Øving 3

11.1

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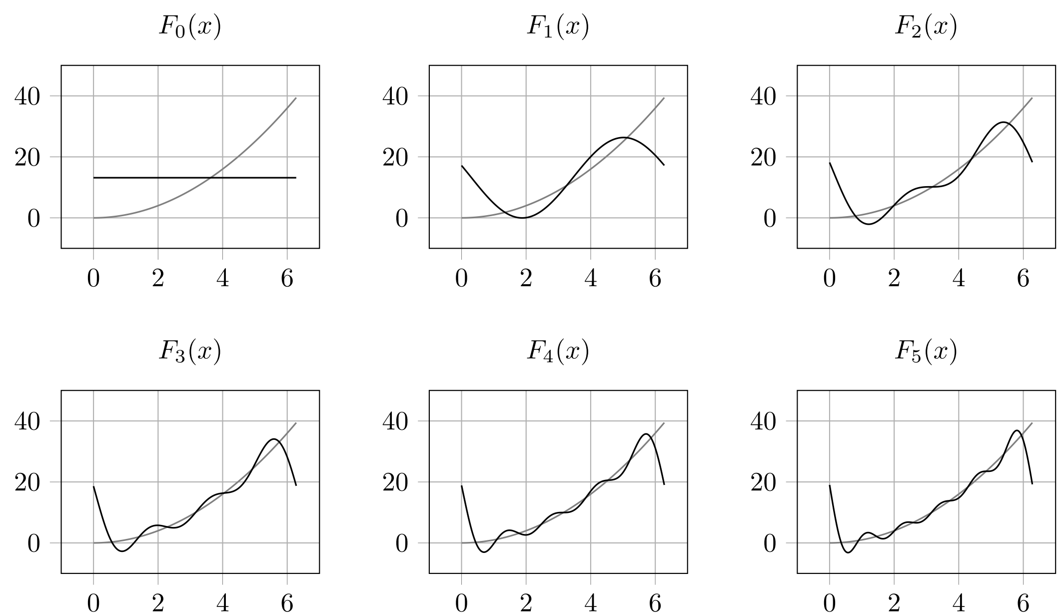
$$f(x) = x^2 \quad (0 < x < 2\pi)$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{1}{2\pi}\left[\frac{1}{3}x^3\right]_0^{2\pi} = \frac{8\pi^3}{6\pi} = \frac{4}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx \; dx = \frac{1}{\pi}\left[\frac{1}{n}x^2 \sin nx + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\sin nx\right]_0^{2\pi} =$$
$$\frac{1}{\pi}\left(\frac{4\pi^2}{n}\cancel{\sin 2\pi n} + \frac{4\pi}{n^2}\cos 2\pi n - \frac{2}{n^3}\cancel{\sin 2\pi n}\right) - \frac{1}{\pi}\left(\frac{2}{n^3}\cancel{\sin 0}\right) = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx \; dx = \frac{1}{\pi}\left[-\frac{1}{n}x^2 \cos nx + \frac{2}{n^2}x\sin nx + \frac{2}{n^3}\cos nx\right]_0^{2\pi} =$$
$$\frac{1}{\pi}\left(-\frac{4\pi^2}{n}\cos 2\pi n + \cancel{\frac{2}{n^3}\cos 2\pi n - \frac{2}{n^3}\cos 0}\right) = -\frac{4\pi}{n}$$

$$f_N(x) = \frac{4}{3}\pi^2 + \sum_{n=1}^{N}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$



