2014 Notes.

Section 7

Magnetic confinement: equilibria

In this section, we will learn about equilibria for magnetically confined plasmas

- What do we mean by equilibrium?
- Cylindrical plasma equilibria
- Tokamak plasma equilibria
    - toroidal and poloidal B-field
    - pressure distribution
    - plasma current distribution
    - Grad-Shafranov equation
    - relation to experimental measurements

Equilibrium: what is it?

The equilibrium describes the steady, basic state of the plasma

Each element of plasma is (at least very nearly) in force-balance

$\boxed{\underset{\sim}{J} \times \underset{-}{B} = \underset{\sim}{\nabla} p}$ [see Section 6]

Equilibria can be unstable, eg if too much pressure or two much current is induced into the plasma $\Rightarrow$ loss of confinement

Such instabilities can be small-scale, driving plasma turbulence

– This turbulence degrades confinement (increases the rate of heat and particle loss, although equilibrium is maintained)

– This physics is extremely complicated – both experimentally and theoretically

Instabilities can also be large scale

– These typically set limits to the plasma pressure and therefore fusion performance

Instabilities arise from one of a number of classes of waves that can exist in the plasma

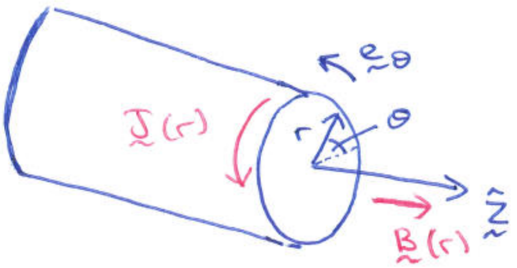
– we shall consider some of the types of wave that can exist in future sections

Cylindrical plasma equilibria: θ pinch

We first consider 1-D equilibria: cylindrical plasma

– we neglect plasma flows, $\underset{\sim}{u}$

In a $\boxed{\text{theta pinch}}$, the B-field is along the axis, and the current density, *J*, is in the azimuthal direction



$$\underset{\sim}{J} = J(r)\,\underset{\sim}{e}_\theta$$

$$\underset{\sim}{B} = B(r)\,\underset{\sim}{e}_z$$

[$\underset{\sim}{e}_\theta$ & $\underset{\sim}{e}_z$ are unit vectors]
[$\underset{\sim}{e}_z \equiv \hat{\underset{\sim}{z}}$]

Of course, *B* and *J* are related through Ampère's law:

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J}$$

In cylindrical coords ~~[illegible]~~

$$\Rightarrow \quad J = -\frac{1}{\mu_0}\frac{\partial B}{\partial r}$$

if $\underset{\sim}{B} = B\,\hat{\underset{\sim}{z}} \Rightarrow \underset{\sim}{\nabla} \times \underset{\sim}{B} = -\frac{\partial B}{\partial r}\underset{\sim}{e}_\theta$

In addition, knowing *B* and *J*, we also know pressure, *p*

$$\underset{\sim}{J} \times \underset{\sim}{B} = \underset{\sim}{\nabla} p$$

$$\Rightarrow \quad (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B} = \mu_0 \underset{\sim}{\nabla} p$$

Vector identity $(\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B} = (\underset{\sim}{B}.\underset{\sim}{\nabla})\underset{\sim}{B} - \frac{1}{2}\underset{\sim}{\nabla} B^2$

Symmetry in $\hat{\underset{\sim}{z}}$ direction $\Rightarrow (\underset{\sim}{B}.\underset{\sim}{\nabla})\underset{\sim}{B} = 0$

$$\Rightarrow \quad \underset{\sim}{\nabla}(\mu_0 p + B^2/2) = 0$$

$$\Rightarrow \quad \mu_0 p + B^2/2 = \text{constant.}$$

$$\underset{\sim}{J} = -\frac{1}{\mu_0}\frac{\partial B}{\partial r}\underset{\sim}{e}_\theta \qquad \underset{\sim}{B} = B\,\underset{\sim}{e}_z$$

$$\underset{\sim}{J} \times \underset{\sim}{B} = -\frac{1}{\mu_0} B \frac{\partial B}{\partial r}\,\underset{\sim}{e}_\theta \wedge \underset{\sim}{e}_z$$

