In [ ]:
```
import numpy as np
from matplotlib import pyplot as plt
from scipy import sparse
from scipy.sparse import csr_matrix
```

a) See attached picture for calculations.

In [ ]:
```
u = lambda x: np.exp(-800*(x-0.4)**2) + 0.25 * np.exp(-40*(x-0.8)**2)
import sympy as sp

x = sp.symbols('x')
u_sym = sp.exp(-800*(x-0.4)**2) + 0.25 * sp.exp(-40*(x-0.8)**2)


u_double_diff = sp.diff(u_sym, x, 2)
u_double_diff - u_sym
```

Out[ ]: $1600.0(x-0.8)^2e^{-40(x-0.8)^2}+2560000(x-0.4)^2e^{-800(x-0.4)^2}-1601e^{-800(x-0.4)^2}-20.25e^{-40(x-0.8)^2}$

b) We use the same strategy as in the previous exercise, though some of the simplifications are no longer valid because the lines do not intersect in the end points of the integrals.

In [ ]:
```
def compute_error_decrease(uc,uf,VXc,VXf,Old2New):

    N = len(VXc)-1
    err = np.zeros(N)

    for n, triple in enumerate(Old2New):

        i = triple[0]
        j = triple[2]
        k = triple[1]

        xi   = VXc[i]
        xj = VXc[j]
        xk  =  xi + (xj - xi)/2

        uci  = uc[i]
        ucj =  uc[j]

        ufi = uf[i]
        ufk = uf[k]
        ufj = uf[j]

        a  = (ucj - uci) / (xj - xi)
        a1 = (ufk - ufi) / (xk - xi)
        a2 = (ufj - ufk) / (xj - xk)

        b = uci - a * xi
        b1 = ufi - a1 * xi
        b2 = ufj - a2 * xj

        int1 = ((a-a1)**2/3) * (xk**3 - xi**3) + (b-b1)**2 * (xk - xi) + (a-a1)*(b-b1) * (xk**2 - xi**2)
        int2 = ((a-a2)**2/3) * (xj**3 - xk**3) + (b-b2)**2 * (xj - xk) + (a-a2)*(b-b2) * (xj**2 - xk**2)

        err[n] = np.sqrt(int1 + int2)

    return err
```

Test case

In [ ]:
```
uc = np.array([0,0])
uf = np.array([0,0,1/2])

VXc = np.array([0,1])
VXf = np.array([0,1/2,1])

Old2New = np.array([[0,2,1]])

compute_error_decrease(uc,uf,VXc,VXf,Old2New)
```

Out[ ]:
```
array([0.28867513])
```

c) Same as in 1.6.

In [ ]:
```
def refine_marked(EToVcoarse, xcoarse, idxMarked):
    N = len(EToVcoarse[:,0]) + 1
    Old2New = np.zeros((len(idxMarked),3),dtype=int)

    EToVfine = EToVcoarse.copy()
    xfine = xcoarse.copy()

    for i,idx in enumerate(idxMarked):
```

```
        xi   = xfine[EToVcoarse[idx][0]]
        xip1 = xfine[EToVcoarse[idx][1]]
        xih  =  xi + (xip1 - xi)/2

        xfine = np.hstack((xfine,[xih]))

        M = EToVfine[idx][1]
        EToVfine[idx][1] = N
        EToVfine = np.vstack((EToVfine,[N,M]))

        Old2New[i] = [EToVfine[idx][0], N, M]

        N += 1

    return EToVfine, xfine, Old2New
```

d)

We obtain the weak formulation

$$\int_\Omega v'u'\,dx + \int_\Omega vudx = -\int_\Omega vfdx \tag{1}$$

and see that the left hand side is the same is in problem 1.1. Thus the elemental contributions to A are the same.

In the exercise it is given that

$$\int_0^L f(x)v(x)dx \approx \sum_{j=1}^M \hat{f}_j \int_0^L N_i(x)N_j(x)dx$$

which we may reduce since N_i, N_j only overlap for $j = i-1, i, i+1$. We note that these integrals were part of the computation of the K_i matrix earlier. Thus we have

$$\bm{b}_i = -\begin{pmatrix} \hat{f}_{i-1} & \hat{f}_i & \hat{f}_{i+1} \end{pmatrix} \begin{pmatrix} \frac{h}{6} \\ \frac{2h}{3} \\ \frac{h}{6} \end{pmatrix} \tag{2}$$

