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Things to Note

Matrix Properties

1. $\boldsymbol{AB} \neq \boldsymbol{BA}$
2. $\boldsymbol{AB} = \boldsymbol{0} \not\Longrightarrow \boldsymbol{A} = \boldsymbol{0} \vee \boldsymbol{B} = \boldsymbol{0}$
3. $(\boldsymbol{AB})^n \neq \boldsymbol{A}^n \boldsymbol{B}^n$
4. $\boldsymbol{A} \operatorname{adj}(\boldsymbol{A}) = \det \boldsymbol{A} \cdot \boldsymbol{I}$
5. REF has zero row $\implies$ not invertible

Cramer's rule

$$\boldsymbol{x} = \frac{1}{\det \boldsymbol{A}} \begin{pmatrix} \det \boldsymbol{A}_1 \\ \det \boldsymbol{A}_2 \\ \vdots \\ \det \boldsymbol{A}_n \end{pmatrix}$$

where $\boldsymbol{A}_i = \boldsymbol{A}$ with the i-th column replaced by $\boldsymbol{b}$

Spans, Subspaces, Linear Independence

1. $\forall S \subseteq V, \dim V = n$
   (a) $|S| = k < n \implies \operatorname{span} S \neq V$
   (b) $|S| = k > n \implies S$ is linearly dependent
2. check subset V is subspace
   (a) must contain zero vector
   (b) $\forall \boldsymbol{u}, \boldsymbol{v} \in V \implies c\boldsymbol{u} + d\boldsymbol{v} \in V$
3. $\boldsymbol{0} \in S$ then S is linearly dependent
4. $\{\boldsymbol{0}\}$ is a subspace of $\mathbb{R}^3$, with basis $\varnothing$

Ranks

1. $\operatorname{rank} \boldsymbol{A} = \operatorname{rank}(\boldsymbol{A} \mid \boldsymbol{b}) \implies (\boldsymbol{A} \mid \boldsymbol{b})$ is consistent
2. $\operatorname{rank} \boldsymbol{AB} \leqslant \min\{\operatorname{rank} \boldsymbol{A}, \operatorname{rank} \boldsymbol{B}\}$
3. $\boldsymbol{B}$ is invertible $\implies \operatorname{rank} \boldsymbol{BA} = \operatorname{rank} \boldsymbol{A}$ (converse is false)

Orthogonal

1. orthogonal set $\implies$ linearly independent set
2. $\boldsymbol{p}$ is the projection of $\boldsymbol{u}$ onto V if $\boldsymbol{u} - \boldsymbol{p}$ is orthogonal to V
3. orthogonal matrix means the transpose is its inverse
4. rows/cols of orthogonal matrix is a basis for $\mathbb{R}^n$

Diagonalization

1. matrix is diagonlizable $\iff$ it has n linearly independent eigenvectors $\iff$ $\dim(E_{\lambda_i}) = r_i$

Recurrence Relation

1. recurrence matrix $= \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ p & q \end{pmatrix}$
2. orthogonally diagonalizable $\iff$ symmetric

Questions

Chapter 1

1. relative positions of three lines if there are
   (a) no soln: either all parallel and not all are the same or two intersect at a point not on the third line
   (b) one soln: all different and intersect at a point or two same lines intersecting with third line at a point
   (c) infinite solns: all lines are the same
2. relative positions of three planes if there are
   (a) no soln: either all parallel and not all are the same or two intersect at a line parallel and not on the third plane
   (b) one soln: three planes intersect at a single point
   (c) infinite solns: two same planes intersect at a line on third plane or all three planes are the same
3. inconsistent linear system can have more unknowns than equations
4. a linear system with unique solution: no. of equations $\geqslant$ no. of unknowns
5. linear system with infinite solutions can have more equations than unknowns
6. matrix with last column as pivot column can be considered as RREF
7. non-homogeneous system cannot have trivial solution

Chapter 2

1. $\boldsymbol{Ax} = \boldsymbol{0}$ has non-trivial soln $\implies \boldsymbol{Ax} = \boldsymbol{b}$ has either no soln or infinitely many soln
2. $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible $\not\Longrightarrow \boldsymbol{A} + \boldsymbol{B}$ is invertible
3. $\boldsymbol{A}$ and $\boldsymbol{B}$ are singular $\not\Longrightarrow \boldsymbol{A} + \boldsymbol{B}$ is singular
4. if $\boldsymbol{A}$, $\boldsymbol{B}$ are singular square matrices, $\boldsymbol{AB}$, $\boldsymbol{BA}$ are singular
5. for $\boldsymbol{A}_{m\times n}, \boldsymbol{B}_{n\times m}$, $m > n \implies \boldsymbol{AB}$ is singular

Chapter 3

Q20 $\operatorname{span}(S_1 \cap S_2) \neq \operatorname{span} S_1 \cap \operatorname{span} S_2$

