DM549/DS(K)820/MM537/DM547

Lecture 7: More on Sets; Functions

Kevin Schewior
Email: kevs@sdu.dk

University of Southern Denmark

September 23, 2023

Last Time: Sets

Definition (Definition 2.1.1)

A *set* (mængde) is an unordered collection of different objects, called *elements*. If object x is an element of set A, we write $x \in A$; otherwise we write $x \notin A$.

Notation: explicit enumeration, enumeration with "$\dots$" or set-builder notation.

Definition

For a finite set A, the *cardinality* of A is the number of elements in A, denoted $|A|$.

Intervals: $[3, 5]$, $(3, 7)$, $(2, 4]$, etc.

Last Time: Subsets, Supersets, and Power Sets

Definition (Definition 2.1.3)

A set A is a *subset* (delmængde) of another set B if, for all $x \in A$, it holds that $x \in B$. In that case, we write $A \subseteq B$. We also say that B is a *superset* of A. If that is not the case, we write $A \not\subseteq B$.

If $A \subseteq B$ and $A \neq B$, we say that A is a *proper subset* (ægte delmængde) of B and write $A \subset B$. If that is not the case, we write $A \not\subset B$.

Definition (Definition 2.1.6)

For a set A, its *power set* (potensmængde) is

$$\mathcal{P}(A) = \{S \mid S \subseteq A\},$$

the set of all subsets of A.

Last Time: Set Operations

We learned:

- $A \cap B = \{x \mid x \in A \wedge x \in B\}$
  - A and B are called disjoint if $A \cap B = \emptyset$
- $A \cup B = \{x \mid x \in A \vee x \in B\}$
- $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$
- $\overline{A} = U - A$
  - U is the universe

A Quiz

Go to pollev.com/kevs

De Morgan's Law for Sets

Theorem (Example 2.2.10)

For any two sets A, B, it holds that

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \text{ and } \overline{A \cap B} = \overline{A} \cup \overline{B}.$$

Remark: Notice the correspondence between

- $\cup$ and $\vee$,
- $\cap$ and $\wedge$,
- $\overline{}$ and $\neg$.

Tuples (vs. Sets)

Definition (Definition 2.1.7)

An n-tuple (n-tupel) is an ordered collection of not necessarily different objects, denoted as $(a_1, a_2, \ldots, a_n)$ where a_i is the object at the i-th position of the tuple, for $i \in \{1, \ldots, n\}$.

Sets:

- the order does not matter,
- objects are all different,
- we use $\in$.

Tuples:

- the order matters,
- objects can be identical,
- we do not use $\in$ (there is no proper notation).

Naming:

- 2-tuples are called *pairs*,
- 3-tuples are called *triples*,
- 4-tuples are called *quadruples*,
- etc.

The Cartesian Product

Definition (Definitions 2.1.8 and 2.1.9)

For two sets A, B, their *Cartesian product* (Kartesisk produkt) is

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\},$$

the set containing all pairs where an element of A is at the first position and an element of B is at the second position.

Likewise, for sets $A_1, A_2 \ldots, A_n$,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1 \wedge a_2 \in A_2 \wedge \cdots \wedge a_n \in A_n\}.$$

Further,

$$\underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$$

for $n \in \mathbb{Z}^+$ is denoted A^n.

A Quiz

Go to `pollev.com/kevs`

Functions

Definition (Definitions 2.3.1 and 2.3.2)

Let A and B be nonempty sets. A *function* (funktion) f from A to B, for each $x \in A$, assigns precisely one element $f(x) \in B$ to x.

We write $f : A \to B$, call A the *domain* (definitionsmængden) of f, and call B the *codomain* (sekundarmængden) of f.

Remarks:

- To specify a function, the domain and codomain of the function must be specified.
- The word "precisely" is important in the definition.
- Sometimes, functions are referred to as mappings or transformations.

The Image of a Function

Definition (Definition 2.3.4)

Let $f : A \to B$ be a function. The *image* or *range* (værdimængden eller billedmængden) of f is

$$\mathrm{Im}(f) = \{f(x) \mid x \in A\} = \{y \in B \mid \exists x \in A : f(x) = y\},$$

the set of all possible values $f(x)$ for $x \in A$.

Injective, Surjective, Bijective

Definition (Definition 2.3.5)

A function $f : A \to B$ is called *injective* or *one-to-one* (injektiv eller en-til-en) if

$$\forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2),$$

that is, f assigns any value $y \in B$ to at most one $x \in A$.

Definition (Definition 2.3.7)

A function $f : A \to B$ is called *surjective* or *onto* (surjektiv eller på) if

$$\forall y \in B : \exists x \in A : f(x) = y,$$

that is, $\mathrm{Im}(f) = B$.

Definition (Definition 2.3.8)

A function $f : A \to B$ is called *bijective* or a *one-to-one correspondence* (bijektion eller en-til-en-korrespondance) if it is both injective and surjective.

Injective, Surjective, Bijective: An Alternative View

A function f is called...

- *injective* if each "y value" is "hit" by *at most one* "x value",
- *surjective* if each "y value" is "hit" by *at least one* "x value",
- *bijective* if each "y value" is "hit" by *exactly* "x value".

A Quiz

Go to `pollev.com/kevs`