DM549 and DS(K)820

Lecture 21: Counting with Repetition and Indistinguishability

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25+28 November, 2024

Last Time: Permutations

Definition (Permutations)

Let S be a finite set. For any integer r with $0 \le r \le |S|$, an *r-permutation* of S is an ordered arrangement of r distinct objects from S.

An $|S|$-permutation of S is simply called a permutation of S.

Computing the number of permutations (Theorem 6.3.1)

Let n and r be integers with $0 \le r \le n$. The number of r-permutations of a set with cardinality n is

$$P(n, r) = \frac{n!}{(n-r)!}.$$

Last Time: Combinations

Definition (Combinations)

Let S be a finite set. For any integer r with $0 \leq r \leq |S|$, an *r-combination* of S is a subset S' of S with $|S'| = r$.

Computing the number of combinations (Theorem 6.3.2)

Let n and r be integers with $0 \leq r \leq n$. The number of r-combinations of a set with cardinality n is

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r! \cdot (n-r)!}.$$

Note: We also denote $C(n, r)$ as

$$\binom{n}{r} \quad \text{(read: "}n\text{ choose }r\text{")}$$

and call it *binomial coefficient*.

Last Time: Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
 & & & & + & & & & & & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

Pascal's identity (Theorem 6.4.2)

Let n, k be integers with $1 \leq k \leq n$. Then it holds that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Last Time: More Identities

Corollary 6.3.2

Let n, r be integers with $0 \leq r \leq n$. Then it holds that

$$\binom{n}{r} = \binom{n}{n-r}.$$

Corollary 6.4.1

Let $n \geq 0$ be an integer. Then it holds that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Why "Binomial" Coefficients?

Question: How to compute $(x+y)^n$ in general?

The Binomial Theorem (Theorem 6.4.1)

Let x and y be variables and, let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} \cdot x^{n-j} y^j.$$

Note: The right-hand side is the same as

$$\binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1} y + \cdots + \binom{n}{n-1} \cdot x y^{n-1} + \binom{n}{n} \cdot y^n.$$

Topics Today

Recall the following aspects of how we defined permutations and combinations:

- No repetitions of elements are allowed.
- All elements of the ground set are distinguishable.

Indeed, if we speak simply of "permutation" or "combination", you can assume that this is the case.

In this lecture, we will look at permutations and combinations with repetition and indistinguishable objects.

This corresponds to Section 6.5 in Rosen's book.

Towards Permutations with Repetition

These are easy questions by now:

- How many different length-8 strings consisting of distinct English upper case letters are there?
- How many different length-8 strings consisting of ~~distinct~~ English upper case letters are there?

Permutations with Repetition

Definition (Regular Permutations)

Let S be a finite set. For any integer r with $0 \leq r \leq |S|$, an *r-permutation of S* is an ordered arrangement of r distinct objects from S.

Definition (Permutations with Repetition)

Let S be a finite set. For any integer r with $0 \leq r \leq |S|$, an *r-permutation of S with repetition* is an ordered arrangement of r ~~distinct~~ objects from S.

Theorem (Theorem 6.5.1)

Let n and r be integers with $0 \leq r \leq n$. The number of r-permutations with repetition of a set with cardinality n is n^r.

Towards Combinations with Repetition

Choosing Fruits:

- Suppose there are a large number of apples, of pears, and oranges.
  - All fruits of the same kind are considered indistinguishable.
- We want to choose exactly four fruits.
  - Some (or all) of them may be the same.
- In how many different ways can we choose the fruits?
  - Like with combinations, it does not matter in which order we choose.

Combinations with Repetition

Observation (Combinations, Alternative Definition)

Let S be a finite set with n elements, and let r be an integer with $0 \leq r \leq n$. There is a one-to-one correspondence between

- the set of r-combinations of S and
- the set of different assignments to the variables $x_1, \ldots, x_n$ such that $x_1, \ldots, x_n \in \{0, 1\}$ and $\sum_{i=1}^{n} x_i = r$.

Definition (Combinations with Repetition)

Let S be a finite set with n elements, and let r be an integer with $0 \leq r \leq n$. An *r-combination of S with repetition* is an assignment to the variables $x_1, \ldots, x_n$ such that $x_1, \ldots, x_n \in \{0, 1, \ldots, r\}$ and $\sum_{i=1}^{n} x_i = r$.

