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2.8

a) Formulation of boundary value problem

To find the Neumann boundary condition on the left and bottom edge we use the divergence theorem. The divergence theorem i given by

$$\iint_\Omega \nabla \cdot F \, dxdy = \int_\Gamma n \cdot F ds$$

where F is some vector field, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^\top$ and $n(x,y)$ is the normal vector at $(x,y) \in \Gamma$. The PDE we want to solve is given by

$$u_{xx} + u_{yy} = -\tilde{q}$$

Note that $u_{xx} + u_{yy} = \nabla \cdot \nabla u$ such that

$$\nabla \cdot \nabla u = -\tilde{q}$$

We integrate both sides over the domain Ω

$$\iint_\Omega \nabla \cdot \nabla u \, dxdy = \iint_\Omega (-\tilde{q}) \, dxdy$$

We see that we can use the divergence theorem on the left hand side. To do it on the right hand side aswell we search for a function g such that $\nabla \cdot g = -\tilde{q}$. Then we have

$$\iint_\Omega \nabla \cdot \nabla u \, dxdy = \iint_\Omega \nabla \cdot g \, dxdy$$

$$\Rightarrow \int_\Gamma n \cdot \nabla u \, ds = \int_\Gamma n \cdot g \, ds$$

We can now skip the integral to get

$$n \cdot \nabla u = n \cdot g$$

The left hand side is the directional derivative in direction n, which is the one needed for the Neumann boundary condition. We now find the function g, we know that

$$\nabla \cdot g = -\tilde{q} = -2\pi \cos(\pi x)\cos(\pi y)$$

One can show that

$$g(x,y) = \begin{bmatrix} -\pi \sin(\pi x)\cos(\pi y) \\ -\pi \cos(\pi x)\sin(\pi y) \end{bmatrix}$$

will satisfy the equation above. Then we have the directional derivative

$$n \cdot \nabla u = n \cdot \begin{bmatrix} -\pi \sin(\pi x)\cos(\pi y) \\ -\pi \cos(\pi x)\sin(\pi y) \end{bmatrix}$$

on the left boundary $n = (-1, 0)^\top$ and $x = 0$ in this case $n \cdot \nabla u = 0$, and on the bottom boundary we have $n = (0, -1)^\top$ and $y = 0$ in this case $n \cdot \nabla u = 0$ hence

$$n \cdot \nabla u = 0 \text{ on } \Gamma_1$$

In this case, $q = 0$ for any $(x,y) \in \Gamma_1$ as the function is symmetric about both x and y axis, so we don't implement the Neumann conditions in the driver functions to save computation time. An implementation using Neumann conditions is found in **`FEMsolver2d`**.

b) Solve problem for $[0,1] \times [0,1]$

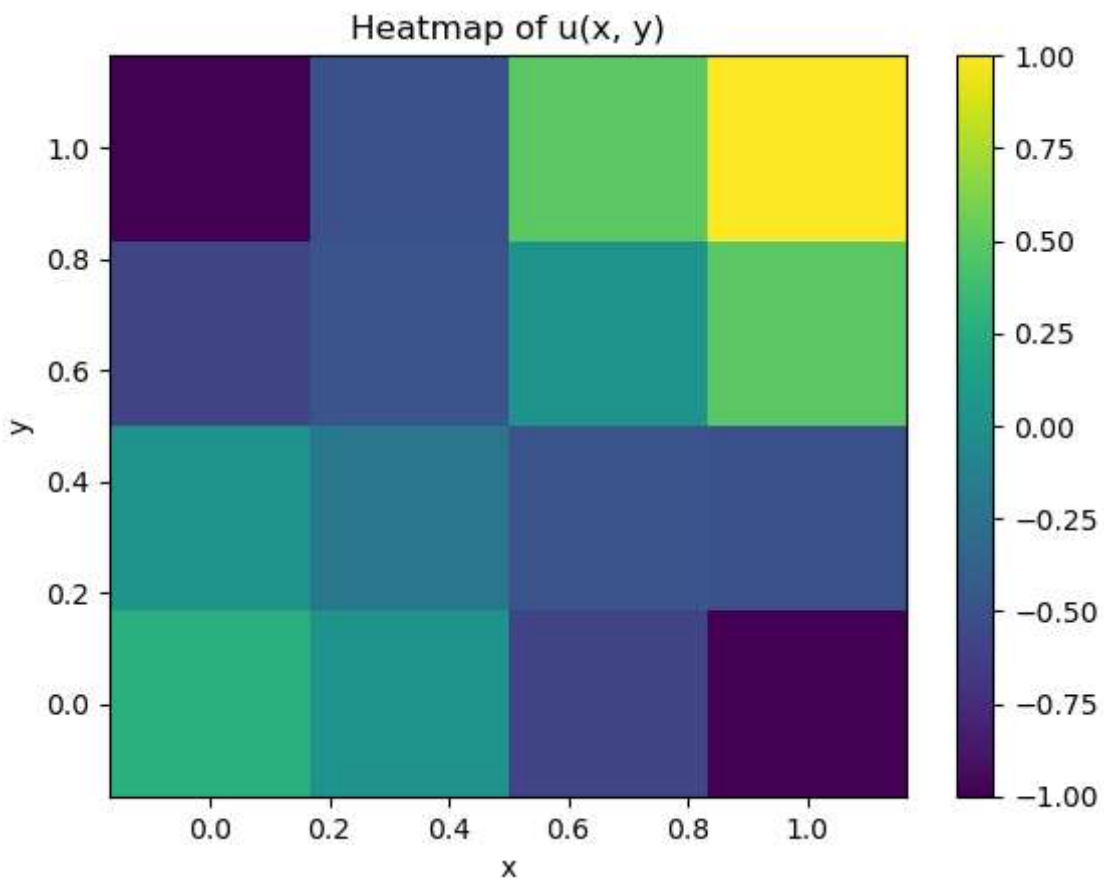
In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt

