DM549/DS(K)820/MM537/DM547

Lecture 8: More on Functions and Cardinality

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Last Time: Functions

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Definition (Definitions 2.3.1 and 2.3.2)

Let A and B be nonempty sets. A *function* (funktion) f from A to B, for each $x \in A$, assigns precisely one element $f(x) \in B$ to x.

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the set of all possible values $f(x)$ for $x \in A$.

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- *surjective* if each "y value" is "hit" by *at least one* "x value",
- *bijective* if each "y value" is "hit" by *exactly* "x value".

Inverting Functions

Definition (Definition 2.3.9)

Let $f : A \to B$ be bijective. The *inverse function* (den inverse funktion) of f is the unique function $f^{-1} : B \to A$ such that, for all $x \in A$,

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Remark:

- That means that, for $f : A \rightarrow B$ bijective and any $x \in A$ with $f(x) = y$,
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  - $f^{-1}(f(x)) = x$ and
  - $f(f^{-1}(y)) = y$.
- There does not exist an inverse function of f (we also say that f is not invertible) if f is not a one-to-one correspondence.

Combining Functions into New Functions

Definition (Definition 2.3.3)

Let $f : A \to B$ and $g : A \to B$ be functions. Then $(f + g) : A \to B$ and $(f \cdot g) : A \to B$ are functions with

$$(f + g)(x) = f(x) + g(x),$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$

for all $x \in A$.

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Let $f : A \to B$ and $g : B \to C$ be functions. Then the *composition* of g and f, $(g \circ f) : A \to C$, is a function with

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Remark: Note that the codomain of f has to match the domain of g.

Increasing, Decreasing, and More

Definition (Definition 2.3.6)

Let $f : A \to B$. If, for all $x_1, x_2 \in A$ with $x_1 < x_2$, it holds that

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If f is increasing or decreasing, it is called *monotone* (monoton).

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Observe:

- A function f is injective if it is strictly increasing or strictly decreasing.

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Observe:

- A function f is injective if it is strictly increasing or strictly decreasing.
- A continuous function f is injective iff it is strictly increasing or strictly decreasing.

A Quiz

Go to `pollev.com/kevs`

Cardinality of Sets in General

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Definition (Definition 2.5.1)

Two sets A, B have the same *cardinality* if there exists a bijection from A to B.

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Note: This is consistent with the definition of cardinality we have learned for finite sets.

The Cardinality of $\mathbb{Z}^+$

Definition (Definition 2.5.3)

The cardinality of $\mathbb{Z}^+$ is called $\aleph_0$. A set A is called

- *countable* if it is finite or has cardinality $\aleph_0$,

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- *countable* if it is finite or has cardinality $\aleph_0$,
- *countably infinite* if it has cardinality $\aleph_0$,
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Note: The next-larger cardinalities after $\aleph_0$ are called $\aleph_1, \aleph_2, \ldots$, but we will not work with them.

Some Laws

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Proposition (only proof sketch, cf. Example 2.5.1)

Let $S \subseteq \mathbb{Z}^+$ be an infinite set. Then

$$|S| = \aleph_0.$$

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$$|\mathbb{Z}^+ \cup S| = \aleph_0.$$

Proposition (only proof sketch, cf. Theorem 2.5.1)

Let S be a finite set. Then

$$|\mathbb{Z}^+ \times S| = \aleph_0.$$

The Cardinality of $\mathbb{Z}$

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Proposition (Example 2.5.3)

It holds that $|\mathbb{Z}| = \aleph_0$.

The Cardinality of $\mathbb{Q}$

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Theorem (Example 2.5.4)

It holds that $|\mathbb{Q}^+| = \aleph_0$.

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Theorem (Example 2.5.4)

It holds that $|\mathbb{Q}^+| = \aleph_0$.

Corollary

It holds that $|\mathbb{Q}| = \aleph_0$.

A Joke

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Person A: So it is possible to count all integers?

A Joke

Person A: So it is possible to count all integers?
Person B: $\aleph_0$.