where $\hat{f}_i = f(x_i)$

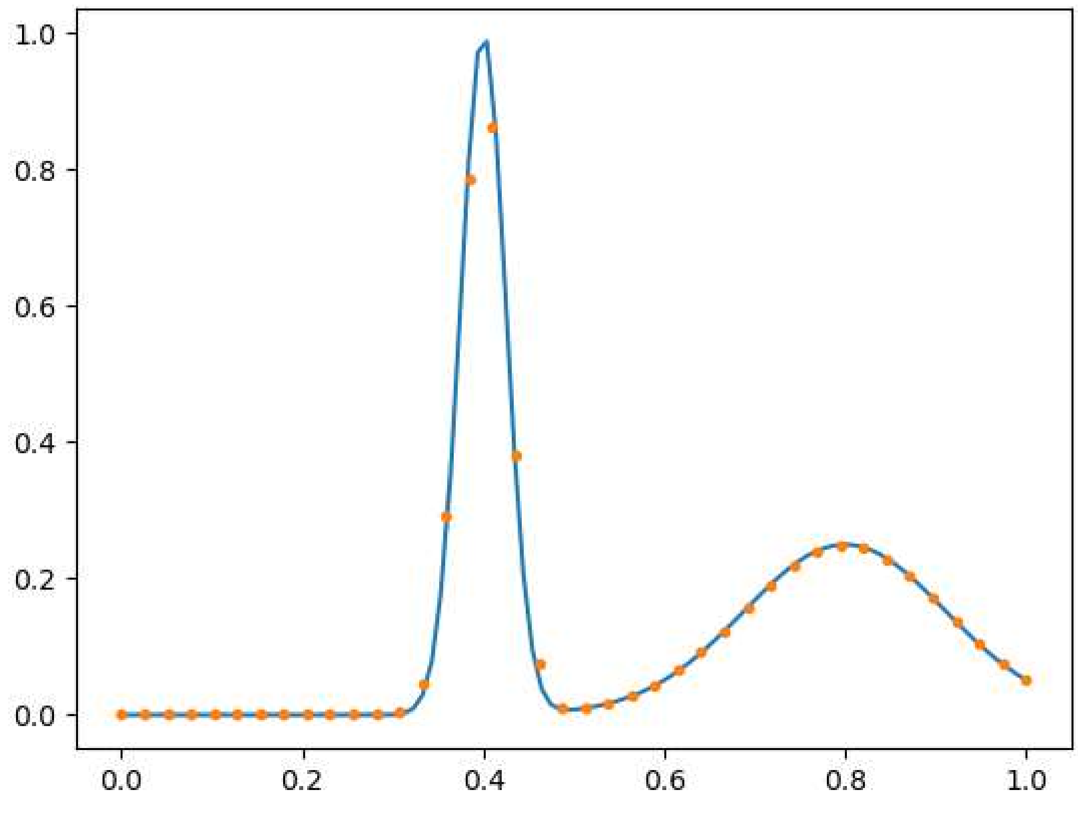
e)

Assembly

```
In [ ]: K = lambda h: np.array([[1/h + h/3, -1/h + h/6], [-1/h + h/6, 1/h + h/3]])

def GlobalAssembly(x,c,d,func):
    M = len(x)
    nnzmax = 4 * M
    ii = np.ones(nnzmax, dtype=int)
    jj = np.ones(nnzmax, dtype=int)
    ss = np.zeros(nnzmax)
    b = np.zeros(M)
    count = 0

    for i in range(M - 1):
        h = x[i+1] - x[i]

        fval = func(x[i])

        if i > 0:
            b[i-1] += h*fval/6
        b[i] += 2*h*fval/3
        b[i+1] += h*fval/6

        Ki = K(h)

        ii[count:count + 4] = [i, i, i + 1, i + 1]
        jj[count:count + 4] = [i, i + 1, i + 1, i]
        ss[count:count + 4] = [
        Ki[0, 0],
        Ki[0, 1],
        Ki[1, 1],
        Ki[1, 0]
        ]
        count += 4

    A = csr_matrix((ss[:count], (ii[:count], jj[:count])), shape=(M, M))
    b = -b

    # Boundary conditions
    b[0] = c
```

```python
    b[1] -= A[0,1]*c

    A[0,0] = 1
    A[0,1] = 0
    A[1,0] = 0

    b[M-1] = d
    b[M-2] -= A[M-1,M-2]*d

    A[M-1,M-1] = 1
    A[M-1,M-2] = 0
    A[M-2,M-1] = 0


    return A, b

def BVP1D(L, x, c, d,func, plot=True):

    if type(x) == int:
        x = np.linspace(0, L, x)

    A,b = GlobalAssembly(x,c,d,func)

    u = sparse.linalg.spsolve(A, b)

    if plot:
        plt.plot(x, u, 'r--',label="FEM solution")
        plt.show()

    return u
```

