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Statements

1. Universal (all, every, any): $\forall x \in D(Q(x))$ is **true** $\iff Q(x)$ is **true** for every $x \in D$

2. Conditional (if...then): $\forall x \in D\ (P(x) \implies Q(x))$ is **true** $\iff$ $({\sim}P(x) \lor Q(x))$ is **true** for every $x \in D$

3. Existential (there exists, there is, some): $\exists x \in D$ such that $Q(x)$ is **true** for at least one $x \in D$

Definitions

Number Theory

1. Divisibility: if $n, d \in \mathbb{Z} \land n \neq 0$,
$$d \mid n \iff \exists k \in \mathbb{Z}(n = dk)$$

2. Congruence: if $a, b \in \mathbb{Z} \land n \in \mathbb{Z}^+$,
$$a \equiv b \pmod n \iff n \mid (a - b)$$

3. Rational:
$$n \in \mathbb{Q} \iff \exists a, b \in \mathbb{Z}\left(n = \frac{a}{b} \land b \neq 0\right)$$

4. Even:
$$n \text{ is even} \iff \exists k \text{ such that } n = 2k$$

5. Odd:
$$n \text{ is odd} \iff \exists k \text{ such that } n = 2k+1$$

6. Prime:
$$\forall r, s \in \mathbb{Z}^+(n = rs \implies (r = 1 \land s = n) \lor (r = n \land s = 1))$$

7. Composite:
$$\exists r, s \in \mathbb{Z}^+((n = rs) \land (1 < r < n) \land (1 < s < n))$$

8. Fraction in lowest term: largest integer that divides numerator and denominator is 1

Propositional Logic

1. Negation of p is ${\sim}p$

2. Conjuction of p and q is $p \land q$

3. Disjunction of p and q is $p \lor q$

4. Statement/Propositional form: expression made of statement variables and logical connectives that becomes a statement when actual statements are substituted for the component statement variables

5. Logical equivalence: statements with identical truth values for each possible substitution of statements for their statement variables

6. Tautology/Contradiction: statement form that is always **true/false** regardless of the truth values of the individual statements substituted for its statement variables

7. Conditional (sufficient condition, only if) of q by p is "if p then q" or "p implies q" denoted as
$$p \implies q$$
p is the hypothesis/antecedent and q is the conclusion/consequent.

8. Contrapositive (sufficient condition, only if):
$${\sim}q \implies {\sim}p \equiv p \implies q$$

9. Converse (necessary condition, if):
$$q \implies p$$

10. Inverse (necessary condition, if):
$${\sim}p \implies {\sim}q \equiv q \implies p$$

11. Biconditional (if and only if):
$$p \iff q \equiv (p \implies q) \land (q \implies p)$$

12. Argument: sequence of statements, where all statements (except the last one) are called premises/assumptions/hypothesis. The final statement is called the conclusion.

   (a) Syllogism: argument with 2 premises and 1 conclusion

   (b) Critical Row: row in truth table where all premises are **true**

   (c) If there is a critical row in which the conclusion is **false**, then the argument form is invalid

   (d) If for all critical rows the conclusion is **true**, then the argument form is valid.

   (e) Sound if and only if it is valid and all its premises are true

13. Predicate: sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables, $P(x_1, x_2, \ldots)$

14. Domain of a predicate variable: set of all values that may be substituted in place of the variable

15. Truth set of $P(x)$: set of all elements of D that make $P(x)$ **true** when substituted for x, $\{x \in D \mid P(x)\}$

Sets

1. Set: unordered collection of objects

   (a) Sets of size 1 are called singletons

   (b) A set is finite if it has finitely many distinct elements

2. Set Equality: sets with all the same elements
$$A = B \iff \forall z(z \in A \iff z \in B)$$

3. Subset: set where all elements are contained by another set (set included by another set)
$$A \subseteq B \iff \forall z(z \in A \implies z \in B)$$

