2014 Notes

Section 6

We will cover:

- The diamagnetic drift and current

- The ideal magneto-hydrodynamic model of the plasma

- The "Frozen in" condition

Drifts in a plasma fluid

We can derive two of the important drifts in a plasma from the force balance equation

This introduces a new drift, the diamagnetic drift, which cannot be derived from the particle picture we employed earlier

$$m_j n_j \left[\frac{\partial \underset{\sim}{u}_j}{\partial t} + (\underset{\sim}{u}_j \cdot \underline{\nabla}) \underset{\sim}{u}_j \right] = -\underset{\sim}{\nabla} p_j + n_j q_j (\underset{\sim}{E} + \underset{\sim}{u}_j \times \underline{B})$$

Neglect inertia (acceleration term)

$q_j = Ze$
$j = i$ (ion) or e (electron)

$\Rightarrow$ OK for small mass

$$\Rightarrow \quad \underset{\sim}{u}_j \times \underline{B} = -\underset{\sim}{E} + \frac{1}{n_j q_j} \underset{\sim}{\nabla} p_j$$

Cross both sides with ***B***:

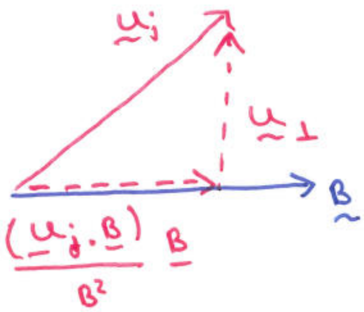
$$(\underset{\sim}{u}_j \times \underline{B}) \times \underline{B} = -\underset{\sim}{E} \times \underline{B} + \frac{1}{n_j q_j} \underline{\nabla} p_j \times \underline{B}$$

Vector identity: $(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{B} \cdot \boldsymbol{C})\boldsymbol{A}$

$$\Rightarrow (\underset{\sim}{u}_j \times \underline{B}) \times \underline{B} = -\left[B^2 \underset{\sim}{u}_j - (\underline{u}_j \cdot \underline{B}) \underline{B} \right] = -B^2 \left[\underline{u}_j - \frac{(\underline{u}_j \cdot \underline{B}) \underline{B}}{B^2} \right]$$

full vector

component parallel to $\underline{B}$



$$\Rightarrow (\underline{u}_j \times \underline{B}) \times \underline{B} = -B^2 \underset{\sim}{u}_{\perp j}$$

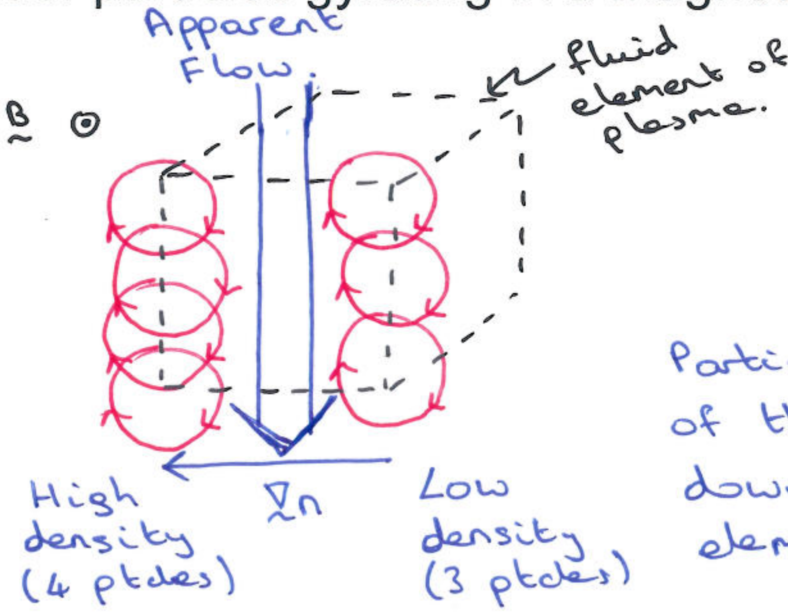
$$\Rightarrow \quad \boxed{\underline{u}_{\perp j} = \frac{(\underset{\sim}{E} \times \underline{B})}{B^2} + \frac{\underline{B} \times \underline{\nabla} p_j}{n_j q_j B^2}}$$

All particles of fluid experience $\underline{E} \times \underline{B}$ drift which gives an $\underline{E} \times \underline{B}$ contribution to the flow

Diamagnetic drift $\Rightarrow$ a fluid flow not experienced by individual particles (a particle cannot have a pressure)

Picture of diamagnetic drift

Consider particles gyrating in a magnetic field (into page)



Particles to the right of the element go up through the element.