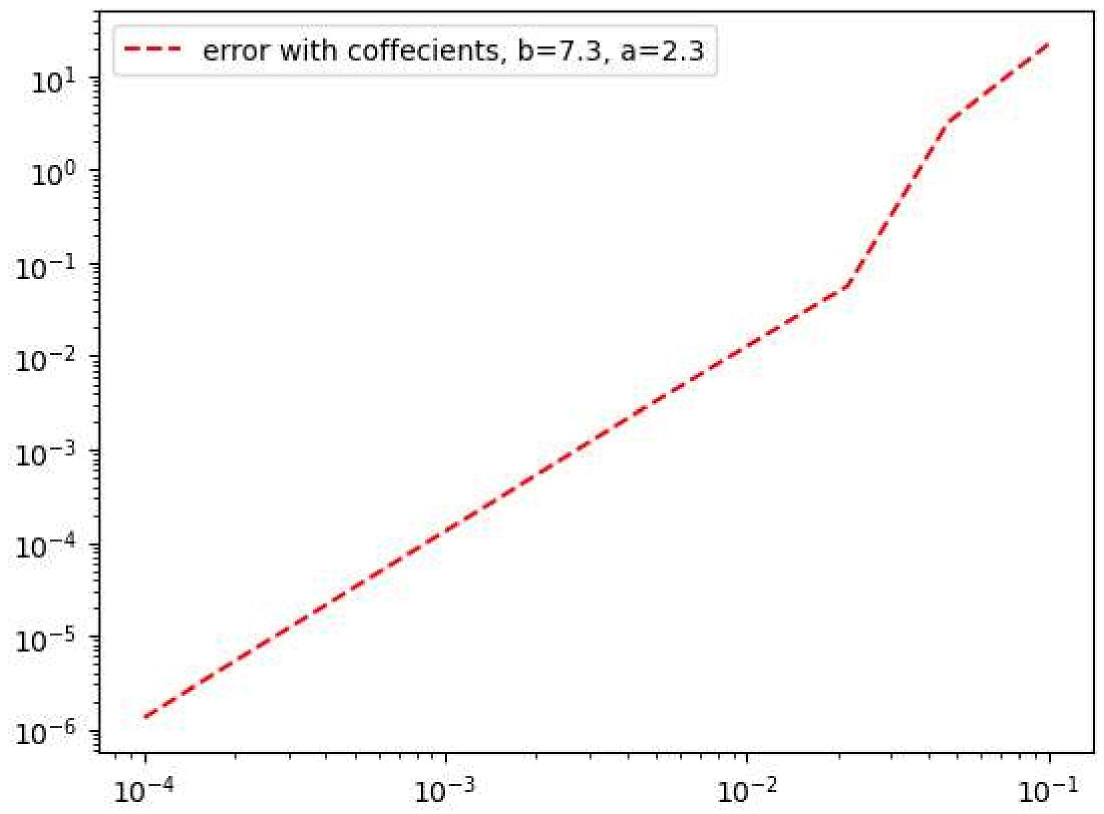
In [ ]:
```python
f = lambda x: (np.exp(-800*(x-0.4)**2)*((-1600*(x-0.4))**2-1601) + 0.25*np.exp(-40*(x-0.8)**2)*((-80*(x-0.8))**2-81))
c = u(0)
d = u(1)

N = 40
uhat = BVP1D(L=1, x=N, c=u(0), d=u(1), func=f, plot=None)

xax = np.linspace(0,1,100)
u = lambda x: np.exp(-800*(x-0.4)**2) + 0.25 * np.exp(-40*(x-0.8)**2)

plt.plot(xax, u(xax),label="Exact solution")
plt.plot(np.linspace(0,1,N), uhat, '.',label="FEM solution")
plt.show()
```



In [ ]:
```python
def convergence_test(h,L,c,d,u,func):
    x = np.linspace(0, L, int(L/h))
    uhat = BVP1D(L, x, c, d,func, plot=False)
    return np.linalg.norm(uhat-u(x),np.inf)
```

In [ ]:
```python
L = 1
H = np.logspace(-1, -4, 10)
error = np.zeros(len(H))
for i,h in enumerate(H):
    error[i] = convergence_test(h,L,c,d,u,f)
```