4. Power set: the set of all subsets, $\mathcal{P}(A)$

5. Cardinality: the number of (distinct) elements in a set, $|A|$

6. Ordered pairs: expressions of the form (x, y)

7. Ordered n-tuples: $(x_1, x_2, \ldots, x_n), n \geqslant 2$
$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \iff x_1 = y_1 \land x_2 = y_2 \land \ldots x_n = y_n$$

8. Cartesian product of sets:
$$A \times B = \{x \in A \land x \in B\}$$

9. Union of sets:
$$A \cup B = \{x \mid x \in A \lor x \in B\}$$

10. Intersection of sets:
$$A \cap B = \{x \mid x \in A \land x \in B\}$$

11. Complement of a set in another set:
$$A - B = A \setminus B = \{x \mid x \in A \land x \notin B\}$$
$$\overline{B} = U \setminus B$$

12. Disjoint sets: $A \cup B = \varnothing$

13. Pairwise/Mutually disjoint sets: for all distinct $i, j \in \{1, 2, \ldots, n\}$,
$$A_i \cap A_j = \varnothing$$

14. Partition of set: $\mathscr{C}$ is a partition of a set A if it is a set of mutually disjoint nonempty subsets (components) of A whose union is A

   (a) $\forall S \in \mathscr{C}(\varnothing \neq S \subseteq A)$

   (b) $\forall x \in A\ \exists S \in \mathscr{C}(x \in S) \land$
$$\forall x \in A\ \forall S_1, S_2 \in \mathscr{C}\ (x \in S_1 \land x \in S_2 \implies S_1 = S_2)$$
$$\therefore \forall x \in A\ \exists! S \in \mathscr{C}(x \in S)$$

Relations

1. Relation:

   (a) $R \subseteq A \times B$

   (b) $x\ R\ y$ for $(x, y) \in R \land x \not R\ y$ for $(x, y) \notin R$
$$R = \{(x, y) \in A \times B \mid x\ R\ y\}$$
$$R^{-1} = \{(y, x) \in B \times A \mid y\ R^{-1}\ x\}$$

2. Binary relation on a set: relation from A to A

3. Reflexive: $\forall x \in A\ (x\ R\ x)$

4. Symmetric: $\forall x, y \in A\ (x\ R\ y \implies y\ R\ x)$

5. Transitive: $\forall x, y, z \in A\ (x\ R\ y \land y\ R\ z \implies x\ R\ z)$

6. Anti-symmetric:
$$\forall x, y \in A\,(x\ R\ y \land y\ R\ x \implies x = y)$$

7. Comparable: $\forall x, y \in A\ (x\ R\ y \lor y\ R\ x)$

8. Equivalence relation: relation that is reflexive, symmetric, and transitive, usually denoted $\sim$

9. Equivalence class of x w.r.t $\sim$: set of all elements that are $\sim$-related to x
$$[x]_\sim = \{y \in A \mid x \sim y\}$$

10. Set of all equivalence classes (quotient of set by relation):
$$A/{\sim} = \{[x]_\sim \mid x \in A\}$$

11. Representative of an equivalence class: element of the equivalence class

Modular Arithmetic and Posets

1. Quotient $\mathbb{Z}/{\sim_n}$ is denoted as $\mathbb{Z}_n$ or $\mathbb{Z}/n\mathbb{Z}$, and addition/multiplication is defined as follows
$$[x] + [y] = [x + y] \qquad [x] \cdot [y] = [x \cdot y]$$

2. (Non-strict) Partial Order: relation that is reflexive, anti-symmetric, and transitive, denoted with $\preccurlyeq$ and $x \prec y \equiv x \preccurlyeq y \land x \neq y$

3. (Non-strict) Total/Linear Order: partial order where every pair of elements is comparable

4. Partially Ordered Set (Poset) refers to the ordered pair (A, R) where R is a partial order on A

5. Hasse Diagram of $\preccurlyeq$: if $x \prec y$ and no $z \in A$ such that $x \prec z \prec y$, then x is placed below y and a line joins x to y, else no line