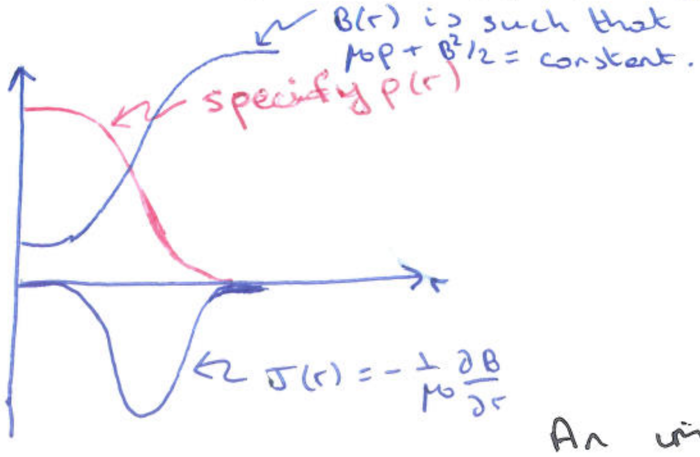
$$= \frac{1}{2\mu_0}\frac{\partial B^2}{\partial r}\underset{\sim}{e}_r = \underset{\sim}{\nabla} p$$

$$p = p(r) \Rightarrow \underset{\sim}{\nabla} p = \frac{dp}{dr}\underset{\sim}{e}_r$$

$$\Rightarrow \frac{\partial}{\partial r}(\mu_0 p + B^2/2) = 0$$

$$\Rightarrow \mu_0 p + B^2/2 = \text{constant.}$$

⇒ Prescribe just one of *p*, *B* and *J*, and the other two are determined, e.g. knowing $p(r) \Rightarrow B(r)$ and $J(r)$



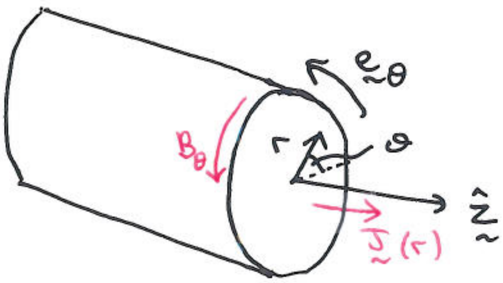
– Note p measures the thermal energy density of the plasma.

– $B^2/2\mu_0$ measures the energy density of magnetic field.

An important parameter $\beta = \frac{2\mu_0 p}{B^2}$ is the ratio of these.

Cylindrical plasma equilibria: Z pinch

In a Z pinch, a current density is driven along the axis:



$\underset{\sim}{J}(r) = J_z(r)\,\hat{\underset{\sim}{z}}$

$\underset{\sim}{B}(r) = B_\theta(r)\,\underset{\sim}{e}_\theta$

The B-field is then determined from J via Ampère's law:

$\mu_0 J_z = (\underset{\sim}{\nabla} \times \underset{\sim}{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(r B_\theta)$ if $\underset{\sim}{B}$ is purely in θ direction.

$\Rightarrow$ Specify $J_z(r)$, then $B_\theta(r)$ is determined

For example, suppose $J_z(r)=J_0$ (ie constant)

$\Rightarrow r B_\theta = \mu_0 \int^r r J_z\, dr = \frac{\mu_0 J_0 r^2}{2} + c$ $\quad c$ = constant of integration

$\Rightarrow B_\theta = \frac{\mu_0 J_0 r}{2} + \frac{c}{r}$ $\quad$ as $\lim_{r\to 0} B_\theta$ must be finite $\Rightarrow c = 0$

$\Rightarrow \boxed{B_\theta = \frac{\mu_0 J_0}{2} r}$

To determine the pressure, consider force balance:

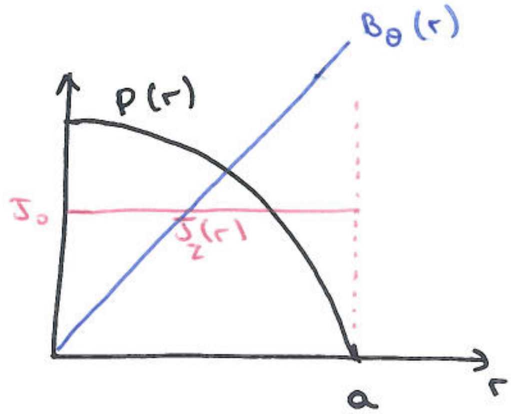
$\underset{\sim}{J} \times \underset{\sim}{B} = \underset{\sim}{\nabla} p = \frac{dp}{dr}\hat{\underset{\sim}{r}}$ $\quad$ as p depends only on r.

