Section 9

Plasma Waves

- In this section we shall demonstrate that plasmas can support waves

- We begin by considering high frequency plasma waves
  - the electron plasma wave, or Langmuir wave
  - the plasma frequency

Waves in plasmas: Langmuir wave

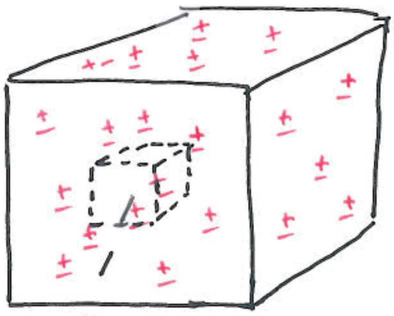
The Langmuir wave is a very high frequency wave

⇒ high frequency means that the heavy ions remain fixed

⇒ the wave arises from the motion of electrons

⇒ it is also called the electron plasma wave

The Physics – consider a box of quasi-neutral plasma
– this is the equilibrium state



Now take a group of electrons near the centre of the box and displace them to the right

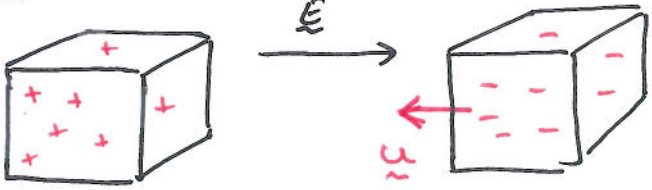
(III) charge separation creates E-field

This is called the perturbed state

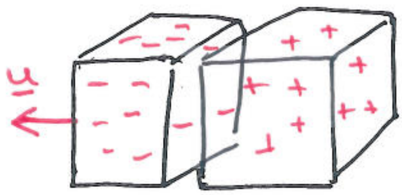
(II) Leaves net positive charge behind

Now release the system

⇒ E field drives the electrons back towards the (fixed) ions



Electrons accelerate towards ions under influence of $\underset{\sim}{E}$- field



Electrons over-shoot the equilibrium state because of their inertia

⇒ $\underset{\sim}{E}$ - field reverses, slowing the electrons and pulling them back towards the ions

⇒ An oscillation at a characteristic frequency – the plasma frequency

Plasma frequency: the maths
(1) Basic equations

Simplifying assumptions

1. No B-field (plays no role in the basic mechanism)
2. No thermal motion (T=0)
3. Uniform ion distribution, fixed in space
4. Plasma has infinite extent
5. Neglect steady state plasma flow (of course we must retain electron oscillatory flow)
6. Restrict our analysis to 1-D
7. Assume ions have charge e [Z=1]

Essential equations:
As we are perturbing a density of electrons, we need an equation for density evolution
$\Rightarrow$ electron continuity

$$(1) \qquad \frac{\partial n_e}{\partial t} + \underset{\sim}{\nabla} \cdot (n_e \underset{\sim}{u}_e) = 0$$

We also need to know about the forces due to an electrostatic field ($\boldsymbol{B}$=0)
$\Rightarrow$ force balance for electrons (retaining inertia)

$$(2) \qquad m_e n_e \left[\frac{\partial \underset{\sim}{u}_e}{\partial t} + (\underset{\sim}{u}_e \cdot \underset{\sim}{\nabla}) \underset{\sim}{u}_e \right] = -n_e e \underset{\sim}{E}$$

– No Lorentz force, as $\underset{\sim}{B} = \underset{\sim}{0}$; No $\underset{\sim}{\nabla} p$ term as $T=0$

We need the self-consistent electric field due to the charge imbalance (Poisson's equation) [the plasma is not locally quasi-neutral]

$$(3) \qquad \varepsilon_0 \underset{\sim}{\nabla} \cdot \underset{\sim}{E} = \rho = (n_i - n_e) e$$

[ρ = charge density]

Plasma frequency: the maths
(2) Linear approximation

We must solve the three equations (1-3) for n_e, $\boldsymbol{u}_e$ and $\boldsymbol{E}$
Suppose that before we perturb our "box" of electrons:

$n_e = n_i = n^{(0)}$ (uniform in space)

$\underset{\sim}{u}_e = \underset{\sim}{u}^{(0)} = \underset{\sim}{0}$ (zero flow assumption)

$\underset{\sim}{E} = \underset{\sim}{E}^{(0)} = \underset{\sim}{0}$ (zero electric field)

This is the equilibrium state, which must satisfy Eqns (1)–(3) $[\partial/\partial t = 0]$ [(1) & (2) trivial; (3) because $n_e = n_i = n^{(0)}$]

This is called the "equilibrium" state

⇒ if we do not perturb it, it will just sit there.

