DM549/DS(K)820/MM537/DM547

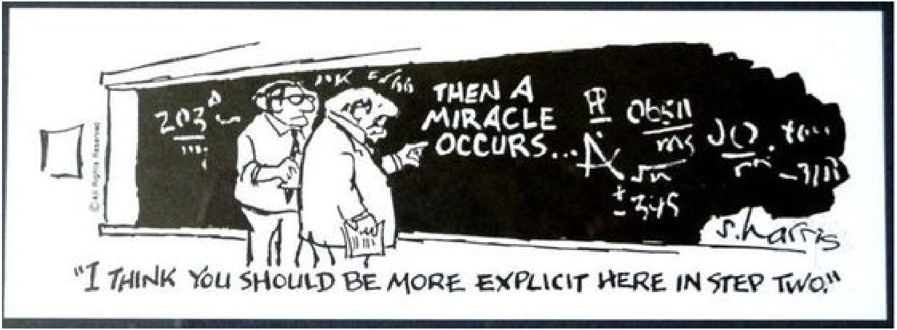
Lecture 5: Proofs by Induction

Kevin Schewior
Email: kevs@sdu.dk

University of Southern Denmark

September 16, 2024

Last Time: Proofs



Last Time: Proof Methods

Last Time: Proof Methods

Overview:

- Direct Proof:
  - We use that $((p \Rightarrow p_1) \wedge (p_1 \Rightarrow p_2) \wedge \cdots \wedge (p_n \Rightarrow q)) \Rightarrow (p \Rightarrow q)$.
  - We also just write $p \Rightarrow p_1 \Rightarrow p_2 \Rightarrow \cdots \Rightarrow p_n \Rightarrow q$.
  - Example: Proof that n odd $\Rightarrow n^2$ odd for all n.

Last Time: Proof Methods

Overview:

- Direct Proof:
  - We use that $((p \Rightarrow p_1) \wedge (p_1 \Rightarrow p_2) \wedge \cdots \wedge (p_n \Rightarrow q)) \Rightarrow (p \Rightarrow q)$.
  - We also just write $p \Rightarrow p_1 \Rightarrow p_2 \Rightarrow \cdots \Rightarrow p_n \Rightarrow q$.
  - Example: Proof that n odd $\Rightarrow n^2$ odd for all n.
- Proof by Contraposition:
  - We use that $(p \Rightarrow q) \equiv (\neg q \Rightarrow \neg p)$.
  - Example: Proof that n^2 odd $\Rightarrow n$ odd for all n.

Last Time: Proof Methods

Overview:

- Direct Proof:
  - We use that $((p \Rightarrow p_1) \wedge (p_1 \Rightarrow p_2) \wedge \cdots \wedge (p_n \Rightarrow q)) \Rightarrow (p \Rightarrow q)$.
  - We also just write $p \Rightarrow p_1 \Rightarrow p_2 \Rightarrow \cdots \Rightarrow p_n \Rightarrow q$.
  - Example: Proof that n odd $\Rightarrow n^2$ odd for all n.
- Proof by Contraposition:
  - We use that $(p \Rightarrow q) \equiv (\neg q \Rightarrow \neg p)$.
  - Example: Proof that n^2 odd $\Rightarrow n$ odd for all n.
- Proof by Contradiction:
  - We use that $(\neg p \Rightarrow F) \Rightarrow p$.
  - Example: Proof that there are two people in this room that were born on the same weekday.

Last Time: Some Tricks

Examples:

- Use that $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$.
  - Example: Proof that n odd $\Leftrightarrow n^2$ odd for all n.

Last Time: Some Tricks

Examples:

- Use that $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$.
  - Example: Proof that n odd $\Leftrightarrow n^2$ odd for all n.
- Use that $p_1 \Leftrightarrow p_2 \Leftrightarrow p_3 \equiv p_1 \Rightarrow p_2 \Rightarrow p_3 \Rightarrow p_1$
  - Example: Exercise 1.7.43 (Sheet 4).

Last Time: Some Tricks

Examples:

- Use that $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$.
  - Example: Proof that n odd $\Leftrightarrow n^2$ odd for all n.
- Use that $p_1 \Leftrightarrow p_2 \Leftrightarrow p_3 \equiv p_1 \Rightarrow p_2 \Rightarrow p_3 \Rightarrow p_1$
  - Example: Exercise 1.7.43 (Sheet 4).
- Make a case distinction.
  - Example: Proof that $\lfloor {}^{(n+1)}/_{2} \rfloor \geq {}^{n}/_{2}$ for all n.

Last Time: Problem Solving

Take-Aways: To solve a Mathematical problem, it may help to...

Last Time: Problem Solving

Take-Aways: To solve a Mathematical problem, it may help to...

- first look at simpler special cases.