In [ ]:
```python
a,b = np.polyfit(np.log(H), np.log(error), 1)

plt.loglog(H, error, 'r--',label=f"error with coffecients, b={np.round(b,1)}, a={np.round(a,1)}")
plt.legend()
plt.show()
```



In [ ]:
```
c = u(0)
d = u(1)
x = np.linspace(0,1,5)
u = lambda x: np.exp(-800*(x-0.4)**2) + 0.25 * np.exp(-40*(x-0.8)**2)

A, b = GlobalAssembly(c=c, d=d, func= u, x=x)
b
```

Out[ ]:
```
array([ 1.90546630e-12, -2.98833896e-04, -1.06198472e-02,  1.61793270e-01,
        5.04741295e-02])
```

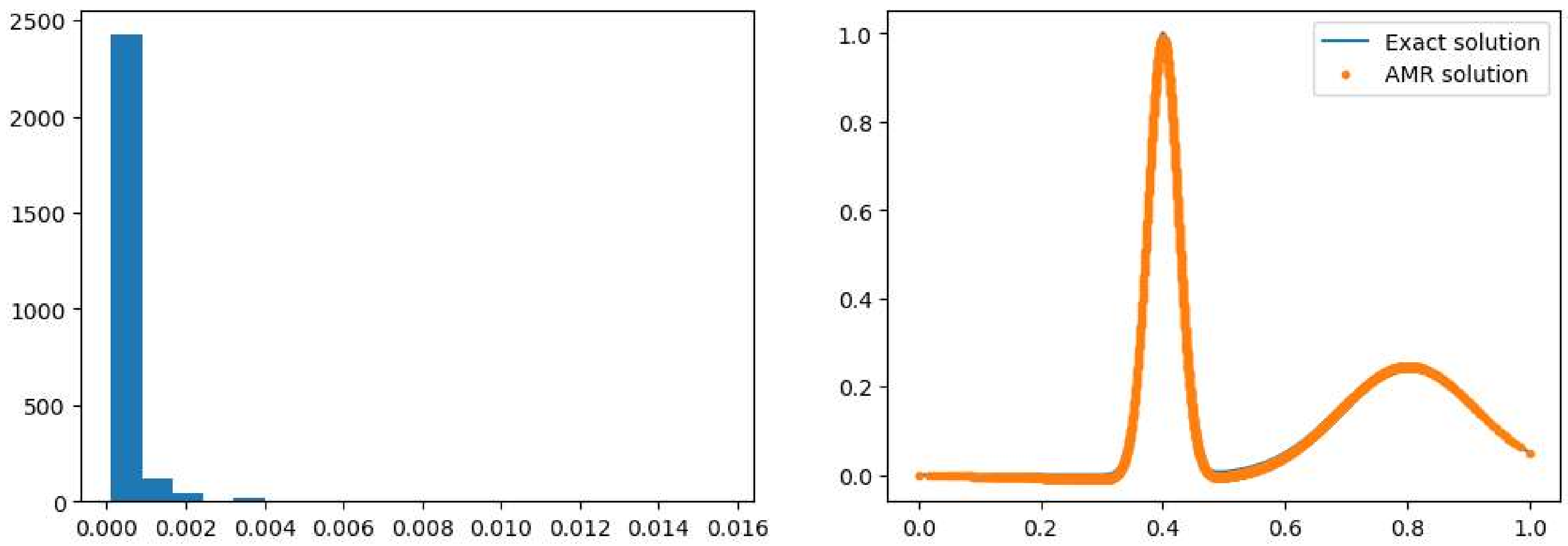
f)

In [ ]:
```
from DriverAMR17 import DriverAMR17
```

In [ ]:
```
VX, uhat, it = DriverAMR17(L,c,d,VXc,f,tol=10**(-4), maxit=100)
print(f"Number of points: {len(VX)}")
print(it)
```

```
Number of points: 2615
16
```

In [ ]:
```
VX = np.sort(VX)

VXdiff = np.diff(VX)

fig, ax = plt.subplots(1,2,figsize=(12,4))

ax[0].hist(VXdiff, bins=20)
ax[1].plot(VX, u(VX),label="Exact solution")
ax[1].plot(VX,uhat,".",label='AMR solution')
ax[1].legend()
plt.show()
```

