2014 Lecture Notes

Section 10

Plasma Waves: Part 2

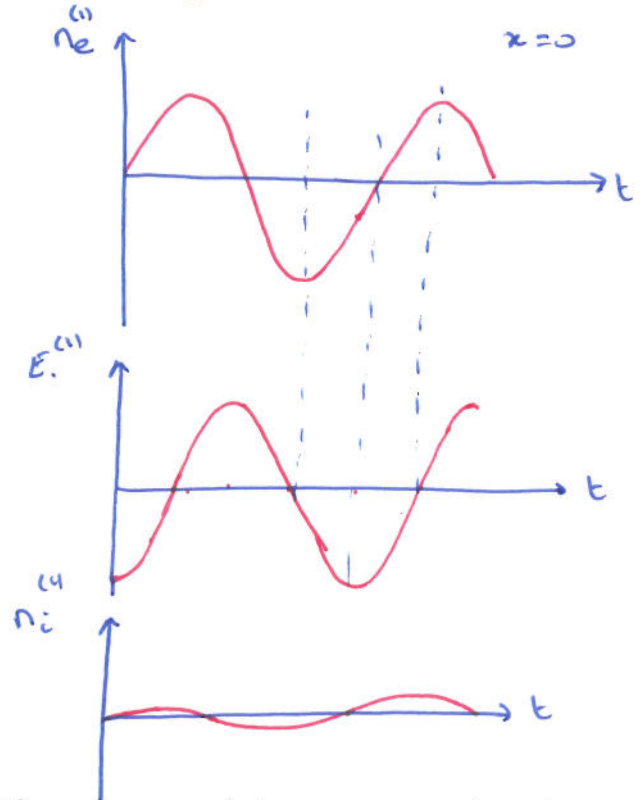
In this section, we will learn about the role of ions in plasma waves: lower frequency waves

- sound waves in plasmas: the ion acoustic wave
- ion plasma frequency (cf electron plasma frequency introduced earlier)
- Ion acoustic and electron plasma waves compared

For lower frequency waves, the ion motion cannot be neglected

Consider a charged particle in an oscillating electric field – if the oscillations are very fast, the heavy particles (ions) will not move significantly before the E-field reverses

Osillate n_e



High frequency oscillation in electrons drives charge separation

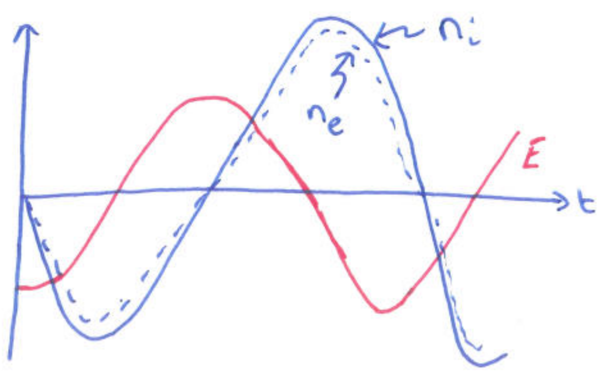
⇒ Oscillating E-field which will act to accelerate both ions and electrons.

⇒ At high frequency, the heavy ions do not have time to react & cannot keep up with electrons ⇒ ion perturbation is small.

Now consider a much slower oscillation of the ions

⇒ Momentary charge imbalance builds E-field

⇒ E-field will drag the electrons with ions to maintain quasi-neutrality (ie $n_e \approx n_i$) – as electrons are light they readily follow the ions.



⇒ ions dominate the dynamics, and electrons just go along for the ride.

Properties of sound waves

Sound waves do propagate (ie $d\omega/dk \neq 0$) and, as we know, do carry information (sound!)

