DM549 and DS(K)820

Lecture 16: Sequences and Summations

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Repetition: Recursive Definition

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- the height of a full binary tree.

Repetition: Structural Induction

We proved statements our recursively defined structures in the following way:

Recipe for Proofs by Structural Induction

To show that $P(S_i)$ holds for all $i \geq 1$, prove:

- Basis step: Prove that $P(S_1)$ holds.
- Inductive step: Prove that

$$\underbrace{P(S_i)}_{\text{inductive hypothesis}} \Rightarrow P(S_{i+1})$$

  for all $i \geq 1$.

That is: Induction on the number of times the recursive step is applied.

Sequences

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Remarks:

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- You can also think of a sequence with domain D as a $|D|$-tuple.

Geometric and Arithmetic Sequences

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An infinite *geometric sequence* (geometrisk følge) is a sequence of the form

$$a_n = c \cdot r^n, \quad n \in \mathbb{N},$$

where $a \in \mathbb{R}$ is the *initial term* (begyndelsesled) and $r \in \mathbb{R}$ is the *common ratio* (fælles faktor).

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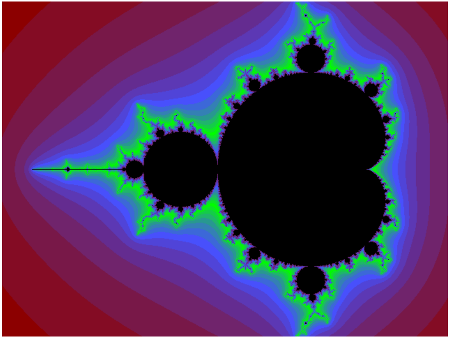
- If $|x_n|$ does not grow arbitrarily large, z belongs to the *Mandelbrot set*.
- The Mandelbrot set in the complex plane is depicted in black:



(Source: Wikipedia)

The colors encode how many iterations it takes for $|x_n|$ to surpass 1000.

Series

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Let

$$a_m, a_{m+1}, \ldots, a_n$$

be a sequence. Then there is an associated *series* (række), the sum of all terms in the sequence. It is denoted by

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- One could also talk about series that are the sum of the infinitely many terms of an infinite sequence.
- Here, we focus on finite sequences.
- Otherwise, to be completely formal, we would need to talk about a concept from calculus called convergence.

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Note: If $|r| < 1$ and we consider the *infinite* geometric series, the term r^{n+1} vanishes as n grows to ∞, so

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}.$$

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Additional formulae: Table 2.4.2.

A Quiz

Go to `pollev.com/kevs`