Take $\hat{\underset{\sim}{r}}$ component $\Rightarrow (\underset{\sim}{J} \times \underset{\sim}{B})_r = J_\theta B_z - J_z B_\theta = -J_z B_\theta$

$\Rightarrow \frac{\mu_0 J_0^2}{2} r = -\frac{dp}{dr}$

Integrate $\Rightarrow$ $p(r) = -\frac{\mu_0 J_0^2}{4} r^2 + c_2$

To fix constant c_2, suppose plasma edge is at $r = a$ and assume $p(r=a) = 0 \Rightarrow c_2 = \frac{\mu_0 J_0^2}{4} a^2$



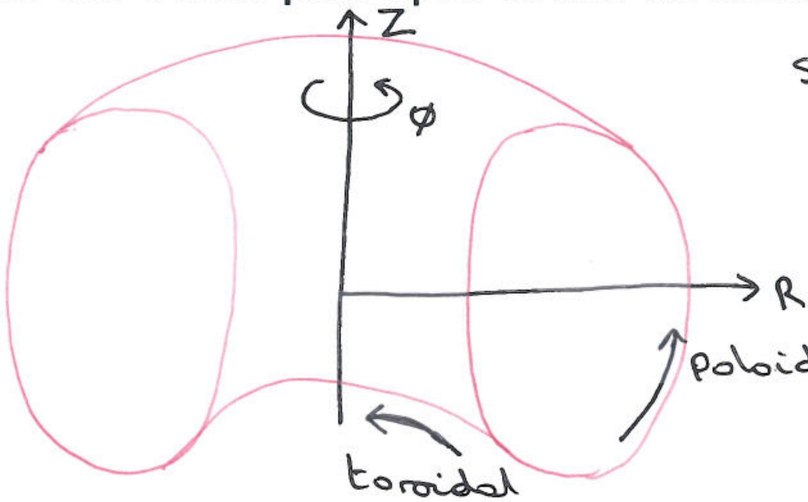
$\Rightarrow \boxed{p(r) = \frac{\mu_0 J_0^2}{4}(a^2 - r^2)}$ $\quad \Rightarrow$ higher current $\Rightarrow$ higher pressure

2-D equilibria: the tokamak

In 1-D, we have seen that we need only specify a profile (ie r-dependence) of one of p, J, B – the other two are then determined.

In 2-D, we shall find that we must specify two profiles

Let us recall the basic principle of the tokamak:



Symmetry in the toroidal (ϕ) direction.

R = major radius coordinate.

Note (R, ϕ, Z) form a cylindrical coordinate system (c.f. (r, θ, z))

1. We apply a toroidal component of the magnetic field with coils or the central copper rod
   ⇒ This would exist in the absence of a plasma, so call it the "vacuum field"

$$B_{vac} \propto \frac{1}{R}$$

2. We apply a current in the toroidal direction, I_p (plasma current)
3. The plasma then generates its own additional currents and fields, necessary to achieve force balance
   ⇒ In solving for the "equilibrium" we solve for the full current and magnetic field (ie applied plus plasma-generated)

Poloidal flux, ψ: $\nabla\cdot\boldsymbol{B}=0$ constraint

The magnetic fields must satisfy $\nabla\cdot\boldsymbol{B}=0$

– Let us work with a cylindrical coordinate system (R, ϕ, z)

$$\Rightarrow \quad \underline{\nabla}\cdot\underline{B} = \frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \underbrace{\cancel{\frac{1}{R}\frac{\partial B_\phi}{\partial\phi}}}_{=0 \text{ by symmetry}} + \frac{\partial B_z}{\partial z}$$

– Because of toroidal symmetry (ie in the φ direction) $\frac{\partial B_\phi}{\partial \phi} = 0$

$$\Rightarrow \boxed{\underline{\nabla}\cdot\underline{B} = \frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{\partial B_z}{\partial z} = 0}$$

– Now introduce a poloidal flux function, $\psi(R,Z)$, such that:

$$B_R = \frac{1}{R}\frac{\partial\psi}{\partial z} \qquad\qquad B_z = -\frac{1}{R}\frac{\partial\psi}{\partial R}$$

$\Rightarrow$ Poloidal component of field $B_p^2 = B_R^2 + B_z^2 = \frac{1}{R^2}\left[\left(\frac{\partial\psi}{\partial z}\right)^2 + \left(\frac{\partial\psi}{\partial R}\right)^2\right] = \frac{|\nabla\psi|^2}{R^2}$

$$\Rightarrow \boxed{RB_p = |\nabla\psi|}$$

– This then ensures $\nabla\cdot\boldsymbol{B}=0$ $= \frac{1}{R}\frac{\partial}{\partial R}\left(\frac{\partial\psi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)$

$$\frac{1}{R}\frac{\partial^2\psi}{\partial R\partial z} \longleftrightarrow \frac{1}{R}\frac{\partial^2\psi}{\partial z\partial R}$$

Cancel.

