DM549 and DS(K)820

Lecture 17: Matrices

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Repetition: Sequences

Definition (Definition 2.4.2)

An infinite *geometric sequence* (geometrisk følge) is a sequence of the form

$$a_n = c \cdot r^n, \quad n \in \mathbb{N},$$

where $a \in \mathbb{R}$ is the *initial term* (begyndelsesled) and $r \in \mathbb{R}$ is the *common ratio* (fælles faktor). We obtain finite geometric sequences by stopping at some point.

Definition (Definition 2.4.3)

An infinite *arithmetic sequence* (aritmetisk følge) is a sequence of the form

$$a_n = b + n \cdot d, \quad n \in \mathbb{N},$$

where $b \in \mathbb{R}$ is the *initial term* (begyndelsesled) and $d \in \mathbb{R}$ is the *common difference* (fælles forskel). We obtain finite arithmetic sequences by stopping at some point.

Repetition: Series

Series: Sum of all numbers in a sequence.

Theorem (Theorem 2.4.1)

For finite *geometric series* (with $c = 1$), the series corresponding to finite geometric sequences, it holds that

$$\sum_{i=0}^{n} r^i = \begin{cases} \frac{r^{n+1}-1}{r-1} & \text{if } r \in \mathbb{R} \setminus \{1\}, \\ n+1 & \text{if } r = 1. \end{cases}$$

Theorem

For finite *arithmetic series*, the series corresponding to finite arithmetic sequences, it holds that

$$\sum_{i=0}^{n} (b + i \cdot d) = b \cdot (n+1) + d \cdot \frac{n \cdot (n+1)}{2}$$

Matrices

Definition (Definition 2.6.1)

A *matrix* (matrix) is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix.

For a matrix A (B, C), we often denote the entry in the i-th row and j-th column of A by $a_{i,j}$ ($b_{i,j}$, $c_{i,j}$).

Applications:

- Solving systems of linear equations
- Computer graphics
- Machine learning
- many more!

Addition of Matrices

Definition (Definition 2.6.3)

Let A and B be $m \times n$ matrices. Then the sum of A and B, $C = A + B$, is the $m \times n$ matrix defined by

$$c_{i,j} = a_{i,j} + b_{i,j}$$

for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

Note: If A and B do not have the same sizes, $A + B$ is not defined!

Multiplication of Matrices

Definition (Definition 2.6.4)

Let A be an $m \times k$ matrix and B be an $k \times n$ matrix. Then the product of A and B, $C = AB$, is an $m \times n$ matrix defined by

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,k}b_{k,j}$$

for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

Remarks:

- If the number of columns of A does not match the number of rows of B, AB is not defined!
- $AB = BA$ does not hold in general (in fact, by the previous remark, it is possible that only one of the products exists).
  - Suitable joke (credit: @Dirque_L on X):
    Q: Why does matrix multiplication work from home?
    A: Because it doesn't commute.
- Associativity holds: $(AB)C = A(BC)$ for matrices A, B, C.

A Quiz

Go to `pollev.com/kevs`

Neutral Matrices

Definition

The *zero matrix of size $m \times n$* is the $m \times n$ matrix $0_{m\times n}$ where the entry in the i-th row and j-th column is 0 for all $i, j \in \{1, \dots, n\}$.

Observation: For any $m \times n$ matrix A, $A + 0_{m\times n} = A$.

Definition (Definition 2.6.5)

The *identity matrix of order n* is the $n \times n$ matrix I_n where the entry in the i-th row and j-th column is

$$\delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

for all $i, j \in \{1, \dots, n\}$.

Observation: For any $m \times n$ matrix A, $AI_n = I_mA = A$.

Transposition of Matrices

Definition (Definition 2.6.6)

Let A be an $m \times n$ matrix. The *transpose* (transponerede) of A is the $n \times m$ matrix A^T where the entry in the i-th row and j-th column is $a_{j,i}$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$.

Definition (Definition 2.6.7)

Let A be a square matrix, i.e., an $n \times n$ for some n. Then A is called *symmetric* (symmetrisk) if $A = A^T$.

Boolean Matrices

Remark: We will interpret "0" as "F" and "1" as "T".

Definition (Definition 2.6.8)

Let A and B be $m \times n$ zero-one matrices. Then $C = A \vee B$ and $D = A \wedge B$ are the $m \times n$ matrices defined by

$$c_{i,j} = a_{i,j} \vee b_{i,j} \text{ and } d_{i,j} = a_{i,j} \wedge b_{i,j}$$

for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

Note: This is analogous to addition ($\vee$ and $\wedge$, respectively, replace $+$).

Definition (Definition 2.6.9)

Let A be an $m \times k$ zero-one matrix and B be an $k \times n$ zero-one matrix. Then the Boolean product of A and B, $C = A \odot B$, is an $m \times n$ zero-one matrix defined by

$$c_{i,j} = (a_{i,1} \wedge b_{1,j}) \vee (a_{i,2} \wedge b_{2,j}) \vee \cdots \vee (a_{i,k} \wedge b_{k,j})$$

for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

Note: This is analogous to the regular product ($\wedge$ replaces $\cdot$ and $\vee$ replaces $+$).

Exponentiation of Matrices

Definition

Let A be an $n \times n$ matrix. Then we define

- $A^0 := I_n$, and,
- for any $r \in \mathbb{Z}^+$,

$$A^r = \underbrace{A \cdot A \cdot \; \cdots \cdot \; A}_{r \text{ times}}.$$

Definition (Definition 2.6.10)

Let A be an $n \times n$ matrix. Then we define

- $A^{[0]} := I_n$, and,
- for any $r \in \mathbb{Z}^+$,

$$A^{[r]} = \underbrace{A \odot A \odot \cdots \odot A}_{r \text{ times}}.$$