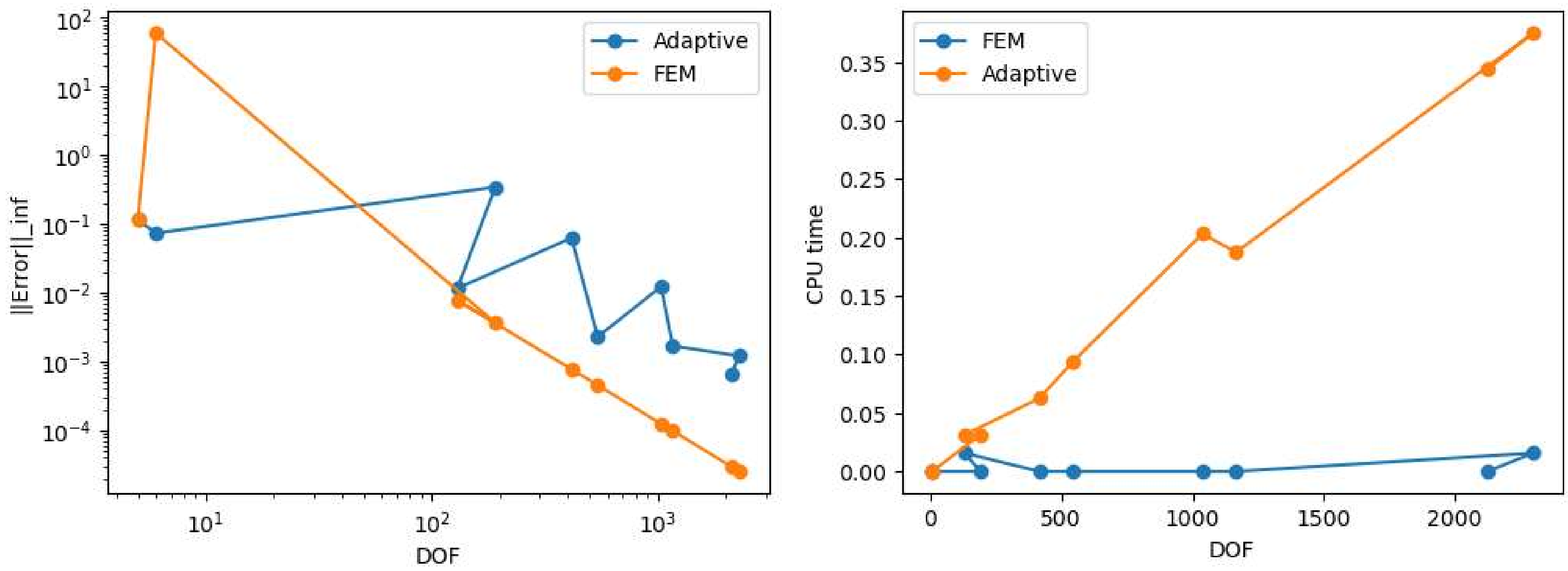


In [ ]:
```
import time
TOL = np.logspace(-1, -5, 10)
M = list(range(10,500,10))
error_AMR = np.zeros(len(TOL))
error_FEM = np.zeros(len(TOL))
Npoints = np.zeros(len(TOL))
CPUtime_AMR = np.zeros(len(TOL))
CPUtime_FEM = np.zeros(len(TOL))
```

```
x = np.linspace(0,1,3)
N = 5
for i,tol in enumerate(TOL):
    start_time = time.process_time()
    VX, uhat, it = DriverAMR17(L,c,d,x,f,tol, maxit=100)
    end_time = time.process_time()

    error_AMR[i] = np.linalg.norm(uhat-u(VX),np.inf)
    Npoints[i] = len(VX)
    CPUtime_AMR[i] += end_time - start_time

for i,n in enumerate(Npoints):
    x = np.linspace(0,1,int(n))
    start_time = time.process_time()
    uhat = BVP1D(L, x, c, d,f, plot=False)
    end_time = time.process_time()
    error_FEM[i] = np.linalg.norm(uhat-u(x),np.inf)
    CPUtime_FEM[i] += end_time - start_time
```

In [ ]:

```
fig, ax = plt.subplots(1,2,figsize=(12,4))

ax[0].loglog(Npoints, error_AMR,"o-",label="Adaptive")
ax[0].loglog(Npoints, error_FEM,"o-",label="FEM")
ax[0].set_xlabel("DOF")
ax[0].set_ylabel("||Error||_inf")
ax[0].legend()

ax[1].plot(Npoints, CPUtime_FEM,"o-",label="FEM")
ax[1].plot(Npoints, CPUtime_AMR,"o-",label="Adaptive")
ax[1].set_xlabel("DOF")
ax[1].set_ylabel("CPU time")
ax[1].legend()

plt.show()
```



In [ ]:

Out[ ]:

```
array([0., 0., 0., 0., 0., 0., 0., 0., 0., 0.])
```