– Note also $\quad \underline{B}\cdot\nabla\psi = B_R\frac{\partial\psi}{\partial R} + B_z\frac{\partial\psi}{\partial z} + \underbrace{\cancel{\frac{B_\phi}{R}\frac{\partial\psi}{\partial\phi}}}_{=0 \text{ by symmetry}}$

$$= \frac{1}{R}\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial R} - \frac{1}{R}\frac{\partial\psi}{\partial R}\frac{\partial\psi}{\partial z} = 0$$

$\Rightarrow \underline{B}\cdot\underline{\nabla}\psi = 0 \quad \Rightarrow \underline{\nabla}\psi$ is perpendicular to $\underline{B}$

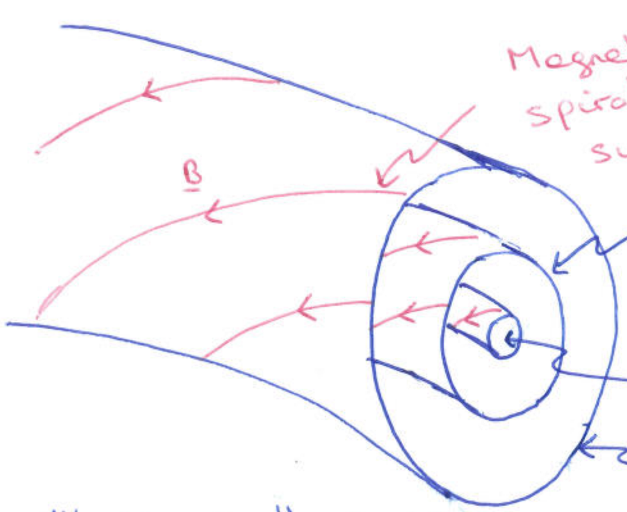
[recall $\underline{\nabla}f$ is a vector ⊥ to the plane of constant f]

⇒ magnetic field lines lie in surfaces of constant ψ

⇒ these surfaces are called <u>flux surfaces</u>

⇒ we note, without proof, that the constant ψ surfaces are a set of nested toroidal surfaces

The idea of flux surfaces



Note:- we can use ψ as a "radial" coordinate, measuring distance from the plasma centre out towards the edge.

Consider force balance:

$$\underset{\sim}{J} \times \underline{B} = \nabla p$$

– Dot both sides with the magnetic field, $\boldsymbol{B}$: $[(\underline{J} \times \underline{B}) \cdot \underline{B} \equiv 0]$

$$\Rightarrow \underline{B} \cdot \underline{\nabla} p = 0$$

$\Rightarrow$ pressure does not vary along the field lines

$\Rightarrow$ pressure does not vary on the flux surfaces [as toroidal symmetry $\Rightarrow$ p on one field line = p on ~~the~~ another in the flux surface.

$\Rightarrow$ pressure only depends on the poloidal flux, ψ

$$\Rightarrow \boxed{p(R, Z) = p(\psi)}$$ – important result.

$\Rightarrow$ pressure is constant on the toroidal flux surfaces.

$\Rightarrow$ a pressure gradient can be maintained across flux surfaces

$\Rightarrow$ allows low pressure at edge, and high (fusion relevant) pressure in centre.

Form for current density and toroidal field

Although the experimentalist drives a current purely in the toroidal direction, the plasma generates its own currents in other directions (eg the diamagnetic current)

Assuming quasi-neutrality: $\nabla \cdot \underline{J} = 0$ ($\underline{J}$ = current density)

[recall from e.m. $\nabla \cdot \underline{J} = \partial \rho / \partial t$, where ρ = charge density]

⇒ we follow the same procedure that we adopted to determine ***B***, and introduce a new scalar field *f(R,Z)*

(1) $$\boxed{J_R = -\frac{1}{\mu_0 R}\frac{\partial f}{\partial Z} \qquad J_Z = \frac{1}{\mu_0 R}\frac{\partial f}{\partial R}}$$

[change of sign & introduction of μ_0 compared to ψ is a convenient normalisation]

$$\Rightarrow \nabla \cdot \underline{J} = \frac{1}{R}\frac{\partial}{\partial R}(R J_R) + \frac{\partial J_Z}{\partial Z} = 0$$

The function *f* is related to the toroidal field:

⇒ Ampère's law: $\nabla \times \underline{B} = \mu_0 \underline{J}$

In cylindrical coordinates,

$$(\nabla \times \boldsymbol{B}) \cdot \boldsymbol{e}_R = \left(\cancel{\frac{1}{R}\frac{\partial B_Z}{\partial \phi}} - \frac{\partial B_\phi}{\partial Z} \right) \qquad (\nabla \times \boldsymbol{B}) \cdot \boldsymbol{e}_Z = \left(\frac{1}{R}\frac{\partial (R B_\phi)}{\partial R} - \cancel{\frac{1}{R}\frac{\partial B_R}{\partial \phi}} \right)$$