from driver28b import Driver28b
from driver28c import Driver28c
from FEMsolver2d import *
```

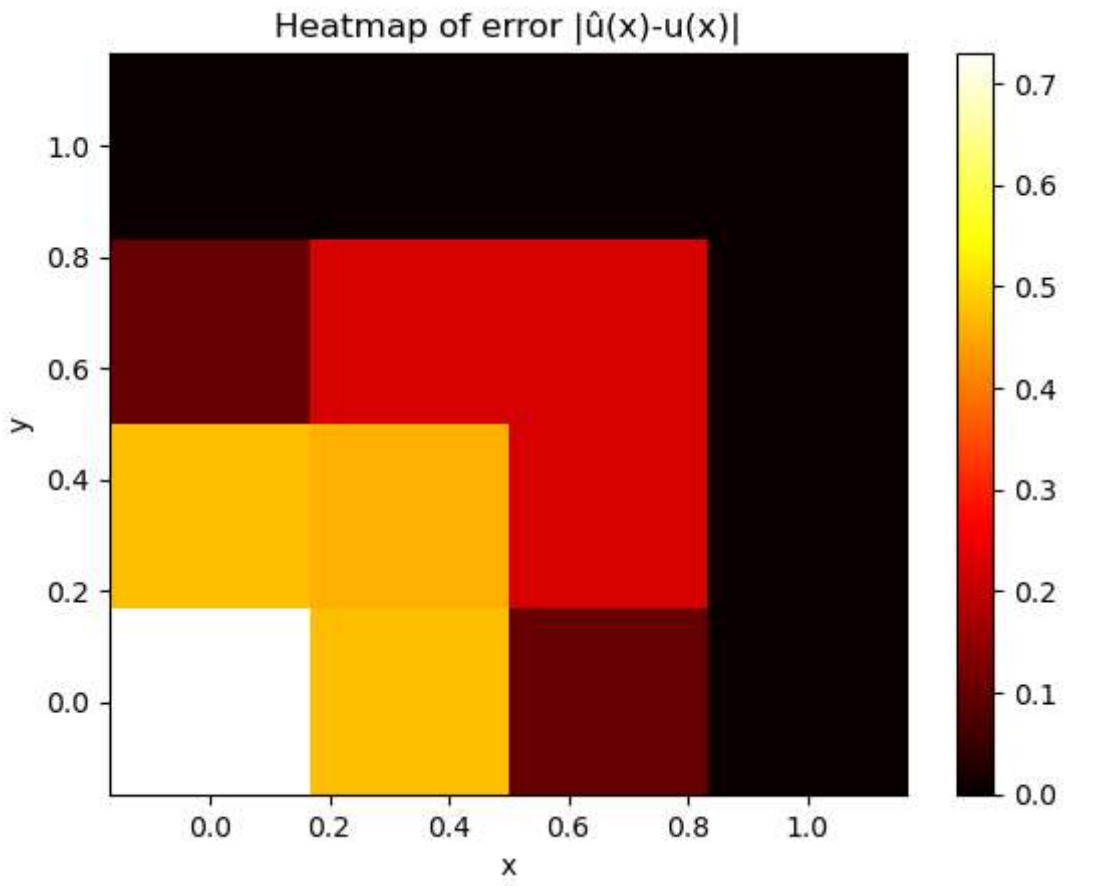
In [ ]:
```
(x0,y0) = (0,0)
L1 = 1
L2 = 1

