator vs Importance Sampling

Attempt to estimate the probality $Z > a$ for $Z \sim N(0, 1)$ for $a = 2, 4$. Do this using a crude monte carlo estimator and with importance sampling.

Using importance sampling set $h(z) = z > a$. We'll choose $g(x)$ to also be a normal distribution, with the same variance as $f(x)$, though with mean a instead. This means g samples around a more often. We thus have to sample fewer points to get an accurate estimate.

Notice $n = 10000$ with importance sampling gives significant digits, while the Crude Estimator needs $n = 100000$ to get a significant digit. A lot of time can be saved.

As long as a is small enough e.g. $a = 2$ Importance Sampling does not seem nescesarry.

In [ ]:

```
def crudeMonteCarloNorm(a,n):
    Us = rnd.normal(size = n)
    return sum(Us > a) / n
n = 10000
n2 = 100000
p1 = crudeMonteCarloNorm(2, n2)
p2 = crudeMonteCarloNorm(4, n2)
```

```
print(f"With {n} samples")
print(f"Crude probability z larger than 2 is {round(p1 * 100,5)}%")
print(f"Crude probability z larger than 4 is {round(p2 * 100,5)}%")

def h(a,x):
    return x > a

sigma1 = 1
a = 2
ys = rnd.normal(loc = a,scale = sigma1,size = n)
fy = stats.norm.pdf(ys)
gy = stats.norm.pdf(ys, loc = a, scale = sigma1)
zs = fy / gy * h(a,ys)
print(f"Probability z larger than 2 with importance sampling {round(np.mean(zs)

a = 4
ys = rnd.normal(loc = a,scale = sigma1,size = n)
fy = stats.norm.pdf(ys)
gy = stats.norm.pdf(ys, loc = a, scale = sigma1)
zs = fy / gy * h(a,ys)

print(f"Probability z larger than 4 with importance sampling {round(np.mean(zs)
```

```
With 10000 samples
Crude probability z larger than 2 is 2.27%
Crude probability z larger than 4 is 0.003%
Probability z larger than 2 with importance sampling 2.3139%
Probability z larger than 4 with importance sampling 0.0031%
```

8)

Analytically set

$$f(x) = \mathbf{1}_{0\leq x\leq 1}$$

$$h(x) = e^x$$

$$g(x) = \lambda e^{-\lambda x}.$$

We want to find the variance of

$$\frac{f(x)h(x)}{g(x)}.$$

Write

$$Var\left(\frac{f(x)h(x)}{g(x)}\right) = E\left(\left(\frac{f(x)h(x)}{g(x)}\right)^2\right) - E\left(\frac{f(x)h(x)}{g(x)}\right)^2$$

$$= \int_{-\infty}^{\infty}\left(\frac{f(x)h(x)}{g(x)}\right)^2 g(x)\,\mathrm{d}x - \left(\int_{-\infty}^{\infty}\frac{f(x)h(x)}{g(x)}g(x)\,\mathrm{d}x\right)^2.$$

The second integral is simply to rewrite

$$\left(\int_{-\infty}^{\infty}\frac{f(x)h(x)}{g(x)}g(x)\,\mathrm{d}x\right)^2 = \left(\int_{-\infty}^{\infty}f(x)h(x)\,\mathrm{d}x\right)^2$$

$$= \left(\int_0^1 f(x) e^x \, \mathrm{d}x \right)^2.$$

The first integral is slightly less pretty

$$\int_{-\infty}^{\infty} \left(\frac{f(x)h(x)}{g(x)} \right)^2 g(x) \, \mathrm{d}x = \int_{-\infty}^{\infty} \frac{f(x)^2 h(x)^2}{g(x)} \, \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \frac{f(x)^2 e^{2x}}{\lambda e^{-\lambda x}} \, \mathrm{d}x = \frac{1}{\lambda} \int_{-\infty}^{\infty} f(x)^2 e^{2x+\lambda x} \, \mathrm{d}x$$

$$= \frac{1}{\lambda} \int_0^1 e^{(2+\lambda)x} \, \mathrm{d}x.$$

Collecting:

$$Var\left(\frac{f(x)h(x)}{g(x)} \right) = \frac{1}{\lambda} \int_0^1 e^{(2+\lambda)x} \, \mathrm{d}x - \left(\int_0^1 f(x) e^x \, \mathrm{d}x \right)^2$$

From Maple:

$$Var\left(\frac{f(x)h(x)}{g(x)} \right) = \frac{-1 + e^{2+\lambda}}{2+\lambda} - (e-1)^2.$$

Using a numerical solver to minimize this expression gives $\lambda \approx 1.354828644$.