Now let us make small perturbations

$n_e = n^{(0)} + n^{(1)}(\underset{\sim}{r}, t)$

$\underset{\sim}{u}_e = \underset{\sim}{u}^{(1)}(\underset{\sim}{r}, t)$

$\underset{\sim}{E} = \underset{\sim}{E}^{(1)}(\underset{\sim}{r}, t)$

[Note that the ions are unperturbed so $n_i = n^{(0)}$ – i.e. $n_i^{(1)} = 0$]

Also $\underset{\sim}{u}_i = \underset{\sim}{0}$]

Where $n^{(1)}$, $\boldsymbol{u}^{(1)}$ and $\boldsymbol{E}^{(1)}$ are all small

⇒ neglect terms quadratic (or higher polynomials) in the perturbations (square a small number ⇒ even smaller number)

⇒ only terms linear in the perturbations remain

⇒ $\boxed{\text{LINEAR THEORY}}$ – Very important in plasma physics.

Linearised equations for the perturbed system

(1) **Continuity**

– First check the equilibrium state satisfies continuity

$$\underbrace{\frac{\partial n_e^{(0)}}{\partial t}}_{=0 \text{ for equilibrium}} + \underbrace{\nabla \cdot (n_e^{(0)} \underline{u}_e^{(0)})}_{=0 \text{ as flow } \underline{u}_e^{(0)}=0} = 0 \quad \Rightarrow \text{satisfied.}$$

Now put in full perturbed density:

$$\frac{\partial}{\partial t}\left(\cancel{n_e^{(0)}} + n_e^{(1)}\right) + \nabla \cdot \left[(n_e^{(0)} + n_e^{(1)})(\underline{u}_e^{(0)} + \underline{u}_e^{(1)}) \right] = 0$$

$$\left[\frac{\partial n_e^{(0)}}{\partial t} = 0 \text{ as no time dependence} \right]$$

$$\frac{\partial n_e^{(1)}}{\partial t} + \nabla \cdot \left[\underbrace{\cancel{n_e^{(0)} \underline{u}_e^{(0)}}}_{=0 \text{ as } \underline{u}_e^{(0)}=0} + n_e^{(0)} \underline{u}_e^{(1)} + \underbrace{n_e^{(1)} \underline{u}_e^{(0)}}_{=0 \text{ as } \underline{u}_e^{(0)}=0} + \underbrace{n_e^{(1)} \underline{u}_e^{(1)}}_{\text{Neglect as quadratic in perturbation} \Rightarrow \text{small.}} \right] = 0$$

But we assumed

1. $u_e^{(0)}$=0 (no steady plasma flow, assumption (5))
2. $n_e^{(0)}$ is uniform in space; assumption (3)

$$\Rightarrow \boxed{\frac{\partial n_e^{(1)}}{\partial t} + n_e^{(0)} \nabla \cdot \underline{u}_e^{(1)} = 0}$$

Linearised equations for the perturbed system

(2) **Force balance** for electrons – first consider equilibrium:

$$m_e\left[\frac{\partial \underline{u}_e^{(0)}}{\partial t} + (\underline{u}_e^{(0)}\cdot\underline{\nabla})\,\underline{u}_e^{(0)}\right] = -e\,\underline{E}^{(0)}$$

As $\underline{u}_e^{(0)} = 0$; $\underline{E}^{(0)} = 0$, this is trivially satisfied.

Same procedure as for continuity ⇒

$$m_e\left[\frac{\partial \underline{u}_e^{(1)}}{\partial t} + \left[(\underbrace{\cancel{\underline{u}_e^{(0)}}}_{=0} + \underline{u}_e^{(1)})\cdot\underline{\nabla}\right](\underbrace{\cancel{\underline{u}_e^{(0)}}}_{=0} + \underline{u}_e^{(1)})\right] = -e(\underbrace{\cancel{\underline{E}^{(0)}}}_{=0} + \underline{E}^{(1)})$$

$$m_e\left[\frac{\partial \underline{u}_e^{(1)}}{\partial t} + \underbrace{(\underline{u}_e^{(1)}\cdot\underline{\nabla})\,\underline{u}_e^{(1)}}_{\text{neglect as quadratic in perturbation}}\right] = -e\,\underline{E}^{(1)}$$

$$\boxed{m_e\frac{\partial \underline{u}_e^{(1)}}{\partial t} = -e\,\underline{E}^{(1)}}$$