Particles to the left of the element go down through the element.

If there is a density gradient, more particles will go down through the element than go up through it

$\Rightarrow$ an apparent flux

$\Rightarrow$ even though individual particles do not drift (which is why we did not predict it with the single particle picture)

NOTE:

Diamagnetic drift:

1) Is perpendicular to $\underset{\sim}{B}$

2) Is perpendicular to $\underset{\sim}{\nabla} n$ [Note - if a temperature gradient, ∇T, ptcles in high T region go faster - enhances flux.

3) Is in opposite direction for ions and electrons.

Diamagnetic current = $\underset{\sim}{J}_D = n_i q_i \underset{\sim}{v}_{Di} + n_e q_e \underset{\sim}{v}_{De}$

$$= \frac{\underset{\sim}{B} \times \underset{\sim}{\nabla} p_i}{B^2} + \frac{\underset{\sim}{B} \times \underset{\sim}{\nabla} p_e}{B^2}$$

$$\Rightarrow \boxed{\underset{\sim}{J}_D = \frac{\underset{\sim}{B} \times \underset{\sim}{\nabla} p}{B^2}}$$

$p = p_e + p_i$ = total pressure.

Ideal Magneto-hydrodynamics (MHD)

Ideal MHD is one of the most important theoretical models for magnetic confinement fusion

It is also important for experimentalists, providing a simple model for interpreting experimental data

The model

- Treats plasma as a single fluid (combines electrons and ions)
- Neglects all dissipation (ie collisional effects such as electrical resistivity, viscosity, etc)

Uses

To evaluate the plasma "equilibrium"

- ie for a given applied magnetic field and current how will the plasma pressure and current density distribute itself?

To evaluate the stability of that equilibrium to small perturbations

For simplicity, we shall assume hydrogenic ions, i.e. $Z = 1$

Quasi-neutrality then implies

$$n_e = n_i = n$$

Ideal MHD: continuity equation

Electron continuity:

$$\frac{\partial n_e}{\partial t} + \underline{\nabla}.(n_e \underline{u}_e) = 0$$

Ion continuity:

$$\frac{\partial n_i}{\partial t} + \underline{\nabla}.(n_i \underline{u}_i) = 0$$

Define a single fluid velocity $\underline{u}$ in terms of momentum, $\underline{u}$

$$\rho \underline{u} = n_i m_i \underline{u}_i + n_e m_e \underline{u}_e$$

where ρ is the mass density of the plasma

$$\rho = n_i m_i + n_e m_e$$

The two continuity equations can be combined to give a single fluid continuity equation

Take $m_e \times$ electron continuity and add $m_i \times$ ion continuity:

$$\frac{\partial}{\partial t}\underbrace{\left[m_e n_e + m_i n_i\right]}_{\rho} + \underline{\nabla}.\underbrace{\left[m_e n_e \underline{u}_e + m_i n_i \underline{u}_i\right]}_{\rho \underline{u}} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla}.(\rho \underline{u}) = 0}$$

Approximate expression for the flow and mass density: $m_e \ll m_i$

$$\rho = n_i m_i + n_e m_e$$
$$\simeq n_i m_i \quad (\text{as } m_e \ll m_i)$$

$n_i = n_e \Rightarrow$

$$\boxed{\rho \simeq n m_i}$$

⇒ mass is mainly in ions

$$\rho \underline{u} = n_i m_i \underline{u}_i + \underbrace{n_e m_e \underline{u}_e}_{\text{neglect}}$$

$$n m_i \underline{u} \simeq n_i m_i \underline{u}_i$$

$$\Rightarrow \boxed{\underline{u} \simeq \underline{u}_i}$$

⇒ momentum is mainly in ions.

Ideal MHD: force balance

Ion force balance:

$$m_i n_i \left[\frac{\partial \underline{u}_i}{\partial t} + (\underline{u}_i . \underline{\nabla}) \underline{u}_i \right] = -\underset{\sim}{\nabla} p_i + n_i q_i (\underset{\sim}{E} + \underset{\sim}{u}_i \times \underline{B})$$

Electron force balance:

$$m_e n_e \left[\frac{\partial \underline{u}_e}{\partial t} + (\underline{u}_e . \underline{\nabla}) \underline{u}_e \right] = -\underline{\nabla} p_e + n_e q_e (\underset{\sim}{E} + \underline{u}_e \times \underline{B})$$

Add these equations, neglecting $m_e << m_i$

$\Rightarrow m_i n_i \simeq \rho \qquad \underset{\sim}{u}_i \simeq \underset{\sim}{u}$; neglect electron inertia

$$\rho \left[\frac{\partial \underline{u}}{\partial t} + (\underline{u} . \underline{\nabla}) \underline{u} \right] = -\underset{\sim}{\nabla} (p_i + p_e) + \underbrace{(n_i q_i + n_e q_e)}_{= 0 \text{ by quasi-neutrality}} \underset{\sim}{E}$$

$$+ (n_i q_i \underset{\sim}{u}_i + n_e q_e \underset{\sim}{u}_e) \times \underset{\sim}{B}$$

Quasi-neutrality $\Rightarrow$ $n_i q_i + n_e q_e = 0$ (no net charge)

Total plasma pressure $p = p_i + p_e$

Current density $\underset{\sim}{J} = n_i q_i \underset{\sim}{u}_i + n_e q_e \underset{\sim}{u}_e$

$$\Rightarrow \boxed{\rho \left[\frac{\partial \underline{u}}{\partial t} + (\underline{u} . \underline{\nabla}) \underline{u} \right] = -\underset{\sim}{\nabla} p + \underset{\sim}{J} \times \underline{B}}$$

hydrodynamic forces due to pressure gradients.

electromagnetic forces is a plasma effect ~~not~~ - a property of conducting fluids.

Ideal MHD: Adiabatic eqn of state

The system is closed with the adiabatic equation of state:

This is an approximation

For an ideal gas, recall from thermodynamics

$$pV^{\gamma} = \text{constant}$$

p = pressure

V = volume

$\gamma = \frac{c_p}{c_v}$ = ratio of specific heat capacities.

Now density $\rho \propto V^{-1}$

$\Rightarrow$ if we can treat plasma as an ideal gas

$$\Rightarrow p\rho^{-\gamma} = \text{constant} \Rightarrow \frac{d}{dt}(p\rho^{-\gamma}) = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\underline{\nabla}\right)(p\rho^{-\gamma}) = 0$$

$$\Rightarrow \cancel{\rho^{-\gamma}}\left(\frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}\right)p - \gamma p\rho^{\cancel{-\gamma}-1}\left(\frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}\right)\rho = 0$$

$$* \quad \frac{\partial p}{\partial t} + \underline{u}\cdot\underline{\nabla}p = \frac{\gamma p}{\rho}\left(\frac{\partial \rho}{\partial t} + \underline{u}\cdot\underline{\nabla}\rho\right)$$

Continuity $\Rightarrow$ $\frac{\partial \rho}{\partial t} = -\underline{\nabla}\cdot(\rho\underline{u}) = -\rho\,\underline{\nabla}\cdot\underline{u} - (\underline{u}\cdot\underline{\nabla})\rho$

$$\Rightarrow \underline{\frac{\partial \rho}{\partial t} + \underline{u}\cdot\underline{\nabla}\rho = -\rho\,\underline{\nabla}\cdot\underline{u}}$$

Substitute in right hand side of *

$$\boxed{\frac{\partial p}{\partial t} + \underline{u}\cdot\underline{\nabla}p = -\gamma p\,\underline{\nabla}\cdot\underline{u}}$$

Ohm's law

Ohm's Law relates the electric field to the current flowing in the plasma

The simplest model is

plasma resistivity

$$\underset{\sim}{E} + \underset{\sim}{u} \times \underline{B} = \eta \underset{\sim}{J}$$

current density

induced e.m.f. due to conductor (plasma) cutting through magnetic field.

This is "resistive MHD". In ideal MHD we neglect the plasma resistivity, η=0, and so

$$\boxed{\underset{\sim}{E} + \underset{\sim}{u} \times \underline{B} = \underline{0}}$$

[Note - $\underset{\sim}{u}$ is then simply the $\underset{\sim}{E} \times \underline{B}$ drift - solve for $\underset{\sim}{u}$]

Equations of Ideal MHD

Continuity

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underline{u}) = 0$$

Force balance

$$\rho \left[\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla}) \underline{u} \right] = - \underset{\sim}{\nabla} p + \underset{\sim}{J} \times \underline{B}$$

Equation of state

$$\frac{\partial p}{\partial t} + (\underline{u} \cdot \underline{\nabla}) p = - \gamma p (\underline{\nabla} \cdot \underline{u})$$

Ohm's law

$$\underset{\sim}{E} + \underset{\sim}{u} \times \underline{B} = 0 \qquad \left[\Rightarrow \underset{\sim}{u}_\perp = \frac{\underset{\sim}{E} \times \underline{B}}{B^2} \right]$$

Combine these with two of Maxwell's equations:

Ampere's law

$$\underset{\sim}{\nabla} \times \underline{B} = \mu_0 \underline{J}$$

(neglecting displacement current)

Faraday's law

$$\frac{\partial \underline{B}}{\partial t} = - \underline{\nabla} \times \underline{E}$$

⇒ 6 equations for the 6 quantities

ρ	mass density	$\underline{B}$	magnetic field
p	pressure	$\underset{\sim}{J}$	current density
$\underset{\sim}{E}$	electric field	$\underline{u}$	fluid flow.

Equilibrium in magnetic confinement

Equilibrium $\Rightarrow$ stationary state

$$\Rightarrow \frac{\partial}{\partial t} \to 0$$

If we also neglect plasma flows, the equilibrium equation is simply

for fusion, we require high pressure gradient.

$$\underset{\sim}{J} \times \underline{B} = \underline{\nabla} p$$

$\Rightarrow$ electromagnetic forces on a fluid element balance the hydrodynamic forces.

$\Rightarrow$ for a given ∇p and $\boldsymbol{B}$ a current density $\boldsymbol{J}$ must flow

$$(\underset{\sim}{J} \times \underline{B}) \times \underline{B} = \underline{\nabla} p \times \underline{B}$$

[cross equilibrium relation with $\underline{B}$]

$$-B^2 \left[\underset{\sim}{J} - \frac{(\underset{\sim}{J} \cdot \underline{B})}{B^2} \underline{B} \right] = \underline{\nabla} p \times \underline{B}$$

$$\Rightarrow \boxed{\underset{\sim}{J}_\perp = \frac{\underline{B} \times \underline{\nabla} p}{B^2}}$$

– same expression as the diamagnetic current!

Note: this is the diamagnetic current, which flows because of the Larmor orbits of the particles

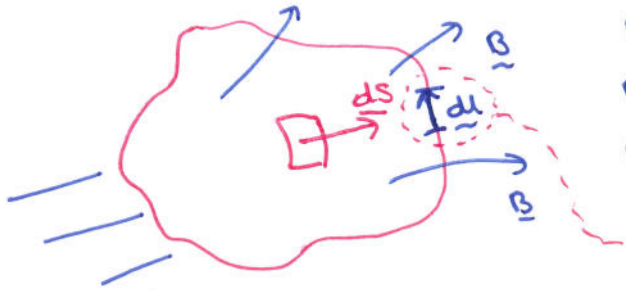
$\Rightarrow$ we do not need to provide this current externally: the plasma automatically produces it for us!

$\Rightarrow$ an example of how, in a plasma, the current density is not just the current driven externally

"Frozen in" magnetic field

In ideal MHD, if plasma fluid elements move the magnetic flux through the surface is conserved: frozen in condition

Consider a surface, S in the plasma



As surface moves, consider motion of line element $\underset{\sim}{dl}$ moving at speed $\underline{u}$ for time dt

area swept $= \underset{\sim}{ds}$

$|\underset{\sim}{ds}| = |\underline{u}\,dt||\underline{dl}|\sin\theta$

$\Rightarrow \boxed{\underset{\sim}{ds} = (\underset{\sim}{u} \times \underset{\sim}{dl})\,dt}$

[recall, $\underset{\sim}{ds}$ points normal to the surface]

The magnetic flux through this surface

$$\Phi = \oint \underline{B} \cdot \underline{ds}$$

Now, suppose this surface moves with the plasma; ie with velocity $\boldsymbol{u}$

Suppose also that $\boldsymbol{B}$ varies in time

rate of change of flux because $\underline{B}$ is evolving

$$\Rightarrow \frac{d\Phi}{dt} = \oint \frac{\partial B}{\partial t} \cdot \underset{\sim}{ds} + \oint \underline{B} \cdot \frac{\underset{\sim}{ds}}{\partial t} = \oint \frac{\partial B}{\partial t} \cdot \underset{\sim}{ds} + \oint \underline{B} \cdot (\underline{u} \times \underset{\sim}{dl})$$

rate of change of flux because surface moves.

$$\oint \underline{B} \cdot (\underset{\sim}{u} \times \underset{\sim}{dl}) = \oint (\underline{B} \times \underline{u}) \cdot \underset{\sim}{dl} \underbrace{=}_{\text{Stokes' theory}} \oint [\underset{\sim}{\nabla} \times (\underline{B} \times \underline{u})] \cdot \underset{\sim}{ds}$$

Faraday $\Rightarrow$ $\dfrac{\partial B}{\partial t} = -\nabla \times \underline{E}$

Combine these $\Rightarrow$ $\dfrac{\partial \Phi}{\partial t} = -\oint \left[\underset{\sim}{\nabla} \times (\underset{\sim}{E} + \underline{u} \times \underline{B}) \right] \cdot \underline{ds}$

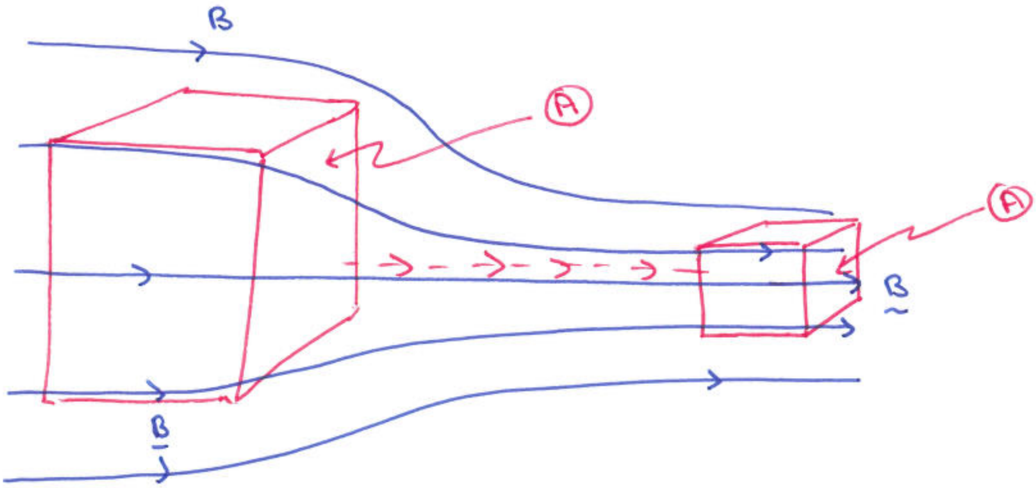
But ideal MHD $\Rightarrow$ $\underline{E} + \underline{u} \times \underline{B} = 0$ $\Rightarrow$ $\boxed{\dfrac{\partial \Phi}{\partial t} = 0}$

$\Rightarrow$ Plasma motions and fields evolve together in a way that conserves the flux

$\Rightarrow$ the magnetic flux is frozen into the plasma.

Picture

Consider a volume of plasma moving from a region of low magnetic field to one of high magnetic field



Flux conservation implies that the same number of magnetic field lines cut through the face A in the low and high field regions

- box must shrink as it moves from low magnetic field to high magnetic field

- it also means the box of plasma will follow field lines

  - the plasma volume and field lines are tied together (if you move the plasma, it pulls the magnetic field lines with it)

  - this is called the **“frozen in”** condition

What you need to know:

You need to know how to derive the ExB and diamagnetic drifts from the fluid force balance equation. You should be able to describe the physical mechanism for the diamagnetic drift, and how it gives rise to a current: the diamagnetic current.

You should understand how to combine the electron and ion continuity equations to deduce the single fluid continuity equation, including an understanding of the approximations made. You should also be able to deduce the ideal MHD force balance equation and understand the physics of the different terms in that equation.

You should be able to write down Ohm's law for a resistive plasma.

You should be able to demonstrate that the diamagnetic current is required for equilibrium (so that electromagnetic forces will balance pressure gradient forces)

You should be able to explain what is meant by the "frozen in" condition for an ideal plasma (ie no resistivity). At a more advanced level, you should be able to prove that the motion of an element of plasma through a magnetic field conserves the magnetic flux through that element. You should understand the consequences of this, and how it leads to the "frozen in" condition.