DM549 and D(K)S820

Lecture 19: The Pigeonhole Principle

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Last Time: The Sum and Subtraction Rules

The Sum Rule

For any finite sets $S_1, S_2, \ldots, S_n$ where $S_i \cap S_j = \emptyset$ for any $i \neq j$, it holds that

$$\underbrace{\left|\bigcup_{i=1}^{n} S_i\right|}_{|S_1 \cup S_2 \cup \cdots \cup S_n|} = \underbrace{\sum_{i=1}^{n} |S_i|}_{|S_1| + |S_2| + \cdots + |S_n|} .$$

The Subtraction Rule

For any finite sets S_1, S_2, it holds that $|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$

Last Time: The Product and Division Rules

The Product Rule

For any finite sets $S_1, S_2, \ldots, S_n$, it holds that

$$\underbrace{\left|\bigtimes_{i=1}^{n} S_i\right|}_{|S_1 \times S_2 \times \cdots \times S_n|} = \underbrace{\prod_{i=1}^{n} |S_i|}_{|S_1| \cdot |S_2| \cdot \cdots \cdot |S_n|} .$$

The Division Rule

Suppose A is a finite set with $A = B_1 \cup B_2 \cup \cdots \cup B_n$ where

- $B_i \cap B_j = \emptyset$ for all $i \neq j$.
- $|B_i| = d$ for all i and

Then $n = |A|/d$.

Tree Diagrams

Some counting problems can be solved by drawing a tree and counting the number of leaves (cf. assigning offices).

Best-of-Five Match:

- Team A plays against Team B several games. Draws are not possible.
- The winner of the match is the first team that has won three games.
- How many possible sequences of "Team A wins" and "Team B wins" are there until the match is over?

Poll: Question 1/4

More bit strings:

- How many bit strings of length eight are there that either start with a 0 or end with a 0 (but not both)?
  - Examples: `01111111`, `10101010`.

Answer at `pollev.com/kevs`

Poll: Question 2/4

Room of love:

- You are in a room with 100 people.
- 78 of these people love Math (and possibly Computer Science).
- 80 of these people love Computer Science (and possibly Math).
- 60 of these people love both Math and Computer Science.
- How many people love neither Math nor Computer Science?

Answer at `pollev.com/kevs`

Poll: Question 3/4

Seating a larger group:

- Suppose five people are to be seated on a cicrular table.
- Two seatings are considered the same if everyone has the same left neighbor and the same right neighbor in both seatings.
- How many seatings that are considered different are there?

Answer at `pollev.com/kevs`

Poll: Question 4/4

No consecutive ones:

- How many bitstrings of length four are there that do not have consecuitve ones?
  - Examples: `0000`, `1001`, `0101`.

Answer at `pollev.com/kevs`

Overview of Today's Lecture

Topics today:

- The Pigeonhole Principle
- The Generalized Pigeonhole Principle

These topics can be found in Sections 6.2 in Rosen's book.

Towards the Pigeonhole Principle

Pigeons and holes:

- A flock of pigeons flies into ten pigeonholes to roost, i.e., each pigeon chooses one of these pigeonholes to roost in.
- How large does the number of pigeons have to be so that there is definitely a hole in which two pigeons are roosting?

The Pigeonhole Principle

The Pigeonhole Principle (Theorem 6.2.1)

Let $k \geq 1$ be an integer. When $k + 1$ or more objects are placed into k boxes, there is at least one box that contains at least two of the objects.

Corollary 6.3.1

A function $f : M \to N$ where $|M|$ is an integer larger than $|N|$ is not one-to-one.

The Pigeonhole Principle: Simple Examples

The Pigeonhole Principle (Theorem 6.2.1)

Let $k \geq 1$ be an integer. When $k+1$ or more objects are placed into k boxes, there exists at least one box that contains at least two of the objects.

Simple Examples:

- In a room with 367 (or more) people, at least two have their birthdays on the same day.
- In any set of 27 English words, at least two need to start with the same letter.
- Among any group of 102 students taking an exam with integer scores in $\{0, \ldots, 100\}$, there must be two with the same score.

