DM549/DS(K)820/MM537/DM547

Lecture 2: Propositional Equivalences and Quantifiers

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September 4, 2024

Last Time

Definition (Definition 1.1.1)

A *proposition* (et udsagn) is a declarative statement (that is, a statement that declares a fact) that is true (sand) or false (falsk) but not both.

We got to know the following **operators** through truth tables:

- the negation $\neg$,
- the conjunction $\wedge$,
- the disjunction $\vee$,
- the implication $\Rightarrow$,
- the bi-implication $\Leftrightarrow$,
- the exclusive or $\oplus$.

Precedence order ("order of evaluation") **of operators:**

- $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$
- There is no consensus on the position of $\oplus$.

Tautologies, Contradictions, and Contingencies

Three possibilities for compound proposition:

- Always true, no matter what values propositional variables take.
  - It is called a *tautology* (tautologi).
- Never true, no matter what values propositional variables take.
  - It is called a *contradiction* (modstrid).
- Neither a tautology nor a contradiction.
  - It is called a *contingency* (kontingens).

Q: How to find out if you are not sure?
A: Construct the truth table (or apply rules that we will see later).

A Quiz

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Logical Equivalences

Definition (Definition 1.3.2)

We call two propositions s, t *logically equivalent*, written $s \equiv t$, if $s \Leftrightarrow t$ is a tautology.

Note:

- In other words: s and t are two ways of saying the same thing.
- To find out whether $s \equiv t$, instead of constructing the truth table for $s \Leftrightarrow t$, one can compare the truth tables for s and t.
- The symbol $\equiv$ is not a logical operator, so $s \equiv t$ is not considered a compound proposition (while $s \Leftrightarrow t$ is).

See Tables 1.3.6–8 for many useful equivalences! We will now see the most important ones.

Distributive Laws

Distributive Laws (Example 1.3.4)

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r), \quad p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

Intuition (first version)**:** For both propositions,

- if p is T, full proposition is T.
- if p is F, proposition is T iff both q and r are T.

Proof (first version)**:**

p	q	r	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

De Morgan's Laws

De Morgan's Laws (Table 1.3.6, line 8)

$$\neg(p \wedge q) \equiv \neg p \vee \neg q, \quad \neg(p \vee q) \equiv \neg p \wedge \neg q$$

Intuition:

- If not both p and q are T, p must be F or q must be F.
- If not at least one of p and q is T, then both p and q must be F.

Proof: Exercises.

Note: This also works for more propositional variables, e.g.:

$$\neg(p \wedge q \wedge r) \equiv \neg p \vee \neg q \vee \neg r.$$

Equivalences Involving Implications (1)

Contraposition (Table 1.3.7, line 2)

$$p \Rightarrow q \equiv \neg q \Rightarrow \neg p$$

Intuition:

- If q is F, then $p \Rightarrow q$ only becomes T if p is F.
- This is what $\neg q \Rightarrow \neg p$ states.

Proof:

p	q	$p \Rightarrow q$	$\neg p$	$\neg q$	$\neg q \Rightarrow \neg p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Equivalences Involving Implications (2)

Formulation only using $\wedge$, $\vee$, $\neg$ (Table 1.3.7, line 1)

$$p \Rightarrow q \equiv \neg p \vee q$$

Intuition:

- If p is F, both propositions are T.
- If p is T, for either proposition to be T, q must be T.

Proof:

p	q	$p \Rightarrow q$	$\neg q$	$\neg p \vee q$
T	T	T	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Equivalences Involving Implications (3)

The Implication and the Bi-implication (Table 1.3.8, line 1)

$$(p \Rightarrow q) \land (q \Rightarrow p) \equiv p \Leftrightarrow q$$

Intuition: The rewritten left-hand side $(p \Rightarrow q) \land (\neg p \Rightarrow \neg q)$ means that both p and q need to have the same truth value.

Proof:

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \land (q \Rightarrow p)$	$p \Leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Note: This justifies the notation of $\Leftrightarrow$ and saying "p if and only if q".

One Last Equivalence

Table 1.3.8, line 5

$$\neg(p \Rightarrow q) \equiv p \wedge \neg q$$

Exercise (not on sheet)**:** Find intuition and truth tables.

Other proof: Blackboard.

A Quiz

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Some Sets of Numbers

Important for this and the next lectues:

- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of *integers* (heltal),
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ is the set of *positive integers*,
- $\mathbb{Z}^- = \{\ldots, -3, -2, -1\}$ is the set of *negative integers*,
- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of *natural numbers*,
  - In some sources, you will find $\{1, 2, 3, \ldots\}$.
- $\mathbb{Q} = \{\frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z}^+\}$ is the set of *rational numbers* (rationale tal),
- $\mathbb{R}$ is the set of *real numbers* (reelle tal), i.e., numbers given by:
  - any non-empty finite sequence of digits before the comma (possibly just 0)
  - and any sequence of digits after the comma (possibly the empty sequence).
- $\emptyset$ is the *empty set* (den tomme mængde).

Remark: We will talk more about sets and real numbers in later lectures!

Open Proposition

Definition

An *open proposition* (propositional function) is a statement in which one (or more) variables occur.

Remarks:

- The variables usually represent numbers.
- When the variables are replaced with actual values, one obtains a proposition.
- For now, we will focus on open propositions with a single variable.

The Universal Quantifier

Definition

For a propositional function $P(x)$, the statement

$$\forall x \in D : P(x)$$

is equivalent to the statement that $P(x)$ is true for all x in the set D. We call $\forall$ the *universal quantifier* (alkvantor).

Remarks:

- Read: "for all x in D, it holds that $P(x)$ (is true)" ("for alle x i D gælder, at $P(x)$ (er sandt)").
- A universal quantification over the empty set is always true.

The Existential Quantifier

Definition

For a propositional function $P(x)$, the statement

$$\exists x \in D : P(x)$$

is equivalent to the statement that there exists at least one x in the set D such that $P(x)$ is true. We call $\exists$ the *existential quantifier* (Eksistenskvantor).

Remarks:

- Read: "there exists x in D such that $P(x)$ (is true)" ("der eksisterer x i D sådan, at $P(x)$ (er sandt)").
- An existential quantification over the empty set is always false.
- The existential quantification is true as long there exists *at least one* x in D with the specified property, not just precisely one.

The Uniqueness Quantifier

Definition

For a propositional function $P(x)$, the statement

$$\exists! x \in D : P(x)$$

is equivalent to the statement that there exists precisely one x in the set D such that $P(x)$ is true. We sometimes call $\exists!$ the *uniqueness quantifier*.

Remarks:

- Read: "there exists precisely one x in D such that $P(x)$ (is true)" ("der eksisterer præcis et x i D sådan, at $P(x)$ (er sandt)").

More on Quantifiers

Remarks:

- We say that the quantifier *binds* variables x.
- In the above statements, we call D the *domain* (domæne) or universe (univers).
- We also say that we *quantify over* (kvantificerer over) D.
- When clear from the context, the domain is sometimes left out.
- Some authors leave out the colon.
- How to memorize?
  - for ∀ll,
  - there ∃xists,
  - "!" looks a bit like "1".
- Quantifiers have a *higher* preference (i.e., they are evaluated earlier) than the operators $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, $\oplus$.