DM549/DS(K)820/MM537/DM547

Lecture 3: More on Quantifiers

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Last Time: Propositional Logic

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More on Quantifiers

Remarks:

- We say that the quantifier *binds* variables x.
- In the above statements, we call D the *domain* (domæne) or universe (univers).
- We also say that we *quantify over* (kvantificerer over) D.
- When clear from the context, the domain is sometimes left out.
- Some authors leave out the colon.
- How to memorize?
  - for $\forall$ll,
  - there $\exists$xists,
  - "!" looks a bit like "1".
- Quantifiers have a *higher* preference (i.e., they are evaluated earlier) than the operators $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, $\oplus$.

A Quiz

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Remarks:

- Read: "... x in D with Q(x) ...".
- Notice the difference in how the quantification with restricted domain can be translated into a quantification without restricted domain.

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De Morgan's Laws for Quantifiers (Table 1.4.2)

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Interpretation: Can pull "$\neg$" to the right, but, by doing so, we "flip" quantifiers ($\forall$ changes to $\exists$ and $\exists$ changes to $\forall$).

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- By adding another quantifier in front (and binding another variable), one obtains a *proposition* (with no variables!).
- The resulting proposition has two *nested* quantifiers.

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  - Also write $\forall x, y : P(x, y)$ and $\exists x, y : P(x, y)$.

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In general: We may exchange consecutive quantifiers if and only if they are of the same type!

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Seriously:

- If you express a mathematical statement in natural language, make sure it is unambiguous!
- When writing down as formal logic, such ambiguities cannot happen.
  - (This is why we are learning about this topic!)

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- etc.
- Or directly: Flip all quantifiers and move the negation all the way inwards.

More Quantifiers

Rule of Thumb

number of quantifier flips (as you read from left to right)

$$\approx$$

complexity of proposition

Test 1

Don't forget that the test has been opened today. Good luck!