- In air it is collisions that result in the transfer of information

- In a plasma, collisions are due to the electromagnetic forces and provide the necessary coupling between different particles' motion

In air, sound waves result in motions of mass

- in a plasma, it is the ions that provide the mass

$\Rightarrow$ to address sound waves in plasmas, we must allow the ions to move

$\Rightarrow$ We are therefore interested in lower frequency waves than Langmuir waves (where ions are essentially fixed)

As the ions move in response to the self- consistent electromagnetic field, they pull the electrons with them

$\Rightarrow$ quasi-neutrality is maintained - ions and electrons oscillate together

$\Rightarrow$ unlike high frequency waves (eg Langmuir wave) where ions are fixed and there is a charge imbalance as electrons oscillate.

Equations for sound waves in plasmas

As ion motion is important, we now require ion force balance to determine the ion velocity, u_i:

$$m_i n_i \left[\frac{\partial \underline{u}_i}{\partial t} + (\underline{u}_i \cdot \underline{\nabla}) \underline{u}_i \right] = n_i e \underline{E} - \underline{\nabla} p_i = -n_i e \nabla \phi - \underline{\nabla} p_i$$

$$n_i = n_i^{(0)} + n_i^{(1)} \qquad p_i = p_i^{(0)} + p_i^{(1)} \qquad \underline{E} = -\underline{\nabla}\phi^{(1)} \qquad \underline{u}_i = \underline{u}_i^{(1)}$$

[assume no equilibrium E-field or flow]

Keeping only linear terms, we write all perturbations in the form: $a^{(1)} = \hat{a}^{(1)} e^{-i\omega t} e^{ikx}$ where $\hat{a}^{(1)}$ is constant.

Linearise as for Langmuir wave (x-component)

$$\Rightarrow \quad m_i n_i^{(0)} (+\cancel{i}\omega) u_i^{(1)} = +\cancel{i} n_i^{(0)} e k \phi^{(1)} + \cancel{i} k p_i^{(1)}$$

Equilibrium satisfies quasi-neutrality $\Rightarrow$ $n_i^{(0)} = n_e^{(0)} = n_0$

$$\Rightarrow \quad \boxed{m_i n_0 \omega u_i^{(1)} = n_0 e k \phi^{(1)} + k p_i^{(1)}}$$

Combine with equation of state to eliminate pressure, $p_i^{(1)}$

$$p_i = C n_i^{\gamma_i} \qquad C = \text{constant.} \qquad \gamma_i = \frac{c_p}{c_v}$$

Equilibrium $\Rightarrow$ $p_i^{(0)} = C n_0^{\gamma_i}$ $\Rightarrow$ $\boxed{C = \dfrac{p_i^{(0)}}{n_0^{\gamma_i}}}$

Now consider perturbed system :-

$$\left(p_i^{(0)} + p_i^{(1)}\right) = \frac{p_i^{(0)}}{n_0^{\gamma_i}} \left[n_0 + n_i^{(1)} \right]^{\gamma_i} = p_i^{(0)} \left[1 + \underbrace{\frac{n_i^{(1)}}{n_0}}_{\ll 1} \right]^{\gamma_i}$$

$$\Rightarrow \quad p_i^{(0)} + p_i^{(1)} \approx p_i^{(0)} \left[1 + \frac{\gamma_i n_i^{(1)}}{n_0} \right] = p_i^{(0)} + \frac{\gamma_i p_0^{(0)}}{n_0} n_i^{(1)}$$

cancel

Note $p_i^{(0)} = n_0 k_B T_i$

$$\Rightarrow \quad \boxed{p_i^{(1)} = \gamma_i k_B T_i n_i^{(1)}}$$

Force balance then becomes: $\boxed{m_i n_0 \omega u_i^{(1)} = n_0 e k \phi^{(1)} + k \gamma_i k_B T_i n_i^{(1)}}$

A little aside: temperature perturbations

The linearised equation of state is

$$p^{(1)} = \frac{\gamma p^{(0)}}{n_0} n^{(1)} = \gamma k_B T^{(0)} n^{(1)} \qquad (1)$$

We have used this to derive the pressure perturbation, but it can instead be used to deduce the temperature perturbation

$$p = n k_B T$$

$$\Rightarrow \quad (p^{(0)} + p^{(1)}) = k_B (n^{(0)} + n^{(1)})(T^{(0)} + T^{(1)})$$

quadratic in perturbation

$$p^{(0)} + p^{(1)} = k_B [n^{(0)} T^{(0)} + n^{(1)} T^{(0)} + n^{(0)} T^{(1)} + \cancel{n^{(1)} T^{(1)}}]$$

cancel

$$\Rightarrow \quad p^{(1)} = k_B [T^{(0)} n^{(1)} + n^{(0)} T^{(1)}] = \gamma k_B T^{(0)} n^{(1)} \qquad \text{(from (1))}$$

$$\Rightarrow \quad \boxed{\frac{T^{(1)}}{T^{(0)}} = (\gamma - 1) \frac{n^{(1)}}{n^{(0)}}}$$

Note that when the ratio of specific heats γ=1, the temperature perturbation $T^{(1)}$=0:

$\Rightarrow$ isothermal perturbations. [i.e. constant temperature].

Sound waves in plasmas (cont)

Back to sound waves:

- We have analysed the ion force balance, and now we must consider the electrons:

$$\underbrace{m_e n_e \left[\frac{\partial \underline{u}_e}{\partial t} + (\underline{u}_e \cdot \underline{\nabla}) \underline{u}_e \right]}_{\text{electron inertia}} = -n_e e \underline{E} - \underline{\nabla} p_e$$

- Now, because electrons respond to ion motion

$\Rightarrow$ $\underline{u}_e \sim \underline{u}_i$ (ie small)

$\Rightarrow$ electron inertia is smaller than ion inertia by a factor $\sim m_e/m_i \Rightarrow$ neglect it.

The electron force balance is therefore simplified:

$$n_e e \underline{\nabla} \phi^{(1)} = \underline{\nabla} p_e^{(1)}$$

$$\Rightarrow n_0 e \phi^{(1)} = p_e^{(1)} = \frac{\gamma_e p_e^{(0)}}{n_0} n_e^{(1)} \quad \text{[using eqn of state for } e^-\text{]}$$

$$\Rightarrow \boxed{\frac{n_e^{(1)}}{n_0} = \frac{e\phi^{(1)}}{\gamma_e k_B T_e}}$$

For isothermal perturbations, γ_e=1, we find that the electron density perturbation is the Boltzmann response, which we derived from the distribution function earlier:

$$\frac{n_e^{(1)}}{n_0} = \frac{e\phi^{(1)}}{k_B T_e}$$

$\Rightarrow$ The Boltzmann response arises because of the balance between pressure gradient and $\underline{E}$-field forces (in a fluid model)

Ion acoustic waves in plasmas (cont)

Let us summarise:

- Ion force balance plus equation of state:

$$\omega m_i n_0 u_i^{(1)} = n_0 e k \phi^{(1)} + \gamma_i T_i k_B k n_i^{(1)}$$

- Electron force balance plus equation of state:

$$\frac{n_e^{(1)}}{n_0} = \frac{e\phi^{(1)}}{\gamma_e k_B T_e}$$

- We now use quasi-neutrality to eliminate $\phi^{(1)}$: $[n_e^{(1)} = n_i^{(1)}]$

$$\Rightarrow \quad \omega m_i n_0 u_i^{(1)} = k k_B [\gamma_e T_e + \gamma_i T_i] n_i^{(1)}$$

To close the system, we need another equation relating $u_i^{(1)}$ and $n_i^{(1)}$:

$\Rightarrow$ ion continuity: $\frac{\partial n_i}{\partial t} + \underline{\nabla} . (n_i \underline{u}_i) = 0$

$$\Rightarrow \boxed{u_i^{(1)} = \frac{\omega}{k}\frac{n_i^{(1)}}{n_0}}$$

(see Section 9 for derivation)

Putting everything together:

$$\frac{m_i}{k}\left[\omega^2 - \frac{(\gamma_e k_B T_e + \gamma_i k_B T_i)}{m_i} k^2\right] n_i^{(1)} = 0$$

Either (1) ni(1)=0, which is simply the equilibrium solution or (2) the plasma supports a wave with frequency:

$$\boxed{\omega = k c_s}$$

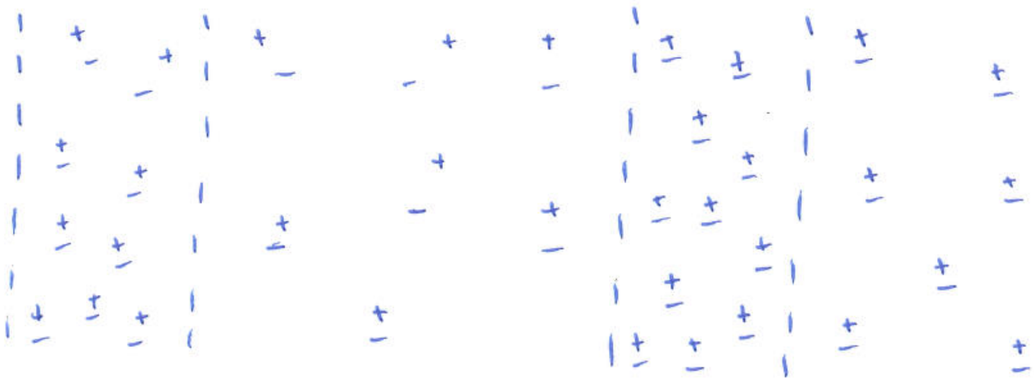
We define the sound speed c_s :-

$$\boxed{c_s^2 = \frac{k_B}{m_i}[\gamma_e T_e + \gamma_i T_i]}$$

[similar size to ion thermal speed].

Picture of plasma sound waves

1. Start with regions of dense and rarefied plasma:



2. Release the system:

   $\Rightarrow$ the compressed region will expand

   (a) The ion pressure gradient drives a flow of ions from the dense to rarefied regions

   $\Rightarrow$ this is the origin of the T_i dependence in c_s

   (b) Because of quasi-neutrality, there will be an electron pressure gradient proportional to T_e

   $\Rightarrow$ electron force balance therefore requires an electric field to balance this pressure gradient $\sim T_e$

   $\Rightarrow$ this electric field drives additional ion flow

   $\Rightarrow$ this is the origin of the T_e dependence in c_s

3. There is an overshoot due to the ion inertia $\Rightarrow$

$\Rightarrow$ rarefied regions become more dense, dense regions become rarefied

$\Rightarrow$ a wave – the ion acoustic wave.

Validity of the plasma approximation

Recall that in the derivation of the ion acoustic wave, we employed the plasma approximation of quasi-neutrality ($n_i \approx n_e$)

– When is this valid?

We can replace quasi-neutrality with Poisson's equation (as for the electron plasma wave):

$$\varepsilon_0 \underline{\nabla} \cdot \underline{E} = e(n_i - n_e) \qquad \underline{E} = -\underline{\nabla}\phi$$

$$\Rightarrow n_i^{(0)} = n_e^{(0)} \qquad \underline{\nabla} \cdot \underline{E} = -\nabla^2 \phi = k^2 \phi$$

and

$$\boxed{k^2 \phi^{(1)} = \frac{e}{\varepsilon_0}\left(n_i^{(1)} - n_e^{(1)}\right)}$$

⇒ Quasi-neutrality suggests we need to consider oscillations with sufficiently long wavelength

⇒ can we quantify?

Take $\gamma_e = \gamma_i = 1$ for simplicity

$$\Rightarrow \frac{n_e^{(1)}}{n_0} = \frac{e\phi^{(1)}}{k_B T_e} \qquad \text{[electron force balance]}$$

Poisson $\Rightarrow$ $$n_i^{(1)} = n_e^{(1)} + \frac{\varepsilon_0 k^2}{e}\phi^{(1)}$$

$$= \frac{n_0 e}{k_B T_e}\left[1 + k^2 \frac{\varepsilon_0 k_B T_e}{n_0 e^2}\right]\phi^{(1)}$$

Recall Debye length $\lambda_D = \sqrt{\dfrac{\varepsilon_0 k_B T_e}{n_0 e^2}}$

$$\Rightarrow \boxed{n_i^{(1)} = \frac{n_0 e}{k_B T_e}\left[1 + k^2 \lambda_D^2\right]\phi^{(1)}}$$

Validity of plasma approximation (2)

$$\frac{n_i^{(1)}}{n_0} = [1 + k^2\lambda_D^2]\frac{e\phi^{(1)}}{k_B T_e}$$

We also have, from ion continuity:

$$u_i^{(1)} = \frac{\omega}{k}\frac{n_i^{(1)}}{n_0}$$

And from force balance for ions:

$$\omega m_i n_0 u_i^{(1)} = n_0 e k \phi^{(1)} + k_B k T_i n_i^{(1)}$$

$$\Rightarrow \quad \frac{m_i}{k}\left[\omega^2 - k^2\left(\frac{k_B T_e}{m_i(1+k^2\lambda_D^2)} + \frac{k_B T_i}{m_i}\right)\right] n_i^{(1)} = 0$$

$$\Rightarrow \quad \boxed{\frac{\omega}{k} = \left[\frac{k_B T_e}{m_i(1+k^2\lambda_D^2)} + \frac{k_B T_i}{m_i}\right]^{1/2}}$$

Assumption of quasi-neutrality requires the waves to have a wavelength >> Debye length, ie $\boxed{k^2\lambda_D^2 << 1}$ $\left[k = \frac{2\pi}{\lambda}\right]$

- Note that in the short wave-length limit, $k\lambda_D$>>1

$$\frac{\omega^2}{k^2} = \frac{k_B T_e}{k^2 m_i \lambda_D^2} + \frac{k_B T_i}{m_i} = \frac{\cancel{k_B}\cancel{T_e}}{k^2 m_i}\frac{n_o e^2}{\varepsilon_0 \cancel{k_B}\cancel{T_e}} + \frac{k_B T_i}{m_i}$$

Recall the electron plasma frequency: $\omega_{pe}^2 = \dfrac{n_e e^2}{\epsilon_0 m_e}$

⇒ We can also define the ion plasma frequency:

$$\boxed{\omega_{pi}^2 = \frac{n_i e^2}{\varepsilon_0 m_i}} \quad (\text{for charge } Z = 1)$$

⇒ for $k^2\lambda_D^2 >> 1$ we have

$$\boxed{\omega^2 = \omega_{pi}^2 + \frac{k^2 V_{thi}^2}{2}} \qquad \boxed{V_{thi}^2 = \frac{2k_B T_i}{m_i}}$$

Ion acoustic and electron plasma waves compared

Ion acoustic: $$\omega^2 = k^2\left[\frac{k_B T_e}{m_i(1+k^2\lambda_D^2)} + \frac{k_B T_i}{m_i}\right]$$

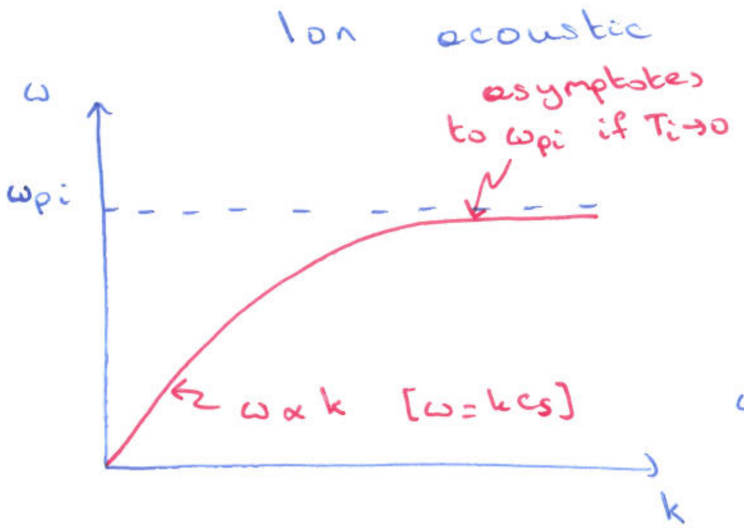
$$\lim_{k\lambda_D\to\infty} \quad \omega^2 = \omega_{pi}^2 + k^2 v_{thi}^2/2$$

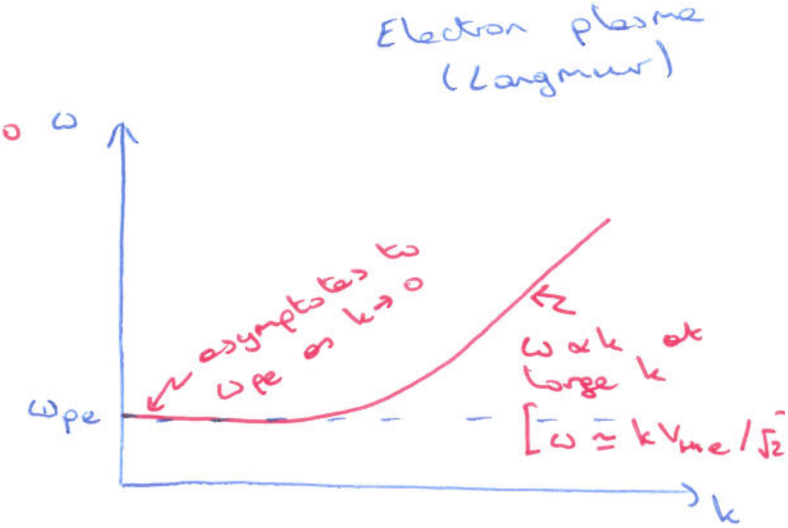
$$\lim_{k\lambda_D \to 0} \quad \omega^2 = k^2 c_s^2 \qquad c_s^2 = \frac{k_B T_e}{m_i} + \frac{k_B T_i}{m_i}$$

Electron plasma:

$$\omega^2 = \omega_{pe}^2 + k^2 v_{the}^2/2$$

Consider cold ions, $T_i \to 0$ (but hot electrons)





A final point:

We have found ω to be real ⇒ purely oscillatory motion (ie a wave)

In more realistic situations, where the equilibrium has a pressure gradient for example, ω can be complex

Recall that perturbations $\sim e^{-i\omega t}$ ⇒ if Im(ω)>0, there is exponential growth (ie the system is unstable)

This same procedure (ie linearisation) is used to calculate the stability of plasmas: very important!

Summary: what you need to know

You should understand the differences between high frequency and low frequency waves, including why quasi-neutrality is not satisfied for high frequency waves, but is for lower frequency waves. You should be able to sketch the electron and ion densities and electric field as a function of time for the Langmuir wave and the ion sound wave.

You should be able to linearise the 2-fluid equations that describe ion sound waves, and derive its frequency. You should know what is meant by the "Boltzmann response", and demonstrate it for the electrons when their inertia is not retained (for an isothermal response, where the temperature perturbation can be neglected).

You should be able to explain why both electron and ion temperatures play a role in the physics of the wave.

You should be able to employ Poisson's equation to prove that quasi-neutrality is only a valid approximation for the ion sound wave when its wavelength is longer than the Debye length. You should also be able to derive the wave frequency in the limit of short wavelength compared to the Debye length.

You should be able to sketch the dispersion relation (ie ω as a function of k) for the ion acoustic (sound) wave and electron plasma (Langmuir) wave in the limit of cold ions, and contrast the two.