Using toroidal symmetry, we have $[\partial / \partial \phi = 0]$

(2) $$J_R = -\frac{1}{\mu_0}\frac{\partial B_\phi}{\partial Z} \qquad J_Z = \frac{1}{\mu_0 R}\frac{\partial}{\partial R}(R B_\phi)$$

Comparing (1) and (2) we see $f(R,Z) = R B_\phi$

OR

$$\boxed{B_\phi = \frac{f(R,Z)}{R}}$$

B_ϕ only proportional to R^{-1} if $f(R,Z)$ = constant

⇒ poloidal currents represented by $f(R,Z)$ modify the toroidal field.

The toroidal field function, f, is also a flux surface quantity

$$J_R = -\frac{1}{\mu_0 R}\frac{\partial f}{\partial Z} \qquad J_Z = \frac{1}{\mu_0 R}\frac{\partial f}{\partial R} \qquad B_\phi = \frac{f(R,Z)}{R}$$

Note: Outside the plasma

$\Rightarrow J_R=J_Z=0$ (no plasma, no current, of course)

$\Rightarrow f=$ constant, and $B_\phi \propto 1/R$

In the plasma, f is not constant in general:

$\Rightarrow$ poloidal currents in the plasma modify B_ϕ, but the form of f is constrained

To see this, consider the force balance equation:

$$\underset{\sim}{J} \times \underline{B} = \underline{\nabla} p$$

$$\Rightarrow \quad \underset{\sim}{J} \cdot \underline{\nabla} p = 0 \qquad [\text{as } \underline{J}\cdot(\underline{J}\times\underline{B}) \equiv 0]$$

But $p = p(\psi)$, so $\underline{\nabla} p = \frac{dp}{d\psi}\underline{\nabla}\psi \Rightarrow \boxed{\underline{J}\cdot\underline{\nabla}\psi = 0}$

$\Rightarrow$ $\underline{J}$ is perpendicular to $\underline{\nabla}\psi$, so $\underset{\sim}{J}$ flows in the flux surfaces.

Using above relations
$$\underline{J}\cdot\underline{\nabla}\psi = J_R\frac{\partial\psi}{\partial R} + J_Z\frac{\partial\psi}{\partial Z}$$
$$= \frac{1}{\mu_0 R}\left[-\frac{\partial f}{\partial Z}\frac{\partial\psi}{\partial R} + \frac{\partial f}{\partial R}\frac{\partial\psi}{\partial Z}\right]$$

If we take $f(R,Z) = f(\psi) \Rightarrow \frac{\partial f}{\partial Z} = \frac{df}{d\psi}\frac{\partial\psi}{\partial Z}$ and $\frac{\partial f}{\partial R} = \frac{df}{d\psi}\frac{\partial\psi}{\partial R}$

$$\Rightarrow \underline{J}\cdot\underline{\nabla}\psi = \frac{1}{\mu_0 R}\frac{df}{d\psi}\left[-\frac{\partial\psi}{\partial Z}\frac{\partial\psi}{\partial R} + \frac{\partial\psi}{\partial R}\frac{\partial\psi}{\partial Z}\right] = 0$$ As required.

$$\Rightarrow \boxed{f(R,Z) = f(\psi)} \qquad \Rightarrow \boxed{RB_\phi = f(\psi)}$$

$\Rightarrow$ The quantity RB_ϕ is constant on a flux surface.

Vector forms for $\boldsymbol{B}$ and $\boldsymbol{J}$

In cylindrical coordinates, we have:

$$\boldsymbol{B} = B_\phi \boldsymbol{e}_\phi + B_R \boldsymbol{e}_R + B_Z \boldsymbol{e}_Z$$

$$B_R = \frac{1}{R}\frac{\partial \psi}{\partial Z}$$

$$B_Z = -\frac{1}{R}\frac{\partial \psi}{\partial R}$$

where $\boldsymbol{e}_\phi$, $\boldsymbol{e}_R$, $\boldsymbol{e}_Z$ are orthogonal unit vectors in the R, ϕ, Z directions [$\boldsymbol{e}_R \times \boldsymbol{e}_\phi = \boldsymbol{e}_R$]

$$\nabla\phi = \frac{1}{R}\boldsymbol{e}_\phi \qquad \nabla R = \boldsymbol{e}_R \qquad \nabla Z = \boldsymbol{e}_Z$$