Last Time: Problem Solving

Take-Aways: To solve a Mathematical problem, it may help to...

- first look at simpler special cases.
- draw suitable pictures.

There is never just a single proof!

"I am not qualified to say whether or not God exists. I kind of doubt He does. Nevertheless I'm always saying that [God] has this transfinite book that contains the best proofs of all mathematical theorems, proofs that are elegant and perfect."

"You don't have to believe in God, but you should believe in The Book."

– Paul Erdős (1913–1996)

There is never just a single proof!

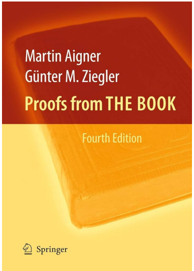
"I am not qualified to say whether or not God exists. I kind of doubt He does. Nevertheless I'm always saying that [God] has this transfinite book that contains the best proofs of all mathematical theorems, proofs that are elegant and perfect."

"You don't have to believe in God, but you should believe in The Book."

– Paul Erdős (1913–1996)

See also:

Martin Aigner und Günter M. Ziegler.
Proof from THE BOOK
4th Edition, Springer, 2014



Some Terminology

Some Terminology

Types of Propositions:

- theorem (sætning): important proposition that is proven to be true.

Some Terminology

Types of Propositions:

- theorem (sætning): important proposition that is proven to be true.
- conjecture (påstand): unproven important proposition that is believed to be true.

Some Terminology

Types of Propositions:

- theorem (sætning): important proposition that is proven to be true.
- conjecture (påstand): unproven important proposition that is believed to be true.
- lemma (lemma): auxiliary theorem.

Some Terminology

Types of Propositions:

- theorem (sætning): important proposition that is proven to be true.
- conjecture (påstand): unproven important proposition that is believed to be true.
- lemma (lemma): auxiliary theorem.
- corollary (korollar): proposition that immediately follows from a theorem.

Introducing Another Proof Method

Theorem (Example 5.1.4)

For all $n \in \mathbb{N}$, it holds that

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

Introducing Another Proof Method

Theorem (Example 5.1.4)

For all $n \in \mathbb{N}$, it holds that

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

Proof idea: Proof by induction:

- First show $\sum_{i=0}^{0} 2^i = 2^1 - 1$. (basis step)
- Then show that

$$\forall k \in \mathbb{Z}^+ : \left(\sum_{i=0}^{k-1} 2^i = 2^k - 1 \Rightarrow \sum_{i=0}^{k} 2^i = 2^{k+1} - 1 \right).$$

(inductive step)

An Interpretation with Dominoes

Interpretation:

- Imagine an infinite row of dominoes, say, one for each positive integer.

An Interpretation with Dominoes

Interpretation:

- Imagine an infinite row of dominoes, say, one for each positive integer.
- Suppose you have to explain to someone not familiar with the concept why *all* of the dominoes have fallen over.

An Interpretation with Dominoes

Interpretation:

- Imagine an infinite row of dominoes, say, one for each positive integer.
- Suppose you have to explain to someone not familiar with the concept why *all* of the dominoes have fallen over.
- Possible explanation:
  - Somebody has knocked over the first domino.

    (Corresponds to basis step.)

An Interpretation with Dominoes

Interpretation:

- Imagine an infinite row of dominoes, say, one for each positive integer.
- Suppose you have to explain to someone not familiar with the concept why *all* of the dominoes have fallen over.
- Possible explanation:
  - Somebody has knocked over the first domino.
    (Corresponds to basis step.)
  - If a domino falls over, then also the next domino falls over.
    (Corresponds to inductive step.)

Induction: A Recipe

Recipe 1 for Proofs by (Simple) Induction

Induction: A Recipe

Recipe 1 for Proofs by (Simple) Induction

To show that $P(n)$ holds for all $n \geq m$, prove:

Induction: A Recipe

Recipe 1 for Proofs by (Simple) Induction

To show that $P(n)$ holds for all $n \geq m$, prove:

- Basis step: Prove that $P(m)$ holds.
- Inductive step: Prove that

  $$\underbrace{P(k)}_{\text{inductive hypothesis}} \Rightarrow P(k+1)$$

  for all $k \geq m$.

Induction: A Recipe

Recipe 1 for Proofs by (Simple) Induction

To show that $P(n)$ holds for all $n \geq m$, prove:

- Basis step: Prove that $P(m)$ holds.
- Inductive step: Prove that

$$\underbrace{P(k)}_{\text{inductive hypothesis}} \Rightarrow P(k+1)$$

  for all $k \geq m$.

Some vocabulary:

- (Mathematical) induction (induktion)
- basis step (basisskridt)
- inductive step (induktionsskridt)
- inductive hypothesis (induktionshypotese)

Some sources use slightly different terms.