(3) **Poisson's equation**

$\underline{E}^{(0)} = 0$ $\quad$ $n_i^{(0)} = n_e^{(0)}$

Equilibrium $\quad \varepsilon_0\,\underline{\nabla}\cdot\underline{E}^{(0)} = (n_i^{(0)} - n_e^{(0)})e = 0$ ✓

Note: charge separation is the key physics for this wave (ions cannot move fast enough to maintain quasi-neutrality)

⇒ perturbations do not satisfy quasi-neutrality

As ions are fixed, $n_i^{(1)}=0$ (ion density is not perturbed)

$$\varepsilon_0\,\underline{\nabla}\cdot(\cancel{\underline{E}^{(0)}} + \underline{E}^{(1)}) = (n_i^{(0)} + \underbrace{\cancel{n_i^{(1)}}}_{=0} - n_e^{(0)} - n_e^{(1)})e$$

$\underline{E}^{(0)} = 0$

$= 0$ as ions are not perturbed

$n_i^{(0)}$ and $n_e^{(0)}$ cancel

$$\Rightarrow\ \boxed{\varepsilon_0\,\underline{\nabla}\cdot\underline{E}^{(1)} = -e\,n_e^{(1)}}$$

Solution of linearised system

(1) **Continuity** $\frac{\partial n_e^{(1)}}{\partial t} + n_e^{(0)} \underline{\nabla} \cdot \underline{u}_e^{(1)} = 0$

(2) **Force balance** $m_e \frac{\partial \underline{u}_e^{(1)}}{\partial t} = -e\,\underline{E}^{(1)}$

(3) **Poisson** $\varepsilon_0 \underline{\nabla} \cdot \underline{E}^{(1)} = -e\, n_e^{(1)}$

Note: as coefficients of the perturbed quantities are independent of time, perturbed quantities can be written in the form: $p^{(1)}(\underline{r}, t) = \hat{p}^{(1)}(\underline{r}) e^{-i\omega t}$

[$p^{(1)}$ is any of the perturbed fields]

$$\Rightarrow \frac{\partial p^{(1)}}{\partial t} = -i\omega \hat{p}^{(1)} e^{-i\omega t} = -i\omega p^{(1)}$$

Similarly, as coefficients are also independent of position, we can write $p^{(1)}(\underline{r}, t) = \bar{p}^{(1)} e^{-i\omega t} e^{i\underline{k}\cdot\underline{r}}$

[$\bar{p}^{(1)}$ is independent of position & time]

$$\Rightarrow \underline{\nabla} p^{(1)} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) p^{(1)} = (ik_x, ik_y, ik_z) p^{(1)}$$

$$\Rightarrow \boxed{\underline{\nabla} p^{(1)} = i\underline{k}\, p^{(1)}}$$

[$\underline{k}\cdot\underline{r} = k_x x + k_y y + k_z z$]

For 1-D perturbations (say x-direction), this simplifies:

$$p^{(1)}(x, t) = \bar{p}^{(1)} e^{-i\omega t} e^{ikx}$$

$k = |\underline{k}| = k_x$

$\underline{k} = (k_x, 0, 0)$

Here $p^{(1)}$ represents $n_e^{(1)}$, or any of the components of $\boldsymbol{E}^{(1)}$ or $\boldsymbol{u}^{(1)}$

Continuity: $-i\omega n_e^{(1)} + n_e^{(0)} (i\underline{k} \cdot \underline{u}_e^{(1)}) = 0$

For 1-D motion in x-direction $\underline{u}_e^{(1)} = (u_x^{(1)}, 0, 0)$

$\underline{k} = (k, 0, 0)$ $\quad k = k_x$

$$\Rightarrow -i\omega n_e^{(1)} = -i n_e^{(0)} k u_x^{(1)}$$

$$\Rightarrow \boxed{n_e^{(1)} = \frac{n_e^{(0)} k}{\omega} u_x^{(1)}}$$

Solution of linearised system (2)

Force balance $m_e \frac{\partial \boldsymbol{u}_e^{(1)}}{\partial t} = -e\boldsymbol{E}^{(1)}$