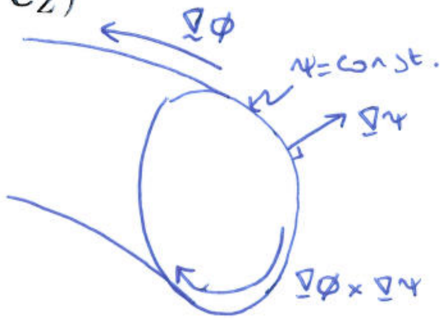
$\Rightarrow$ Eq (1) $\quad \boldsymbol{B} = RB_\phi \nabla\phi + \frac{1}{R}\frac{\partial \psi}{\partial Z}\nabla R - \frac{1}{R}\frac{\partial \psi}{\partial R}\nabla Z$

Now

$$\nabla\phi \times \nabla\psi = \nabla\phi \times \left[\frac{\partial \psi}{\partial R}\nabla R + \underbrace{\cancel{\frac{\partial \psi}{\partial \phi}}}_{=0 \text{ by symmetry}}\nabla\phi + \frac{\partial \psi}{\partial Z}\nabla Z\right]$$

$$= \frac{1}{R}\frac{\partial \psi}{\partial R}(\boldsymbol{e}_\phi \times \boldsymbol{e}_R) + \frac{1}{R}\frac{\partial \psi}{\partial Z}(\boldsymbol{e}_\phi \times \boldsymbol{e}_Z)$$

$$= -\frac{1}{R}\frac{\partial \psi}{\partial R}\boldsymbol{e}_Z + \frac{1}{R}\frac{\partial \psi}{\partial Z}\boldsymbol{e}_R$$



$$\boxed{\boldsymbol{B} = \underbrace{f(\psi)\nabla\phi}_{\text{Toroidal field } f(\psi)=RB_\phi} + \underbrace{\nabla\phi \times \nabla\psi}_{\text{poloidal field}}}$$

Similarly, for the current density

$$\boldsymbol{J} = RJ_\phi \nabla\phi - \frac{1}{\mu_0 R}\frac{\partial f}{\partial Z}\boldsymbol{e}_R + \frac{1}{\mu_0 R}\frac{\partial f}{\partial R}\boldsymbol{e}_Z$$

$$J_R = -\frac{1}{\mu_0 R}\frac{\partial f}{\partial Z}$$

$$J_Z = \frac{1}{\mu_0 R}\frac{\partial f}{\partial R}$$

Comparison with Eq (1) for $\boldsymbol{B}$:

$$B_\phi \to J_\phi \qquad \psi \to -\frac{1}{\mu_0} f$$

$$\boldsymbol{J} = RJ_\phi \nabla\phi - \frac{1}{\mu_0}\nabla\phi \times \nabla f \qquad \nabla f = \frac{df}{d\psi}\nabla\psi$$

$$\boxed{\boldsymbol{J} = RJ_\phi \nabla\phi - \frac{1}{\mu_0}\frac{df}{d\psi}\nabla\phi \times \nabla\psi}$$

Note $\boldsymbol{J}\cdot\nabla\psi = 0 \Rightarrow$ no current density flows across flux surfaces – has toroidal and poloidal components.

Grad Shafranov Equation

The Grad-Shafranov equation is the force balance equation that determines $\psi(R,Z)$

$$\begin{aligned} \boldsymbol{J}\times\boldsymbol{B} &= \left[RJ_\phi\nabla\phi - \frac{1}{\mu_0}\frac{df}{d\psi}\nabla\phi\times\nabla\psi\right]\times\left[f\nabla\phi+\nabla\phi\times\nabla\psi\right] \\ &= -RJ_\phi\left(\nabla\phi\times\nabla\psi\right)\times\nabla\phi - \frac{1}{\mu_0}f\frac{df}{d\psi}\left(\nabla\phi\times\nabla\psi\right)\times\nabla\phi \end{aligned}$$

Vector identity: $(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C} = (\boldsymbol{A}\cdot\boldsymbol{C})\,\boldsymbol{B} - (\boldsymbol{B}\cdot\boldsymbol{C})\,\boldsymbol{A}$

$$\Rightarrow\quad \begin{aligned} \left(\nabla\phi\times\nabla\psi\right)\times\nabla\phi &= |\nabla\phi|^2\nabla\psi - \underbrace{\left(\nabla\phi\cdot\nabla\psi\right)}_{=0\ \text{(orthogonal)}}\nabla\phi \\ &= \frac{1}{R^2}\nabla\psi \end{aligned}$$

$$\Rightarrow\quad \boldsymbol{J}\times\boldsymbol{B} = -\left[\frac{J_\phi}{R}+\frac{1}{\mu_0 R^2}f\frac{df}{d\psi}\right]\nabla\psi = \nabla p = \frac{dp}{d\psi}\nabla\psi$$

$$\boxed{J_\phi = -R\frac{dp}{d\psi} - \frac{f}{\mu_0 R}\frac{df}{d\psi}} \Rightarrow \left\{\begin{array}{l}\text{Provides a link} \\ \text{between current,} \\ \text{field and pressure}\end{array}\right.$$