In [ ]:
```python
import matplotlib.pyplot as plt
lambda_opt = 1.354828644

def var_anal(lam):
    return (-1 + np.exp(2 + lam))/(lam*(2 + lam)) - (np.exp(1) - 1)^2

def variance(lam, n):
    Us1 = rnd.uniform(size = n)
    Us2 = rnd.uniform(size = n)
    return 1/lam * np.mean(np.exp((2+lam)*Us1)) - np.mean(np.exp(Us2))**2

lams = np.linspace(0.09, 5, 1000)
ys = np.zeros(1000)
for i in range(len(lams)):
    ys[i] = variance(lams[i], 10000)
print("Theoretical variance at optimal lambda is", np.min(ys))



plt.plot(lams, ys)

def g(x,lam):
    return lam * np.exp(-lam * x)

def f(x):
    b1 = x > 0
    b2 = x < 1
    return b1 * b2
def h(x):
```

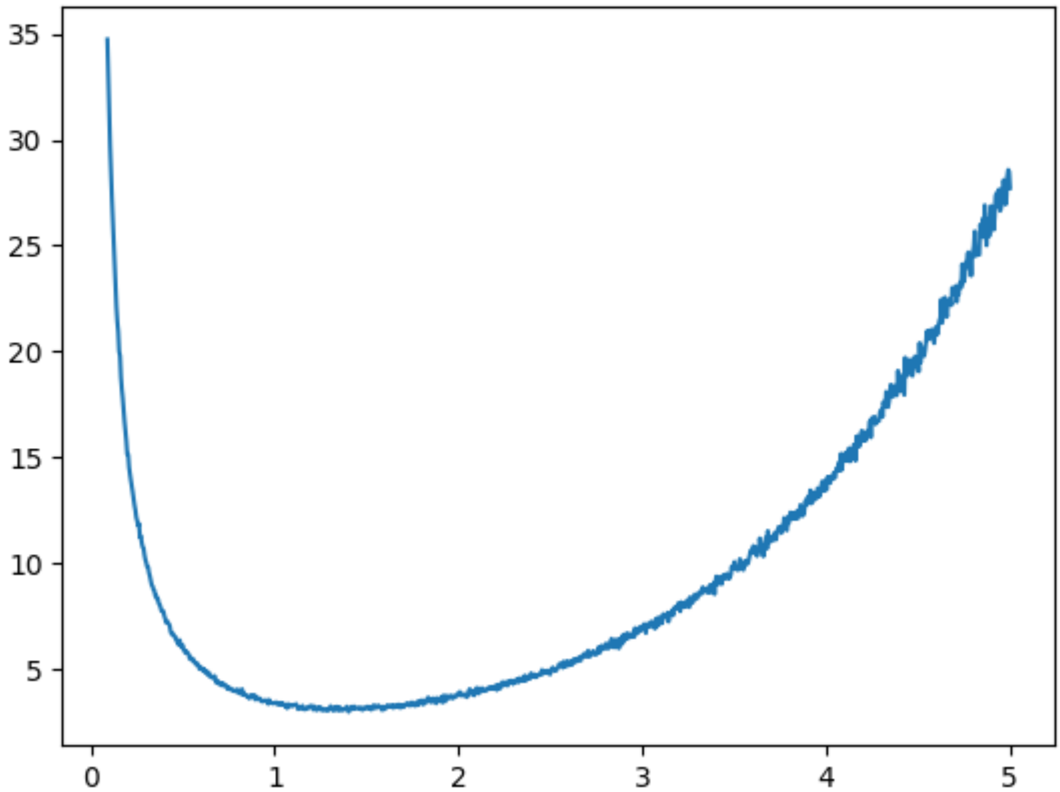
```
    return np.exp(x)

n = 10000
lam = lambda_opt



n = 10000
Xs_crude = crudeMC(n)
mean = np.mean(Xs_crude)
var = np.var(Xs_crude)
s = np.sqrt(var / len(Xs_crude))
a = stats.norm.ppf(alpha/2)
b = stats.norm.ppf(1 - alpha/2)
print("Using Crude Monte Carlo")
print(f"Mean is {round(mean,4)}, with confidence interval [{round(mean + s * a,3
print(f"Width of CI = {round(abs(mean + s * a - mean - s*b),3)}")
```

```
Theoretical variance at optimal lambda is 2.9803500419443325
Actual variance of sample at optimal lambda is 3.1416055177375855
Using importance sampling
Mean is 1.7106, with confidence interval [1.676,1.745]
Width of CI = 0.069
Using Crude Monte Carlo
Mean is 1.7172, with confidence interval [1.708,1.727]
Width of CI = 0.019
```



We note here that the variance of our sample, when using the optimal lambda, is close to the variance we'ed expect given the plot. Using importance sampling with this $g(x)$ we would not expected to find a lower variance.