6. Minimal Element: $\forall x \in A(x \preccurlyeq c \implies c = x)$

7. Maximal Element: $\forall x \in A(c \preccurlyeq x \implies c = x)$

8. Smallest Element: $\forall x \in A(c \preccurlyeq x)$

9. Largest Element: $\forall x \in A(x \preccurlyeq c)$

10. Linearization of $\preccurlyeq$ is a total order $\preccurlyeq^*$ such that
$$\forall x, y \in A(x \preccurlyeq y \implies x \preccurlyeq^* y)$$

Functions

1. $f : A \to B$: assignment of each element in A to exactly one element in B,

   (a) If $x \in A$ then $f(x)$ is the image of x under f, if $y = f(x)$ then f maps x to y, denoted as $f : x \mapsto y$

(b) A is the domain of f, B is the codomain of f

2. Identity function: id: $A \to A$, which satisfies $\forall x \in A\,(\mathrm{id}_A(x) = x)$

Cardinality

1. (Cantor): for **any** set A, B ($\exists$ bijection $f : A \to B \iff |A| = |B|$)
2. (Cantor): A is finite $\vee$ $|A| = |\mathbb{Z}_{\geqslant 0}| \implies A$ is countable

Counting and Probability

1. Sample space: set of all possible outcomes of a random process
2. event: subset of sample space
3. let S be finite sample space where all outcomes are equally likely and E be an event in S, then probability of E is denoted as $P(E) = \dfrac{|E|}{|S|}$
4. r-permutation: ${}_nP_r$, ordered selection of r elements from n elements
5. r-combination: ${}_nC_r$, number of subsets of size r from n elements
6. Generalized Pigeonhole Principle: $f : X \to Y$ where $|X| = n, |Y| = m$, $\exists k \in \mathbb{Z}_{>0}$ such that $k < \frac{m}{n}$ then $\exists y \in Y$ such that y is the image of at least $k + 1$ distinct elements of X
7. Generalized Pigeonhole Principle (contrapositive): $f : X \to Y$ where $|X| = n, |Y| = m$, $\exists k \in \mathbb{Z}_{>0}$ such that $\forall y \in Y, f^{-1}(y)$ has at most k elements, then X has at most km elements
8. conditional probability: $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

	order matters	order does not matters
repetition allowed	n^k	$\binom{k+n-1}{k}$
repetition not allowed	${}_nP_k$	$\binom{n}{k}$

Order of Operations

1. $\sim$
2. $\wedge$ and $\vee$
3. $\implies$ and $\iff$

Argument Forms

Rules of Inferences

$$\text{premise}_1 \wedge \ldots \wedge \text{premise}_n \implies \text{conclusion}$$

1. Modus Ponens and Universal Modus Ponens
$$\begin{array}{c} p \implies q \\ p \\ \therefore q \end{array}$$
2. Modus Tollens and Universal Modus Tollens
$$\begin{array}{c} p \implies q \\ \sim q \\ \therefore \sim p \end{array}$$
3. Generalization
$$\begin{array}{c} p \\ \therefore p \vee q \end{array} \qquad \begin{array}{c} q \\ \therefore p \vee q \end{array}$$
4. Specialization
$$\begin{array}{c} p \wedge q \\ \therefore p \end{array} \qquad \begin{array}{c} p \wedge q \\ \therefore q \end{array}$$
5. Conjunction
$$\begin{array}{c} p \\ q \\ \therefore p \wedge q \end{array}$$
6. Elimination
$$\begin{array}{c} p \vee q \\ \sim q \\ \therefore p \end{array} \qquad \begin{array}{c} p \vee q \\ \sim p \\ \therefore q \end{array}$$
7. Transitivity
$$\begin{array}{c} p \implies q \\ q \implies r \\ \therefore p \implies r \end{array}$$
8. Proof by division into cases
$$\begin{array}{c} p \vee q \\ p \implies r \\ q \implies r \\ \therefore r \end{array}$$
9. Contradiction rule
$$\begin{array}{c} \sim p \implies \textbf{false} \\ \therefore p \end{array}$$