Consider toroidal component of Ampère's law: $(\nabla\times\boldsymbol{B})_\phi = \mu_0 J_\phi$

$$\begin{aligned} \mu_0 J_\phi &= \frac{\partial B_R}{\partial Z} - \frac{\partial B_Z}{\partial R} \\ &= \frac{1}{R}\frac{\partial^2\psi}{\partial Z^2} + \frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial\psi}{\partial R}\right] \end{aligned}$$

$B_R = \frac{1}{R}\frac{\partial\psi}{\partial Z}$

$B_Z = -\frac{1}{R}\frac{\partial\psi}{\partial R}$

Equating the two forms for J_ϕ provides a 2D partial differential equation for ψ – the Grad Shafranov equation

$$\boxed{R\frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial\psi}{\partial R}\right] + \frac{\partial^2\psi}{\partial Z^2} = -\mu_0 R^2\frac{dp}{d\psi} - f\frac{df}{d\psi}}$$

To solve this we require a boundary condition; the pressure profile, $p(\psi)$ and the toroidal field function $f(\psi)$ (or current density)

Importance of Grad-Shafranov Equation

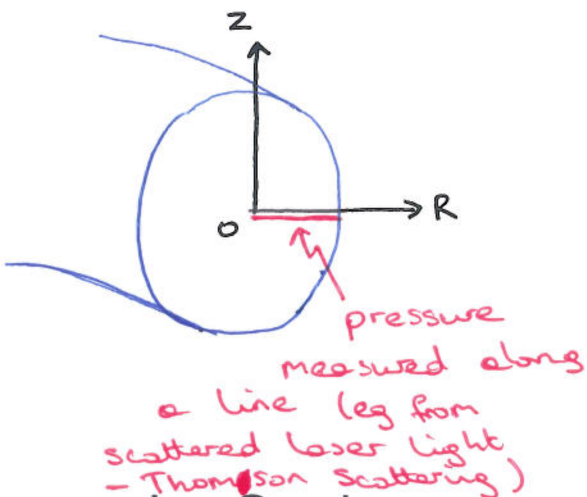
We need to specify two "free" profiles: $p(\psi)$ and $f(\psi)$

$p(\psi) \Rightarrow$ Determined by a balance between heat/particle sources and losses (diffusion)

$f(\psi) \Rightarrow$ Related to the current distribution

Then, if we know $\psi(R,Z)$, $\Rightarrow$ we can evaluate $p(R,Z)$ and $J_\phi(R,Z)$ [pressure and current density]

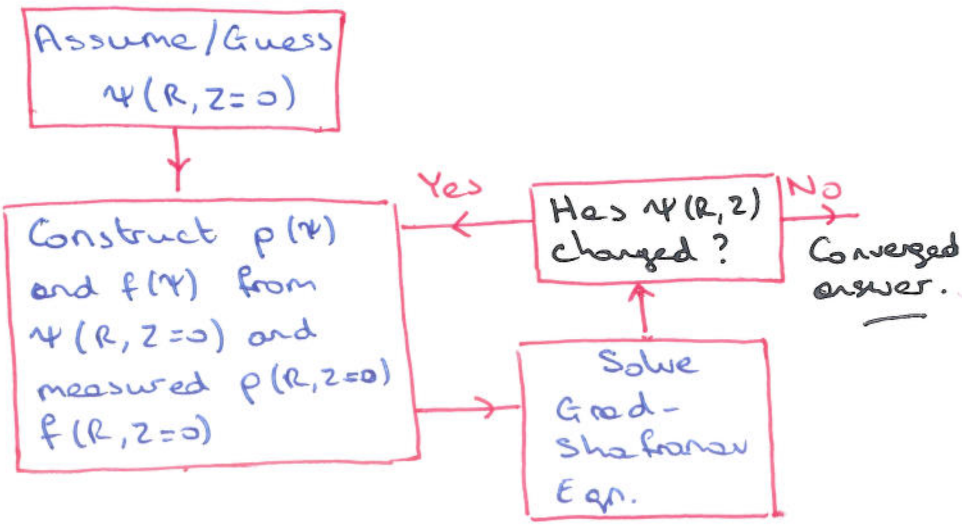
Importance from an experimenal point of view
An experimentalist would typically measure:



1. The plasma boundary
   – A flux surface, which serves as a boundary condition for G-S eqn
2. Pressure along a single chord like that shown $\Rightarrow p(R,Z\!=\!0)$
3. (If lucky!) current density along the chord $\Rightarrow J(R,Z\!=\!0) \Rightarrow f(R,Z\!=\!0)$