noelms1 = 3
noelms2 = 3

lam1 = 1
```

```
lam2 = 1

qt = lambda x,y: 2*np.pi*np.pi*np.cos(np.pi*x)*np.cos(np.pi*y)
f = lambda x,y: np.cos(np.pi*x)*np.cos(np.pi*y)

VX, VY, EToV, u = Driver28b(x0, y0, L1, L2, noelms1, noelms2, lam1, lam2, f, qt)
```

2D plot of the solution:

In [ ]:
```
ufun = lambda x,y: np.cos(np.pi*x)*np.cos(np.pi*y)
plot_heatmap(u,VX,VY,ufun)
```

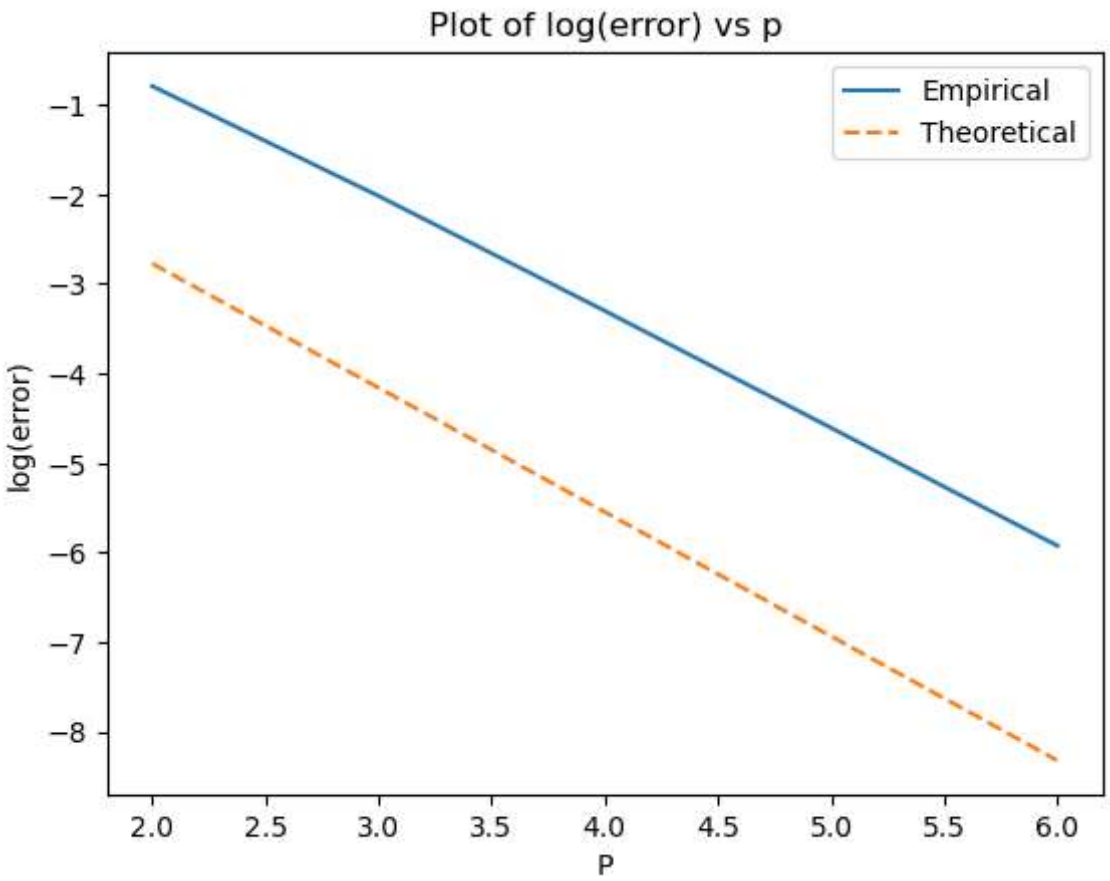




Finally, we perform a convergence test.

In [ ]:
```
P = list(range(2,7))
error = np.zeros(len(P))
ufun = lambda x,y: np.cos(np.pi*x)*np.cos(np.pi*y)