Take x-component to derive $u_x^{(1)}$

$$\frac{\partial u_x^{(1)}}{\partial t} = -i\omega u_x^{(1)}$$

$$-i\omega m_e u_x^{(1)} = -e E_x^{(1)}$$

$$\Rightarrow \boxed{u_x^{(1)} = -\frac{ie}{m_e\omega} E_x^{(1)}}$$

Combine with continuity $\Rightarrow n_e^{(1)} = \frac{n_e^{(0)} k}{\omega} u_x^{(1)}$

$$* \quad \Rightarrow n_e^{(1)} = -i \frac{n_e^{(0)} ek}{m_e\omega^2} E_x^{(1)}$$

To eliminate $E_x^{(1)}$, use Poisson:

$$\underline{\nabla} . \underline{E}^{(1)} = -\frac{n_e^{(1)} e}{\varepsilon_0}$$

$$\underline{\nabla} . \underline{E}^{(1)} = \frac{\partial E_x^{(1)}}{\partial x} + \frac{\partial E_y^{(1)}}{\partial y} + \frac{\partial E_z^{(1)}}{\partial z}$$

but only E_x term survives as no y or z variation.

$$\Rightarrow \underline{\nabla} . \underline{E}^{(1)} = \frac{\partial E_x^{(1)}}{\partial x} = ik E_x^{(1)}$$

$$ik E_x^{(1)} = -\frac{e}{\varepsilon_0} n_e^{(1)}$$

$$\Rightarrow E_x^{(1)} = \frac{ie}{k\varepsilon_0} n_e^{(1)}$$

Substitute in * above for $E_x^{(1)}$

$$n_e^{(1)} = \left[-i\frac{n_e^{(0)} ek}{m_e\omega^2}\right]\left[\frac{ie}{k\varepsilon_0} n_e^{(1)}\right]$$

$$\Rightarrow \left[1 - \frac{n_e^{(0)} e^2}{\varepsilon_0 m_e \omega^2}\right] n_e^{(1)} = 0 \quad \Rightarrow \text{either } n_e^{(1)} = 0$$ [the equilibrium solution]

OR $\boxed{\omega^2 = \frac{n_e e^2}{\varepsilon_0 m_e}}$

[note, as $n_e^{(1)} \ll n_e^{(0)}$, we can write $n_e \simeq n_e^{(0)} + n_e^{(1)} \simeq n_e^{(0)}$]

$\Rightarrow$ A wave, the electron plasma wave, can exist with a frequency $\boxed{\omega = \left(\frac{n_e e^2}{\varepsilon_0 m_e}\right)^{1/2}}$

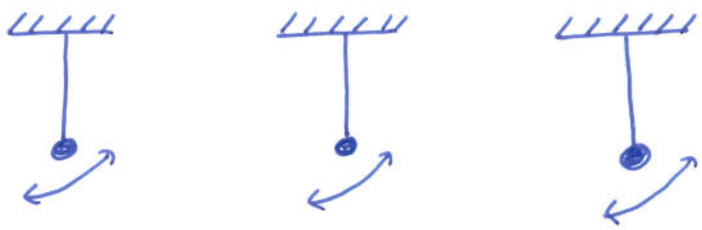
Characteristics of electron plasma wave

The wave oscillates with frequency

$\omega = \omega_{pe}$ where $\omega_{pe} = \left(\frac{n_e e^2}{\varepsilon_0 m_e}\right)^{1/2}$ is called the plasma frequency. One can also define $\omega_{pi} = \left(\frac{n_i e^2}{\varepsilon_0 m_i}\right)^{1/2}$

⇒ independent of wavelength (ie wave-number, k)
⇒ Group velocity =0 (recall group velocity $V_g = \partial\omega/\partial k$
⇒ no information propagates via the wave

An analogy
Consider a number of identical pendulums:



Each oscillates with the same characteristic frequency, but they are independent of each other
⇒ information is not communicated from one to the next
⇒ only when the pendulums are coupled (eg via a spring) does information get transferred

Thermal effects couple the electron plasma wave oscillations

At finite temperature (ie relax assumption (2)), electrons move, on average, with a thermal velocity

$\Rightarrow$ they can move from one oscillating cell to another

$\Rightarrow$ this then couples the oscillations in two adjacent cells, and the wave propagates

$\Rightarrow$ This is the electron plasma wave, or Langmuir wave

The maths: [1-D motion]   perturbations $\sim e^{-i\omega t} e^{ikx}$

(1) $\frac{\partial n_e}{\partial t} + \nabla . (n_e \underline{u}_e) = 0 \quad \Rightarrow \quad +\cancel{i}\omega n_e^{(1)} = +\cancel{i} k n_e^{(0)} u_x^{(1)}$

