Øving 7

12.4

19

$$u_{tt} = c^2 u_{xx} \qquad u(0,t) = 0 \qquad u_x(L,t) = 0$$

$$u_n(x,t) = F(x)G(t)$$

$$F(x)G''(t) = c^2 F''(x)G(t)$$
$$\frac{F''(x)}{F(x)} = \frac{G''(t)}{c^2 G(t)} = k = -p^2$$
$$F''(x) + p^2 F(x) = 0 \qquad G''(t) + p^2 c^2 G(t) = 0$$

$$F(x) = a\cos px + b\sin px$$
$$F(0) = 0 = a \Rightarrow F(x) = \sin px$$
$$F'(x) = p\cos px$$
$$F'(L) = 0 = \cos px$$
$$pL = n\pi + \frac{\pi}{2}, \qquad n \in \mathbb{N}$$
$$p = \frac{(2n+1)\pi}{2L}$$

$$G(t) = A\cos pct + B\sin pct$$
$$G_t(t) = -Apc\sin pct + Bpc\sin pct$$
$$G_t(0) = 0 = Bpc \Rightarrow G(t) = A\cos pct$$

$$u_n(x,t) = A_n \sin p_n x \ \cos p_n ct$$
$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin p_n x \ \cos p_n ct$$
$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \sin p_n x \Rightarrow A_n = \frac{2}{L}\int_0^L f(x)\sin p_n x \ dx$$

12.6

21

$$u_{xx} + u_{yy} = 0$$
$$u(x,24) = 25 \qquad u(x,0) = 0 \qquad u(0,y) = 0 \qquad u(24,y) = 0$$

$$u_n = F(x)G(y) \qquad -\frac{F''(x)}{F(x)} = \frac{G''(y)}{G(y)} = k$$
$$F''(x) + kF(x) = 0 \qquad G''(y) - kG(y) = 0 \qquad k = \left(\frac{n\pi}{24}\right)^2$$
$$F(x) = \sin\left(\frac{n\pi}{24}x\right) \qquad G(y) = A_n \sinh\left(\frac{n\pi}{24}y\right)$$
$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{24}y\right) \sin\left(\frac{n\pi}{24}x\right)$$
$$u(x,24) = 25 = \sum_{n=1}^{\infty} A_n \sinh(n\pi) \sin\left(\frac{n\pi}{24}x\right)$$
$$A_n \sinh(n\pi) = \frac{2}{24}\int_0^{24} 25 \sin\left(\frac{n\pi}{24}x\right) dx$$
$$A_n = \frac{25}{2\sinh(n\pi)}\int_0^{24} \sin\left(\frac{n\pi}{24}x\right) dx = \frac{300\,(1-\cos n\pi)}{n\pi \sinh n\pi}$$

$$u(x,y) = \sum_{n=1}^{\infty} \frac{300\,(1-\cos n\pi)}{n\pi \sinh n\pi} \sinh\left(\frac{n\pi}{24}y\right) \sin\left(\frac{n\pi}{24}x\right)$$

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$$u_{xx} + u_{yy} = 0$$
$$u(0,y) = 0 \qquad u(24,y) = f(y) \qquad u_y(x,0) = 0 \qquad u_y(x,24) = 0$$

$$u_n = F(x)G(y) \qquad -\frac{F''(x)}{F(x)} = \frac{G''(y)}{G(y)} = k$$
$$F''(x) + kF(x) = 0 \qquad G''(y) - kG(y) = 0 \qquad k = -\left(\frac{n\pi}{24}\right)^2$$
$$F(x) = \sinh\left(\frac{n\pi}{24}x\right) \qquad G(y) = A\cos\left(\frac{n\pi}{24}y\right)$$
$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi}{24}x \cos\frac{n\pi}{24}y$$
$$A_n = \frac{1}{12\sinh n\pi}\int_0^{24} f(y)\cos\frac{n\pi}{24}y \; dy$$

12.7

4

$$u_t = c^2 u_{xx} \qquad u(x,0) = f(x) = e^{-|x|}$$

$$\mathscr{F}(u_t) = c^2 \mathscr{F}(u_{xx})$$
$$\hat{u}_t = -c^2\omega^2\hat{u}$$
$$\hat{u}(\omega,t) = C(\omega)e^{-c^2\omega^2 t}$$
$$\hat{u}(\omega,0) = \hat{f}(\omega) = C(\omega)$$
$$\hat{u}(\omega,t) = \hat{f}(\omega)e^{-c^2\omega^2 t} =$$
$$\frac{\sqrt{2}}{\sqrt{\pi}}\frac{1}{1+\omega}e^{-c^2\omega^2 t}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\sqrt{2}}{\sqrt{\pi}}\frac{1}{1+\omega}e^{-c^2\omega^2 t}d\omega =$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+\omega}e^{ix\omega - c^2 t\omega^2}d\omega$$

13.1

2

$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$$
$$i = 0 + i\cdot 1 = 1\cdot e^{\frac{\pi}{2}i} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$
$$z\cdot i = re^{i\theta}\cdot e^{\frac{\pi}{2}i} = re^{\left(\theta+\frac{\pi}{2}\right)i}$$

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$$z = x + iy$$
$$z \text{ pure imaginary} \Leftrightarrow x = 0$$
$$\bar{z} = -z \Leftrightarrow x - iy = -x - iy \Leftrightarrow$$
$$x = -x \Leftrightarrow x = 0$$

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$$z_1 = r_1 e^{i\theta_1}, \qquad z_2 = r_2 e^{i\theta_2}$$
$$z_1\cdot z_2 = r_1\cdot r_2 e^{i(\theta_1+\theta_2)} = 0 \Rightarrow$$
$$|z_1\cdot z_2| = r_1\cdot r_2 = |0| = 0 \Rightarrow$$
$$r_1 = 0 \vee r_2 = 0$$

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$$z_1 = -2 + 5i$$
$$z_1^2 = (-2+5i)^2 = 2^2 - 2 \cdot 5i + (5i)^2 = 4 - 25 - 10i = -21 - 10i$$
$$\mathrm{Re}\left(z_1^2\right) = \mathrm{Re}(-21 - 10i) = -21$$

$$(\mathrm{Re}\ z_1)^2 = (-2)^2 = 4$$

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$$z_2 = 3 - i$$
$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{-2-5i}{3+i} = \frac{-2-5i}{3+i} \cdot \frac{3-i}{3-i} = \frac{-6+2i-15i-5}{9+1} = \frac{-11-13i}{10} = -1.1 - 1.3i$$

$$\frac{z_1}{z_2} = \frac{-2+5i}{3-i} = -1.1 + 1.3i$$
$$\left(\frac{\bar{z_1}}{z_2}\right) = -1.1 - 1.3i$$

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$$\frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$
$$\mathrm{Im}\left(\frac{1}{z}\right) = \frac{y}{x^2+y^2}$$

$$\frac{1}{z^2} = \frac{1}{x^2-y^2+2iy} = \frac{1}{x^2-y^2+2iy} \cdot \frac{x^2-y^2-2iy}{x^2-y^2-2iy} =$$
$$\frac{x^2-y^2-2iy}{\left(x^2-y^2\right)^2+4y^2} = \frac{x^2-y^2-2iy}{x^4-2x^2y^2+y^4+4y^2}$$
$$\mathrm{Im}\left(\frac{1}{z^2}\right) = \frac{2y}{x^4-2x^2y^2+y^4+4y^2}$$