If we knew $p(\psi)$ and $f(\psi)$, we could then solve Grad-Shafranov equation to derive pressure and f (and therefore current density) everywhere in the plasma
– but we do know $p(R,Z\!=\!0)$, $f(R,Z\!=\!0)$ from measurement

– A possible procedure

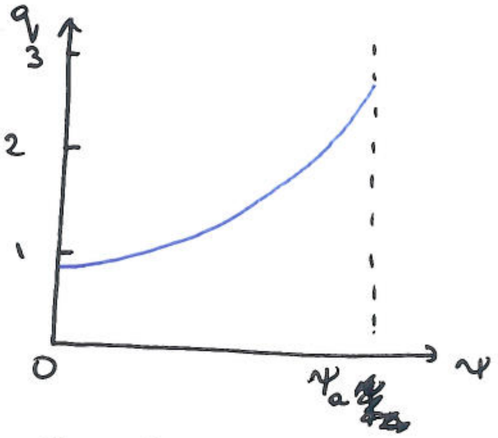


Some tokamak jargon

Safety factor
Denoted by the variable q
• A field line migrates both poloidally and toroidally as it winds around a flux surface

Definition of q: the number of times a magnetic field line traverses around the tokamak toroidally to wrap around once in the poloidal direction

Typical q-profile in a conventional tokamak

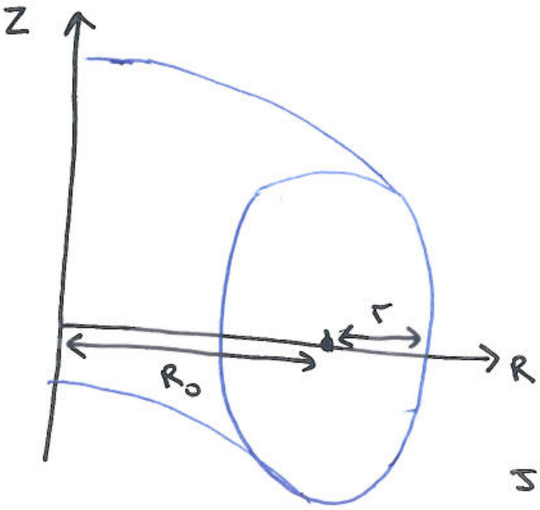


$q(\psi=0)$ is typically ~ 1

$q(\psi=\psi_a)$ is typically ~ 3 upwards.

$q \propto \dfrac{B}{I_p}$ ← magnetic field (B); ← total plasma current (I_p).

Aspect ratio, A
Definition: ratio of the major radius, R_0, to minor radius, r



Aspect ratio $A = \dfrac{R_0}{r}$

Inverse aspect ratio.

$$\varepsilon = \frac{1}{A} = \frac{r}{R_0}$$

JET and ITER, $A \simeq 3$

MAST $A \simeq 1.4$.

Note, we have used ψ as a measure of minor radius

What you need to know:

You need to learn what is meant by a theta pinch and a Z-pinch. You should also be able to derive the relationships between current density, pressure and magnetic field for these devices using Ampere's law and the MHD force balance relationship

For a tokamak equilibrium, you must learn the definition of the poloidal flux, ψ, and how it is related to the two components of poloidal magnetic field, B_R and B_Z. You should be able to demonstrate that the magnetic field lines lie on the surfaces of constant ψ, and that the resulting magnetic field is divergence-free.

You should be able to prove that the pressure is constant on the flux surfaces.

You should be able to define the toroidal field function, f, in terms of the R and Z components of current density in a tokamak and understand how it is related to the toroidal component of magnetic field. You should be able to prove that f is constant on a flux surface.

You do not need to learn the derivations of the magnetic field and current density in a tokamak, nor do you need to learn their forms. Given the equations, however, you would need to be able to describe the meaning of the different terms.

You need to know that the Grad Shafranov equation provides the poloidal flux ψ as a function of R and Z inside and outside the plasma. As such, it describes the tokamak plasma equilibrium. You need to know that to solve this equation requires a boundary condition (i.e. the R and Z coordinates of one flux surface, typically that at the plasma boundary) and the forms of $p(\psi)$ and $f(\psi)$ to be specified. These so-called profiles are equivalent to specifying the plasma pressure and current density distribution. You do not need to remember the form of the Grad-Shafranov equation, nor how to derive it. You do need to be able to describe how it can be used to calculate the pressure and current density everywhere in the plasma, given a measurement of pressure and current density along one chord in the plasma.

You need to learn the definition of safety factor, aspect ratio, minor radius and major radius.