DM549/DS(K)820/MM537/DM547

Lecture 6: Some More Induction and Sets

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Last Time: Induction

Recipe 1 for Proofs by (Simple) Induction

To show that $P(n)$ holds for all $n \geq m$, prove:

- Basis step: Prove that $P(m)$ holds.
- Inductive step: Prove that

$$\underbrace{P(k)}_{\text{inductive hypothesis}} \Rightarrow P(k+1)$$

  for all $k \geq m$.

Recipe 2 for Proofs by (Simple) Induction

To show that $P(n)$ holds for all $n \geq m$, prove:

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- Inductive step: Prove that

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  for all $k \geq m+1$.

What a Proof by Induction is *NOT* (at SDU)

What a Proof by Induction is *NOT* (at SDU)

Professor Schmidt demonstrates
the concept of proof by induction.

(This joke is all Zach Weinersmith's fault, but I also apologize.)

A Strange Proof by Induction

"Theorem" (Example 5.1.15)

All apples have the same color.

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- In general, one may find a (not necessarily correct) proof for a statement that is neither obviously true nor obviously false.

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"Theorem" (Example 5.1.15)

All apples have the same color.

Remark:

- This statement is obviously false.
- It was still possible to overlook the mistake in the proof.
- In general, one may find a (not necessarily correct) proof for a statement that is neither obviously true nor obviously false.
- One needs to carefully check whether the proof is correct.

Today: Sets

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NOT this game:

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See also:
Fábio Botler, Andrés Cristi, Ruben Hoeksma, Kevin Schewior, Andreas Tönnis: SUPERSET: A (Super)Natural Variant of the Card Game SET. FUN 2018: 12:1–12:17

Sets

Definition (Definition 2.1.1)

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Different notations for sets:

- Explicit enumeration of finitely many elements: List all elements as

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- Set-builder notation: For a propositional function proposition $P(x)$ and domain D (also a set!), the set $\{x \in D \mid P(x)\}$ is the set of all elements x of D such that $P(x)$ is true.

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A special set: $\emptyset$ (also $\{\}$) is the empty set, i.e., the one containing no elements.

Venn Diagrams

Visual representation of sets and their relationships:

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Visual representation of sets and their relationships:

- A rectangle for the universal set U containing all objects under consideration.
- Inside the rectangle, circles for the different sets.
- Inside the respective circles, points for elements.
- Can usually be clearly arranged for up to 3 sets.

Finite Sets, Infinite Sets, and Cardinality

Two types of sets:

- finite sets, i.e., sets that have finitely many elements.

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Note: The definition of the cardinality of infinite sets is more complicated; we will talk about this in a later lecture!

Intervals

A special type of sets: For $a, b \in \mathbb{R}$:

- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ is called the *closed interval from a to b*,

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Note: These sets are infinite if and only if $a < b$!

A Quiz

Go to `pollev.com/kevs`

Subsets and Supersets

Definition (Definition 2.1.3)

A set A is a *subset* (delmængde) of another set B if, for all $x \in A$, it holds that $x \in B$. In that case, we write $A \subseteq B$. We also say that B is a *superset* of A. If that is not the case, we write $A \not\subseteq B$.

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If $A \subseteq B$ and $A \neq B$, we say that A is a *proper subset* (ægte delmængde) of B and write $A \subset B$. If that is not the case, we write $A \not\subset B$.

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Observe:

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  - $A \subseteq A$ (but $A \not\subset A$),
  - $\emptyset \subseteq A$.

Power Sets

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For a set A, its *power set* (potensmængde) is

$$\mathcal{P}(A) = \{S \mid S \subseteq A\},$$

the set of all subsets of A.

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Note: We will see later that $|\mathcal{P}(A)| = 2^{|A|}$.

Set Operators: Intersection and Union

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Definition (Definition 2.1.2)

For two sets A, B, their *intersection* (fællesmængde/snitmængde) is

$$A \cap B = \{x \mid x \in A \wedge x \in B\},$$

the set containing all objects that are elements of A and of B.

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Definition (Definition 2.1.1)

For two sets A, B, their *union* (foreningsmængde) is

$$A \cup B = \{x \mid x \in A \vee x \in B\},$$

the set containing all objects that are elements of A or of B.

Set Operators: Set Difference and Complement

Definition (Definition 2.1.4)

For two sets A, B, the *(set) difference of A and B* is

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Note: The book uses "$-$" instead of "$\setminus$".

Definition (Definition 2.1.2)

For a set A and a universal set U, the *complement* (komplement) of A is

$$\overline{A} = U \setminus A.$$

A Quiz

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De Morgan's Law for Sets

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Theorem (Example 2.2.10)

For any two sets A, B, it holds that

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Remark: Notice the correspondence between

- $\cup$ and $\vee$,
- $\cap$ and $\wedge$,
- $\overline{}$ and $\neg$.

Tuples (vs. Sets)

Definition (Definition 2.1.7)

An n-tuple (n-tupel) is an ordered collection of not necessarily different objects, denoted as $(a_1, a_2, \ldots, a_n)$ where a_i is the object at the i-th position of the tuple, for $i \in \{1, \ldots, n\}$.

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Naming:

- 2-tuples are called *pairs*,
- 3-tuples are called *triples*,
- 4-tuples are called *quadruples*,
- etc.

The Cartesian Product

Definition (Definitions 2.1.8 and 2.1.9)

For two sets A, B, their *Cartesian product* (Kartesisk produkt) is

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\},$$

the set containing all pairs where an element of A is at the first position and an element of B is at the second position.

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Likewise, for sets $A_1, A_2 \ldots, A_n$,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1 \wedge a_2 \in A_2 \wedge \cdots \wedge a_n \in A_n\}.$$

Further,

$$\underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$$

for $n \in \mathbb{Z}^+$ is denoted A^n.

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