2) $n m_e \left(\frac{\partial \underline{u}_e}{\partial t} + (\underline{u}_e . \nabla) \underline{u}_e \right) = -n_e \underline{E} - \nabla p_e \Rightarrow \underbrace{+ i m_e n_e^{(0)} \omega u_x^{(1)} = + n_e^{(0)} e E_x + ik p_e^{(1)}}_{x\text{-component force balance.}}$

(3) $\nabla . \underline{E} = \frac{1}{\varepsilon_0} (n_i - n_e) e \quad \Rightarrow \quad ik E_x^{(1)} = - \frac{n_e^{(1)} e}{\varepsilon_0}$

We need another expression for the pressure perturbation

$\Rightarrow \quad p_e = C n_e^{\gamma}$ Equilibrium $\Rightarrow \quad p_e^{(0)} = C n_e^{(0)\gamma} \Rightarrow \boxed{C = \frac{p_e^{(0)}}{n_e^{(0)\gamma}}}$

$$(p_e^{(0)} + p_e^{(1)}) = \frac{p_e^{(0)}}{n_e^{(0)\gamma}} (n_e^{(0)} + n_e^{(1)})^{\gamma} = p_e^{(0)} \left(1 + \frac{n_e^{(1)}}{n_e^{(0)}}\right)^{\gamma} \approx p_e^{(0)} \left(1 + \frac{\gamma n_e^{(1)}}{n_e^{(0)}}\right)$$

$\Sigma << 1 \Rightarrow$ expand.

(4) $\Rightarrow \boxed{p_e^{(1)} = \frac{\gamma p_e^{(0)}}{n_e^{(0)}} n_e^{(1)}}$

Sub for $u_x^{(1)}$ in Eq (1) using Eq (2) $\Rightarrow \quad \omega^2 n_e^{(1)} = \frac{k \cancel{n_e}}{i m_e \cancel{n_e^{(0)}} \cancel{\omega}} \left[n_e^{(0)} e E_x + ik p_e^{(1)} \right]$

Eliminate $E_x^{(1)}$ using (3) and $p_e^{(1)}$ using (4)

$$\Rightarrow \left[\omega^2 + \frac{i\cancel{k}}{m_e} \left(n_e^{(0)} e \left(\frac{ie}{\varepsilon_0 \cancel{k}} \right) + ik^2 \gamma \underbrace{\frac{p_e^{(0)}}{n_e^{(0)}}}_{k_B T_e} \right) \right] n_e^{(1)} = 0$$

Recall $p_e = n_e k_B T_e$

$$\Rightarrow \left[\omega^2 - \frac{n_e e^2}{\varepsilon_0 m_e} - \gamma k^2 \underbrace{\frac{k_B T_e}{m_e}}_{V_{the}^2/2} \right] n_e^{(1)} = 0$$

$V_{the}^2 = \frac{2 k_B T_e}{m_e}$

$$\Rightarrow \boxed{\omega^2 = \omega_{pe}^2 + \frac{\gamma k^2}{2} V_{the}^2}$$

Note group velocity $\boxed{V_g = \frac{\partial \omega}{\partial k} = \frac{\gamma V_{the}^2}{2 V_p}}$

Phase velocity $\boxed{V_p = \omega / k.}$

What you should know

You should be able to describe the physical mechanism that underlies the Langmuir or electron plasma wave. You should also understand the key parameters that influence its frequency (density, charge and electron mass). You need to learn the expression for the electron plasma frequency.

You should understand how to linearise the 2-D fluid equations. You should understand that the equilibrium must be a time-independent solution to these equations, and then be able to derive the linearised equations describing the time-dependent perturbation to this equilibrium.

You should understand how to express the solution in terms of plane waves, ie $\sim e^{i\omega t}e^{i\mathbf{k}.\mathbf{r}}$, and how to act on this with the differential operators to produce a set of algebraic equations for the perturbed quantities

You should understand how to manipulate these equations to derive the expression for the frequency of the electron plasma wave.

You should be able to explain physically why no information is transferred in the limit of zero temperature, but how a finite temperature does lead to transfer of information (ie non-zero group velocity)

You should be able to linearise the adiabatic pressure law, and combine this with the other equations to derive the frequency of the Langmuir wave for finite temperature to demonstrate that it does have finite group velocity in this limit.