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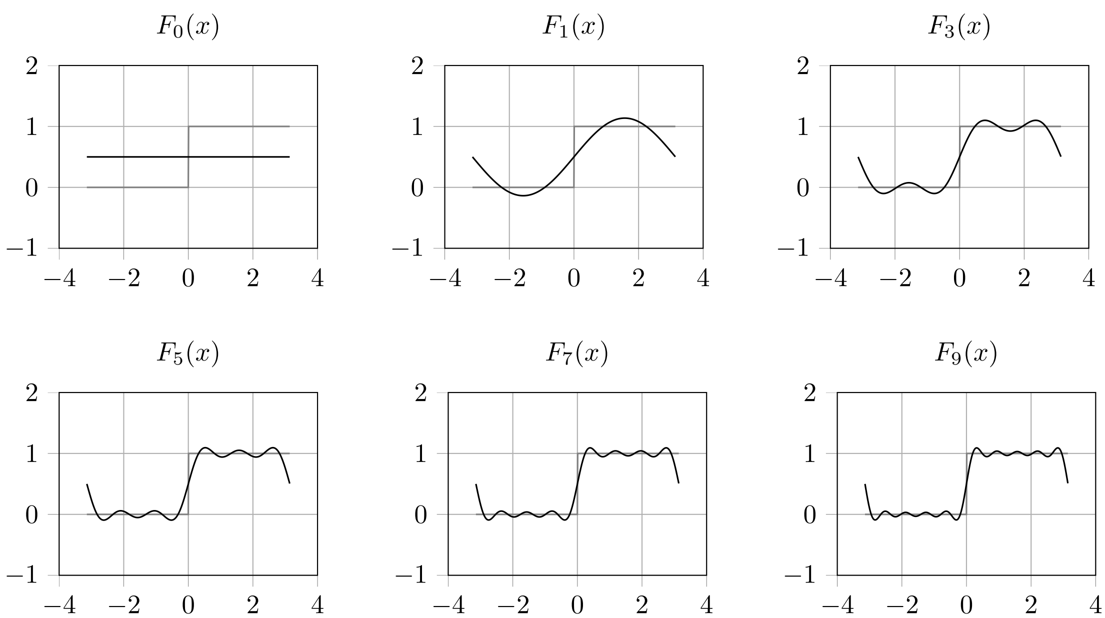
$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \quad -\pi < x < \pi$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; dx = \frac{1}{\pi} \int_{0}^{\pi} \cos nx \; dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \; dx = \frac{1}{\pi} \int_{0}^{\pi} \sin nx \; dx = -\frac{1}{\pi n} \left[\cos nx\right]_0^{\pi} =$$
$$-\frac{1}{\pi n} \left(\cos n\pi - \cos 0\right) = -\frac{1}{\pi n} \left((-1)^n - 1\right)$$
$$b_n = \begin{cases} \frac{2}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$f_N(x) = \frac{1}{2} + \sum_{n=1}^{\lceil N/2 \rceil} \left(\frac{2}{\pi(2n-1)} \sin nx \right)$$



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$$f(x) = \begin{cases} -\frac{\pi}{2} & x < -\frac{\pi}{2} \\ x & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ \frac{\pi}{2} & \frac{\pi}{2} < x \end{cases}$$

$$f(x) \text{ odd} \Rightarrow a_n = 0 \quad \forall \quad n \in \mathbb{N}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \; dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx \; dx =$$