The Pigeonhole Principle: The Existential Quantifier

The Pigeonhole Principle (Theorem 6.2.1)

Let $k \geq 1$ be an integer. When $k+1$ or more objects are placed into k boxes, there exists at least one box that contains at least two of the objects.

Notice the difference:

- How many people do I need to consider to definitely find two people that have their birthdays on the same day?
- How many people do I need to consider to definitely find two people that have birthday on February 13 (Dirichlet's birthday)?

The Pigeonhole Principle: Perhaps surprising Example

The Pigeonhole Principle (Theorem 6.2.1)

Let $k \geq 1$ be an integer. When $k+1$ or more objects are placed into k boxes, there exists at least one box that contains at least two of the objects.

Subsequences:

- A sequence $\{a'_n\}$ is called a subsequence of a sequence $\{a_n\}$ if $\{a'_n\}$ emerges from $\{a_n\}$ by deleting terms.
- Let $n \geq 1$ be an integer.
- Claim: Any sequence of n^2+1 distinct numbers contains a strictly increasing subsequence or a strictly decreasing subsequence of length $n+1$.
  - Example 1: 5, 2, 0, 7, 1, 4, 9, 3, 8, 6
  - Example 2: 10, 6, 12, 3, 14, 7, 5, 16, 1, 4, 13, 11, 0, 9, 2, 15, 8

The Pigeonhole Principle: Perhaps surprising Example 2

The Pigeonhole Principle (Theorem 6.2.1)

Let $k \geq 1$ be an integer. When $k+1$ or more objects are placed into k boxes, there exists at least one box that contains at least two of the objects.

"Binary" Numbers:

- Note that:
  - $2 \cdot 5 = 10$,
  - $3 \cdot 37 = 111$,
  - $4 \cdot 25 = 100$,
  - $5 \cdot 2 = 10$,
  - $6 \cdot 185 = 1110$,
  - $7 \cdot 143 = 1001$.
- Claim: For every integer $n \geq 1$, there exists an integer $k \geq 1$ such that the decimal representation of $k \cdot n$ consists of only 0s and 1s.

Towards the Generalized Pigeonhole Principle

Again pigeons and holes:

- A flock of pigeons flies into ten pigeonholes to roost, i.e., each pigeon chooses one of these pigeonhole to roost in.
- How large does the number of pigeons have to be so that there is definitely a hole in which three pigeons are roosting? How large does the number of pigeons have to be so that there is definitely a hole in which four pigeons are roosting?

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle (Theorem 6.2.2)

Let $N, k \geq 1$ be integers. When N or more objects are placed into k boxes, there is at least one box that contains at least $\lceil N/k \rceil$ of the objects.

The Generalized Pigeonhole Principle: Examples

The Generalized Pigeonhole Principle (Theorem 6.2.2)

Let $N, k \geq 1$ be integers. When N or more objects are placed into k boxes, there is at least one box that contains at least $\lceil N/k \rceil$ of the objects.

Examples:

- In a room with 100 people, there are nine of them that have their birthdays in the same month.
- Among a set of nine cards from a standard deck of cards, there must be three of the same suit.

Poll

Exam:

- Suppose there is an exam with scores in $\{0, \ldots, 100\}$.
- What is the minimum number of students to take the exam for there to be guaranteed to exist at least six of them that get the same score?

Answer at `pollev.com/kevs`

Summary: The Pigeonhole Principle and its Generalization

The Pigeonhole Principle (Theorem 6.2.1)

Let $k \geq 1$ be an integer. When $k + 1$ or more objects are placed into k boxes, there exists at least one box that contains at least two of the objects.

The Generalized Pigeonhole Principle (Theorem 6.2.2)

Let $N, k \geq 1$ be integers. When N or more objects are placed into k boxes, there exists at least one box that contains at least $\lceil N/k \rceil$ of the objects.