a) Collocation Legendre method:

$\mathcal{L}_n u = f$

with BC:

$\tilde{\mathcal{L}}_n u = \tilde{f}$

D comes from vandermonde because $D = V_x V^{-1}$ and "$V = \phi_i$" ↖ $\alpha = \beta = 0$

$\mathcal{L}_n = -aD^2 + BD + C$ (because a is constant)

$\Rightarrow \varepsilon D^2 - D = \mathcal{L}_n \in \mathbb{R}^{N+1 \times N+1}$

$f = \underline{1} \in \mathbb{R}^{N+1}$

boundary conditions Add 1 row and last:

0: $[1 \; -0-] \, u = 0$
N+1: $[-0- \; 1] \, u = 0$

solve depending on N for:
$\varepsilon = [0.1, 0.01, 0.001]$.

Legendre Tau method:

represent solution as:

$$u_N = \sum_{n=0}^{N} \hat{u}_n \phi_n(x) \quad , \quad \hat{u}_n = \frac{(u_N, \phi_n)_w}{(\phi_n, \phi_n)_w}$$

* maybe change of variables since $x \in [-1, 1]$

boundary condition are handled in expansion coefficients as $\sum_{n=0}^{N} \hat{u}_n \phi_n(\pm 1) = 0$

Set up of differential equation is

$\mathcal{L} u = f$

TAU method $\Rightarrow (\mathcal{L} u_N, \phi_n)_w = (f, \phi_n)_w$, $n = 0, 1, \dots, N-2$

$$\frac{\partial^2}{\partial x^2} u = \sum_{n=0}^{\infty} \hat{u}_n \frac{\partial^2}{\partial x^2} \phi_n = \sum_{n=0}^{\infty} \hat{u}_n^{2} \phi_n$$

$$\frac{\partial}{\partial x} u = \sum_{n=0}^{\infty} \hat{u}_n \frac{\partial}{\partial x} \phi_n = \sum_{n=0}^{\infty} \hat{u}_n^{1} \phi_n$$

insert in differential equation:

$$-\varepsilon \sum_{n=0}^{\infty} \hat{u}_n^{2} \phi_n - \sum_{n=0}^{\infty} \hat{u}_n^{1} \phi_n = 1$$

multiply with ϕ_i and integrate:

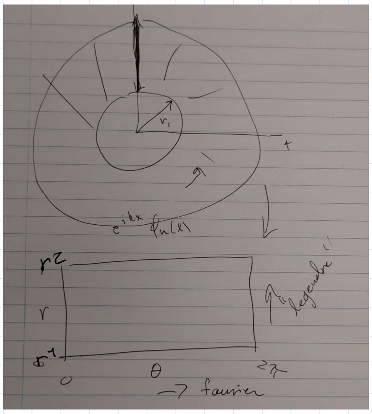
$$-\varepsilon \int_{-1}^{1} \sum_{n=0}^{\infty} \hat{u}_n^{2} \phi_n \phi_i \, dx - \int_{-1}^{1} \sum_{n=0}^{\infty} \hat{u}_n^{1} \phi_n \phi_i \, dx = \int_{-1}^{1} \phi_i \, dx$$

$$\Rightarrow -\varepsilon \sum_{n=0}^{\infty} \hat{u}_n^{2} \int_{-1}^{1} \phi_n \phi_i \, dx - \sum_{n=0}^{\infty} \hat{u}_n^{1} \int_{-1}^{1} \phi_n \phi_i \, dx = \int_{-1}^{1} \phi_i \, dx$$

$\Rightarrow$

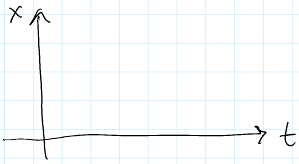
b) $\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0$

$(r, \theta) \in [r_1, \infty] \times [0, 2\pi]$, $r_1 > 0$



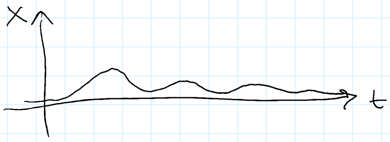
Solve using a multidimensional grid.
see slides

c) Multidimensional grid:



maybe: $\sum_i \hat{u}_i(t) e^{ikx}$

Solution maybe understood as:



$$-\xi \sum_{n=0} \hat{u}_n^{2} \int_{-1} \varphi_n \varphi_i \, dx - \sum_{n=0} \hat{u}_n^{1} \int_{-1} \varphi_n \varphi_i \, dx = \int_{-1} \varphi_i \, dx$$

$$\Rightarrow -\xi \hat{u}_i^{2} - \hat{u}_i^{1} = \int_{-1}^{1} \varphi_i \, dx = \tilde{f}$$

this is the ith equation in the system.

needs to be computed