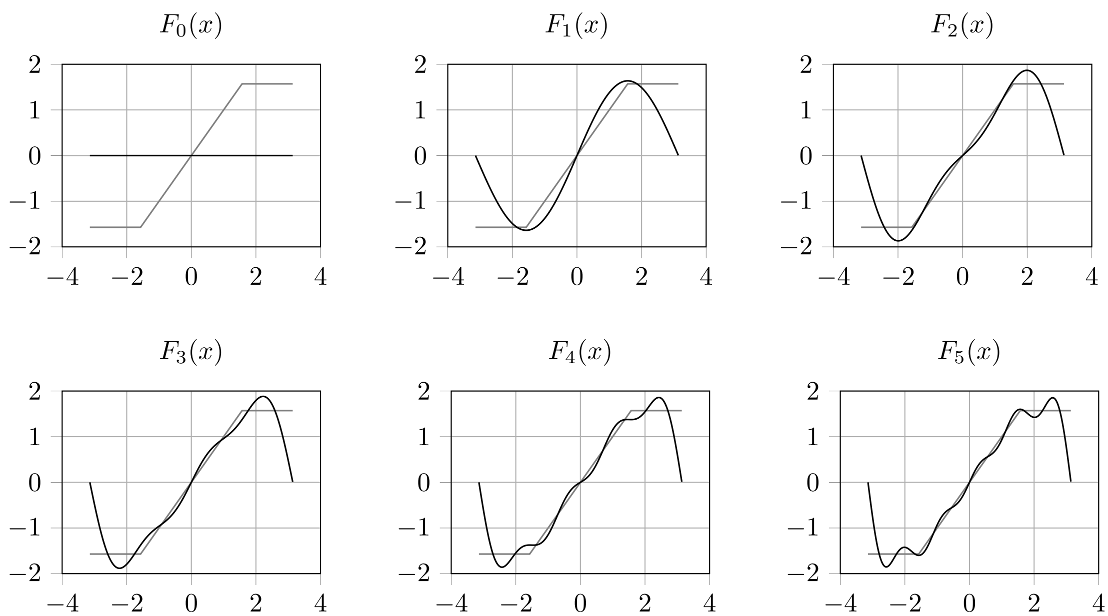
$$\frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} x \sin nx \; dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}\sin nx \; dx =$$

$$\frac{2}{\pi}\left[-\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\frac{\pi}{2}} + \left[-\frac{1}{n}\cos nx\right]_{\frac{\pi}{2}}^{\pi} =$$

$$\frac{2}{\pi}\left(\cancel{-\frac{\pi}{2n}\cos\frac{n\pi}{2}} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right) + \cancel{\frac{1}{n}\cos\frac{n\pi}{2}} - \frac{1}{n}\cos n\pi$$

$$b_n = \begin{cases} -\frac{1}{n} & n \text{ even} \\ \frac{2(-1)^{\frac{n-1}{2}}}{\pi n^2} + \frac{1}{n} & n \text{ odd} \end{cases}$$

$$f_N(x) = \sum_{n \text{ odd}}^{N}\left(\frac{2(-1)^{\frac{n-1}{2}}}{\pi n^2} + \frac{1}{n}\right)\sin nx - \sum_{n \text{ even}}^{N}\frac{1}{n}\sin nx$$



11.2

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$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

$$f(-x) = \begin{cases} -1 & (-x) < 0 \\ 1 & (-x) > 0 \end{cases} = \begin{cases} -1 & x > 0 \\ 1 & x < 0 \end{cases} =$$

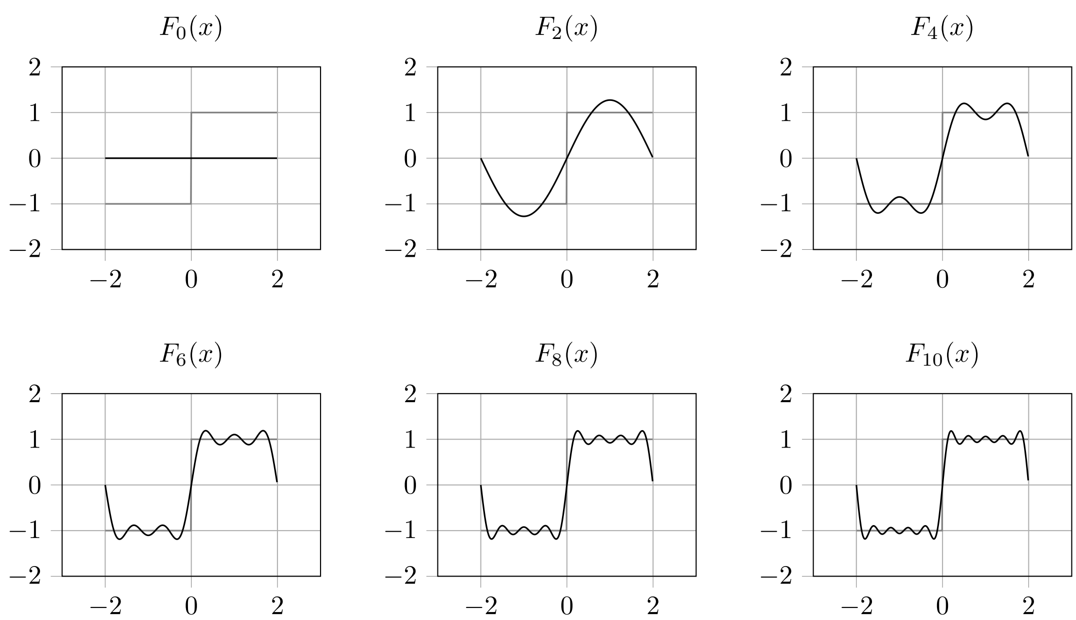
$$\begin{cases} 1 & x < 0 \\ -1 & x > 0 \end{cases} = -\begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases} = -f(x) \Rightarrow$$

