Øving 8

13.3

15

$$z = x + iy |z|^2 Im\left(\frac{1}{z}\right) = (x^2 + y^2)\frac{y}{x^2 + y^2} = y$$

$$z = 0 \Rightarrow y = 0 \Rightarrow |z|^2 \text{ Im}\left(\frac{1}{z}\right) = 0 \Rightarrow f(z) \text{ is continous}$$

16

$$\frac{\text{Im } z^2}{|z|^2} = \frac{2xy}{x^2 + y^2}$$

$$\text{Limit along } x = y :$$

$$\lim_{x \to 0} \frac{2x^2}{x^2 + x^2} = 1 \neq 0 \Rightarrow f(z) \text{ is discontinous}$$

13.4

10

$$f(z) = \ln|z| + i \text{ Arg } z = \ln r + i\theta = u(r,\theta) + iv(r,\theta)$$

$$u_r = \frac{1}{r} = \frac{1}{r}v_\theta$$

$$u_\theta = 0 = v_r$$

$$f(z) \text{ is therefore analytic}$$

18

$$u = x^3 - 3xy^2$$

$$u_{xx} + u_{yy} = 6x - 6x = 0 \Rightarrow u(x, y) \text{ is harmonic}$$

$$u_x = 3x^2 - 3y^2 = v_y$$

$$v = \int \left(3x^2 - 3y^2\right) dy = 3x^2 y - y^3 + C(x)$$

$$u_y = -6xy = -v_x$$

$$v = \int 6xy dx = 3x^2 y + C(y)$$

$$v = 3x^2 y - y^3 + C$$

19

$$v = e^{-x}\sin 2y \quad v_{xx} + v_{yy} = e^{-x}\sin 2y - 4e^{-x}\sin 2y = -3e^{-x}\sin 2y \neq 0$$
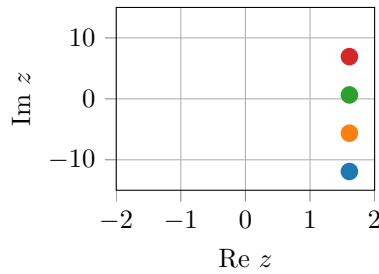
$$v(x, y) \text{ is therefore not harmonic}$$

13.5

20

$$e^z = 4 + 3i$$
$$z = \ln(4 + 3i) = \ln 5 + i(\operatorname{Arg}(4 + 3i) + 2\pi n)$$

13.6

10

$$\sinh(3 + 4i) = \frac{1}{2}\left(e^{3+4i} - e^{-3-4i}\right) = \frac{1}{2}\left(e^3 e^{4i} - e^{-3}e^{-4i}\right) =$$
$$\frac{1}{2}\left(e^3(\cos 4 + i\sin 4) - e^{-3}(\cos 4 - i\sin 4)\right) =$$
$$\cos 4 \cdot \frac{1}{2}\left(e^3 - e^{-3}\right) + i\sin 4 \cdot \frac{1}{2}\left(e^3 + e^{-3}\right) =$$
$$\sinh 3 \cos 4 + i\cosh 3 \sinh 4$$

$$\cosh(3 + 4i) = \cosh 3 \cos 4 + i\sinh 3 \sin 4$$

16

$$\sin z = 100 \Rightarrow \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = 100$$
$$e^{iz} - e^{-iz} = 200i$$
$$e^{2iz} - 1 = 200ie^{iz} \Rightarrow \left(e^{iz}\right)^2 - 200ie^{iz} - 1 = 0$$
$$e^{iz} = \frac{200i \pm \sqrt{(200i)^2 - 4 \cdot 1 \cdot (-1)}}{2} = \frac{200i \pm \sqrt{-39996}}{2} =$$
$$100i \pm 3i\sqrt{1111} \approx 199.995 \vee 0.005i$$
$$iz = \ln 199.995i = \pm\ln 199.995 + \ln i = \pm 5.298 + i\left(\frac{\pi}{2} + 2\pi n\right)$$
$$z = \frac{\pi}{2} + 2\pi n \pm 5.298i$$

19

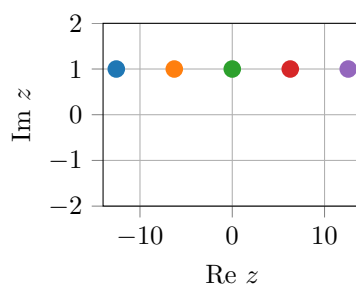$$\sinh z = 0 \Rightarrow \frac{1}{2}\left(e^z - e^{-z}\right) = 0$$
$$e^{2z} - 1 = 0$$
$$2z = \ln 1 = 2\pi n i$$
$$z = \pi n i$$

13.7

15

$$\ln e^i = i \ln e = i - 2\pi n$$

17

$$\ln\left(4 - 3i\right) = \ln 5 + i\,\mathrm{Arg}\left(4 - 3i\right) = \ln 5 + i\mathrm{atan}\frac{-3}{4} + 2i\pi n$$
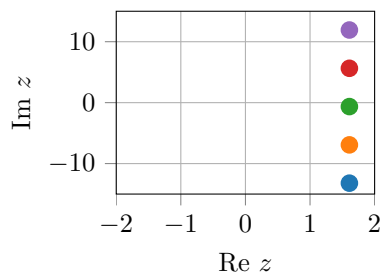
26

$$i^{\frac{i}{2}} = e^{\frac{i}{2}\ln i} = e^{\frac{i}{2}\cdot i\frac{\pi}{2}} = e^{-\frac{\pi}{4}} \approx 0.4559$$