Øving 11

15.2.16

$$a_n = \frac{(3n)!}{2^n(n!)^3}, \qquad a_{n+1} = \frac{(3(n+1))!}{2^{n+1}\left((n+1)!\right)^3} = \frac{(3n+1)(3n+2)(3n+1)(3n)!}{2 \cdot 2^n(n+1)^3(n!)^3}$$

$$\frac{a_{n+1}}{a_n} = \frac{2^n(n!)^3}{(3n)!} \cdot \frac{(3n+1)(3n+2)(3n+1)(3n)!}{2 \cdot 2^n(n+1)^3(n!)^3} = \frac{1}{2} \cdot \frac{(3n+1)(3n+2)(3n+1)}{(n+1)^3}$$

$$\lim_{n\to\infty} \frac{1}{2} \cdot \frac{(3n+1)(3n+2)(3n+1)}{(n+1)^3} = \frac{27}{2} \Rightarrow$$

$$\text{Center: } z = 0, \qquad \text{Radius: } R = \frac{2}{27} \approx 0.074$$

15.3

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$$\frac{2n(2n-1)}{n^n} z^{2n-2} = \left(\frac{1}{n^n} z^{2n}\right)''$$

$$\frac{1}{n^n} z^{2n} = b_n z^{2n} = b_n w^n, \quad w = z^2$$

$$\frac{b_{n+1}}{b_n} = \frac{n^n}{(n+1)^{n+1}} = \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{1}{n+1}\left(1+\frac{1}{n}\right)^{-n}$$

$$\lim_{n\to\infty} \frac{1}{n+1}\left(1+\frac{1}{n}\right)^{-n} = \frac{1}{\infty} \cdot e^{-1} = 0$$

$$R_w = \infty \Rightarrow R_z = \sqrt{\infty} = \infty$$

$$\frac{2n(2n-1)}{n^n} z^{2n-2} = a_n z^{2n} \cdot z^2 = a_n w^n \cdot w, \quad w = z^2$$

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{2n(2n-1)(n+1)}\left(1+\frac{1}{n}\right)^{-n} \sim \frac{1}{n} \to 0$$

$$R_w = \infty \Rightarrow R_z = \infty$$

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$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ...$$
$$f(-z) = a_0 - a_1 z + a_2 z^2 - a_3 z^3 + ...$$
$$f(z) - f(-z) = 0 = 2a_1 z + 2a_3 z^3 + ... \Rightarrow a_{2n+1} = 0$$

$$\cos z = \sum_{n \text{ even}}^{\infty} \frac{(-1)^{\frac{n}{2}}}{n!} z^n$$

15.4

3

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$
$$\sin \frac{z^2}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{z^2}{2}\right)^{2n+1}$$

$$\frac{(-1)^n}{(2n+1)!} \left(\frac{z^2}{2}\right)^{2n+1} = \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} z^{4n+2} = a_n z^{4n+2}$$
$$|a_{n+1}| = \frac{1}{(2n+3)! 2^{2n+3}} = \frac{1}{4(2n+3)(2n+2)(2n+1)! \cdot 2^{2n+1}}$$
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(2n+3)! 2^{2n+3}} = \frac{1}{4(2n+3)(2n+2)} \to 0$$

$$\sin \frac{z^2}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} z^{4n+2} \quad \forall \quad z \in \mathbb{C}$$

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$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad e^{-z^2} = \sum_{n=0}^{\infty} \frac{(-z^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!}$$

$$\frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}$$

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$$e^{z-1} = e^{-1} \cdot e^z = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \quad \forall \quad z \in \mathbb{C}$$
$$e^z = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

$$e^{z(z-2)} = e^z = e \sum_{n=0}^{\infty} \frac{(z(z-2)-1)^n}{n!}$$

15.5

5

Power series are uniformely convergen for all z within the radius of convergence.

$$a_n = \binom{n}{2} = \frac{n!}{(n-2)! \cdot 2}$$

$$a_{n+1} = \frac{(n+1)!}{(n-1)! \cdot 2} = \frac{n+1}{n-1} \cdot \frac{n!}{(n-2)! \cdot 2}$$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n-1} \to 1 \Rightarrow R_{4z} = 1 \Rightarrow R_z = \frac{1}{4}$$

Uniformely convergen for all z such that $\left| z + \frac{i}{2} \right| < \frac{1}{4}$

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Series is power series, so have to check radius of convergence.

$$a_n = \frac{(n!)^2}{(2n)!}, \quad a_{n+1} = \frac{((n+1)!)^2}{(2(n+1))!} = \frac{(n+1)^2}{(2n+2)(2n+1)} \cdot \frac{(n!)^2}{(2n)!}$$
$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(2n+2)(2n+1)} \to \frac{1}{4} \Rightarrow R = 4 > 3 \Rightarrow$$

Series is uniformely convergent in disk of radius 3