Q20/23 $\operatorname{span}(S_1 \cup S_2) = \operatorname{span} S_1 + \operatorname{span} S_2 \neq \operatorname{span} S_1 \cup \operatorname{span} S_2$

Q24 let V, W be subspaces of $\mathbb{R}^n$
1. $V \cap W$ is a subspace of $\mathbb{R}^n$
2. $V \cup W$ is a subspace of $\mathbb{R}^n \iff V \subseteq W \vee W \subseteq V$

Q26 for a nonzero matrix in REF, the nonzero rows are always linearly independent

Q30 $\boldsymbol{Pu}_1, \boldsymbol{Pu}_2, \dots, \boldsymbol{Pu}_k$ linearly independent $\implies \boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k$ linearly independent.

Q30 $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k$ linearly independent

P invertible $\implies \boldsymbol{Pu}_1, \boldsymbol{Pu}_2, \dots, \boldsymbol{Pu}_k$ linearly independent

P is singular $\not\Longrightarrow \boldsymbol{Pu}_1, \boldsymbol{Pu}_2, \dots, \boldsymbol{Pu}_k$ linearly independent/dependent

Q41 S is a finite subset of V such that

$\operatorname{span} S = V \implies \exists S' \subseteq S$ such that S' is a basis for V

S is linearly independent $\implies \exists S'$ a basis for V such that $S' \supseteq S$

Q42 $\dim V = n \implies \exists n+1$ vectors such that $\forall \boldsymbol{v} \in V$, $\boldsymbol{v}$ can be expressed as a linear combination of $n+1$ vectors

Q43 $\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W)$

Chapter 4

Q17 In $\mathbb{R}^3$,
1. rank $= 0 \implies$ soln set is $\mathbb{R}^3$
2. rank $= 1 \implies$ soln set is plane in $\mathbb{R}^3$ passing through origin
3. rank $= 2 \implies$ soln set is line in $\mathbb{R}^3$ passing through origin
4. rank $= 3 \implies$ soln set is $\{\boldsymbol{0}\}$

Q20 $\boldsymbol{AB} = 0 \implies \operatorname{col}(B) \subseteq \operatorname{null}(A)$

Q21 row space and nullspace of a matrix cannot contain the same non-zero vector

Q23 $\operatorname{rank}(\boldsymbol{A} + \boldsymbol{B}) \leqslant \operatorname{rank} \boldsymbol{A} + \operatorname{rank} \boldsymbol{B}$

Q25 $\operatorname{rank} \boldsymbol{A} = \operatorname{rank} \boldsymbol{A}^T\boldsymbol{A} = \operatorname{rank} \boldsymbol{AA}^T$

nullity $\boldsymbol{A} =$ nullity $\boldsymbol{A}^T\boldsymbol{A} \neq$ nullity $\boldsymbol{AA}^T$

Chapter 5

Cauchy-Schwarz $\|\boldsymbol{u} \cdot \boldsymbol{v}\| \leqslant \|\boldsymbol{u}\| \|\boldsymbol{v}\|$

Triangle $\|\boldsymbol{u} + \boldsymbol{v}\| \leqslant \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$

$d(\boldsymbol{u}, \boldsymbol{w}) \leqslant d(\boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{v}, \boldsymbol{w})$

Q31 S, T orthonormal bases for V $\implies \forall \boldsymbol{u}, \boldsymbol{v} \in V, (\boldsymbol{u})_S \cdot (\boldsymbol{v})_S = (\boldsymbol{u})_T \cdot (\boldsymbol{v})_T$

Q32 $\boldsymbol{A}$ is orthogonal $\implies$
- $\|\boldsymbol{u}\| = \|\boldsymbol{Au}\|$
- $d(\boldsymbol{u}, \boldsymbol{v}) = d(\boldsymbol{Au}, \boldsymbol{Av})$
- angle between $\boldsymbol{u}, \boldsymbol{v}$ = angle between $\boldsymbol{Au}, \boldsymbol{Av}$

Q33 $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n\}$ is a basis for $\mathbb{R}^n \implies T = \{\boldsymbol{Au}_1, \boldsymbol{Au}_2, \dots, \boldsymbol{Au}_n\}$ is a basis for $\mathbb{R}^n$

Q33 S orthogonal/orthonormal $\implies T$ orthogonal/orthonormal

Chapter 6

Q3 $\boldsymbol{A}$ is a square matrix, $\boldsymbol{Ax} = \lambda\boldsymbol{x} \implies \boldsymbol{A}^T\boldsymbol{u} = \lambda\boldsymbol{u}$