for i,p in enumerate(P):
    noelms1 = 2**p
    noelms2 = 2**p

    VX, VY, EToV, u = Driver28b(x0, y0, L1, L2, noelms1, noelms2, lam1, lam2, f, qt)
    u_exact = ufun(VX,VY)
    error[i] = np.linalg.norm(u-u_exact, np.inf)


plt.plot(P,np.log(error), label='Empirical')
plt.plot(P,-2*np.log(2)*np.array(P),'--', label='Theoretical')
plt.title(r"Plot of log(error) vs p")
plt.xlabel('P')
plt.ylabel('log(error)')
```

```
plt.legend()
plt.show()
```



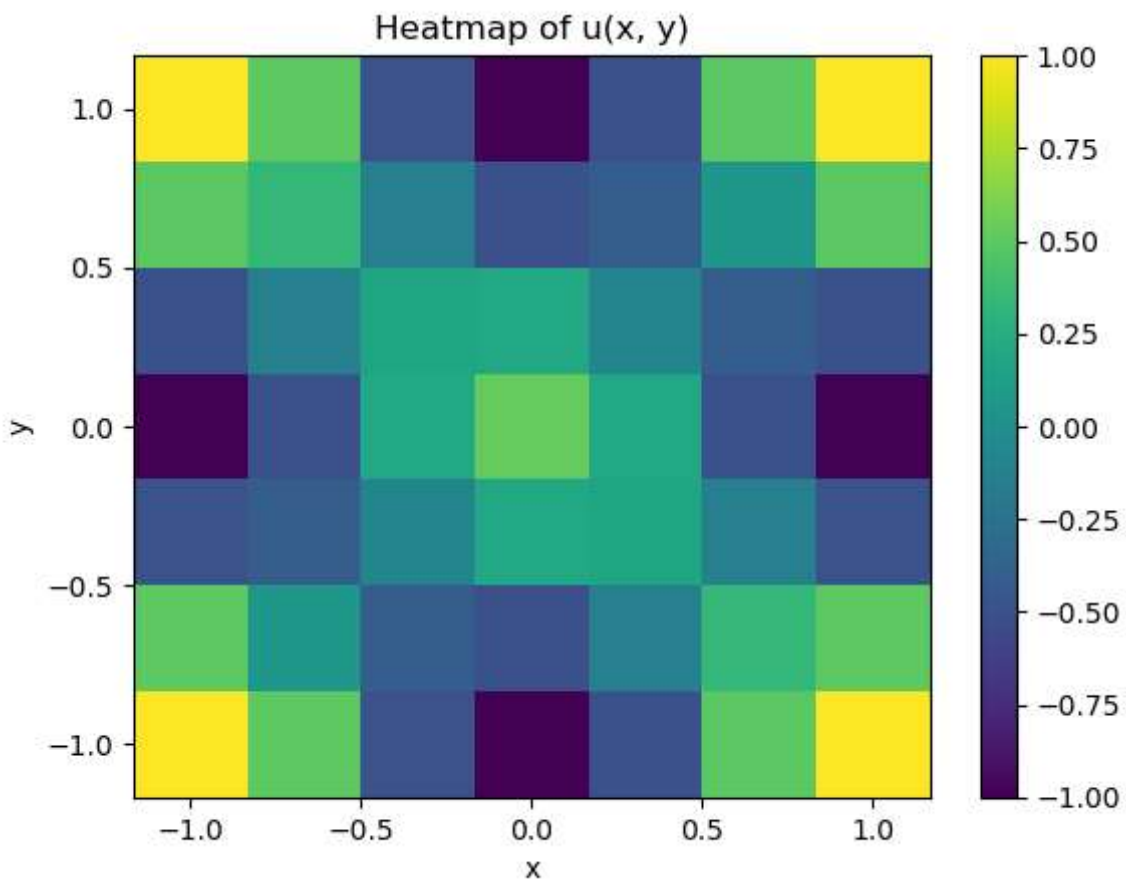
Everything appears to be in order.

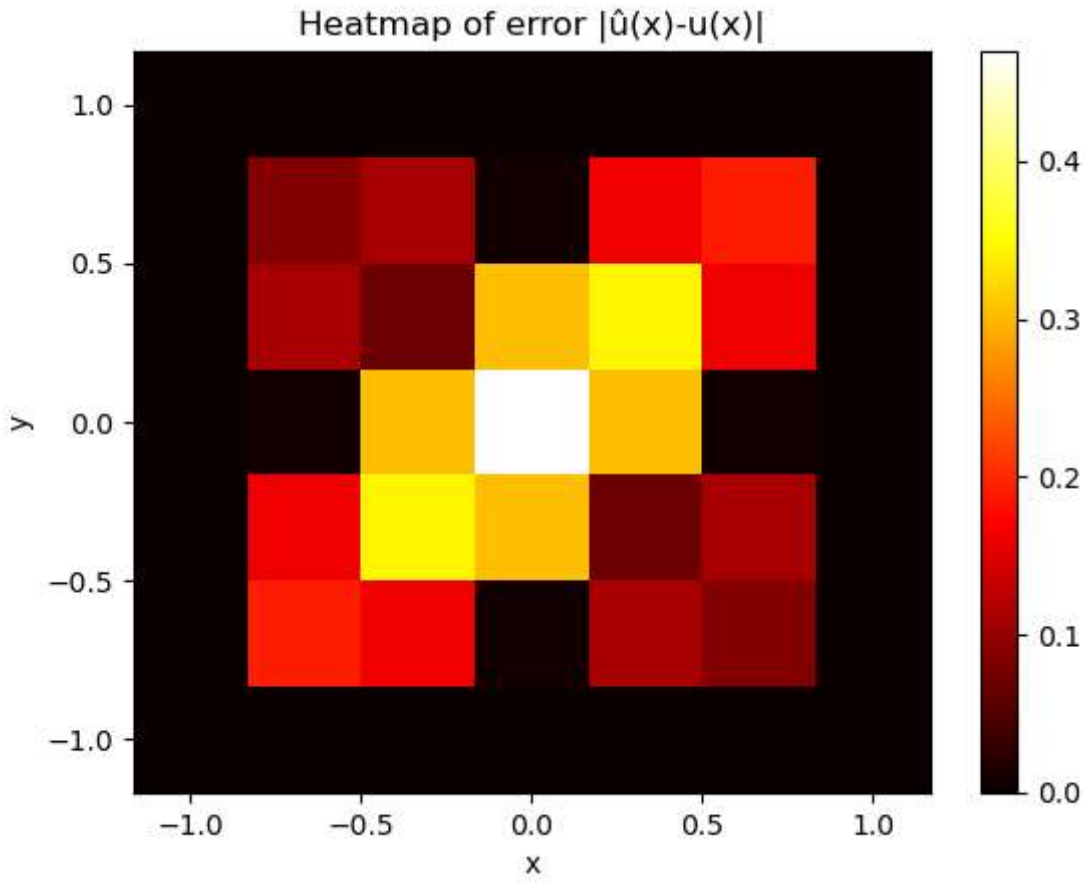
c) Solve original problem

In [ ]:
```
(x0,y0) = (-1,-1)
L1 = 2
L2 = 2