Rules of Inference for Quantified Statements

1. Universal Modus Ponens
$$\begin{array}{c} \forall x \in D(P(x) \implies Q(x)) \\ P(a) \text{ for a particular a} \in D \\ \therefore Q(a) \end{array}$$
2. Universal Modus Tollens
$$\begin{array}{c} \forall x \in D(P(x) \implies Q(x)) \\ \sim Q(a) \text{ for a particular a} \in D \\ \therefore \sim P(a) \end{array}$$
3. Universal Transitivity
$$\begin{array}{c} \forall x \in D(P(x) \implies Q(x)) \\ \forall x \in D(Q(x) \implies R(x)) \\ \therefore \forall x(P(x) \implies R(x)) \end{array}$$
4. Universal Instantiation
$$\begin{array}{c} \forall x \in D(P(x)) \\ \therefore P(a) \text{ if } a \in D \end{array}$$
5. Universal Generalization
$$\begin{array}{c} \forall P(a) \text{ for every } a \in D \\ \therefore \forall x \in D(P(x)) \end{array}$$
6. Existential Instantiation
$$\begin{array}{c} \exists x \in D(P(x)) \\ \therefore P(a) \text{ for some } a \in D \end{array}$$
7. Existential Generalization
$$\begin{array}{c} P(a) \text{ for some } a \in D \\ \therefore \exists x \in D(P(x)) \end{array}$$

Common Fallacies

1. Ambiguous premises
2. Circular reasoning
3. Jumping to a conclusion
4. Converse/Inverse error

Proof Types

1. Direct Proof: using algebra/definitions to construct an argument
2. By Construction: form of direct proof which comes up with a specific example to prove/disprove the statement
3. By Contradiction: assume the negation of the statement and arrive to the conclusion that the negation is false, and since every step is logically correct, the assumption must be false
4. By Exhaustion: list all possible scenarios (for finite cases)

(T2.1.1) Logical Equivalences

1. Commutative Law
$$p \wedge q \equiv q \wedge p \qquad p \vee q \equiv q \vee p$$
2. Associative laws
$$p \wedge q \wedge r \equiv (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$
$$p \vee q \vee r \equiv (p \vee q) \vee r \equiv p \vee (q \vee r)$$
3. Distributive laws
$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (q \wedge r)$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (q \vee r)$$
4. Identity laws
$$p \wedge \textbf{true} \equiv p \qquad p \vee \textbf{false} \equiv p$$
5. Negation laws
$$p \vee \sim p \equiv \textbf{true} \qquad p \wedge \sim p \equiv \textbf{false}$$
6. Double negative law
$$\sim(\sim p) \equiv p$$
7. Idempotent laws
$$p \wedge p \equiv p \qquad p \vee p \equiv p$$
8. Universal bound laws
$$p \vee \textbf{true} \equiv \textbf{true} \qquad p \wedge \textbf{false} \equiv \textbf{false}$$
9. De Morgan's laws
$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$
$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$
10. Absorption laws
$$p \vee (p \wedge q) \equiv p \qquad p \wedge (p \vee q) \equiv p$$
11. Negation of true and false
$$\sim\textbf{true} \equiv \textbf{false} \qquad \sim\textbf{false} \equiv \textbf{true}$$

(T5.3.5) Set Identities

1. Identity Law
$$A \cup \varnothing = A \qquad A \cap U = A$$
2. Universal Bound Law
$$A \cup U = U \qquad A \cap \varnothing = \varnothing$$
3. Idempotent Law*
4. Double Complement Law
$$\overline{(\overline{A})} = A$$
5. Commutative Law*
6. Associative Law*
7. Distributive Law*
8. De Morgan's Law*
9. Absorption Law*
10. Complement Law
$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \varnothing$$
11. Set Difference Law
$$A \setminus B = A \cap \overline{B}$$
12. Top and Bottom Law
$$\overline{\varnothing} = U \qquad \overline{U} = \varnothing$$

