DM549/DS(K)820/MM537/DM547
Lecture 9: More on Cardinality; Recursive Definitions and Strong Induction

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Last Time: Cardinality

Definition (Definition 2.5.1)

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The cardinality of $\mathbb{Z}^+$ is called $\aleph_0$. A set A is called

- *countable* if it is finite or has cardinality $\aleph_0$,
- *countably infinite* if it has cardinality $\aleph_0$,
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Proposition (Example 2.5.4)

It holds that $|\mathbb{Q}| = \aleph_0$.

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Theorem (Example 2.5.5)

The set $\mathbb{R}$ is uncountable.

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Remark: This argument is known as Cantor's diagonalization argument.

A Quiz

Go to `pollev.com/kevs`

Some Surprises (not relevant for exam, but for life)

Russell's Paradox: The set $S = \{x \text{ is a set} \mid x \notin x\}$ cannot exist.

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Surprising status: One can *prove* that it is neither possible to

- prove the continuum hypothesis from axioms of Zermelo–Fraenkel set theory, nor to
- disprove the continuum hypothesis from axioms of Zermelo–Fraenkel set theory

(unless axioms contradict each other already, which is *provably* impossible to disprove).

Rabbits

Simplified assumptions:

- A pair of rabbits of the same age reproduce in the following way:

  Starting from when they are two month old, they create a new pair of rabbits every month.
- Rabbits never die.

A pair of rabbits is born now and put on an (until then) rabbitless island.

Question: How does the rabbit population on the island develop?

Recursive Definitions

A recursive definition is a self-referential definition, such as:

Definition (Definition 2.4.5)

The Fibonacci Numbers are defined by:

$$\begin{aligned} f_0 &= 0, \\ f_1 &= 1, \\ f_n &= f_{n-1} + f_{n-2}, \quad \text{for } n \geq 2. \end{aligned}$$

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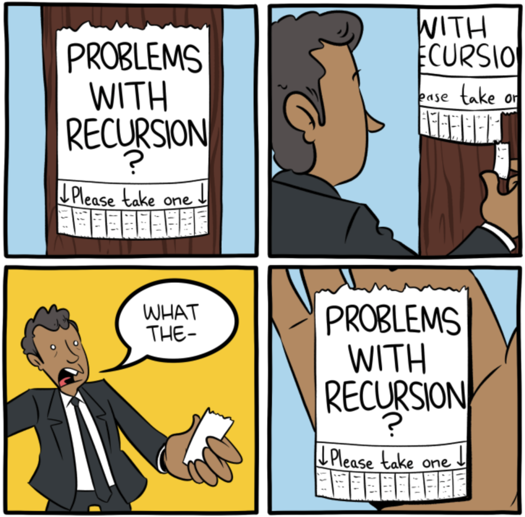

(by Zach Weinersmith)

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Theorem

For all $n \geq 3$, it holds that

$$f_n \geq \varphi^{n-2}.$$

Repetition: Recipe for (Regular) Induction

Recipe 1 for Proofs by (Simple) Induction

To show that $P(n)$ holds for all $n \geq m$, prove:

- Basis step: Prove that $P(m)$ holds.
- Inductive step: Prove that

$$\underbrace{P(k)}_{\text{inductive hypothesis}} \Rightarrow P(k+1)$$

  for all $k \geq m$.

Recipe for Strong Induction

Recipe for Proofs by *Strong* Induction

To show that $P(n)$ holds for all $n \geq m$, prove for some $\ell \geq 0$:

- Basis step: Prove that $P(m), P(m+1), \ldots, P(m+\ell)$ hold.
- Inductive step: Prove that

$$(P(k-\ell) \wedge \cdots \wedge P(k-1) \wedge P(k)) \Rightarrow P(k+1)$$

for all $k \geq m + \ell$.

Another Example

Theorem

For every $n \in \mathbb{N}$ with $n \geq 4$, there exist $a, b \in \mathbb{N}$ such that

$$n = 2a + 5b.$$

Interpretation: Any number of $n \geq 4$ Danish kroner can be given as change with 2 kroner and 5 kroner coins only.

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- It was fun teaching you!
- Have fun in the remaining part of the semester (probably 12 lectures more).