noelms1 = 6
noelms2 = 6

lam1 = 1
lam2 = 1

qt = lambda x,y: 2*np.pi*np.pi*np.cos(np.pi*x)*np.cos(np.pi*y)
f = lambda x,y: np.cos(np.pi*x)*np.cos(np.pi*y)

VX, VY, EToV, u = Driver28c(x0, y0, L1, L2, noelms1, noelms2, lam1, lam2, f, qt)
```

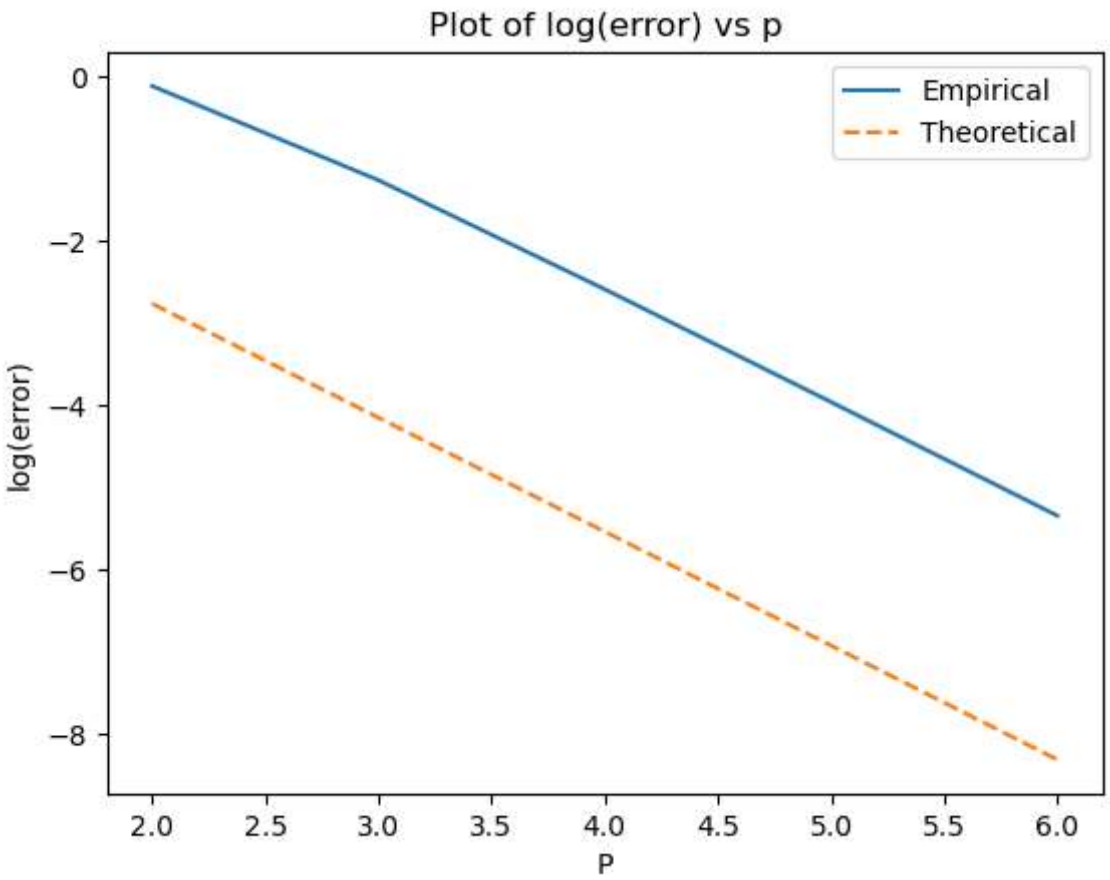
We plot u(x,y) in 2D:

In [ ]:
```
plot_heatmap(u,VX,VY,ufun)
```





In [ ]:
```
ufun = lambda x,y: np.cos(np.pi*x)*np.cos(np.pi*y)
P = list(range(2,7))
error = np.zeros(len(P))

for i,p in enumerate(P):
    noelms1 = 2**p
    noelms2 = 2**p

    VX, VY, EToV, u = Driver28c(x0, y0, L1, L2, noelms1, noelms2, lam1, lam2, f, qt)
    u_exact = ufun(VX,VY)
    error[i] = np.linalg.norm(u-u_exact, np.inf)

DOF = [(2**p+1)**2 for p in P]

#plt.loglog(DOF,error,'r--')
#plt.semilogy(P,error)
plt.plot(P,np.log(error), label='Empirical')
plt.plot(P,-2*np.log(2)*np.array(P),'--', label='Theoretical')
plt.title(r"Plot of log(error) vs p")
plt.legend()
plt.xlabel('P')
plt.ylabel('log(error)')
plt.show()
```



d) Compute approximations for differing meshes.

In [ ]:
```
P = list(range(1,7))
zero_1 = np.zeros(len(P))
zero_2 = np.zeros(len(P))

x0 = 0
y0 = 0
```

```
L1 = 1
L2 = 1

lam1 = 1
lam2 = 1

for i, p in enumerate(P):
    noelms1 = 2**p
    noelms2 = 2**p

    VX, VY, EToV, u1 = Driver28b(x0, y0, L1, L2, noelms1, noelms2, lam1, lam2, f, qt)

    zero_1[i] = u1[noelms1]

    VX, VY, EToV, u2 = Driver28c(x0-1, y0-1, L1+1, L2+1, noelms1*2, noelms2*2, lam1, lam2, f, qt)

    zero_2[i] = u2[int(len(u2)/2)]

print(zero_1)
print(zero_2)
```

```
[-0.36292216  0.54719116  0.86695986  0.96310108  0.99000096  0.99731805]
[0.11685028 0.71905726 0.92545767 0.98108619 0.99525401 0.99881241]
```