Q23 $\boldsymbol{A}$ diagonalizable $\implies \boldsymbol{A}^T$ diagonalizable

Q23 $\boldsymbol{A}, \boldsymbol{B}$ diagonalizable

$\not\Longrightarrow \boldsymbol{A} + \boldsymbol{B}$ diagonalizable

$\not\Longrightarrow \boldsymbol{AB}$ diagonalizable

Q26 for symmetric matrices, eigenvectors of different eigenspaces are orthogonal

Q30 $\boldsymbol{A}, \boldsymbol{B}$ orthogonally diagonalizable

$\implies \boldsymbol{A} + \boldsymbol{B}$ orthogonally diagonalizable

$\not\Longrightarrow \boldsymbol{AB}$ orthogonally diagonalizable

Assignments

A1 Q7 for $\boldsymbol{A}_{m\times n}$, $\boldsymbol{B}_{n\times m}$
1. $\boldsymbol{A}$ singular $\implies \boldsymbol{BA}$ singular
2. $m > n \implies \boldsymbol{AB}$ singular
3. $\boldsymbol{A}$ invertible $\not\Longrightarrow \boldsymbol{AB}, \boldsymbol{BA}$ invertible

A3 Q3 for $\boldsymbol{A}_{m\times n}, \boldsymbol{B}_{n\times k}$
1. $m > n \wedge \operatorname{rank} \boldsymbol{A} = n \implies \operatorname{rank} \boldsymbol{AB} = \operatorname{rank} \boldsymbol{B}$
2. $k > n \wedge \operatorname{rank} \boldsymbol{B} = n \implies \operatorname{rank} \boldsymbol{AB} = \operatorname{rank} \boldsymbol{A}$

A3 Q7 for some subspace V of $\mathbb{R}^n$,
1. $V \cap V^\perp = \{\boldsymbol{0}\}$
2. $\boldsymbol{A}_{n\times k}$ where $\operatorname{col}(\boldsymbol{A}) = V$

$\implies \text{null}(\boldsymbol{A}\boldsymbol{A}^T) = V^\perp$

A4 Q2 eigenvalues of orthogonal matrix are ± 1

A4 Q2 $\boldsymbol{A}$ diagonalizable orthogonal matrix (not orthogonally diagonalizable) then $\boldsymbol{A}^2 = \boldsymbol{I}$

Exams

14/15 S1 Q6 $\forall \boldsymbol{u}, \boldsymbol{v} \neq \boldsymbol{0}$, $\text{span}\{\boldsymbol{u}\} \cap \text{span}\{\boldsymbol{v}\} = \text{span}\{\boldsymbol{u}, \boldsymbol{v}\} \implies \text{span}\{\boldsymbol{u}\} = \text{span}\{\boldsymbol{v}\}$

14/15 S2 Q3 for $\boldsymbol{A}_{m\times n}, \boldsymbol{c}_{m\times 1}$, $n > m \implies \boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}$ always has infinitely many least squares solutions

14/15 S2 Q4 for $\boldsymbol{M}_{n\times n}, \boldsymbol{N}_{n\times n}$ with the same n linearly independent eigenvectors, $\boldsymbol{M}\boldsymbol{N} = \boldsymbol{N}\boldsymbol{M}$

15/16 S2 Q5 $T : \mathbb{R}^n \to \mathbb{R}^m$ such that standard matrix is diagonalizable $\implies \text{R}(T) = \text{R}(T \circ T) \wedge \ker(T) = \ker(T \circ T)$

16/17 S2 Q4 $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $\ker(T) = \ker(T \circ T) \implies \ker(T \circ T) = \ker(T \circ T \circ T)$

16/17 S2 Q6 $\boldsymbol{A}_{n\times n}, \boldsymbol{B}_{n\times n}$
$\implies \text{nullity}\ \boldsymbol{A} + \text{nullity}\ \boldsymbol{B} \geqslant \text{nullity}\ \boldsymbol{A}\boldsymbol{B}$

19/20 S1 Q6 $\boldsymbol{A}_{n\times n}$ such that $\boldsymbol{A}^2 = \boldsymbol{I} \implies \text{rank}(\boldsymbol{I} + \boldsymbol{A}) + \text{rank}(\boldsymbol{I} - \boldsymbol{A}) = n$

19/20 S1 Q6 $\boldsymbol{A}^2 - \boldsymbol{B}^2 = \boldsymbol{A}\boldsymbol{B} \implies \boldsymbol{A}, \boldsymbol{B}$ cannot be orthogonal

20/21 S1 Q2 $\forall \boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{v}$ can be written **uniquely** as $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ where $\boldsymbol{v}_1 \in V, \boldsymbol{v}_2 \in V^\perp$

20/21 S1 Q4 E_0 corresponds to nullspace of matrix

20/21 S1 Q5 $T : \mathbb{R}^n \to \mathbb{R}^n$

$\implies R(T^{k+1}) \subseteq R(T^k)$

$\implies m > n \wedge T^m$ is zero transformation $\implies T^n$ is zero transformation