* — see logical equivalence

Theorems (5th edition)

(T3.2.1) $\sim(\forall x \in D, P(x)) \equiv \exists x \in D$ such that $P(x)$

(T3.2.2) $\sim(\exists x \in D$ such that $P(x)) \equiv \forall x \in D, P(x)$

(T4.3.1) Every integer is a rational number

(T4.3.2) Sum of any two rational number is rational

(C4.2.3) Double of a rational number is rational

(T4.4.1) $\forall a, b \in \mathbb{Z}^+ (a \mid b \implies a \leqslant b)$

(T4.4.2) The only divisors of 1 are 1 and -1

(T4.4.3) $\forall a, b, c \in \mathbb{Z} (a \mid b \wedge b \mid c \implies a \mid c)$

(T4.7.1) There is no greatest integer

(P4.7.4) $\forall n \in \mathbb{Z} (n^2 \text{ is even} \implies n \text{ is even})$

(T4.8.1) $\sqrt{2}$ is irrational

(T5.1.1.7) Empty set is a unique set with no element

(T5.2.4) For finite sets, $|\mathcal{P}(A)| = 2^{|A|}$

(T5.3.12) For (pairwise) disjoint sets,

$$|A \cup B| = |A| + |B|$$
$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$$

(T5.3.13) Inclusion-Exclusion Principle: for finite sets, $|A \cup B| = |A| + |B| - |A \cap B|$

(P6.2.16) $\sim_{\mathscr{C}}$ the same-component relation is an equivalence relation

(P6.3.4) $\sim_n$ the congruence-mod-n relation is an equivalence relation

(L6.4.4) Let $\sim$ be an equivalence relation, then

$$\begin{tikzcd} & x \sim y \arrow[dr, Rightarrow] & \\ {[x]_\sim = [y]_\sim} \arrow[ur, Rightarrow] & & {[x] \cap [y] \neq \varnothing} \arrow[ll, Rightarrow] \end{tikzcd}$$

(T6.4.9) $A/\sim$ is a partition of A

(P7.1.5) Addition/Multiplication is well-defined on $\mathbb{Z}_n$. $\forall n \in \mathbb{Z}^+$,

$$[x_1] = [x_2] \wedge [y_1] = [y_2] \implies$$
$$[x_1] + [y_1] = [x_1 + y_1] = [x_2 + y_2] = [x_2] + [y_2]$$
$$\wedge [x_1] \cdot [y_1] = [x_1 \cdot y_1] = [x_2 \cdot y_2] = [x_2] \cdot [y_2]$$

(P7.4.4) For posets, a smallest element is minimal and there is at most one smallest element

(P7.4.6) For a nonempty finite poset, a minimal element can be found

(T7.4.10) For any partial order on a set, there exists a total order on that set

(T9.1.1) $\forall m, n \in \mathbb{Z}$ ($m \leqslant n \implies \exists n - m + 1$ integers from m to n inclusive)

(T9.2.1) if an operation has k steps independent of each other, the operation can be done in the product of all steps ways

(T9.3.3) inclusion/exclusion rule:

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B|$$
$$- |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

(T9.7.1) Pascal's Formula $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

(Tutorial 4) Division Theorem:

$$\forall n \in \mathbb{Z}, d \in \mathbb{Z}^+ \exists! q, r \in \mathbb{Z} \text{ s.t. }$$
$$n = dq + r \wedge 0 \leqslant r < d$$

(T9.8)

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
2. expected value $= \sum_{k=1}^{n} a_k p_k$
3. for (not necessarily independent) random variables,
$$E\left(\sum_{i=1}^{n} c_i \cdot X_i\right) = \sum_{i=1}^{n} (c_i \cdot E[X_i])$$

(T9.9.1) Bayes' Theorem for mutually disjoint events

$$P(B_k|A) = \frac{P(A|B_k) \cdot P(B_k)}{\sum_{i=1}^{n} P(A|B_i) \cdot P(B_i)}$$

(T10.1.1) Pigeonhole Principle: let A, B be finite sets, $\exists$ injective $f : A \to B \implies |A| \leqslant |B|$

(T10.1.2) Dual Pigeonhole Principle: let A, B be finite sets, $\exists$ surjective $f : A \to B \implies |A| \geqslant |B|$

(T10.1.3) let A, B be finite sets, $\exists$ bijection $f : A \to B \iff |A| = |B|$

(P10.2.3) same cardinality relation is an equivalence relation

(L10.3.5) infinite set A is countable $\iff \exists$ sequence $a_0, a_1, \ldots$ in which every element of A appears

(P10.3.6) any subset of a countable set is countable

(P10.3.7) any infinite set has a countable infinite subset

(P10.4.1) union of countable infinite sets is countable

(T10.4.2) $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ is countable

(T10.4.3) power set of countable infinite set is uncountable

Set Notations

1. Roster notation: $\{x_1, x_2, ..., x_n\}$
2. Set builder notation: $\{x \in U \mid P(x)\}$
3. Replacement notation: $\{f(x) \mid x \in U\}$

Useful

1. $(p \vee q) \wedge \sim(p \wedge q) \equiv p \oplus q$
2. $(A \cap B) \cup (A \setminus B) = A$
3. $A \cap B \subseteq A$
4. R on $\varnothing \implies R$ is an equivalence relation where R is the null relation

Tutorial 4

1. The following are equivalent:
   (a) $\forall x, y \in A (x \; R \; y \implies y \; R \; x)$
   (b) $R = R^{-1}$
   (c) $\forall x, y \in A (x \; R \; y \iff y \; R \; x)$
2. Relation on $\mathbb{Q}$:
   (a) $xy \geqslant 0$ is reflexive and symmetric
   (b) $xy \geqslant 0$ is symmetric and transitive

Tutorial 5

1. for total orders, all minimal elements are smallest

Tutorial 6

1. sum of squares $= \dfrac{n(n+1)(2n+1)}{6}$
2. every positive integer can be written as sum of distinct non-negative integer powers of 2

Tutorial 7

1. $f : A \to A, \forall g \; (g \circ f = g \implies f = \text{id})$
2. f, g injective $\implies f \circ g$ injective
3. $f \circ g$ injective $\implies g$ injective
4. f, g surjective $\implies f \circ g$ surjective
5. $f \circ g$ surjective $\implies f$ surjective
6. f, g bijective $\implies (g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Tutorial 8

1. $\forall [a], [b] \in \mathbb{Z}_n \; (|[a]| = |[b]|)$
2. A countable infinite, B finite $\implies A \cup B$ countable
3. A infinite, B finite $\implies |A \cup B| = |A|$
4. A is infinite $\iff \exists A \subsetneq B$ such that $|A| = |B|$
5. A_i countable $\implies \bigcup_{i \in \mathbb{Z}_{\geqslant 0}} A_i$ countable

Tutorial 10

1. no. of reflexive relations $= \dfrac{2^{n^2-n}}{2^{n^2}} = \dfrac{1}{2^n}$
2. no. of symmetric relations $= \dfrac{2^{\frac{n^2+n}{2}}}{2^{n^2}} = \dfrac{1}{2^{\frac{n^2-n}{2}}}$

Tutorial 11

1. $\forall$ simple graphs G with n vertices, $\forall v \in V(G)$ $(\deg v \geqslant \lfloor \frac{n}{2} \rfloor)$
2. $\forall$ simple graphs G with at least 2 vertices, 2 vertices have the same degree