Induction: Another Recipe

Recipe 2 for Proofs by (Simple) Induction

To show that $P(n)$ holds for all $n \geq m$, prove:

- Basis step: Prove that $P(m)$ holds.
- Inductive step: Prove that
  $$\underbrace{P(k-1)}_{\text{inductive hypothesis}} \Rightarrow P(k)$$
  for all $k \geq m+1$.

Induction: Another Recipe

Recipe 2 for Proofs by (Simple) Induction

To show that $P(n)$ holds for all $n \geq m$, prove:

- Basis step: Prove that $P(m)$ holds.
- Inductive step: Prove that

$$\underbrace{P(k-1)}_{\text{inductive hypothesis}} \Rightarrow P(k)$$

for all $k \geq m+1$.

Note: It does not matter which recipe you use.

Another Proof by Induction

Theorem (Example 5.1.6)

For all $n \in \mathbb{Z}$ with $n \geq 4$, it holds that

$$2^n < n!\,.$$

Yet Another Proof by Induction

Fact

The sum of angles of a triangle is 180°.

Yet Another Proof by Induction

Fact

The sum of angles of a triangle is 180°.

Definition

Let $n \in \mathbb{Z}$ with $n \geq 3$. Then an n-gon is called *convex* if any line segment connecting two of its vertices only runs through the interior and boundary of the n-gon.

Yet Another Proof by Induction

Fact

The sum of angles of a triangle is 180°.

Definition

Let $n \in \mathbb{Z}$ with $n \geq 3$. Then an n-gon is called *convex* if any line segment connecting two of its vertices only runs through the interior and boundary of the n-gon.

Theorem

For all $n \in \mathbb{Z}$ with $n \geq 3$, it holds that for all convex n-gons the sum of its angles is $(n-2) \cdot 180^\circ$.

Yet Another Proof by Induction

Fact

The sum of angles of a triangle is 180°.

Definition

Let $n \in \mathbb{Z}$ with $n \geq 3$. Then an n-gon is called *convex* if any line segment connecting two of its vertices only runs through the interior and boundary of the n-gon.

Theorem

For all $n \in \mathbb{Z}$ with $n \geq 3$, it holds that for all convex n-gons the sum of its angles is $(n-2) \cdot 180^\circ$.

Note:

- Our statement $P(n)$ is (still) a universally quantified statement!

Yet Another Proof by Induction

Fact

The sum of angles of a triangle is 180°.

Definition

Let $n \in \mathbb{Z}$ with $n \geq 3$. Then an n-gon is called *convex* if any line segment connecting two of its vertices only runs through the interior and boundary of the n-gon.

Theorem

For all $n \in \mathbb{Z}$ with $n \geq 3$, it holds that for all convex n-gons the sum of its angles is $(n-2) \cdot 180^\circ$.

Note:

- Our statement $P(n)$ is (still) a universally quantified statement!
- The theorem is also true for non-convex polygons, but the proof becomes more complicated.

Yet Another Proof by Induction

Fact

The sum of angles of a triangle is $180°$.

Definition

Let $n \in \mathbb{Z}$ with $n \geq 3$. Then an n-gon is called *convex* if any line segment connecting two of its vertices only runs through the interior and boundary of the n-gon.

Theorem

For all $n \in \mathbb{Z}$ with $n \geq 3$, it holds that for all convex n-gons the sum of its angles is $(n-2) \cdot 180°$.

Note:

- Our statement $P(n)$ is (still) a universally quantified statement!
- The theorem is also true for non-convex polygons, but the proof becomes more complicated.
  - Cannot always cut off the triangle formed by three consecutive points.

A Strange Proof by Induction

"Theorem" (Example 5.1.15)

All apples have the same color.

A Strange Proof by Induction

"Theorem" (Example 5.1.15)

All apples have the same color.

Remark:

- This statement is obviously false.

A Strange Proof by Induction

"Theorem" (Example 5.1.15)

All apples have the same color.

Remark:

- This statement is obviously false.
- It was still possible to overlook the mistake in the proof.

A Strange Proof by Induction

"Theorem" (Example 5.1.15)

All apples have the same color.

Remark:

- This statement is obviously false.
- It was still possible to overlook the mistake in the proof.
- In general, one may find a (not necessarily correct) proof for a statement that is neither obviously true nor obviously false.

A Strange Proof by Induction

"Theorem" (Example 5.1.15)

All apples have the same color.

Remark:

- This statement is obviously false.
- It was still possible to overlook the mistake in the proof.
- In general, one may find a (not necessarily correct) proof for a statement that is neither obviously true nor obviously false.
- One needs to carefully check whether the proof is correct.

What a Proof by Induction is *NOT* (at SDU)

What a Proof by Induction is *NOT* (at SDU)

Professor Schmidt demonstrates
the concept of proof by induction.

(This joke is all Zach Weinersmith's fault, but I also apologize.)