Theorem (Theorem 6.5.2)

Let n and r be integers with $0 \leq r \leq n$. The number of r-combinations of an n-element set with repetition is

$$\binom{n+r-1}{r}.$$

Combinations with Repetition: More examples

Definition (Combinations with Repetition)

Let S be a finite set with n elements, and let r be an integer with $0 \leq r \leq n$. An *r-combination of S with repetition* is an assignment to the variables $x_1, \ldots, x_n \in \{0, 1, \ldots, r\}$ with $\sum_{i=1}^{n} x_i = r$.

Theorem (Theorem 6.5.2)

Let n and r be integers with $0 \leq r \leq n$. The number of r-combinations with repetition of an n-element set is

$$\binom{n+r-1}{r}.$$

Another example:

- There are five danish bills in my pocket. How many possible combinations of bill values can I have there?
  - There exist bills of values 50 kr., 100 kr., 200 kr., 500 kr., 1000 kr.

Combinations and Permutations w/ and w/o Repetition

Let S be a set of cardinality n, and let r be an integer with $0 \leq r \leq n$.

	number of r-permutations of S	**number of r-combinations of S**
without repetition	$P(n, r)$	$C(n, r)$
with repetition	n^r	$C(n + r - 1, r)$

Note: Sometimes "replacement" is used rather than "repetition" (e.g., `itertools` package in python).

Towards Permutations with Indistinguishable Objects

Rearrangements: In how many ways can the letters of

- UNCOPYRIGHTABLE
- COOKIE
- INDISTINGUISHABILITY

be arranged to obtain a distinct string?

Permutations with Indistinguishable Objects

Theorem (Theorem 6.5.4)

Suppose there are n objects, each of which has exactly one of k types, and there are n_i objects of type i for any $i \in \{1, \ldots, k\}$. Then the number of different permutations of the n objects is

$$\frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!}.$$

Poll: Question 1/2

Integer solutions:

- How many different integer solutions are there to the equation $x + y + z = 6$ where $x, y, z \geq 0$?

Answer at `pollev.com/kevs`

Poll: Question 2/2

ABC Strings:

- How many different strings of length 8 are there that contain precisely 3 As, 3 Bs, and 2 Cs?

Answer at `pollev.com/kevs`

Distinguishable Objects into Distinguishable Boxes

Dealing Cards:

- Assume you are playing a card game with four players.
- The deck has 52 cards.
- Each players receives a hand of five cards in the beginning.
- In how many different ways can the cards be dealt?

Theorem 6.5.4

The number of ways to distribute n distinguishable objects into k distinguishable boxes so that the i-th box receives n_i objects for any $i \in \{1, \ldots, k\}$ is

$$\frac{n!}{n_1! \cdot n_2! \cdot \;\cdots\; \cdot n_k!}.$$

Indistinguishable Objects into Distinguishable Boxes

Santa's Problem:

- How many ways are there to distribute 10 indistinguishable gifts to 8 distinguishable children?

Observation

The number of ways to distribute n indistinguishable objects into k distinguishable boxes equals the number of k-combinations of a set of cardinality n with repetition, i.e., $C(n+k-1,k)$.

Distinguishable Objects into Indistinguishable Boxes

Balls into Bins:

- How many ways are there of assigning 4 distinguishable balls into 3 indistinguishable bins?

Theorem (no proof)

The number of ways to distribute n distinguishable objects into k indistinguishable boxes equals

$$\sum_{j=1}^{k} \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j-i)^n.$$

Indistinguishable Objects into Indistinguishable Boxes

Books into Boxes:

- How many ways are there to pack six copies of the same book into four identical boxes?
- Any box may contain an arbitrary number of books.

Bad news: No simple formula exists for the number of distributing n indistinguishable objects into k indistinguishable boxes!

Objects into Boxes

	k **distinguishable boxes**	k **indistinguishable boxes**
n **distinguishable objects**	$\dfrac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!}$ where $n_1, \dots, n_k$ are the number of objects in boxes $1, \dots, k$	$\displaystyle\sum_{j=1}^{k} \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j-i)^n$
n **indistinguishable objects**	$C(n+k-1, k)$	no formula

Summary: Permutations with Indistinguishable Objects

Theorem (Theorem 6.5.4)

Suppose there are n objects, each of which has exactly one of k types, and there are n_i objects of type i for any $i \in \{1, \ldots, k\}$. Then the number of different permutations of the n objects is

$$\frac{n!}{n_1! \cdot n_2! \cdot \; \cdots \; \cdot n_k!}.$$

Summary: Combinations/Permutations w/ and w/o Rep.

Let S be a set of cardinality n, and let r be an integer with $0 \leq r \leq n$.

	number of r-permutations of S	**number of r-combinations of S**
without repetition	$P(n,r)$	$C(n,r)$
with repetition	n^r	$C(n+r-1,r)$

Happy holidays