Work sheet week 3

lørdag 13. januar 2018 12.34

C.1
Problem:
Given $e^{i\theta} = \cos\theta + i\sin\theta$ prove $\sin^2\theta + \cos^2\theta = 1$

$e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

The sine and cosine functions are respectively odd and even functions so we can use:
$\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin\theta$

This then gives:
$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

We can then multiply these two complex exponentials:
$e^{i\theta} \cdot e^{-i\theta} = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)$
$e^{i\theta - i\theta} = \cos^2\theta - (i\sin\theta)^2$
$e^0 = \cos^2\theta + \sin^2\theta$
$1 = \sin^2\theta + \cos^2\theta$
Q.E.D.

C.2
1.
Using the property of the exponential function that moves numbers from an additive space to a multiplicative space:
$e^{a+b} = e^a \cdot e^b \Rightarrow e^{z+\pi i} = e^z \cdot e^{\pi i}$

$e^{i\pi} = \cos\pi + i\sin\pi = -1 \Rightarrow e^z \cdot e^{\pi i} = -e^z$
Q.E.D.

2.
Let $z = a + bi = re^{i\theta}$ and $\bar{z} = a - bi = re^{-i\theta}$
$\overline{e^z} = \overline{e^{a+bi}} = \overline{e^a \cdot e^{bi}} = e^a \cdot \overline{e^{bi}} = e^a \cdot e^{-bi}$
$e^{\bar{z}} = e^{a-bi} = e^a \cdot e^{-bi}$
We now see that the conjugate of the exponential equals the exponential of the conjugate.
Q.E.D.

3.
Let $z = a + bi$

$e^z = e^{a+bi} = e^a \cdot e^{bi}$
The real number exponential is never zero: $e^a > 0$
$e^{bi} = \cos b + i\sin b$
The cosine and sine function will never take the value zero simultaneously as the point (0, 0) does not lie on the unit circle and the complex exponential will therefore never take the value 0. $x^2 + y^2 = 0^2 + 0^2 \neq 1$
Q.E.D.

4.
let $f(z): \mathbb{C} \to \mathbb{C} = e^z$

$f(0) = e^0 = 1$
$f(2\pi i) = e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$
$2\pi i \neq 0$

$f(z)$ is not a one-to-one function proven by counterexample

5.
Let $z = a + bi$
$e^z = e^a \cdot e^{bi} = e^a(\cos b + i \sin b)$
$e^{-z} = e^{-a} \cdot e^{-bi} = e^{-a}(\cos b - i \sin b)$

$$\frac{1}{e^z} = \frac{1}{e^a(\cos b + i \sin b)} = \frac{1}{e^a(\cos b + i \sin b)} \cdot \frac{\cos b - i \sin \theta}{\cos b - i \sin \theta}$$
$$= \frac{1}{e^a} \cdot \frac{\cos b - i \sin \theta}{(\cos b + i \sin b)(\cos b - i \sin \theta)} =$$

$$e^{-a} \cdot \frac{\cos b - i \sin b}{\cos^2 b + \sin^2 b} = e^{-a} \cdot \frac{\cos b - i \sin b}{1} = e^{-a} \cdot (\cos b - i \sin b) = e^{-z}$$
Q.E.D.

C.3
Fifth roots of unity:
$z = 1 = e^{2\pi n i}$
$z^{\frac{1}{5}} = 1^{\frac{1}{5}} = e^{\frac{2\pi n i}{5}}$
This gives unique values for $n \in \{0, 1, 2, 3, 4\}$
$1^{\frac{1}{5}} \in S = \left\{1, e^{\frac{2\pi}{5}i}, e^{\frac{4\pi}{5}i}, e^{\frac{6\pi}{5}i}, e^{\frac{8\pi}{5}i}\right\}$

The sum of these will trace out the vertices of a regular pentagon in the complex plane, and you will end up at the origin
$$\sum S = 0$$

$$\prod S = 1 \cdot e^{\frac{2+4+6+8}{5}\pi i} = e^{\frac{20}{5}\pi i} = e^{4\pi i} = \cos 4\pi + i \sin 4\pi = 1$$

C.4
A polynomial with real coefficients can always be factored to linear factors $(z - x_0)$ and irreducible second order polynomials $(z^2 + pz + q)$. The linear factors have real zeroes which are their own conjucate. The zeroes of the irredusible quadratic terms will also have conjugate solutions which can be shown with the quadratic formula:

$$z = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

The imaginary term of this will either be the positive or negative root, and the real part does not change depending on the sign of the root, so the two zeroes will always be conjugate.