We can try the same thing, now with $\lambda = 3.5$, where we would expect to see a sample variance aroung 10, which we indeed do.

In [ ]:
```
n = 10000
lam = 3.5
ys = rnd.exponential(scale = 1/lam, size = n)

zs = f(ys) * h(ys) / g(ys, lam)

mean = np.mean(zs)
var = np.var(zs)

print(f"Using lambda = {lam} we instead find a variance of", var)
```

```
Using lambda = 3.5 we instead find a variance of 9.901624437913682
```

9) Deriving the IS estimator for the mean of a Pareto distribution

pdf for a arbritray Pareto distribution is $f(x) = \frac{k}{y}\left(\frac{\beta}{y}\right)^k$. From the lectures we know that the first moment distribution of a Pareto distribution is just a Pareto distribution with paramters β and $k-1$. Thus

$$g(y) = \frac{k-1}{y}\left(\frac{\beta}{y}\right)^{k-1}.$$

As we're estimating the mean, set $h(y) = y$. Now the IS estimator for the mean is

$$\frac{f(y)h(y)}{g(y)} = \frac{\frac{k}{y}\left(\frac{\beta}{y}\right)^k \cdot y}{\frac{k-1}{y}\left(\frac{\beta}{y}\right)^{k-1}} = \frac{yk}{k-1}\frac{\left(\frac{\beta}{y}\right)^k}{\left(\frac{\beta}{y}\right)^{k-1}} = \frac{yk}{k-1}\frac{\beta}{y} = \beta\frac{k}{k-1}.$$

Which we know is just the mean of the original Pareto distribution $f(x)$, i.e. we should be sampling exactly the mean! Which we do, with an exact CI.

In [ ]:
```
k = 2
def g(x):
    return (k-1) /(x) * (1/x)**(k-1)

def f(x):
    return k /(x) * (1/x)**k
def h(x):
    return x

n = 100
lam = lambda_opt
ys = rnd.exponential(scale = 1/lam, size = n)
beta = 1
zs = f(ys) * h(ys) / g(ys)
alpha = 0.05
mean = np.mean(zs)
var = np.var(zs)
s = np.sqrt(var / len(zs))
a = stats.norm.ppf(alpha/2)
b = stats.norm.ppf(1 - alpha/2)
print("Using importance sampling to sample the mean of a pareto distribution usi
print(f"Mean is {round(mean,4)}, with confidence interval [{round(mean + s * a,3
print(f"Width of CI = {round(abs(mean + s * a - mean - s*b),3)}")
```

```
print(f"Exact solution for the Pareto distribution mean {k/(k-1) * beta}")
```

```
Using importance sampling to sample the mean of a pareto distribution using its f
irst moment
Mean is 2.0, with confidence interval [2.0,2.0]
Width of CI = 0.0
Exact solution for the Pareto distribution mean 2.0
```

To do the same in the case of the integral $\int_0^1 e^x \,\mathrm{d}x$ we would need to know a normalising constant s.t. $\frac{1}{c}\int_0^1 e^x \,\mathrm{d}x = 1$, which is simple since we know the actual value of the integral is $e-1$, then $c = e-1$. If we choose $g(x) = \frac{1}{c}e^x$, we would then have IS estimator

$$\frac{e^x}{\frac{1}{c}e^x} = e - 1,$$

and would once again succesfully estimate the mean. The issue is the need for the constant c, which if you know it, makes the whole process redundant, as it requires already knowing the integral, and thus is not really meaningfull.

In [ ]:

```
beta = 0.005
k = 2
def g(x, beta, k):
    return 1/ (np.exp(1) - 1) * np.exp(x)
def f(x):
    return (x > 0) * (x < 1)
def h(x):
    return np.exp(x)

def pareto(beta, k, n = 10000):
    Us = np.random.uniform(size = n)
    Xs = beta * (Us**(-1/k)-1)
    return Xs

ys = pareto(beta, k)

zs = f(ys) * h(ys) / g(ys, beta, k)

mean = np.mean(zs)
var = np.var(zs)
alpha = 0.05
print("Actual variance of sample at optimal lambda is", var)
s = np.sqrt(var / len(zs))
a = stats.norm.ppf(alpha/2)
b = stats.norm.ppf(1 - alpha/2)
print("Using importance sampling")
print(f"Mean is {round(mean,4)}, with confidence interval [{round(mean + s * a,3
print(f"Width of CI = {round(abs(mean + s * a - mean - s*b),3)}")
```

```
Actual variance of sample at optimal lambda is 9.883934104353516e-32
Using importance sampling
Mean is 1.7183, with confidence interval [1.718,1.718]
Width of CI = 0.0
```

In [ ]: