DM549/DS820/MM537/DM547

Lecture 11: Transitive Closure and Equivalence Relations

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Last Time: Relations

Definition (Definition 9.1.1)

Let A, B be sets. A *(binary) relation* (relation) *from A to B* is a subset of $A \times B$.

Definition (Definition 9.1.2)

Let A be a set. A *relation on A* is a relation from A to A, i.e., a subset of $A \times A$.

Representations: As set, graph, or matrix.

Properties of relations:

- Reflexivity
- Irreflexivity
- Symmetry
- Antisymmetry
- Transitivity

Last Time: Closures

Definition (Definition 9.4.1)

Let R be a relation on set A, and let P be a property of relations. Then the *closure* (lukning) of R w.r.t. P is (if it exists) the relation C on A such that

(i) $R \subseteq C$,

(ii) C fulfills property P,

(iii) $C \subseteq S$ for every S that fulfills (i) and (ii) (in place of C).

Exists for these P: Reflexivity, Symmetry, Transitivity

Does not exists for these P: Irreflexivity, Antisymmetry

The Transitive Closure

When P is transitivity:

- We can compute transitive closure by finding $a, b, c \in A$ with $(a, b), (b, c) \in R$ but $(a, c) \notin R$ and add (a, c) to R, until we can no longer find such a, b, c.
- Process must terminate because $A \times A$ is transitive.
- We shall see a different way of computing transitive closure in a bit.

Combining two Relations

Note: For two relations R, S from a set A to a set B, we can consider $R \cup S$, $R \cap S$, $R \setminus S$, etc.

Definition (Definition 9.1.6)

Let A, B, C be sets, R a relation from A to B, and S a relation from B to C. Then

$$S \circ R = \{(a, c) \mid \exists b : (a, b) \in R \land (b, c) \in S\}.$$

If $A = B$, then R^2 denotes $R \circ R$, R^3 denotes $R \circ R \circ R$, etc.

Remarks:

- This is a generalization of the composition of two functions!
- $(a, b) \in R^k$ if and only if, in the graph corresponding to R, one can walk from a to b in *precisely* k steps along edges.

Alternative Characterization of the Transitive Closure

Definition

For a relation R on a set A,

$$R^\star = \underbrace{\bigcup_{i=1}^{\infty} R^i}_{R \cup R^2 \cup R^3 \cup \cdots} .$$

Informally: $R^\star$ is set of pairs $(a, b) \in A \times A$ such that one can walk from a to b in graph corresponding to R along edges (in at least one step).

Theorem (only proof sketch, Theorem 9.4.2)

The transitive closure of a relation R is

$$t(R) = R^\star.$$

Equivalence Relations

Definition (Definitions 9.5.1 and 9.5.2)

A relation R on a set is called an *equivalence relation* (ækvivalensrelation) if it is

- reflexive,
- symmetric, and
- transitive.

If this is the case and $(a, b) \in R$, a and b are called *equivalent* (ækvivalent).

Equivalence Classes

Definition (Definition 9.5.3)

Let R be an equivalence relation on a set A. For $a \in A$,

$$[a]_R = \{b \mid (a, b) \in R\}$$

is the *equivalence class* (ækvivalensklassen) of a with respect to R.

Lemma (Theorem 9.5.1)

Let R be an equivalence relation on a set A, and let $a, b \in A$. The following three statements are equivalent:

(i) $(a, b) \in R$,

(ii) $[a] = [b]$,

(iii) $[a] \cap [b] \neq \emptyset$.

A Characterization of Equivalence Relations

Rep.: R equiv. rel. on A. For $a, b \in A$ equiv.: (i) $(a, b) \in R$, (ii) $[a] = [b]$, (iii) $[a] \cap [b] \neq \emptyset$.

Definition

Let A be a set. Then a set P containing subsets of A as elements is called a *partition* (partitionering) of A if the following holds:

- $\bigcup_{A' \in P} A' = A$
- for all $A', A'' \in P$ with $A' \neq A''$, $A' \cap A'' = \emptyset$.

Theorem (Theorem 9.5.2)

Let A be a set. There is a one-to-one correspondence between equivalence relations on A and partitions of A:

(1) For any equivalence relation R on A, $P = \{[a]_R \mid a \in A\}$ is a partition of A.

(2) For any partition $P = \{A_i \mid i \in I\}$ of A, there exists an equivalence relation R on A such that $\{[a]_R \mid a \in A\} = \{A_i \mid i \in I\}$.