Quiz 3

1. Order of quantifiers can be freely arranged if all quantifiers are of the same type
2. $\forall (P(x) \wedge Q(x)) \iff \forall x P(x) \wedge \forall x Q(x)$
3. $\exists (P(x) \vee Q(x)) \iff \exists x P(x) \vee \exists x Q(x)$

Quiz 5

1. $\mathcal{P}(\varnothing) = \{\varnothing\}$ has 1 element and 2 subsets
2. For all sets, $B \times A \neq A \times B$

Quiz 6

1. non-symmetric relations which are reflexive $\not\Rightarrow$ they are antisymmetric
2. non-symmetric relation which are antisymmetric $\not\Rightarrow$ they are reflexive
3. symmetric relations may be antisymmetric or not
4. antisymmetric relations may be symmetric or not
5. $\min(|R|) = n$ if R is an equivalence relation on A where $|A| = n$

Quiz 7.3/7.4

1. not symmetric may not be antisymmetric
2. for finite posets,
   (a) any minimal element is not necessarily smallest
   (b) any smallest is minimal
   (c) the smallest element is unique
   (d) $\exists$ smallest $\implies$ exactly 1 minimal
   (e) exactly 1 minimal $\not\Rightarrow \exists$ smallest

Quiz 9

1. $h \circ f, f \circ h$ bijective $\implies h, f$ bijective

2017

1. check for isomorphism: label vertices and check whether the degrees can be mapped
2. simple connected graph with n vertices and no cycles $\implies n - 1$ edges

2018

1. sum from 1 to $n = \binom{n}{2}$ and $2^n = \sum_{k=0}^{n} \binom{n}{k}$

2019

1. (Dirac's Theorem) $\forall v \in V(G)$, $\deg v \geqslant \lfloor \frac{n}{2} \rfloor \implies G$ is has a Hamiltonian circuit
2. R is reflexive/transitive/symmetric $\implies R^{-1}$ is reflexive/transitive/symmetric
3. for $f : A \to B$ where $|A| = m, |B| = n$
   (a) $n > m \implies {}_nP_m$ injective functions
   (b) for surjective functions, calculate combinations of mapping m to n and subtract the cases where n has no mapping

$$
\begin{aligned}
S(m,n)n! &= \begin{Bmatrix} m \\ n \end{Bmatrix} n! \\
&= \frac{1}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m
\end{aligned}
$$

2020

1. there are 2^{n^2} directed graphs with n vertices
2. complete graphs have $\binom{n}{2}$ vertices
3. $a \in A$ is smallest $\implies \forall x \in A, x$ is minimal $\implies x = a$
4. G is a bipartite graph $\implies \max |E(G)| = \lfloor \frac{n^2}{4} \rfloor$
5. superset of a reflexive relation is reflexive
6. subset of anti-symmetric relation is anti-symmetric

2020 Semester 2

1. for $f : A \to B$, $\forall X \subseteq A, \forall Y \subseteq B$
   (a) $|f(X)| \leqslant |X|$
   (b) $|f^{-1}(Y)| \nleqslant |Y| \wedge |f^{-1}(Y)| \ngeqslant |Y|$
2. $f = \mathrm{id} \iff \forall$ injective/surjective/bijective g $(g \circ f = g)$
3. $f = \mathrm{id} \iff \mathrm{id} \circ f = \mathrm{id}$
4. countable sets:
   (a) $\mathbb{Z}^*$ of all strings over $\mathbb{Z}$
   (b) set of all simple undirected graphs whose vertex set is a finite subset of $\mathbb{Z}$
5. uncountable sets:
   (a) set of all partitions of $\mathbb{Z}$
   (b) set of all partial orders on $\mathbb{Z}$
   (c) set of all functions $\mathbb{Z} \to \mathbb{Z}$

Assumptions

1. every integer is even or odd, but not both
2. every rational can be reduced to a fraction in its lowest term