$$f(x) \text{ is odd}$$

$$a_n = 0, \quad b_n = \frac{2}{2}\int_0^2 \sin\frac{\pi n x}{2}\ dx = \left[-\frac{2}{\pi n}\cos\frac{\pi n x}{2}\right]_0^2 = \frac{2}{\pi}\left(-\frac{1}{n}\cos n\pi + \frac{1}{n}\right)$$

$$\frac{2}{\pi n}\left(1 - (-1)^{n+1}\right) = \begin{cases} \frac{4}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$f_N(x) = \sum_{n \text{ odd}}^{N} \left(\frac{4}{\pi n}\sin\frac{\pi n x}{2}\right)$$



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$$f(x) = \begin{cases} -x-\pi & x < -\frac{\pi}{2} \\ x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -x+\pi & \frac{\pi}{2} < x \end{cases}$$

$$f(-x) = \begin{cases} x-\pi & -x < -\frac{\pi}{2} \\ -x & -\frac{\pi}{2} < -x < \frac{\pi}{2} \\ x+\pi & \frac{\pi}{2} < -x \end{cases} = \begin{cases} x-\pi & x > \frac{\pi}{2} \\ -x & \frac{\pi}{2} > x > -\frac{\pi}{2} \\ x+\pi & -\frac{\pi}{2} > x \end{cases} =$$

$$\begin{cases} x-\pi & \frac{\pi}{2} < x \\ -x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ x+\pi & x < -\frac{\pi}{2} \end{cases} = \begin{cases} x+\pi & x < -\frac{\pi}{2} \\ -x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ x-\pi & \frac{\pi}{2} < x \end{cases} =$$

$$-\begin{cases} -x-\pi & x < -\frac{\pi}{2} \\ x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -x+\pi & \frac{\pi}{2} < x \end{cases} = -f(x) \Rightarrow$$

$$f(x) \text{ is odd}$$

$$a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \; dx =$$

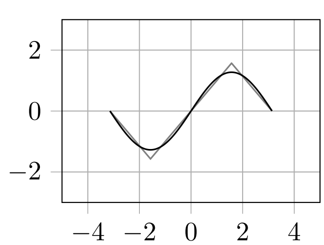
$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin nx \; dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} (\pi\sin nx - x\sin nx)\, dx =$$

$$\frac{2}{\pi}\left(\left[\frac{1}{n^2}\sin nx - \frac{x}{n}\cos nx\right]_0^{\frac{\pi}{2}} + \left[-\frac{\pi}{n}\cos nx + \frac{x}{n}\cos nx - \frac{1}{n^2}\sin nx\right]_{\frac{\pi}{2}}^{\pi}\right) =$$

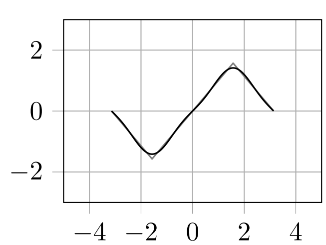
$$\frac{2}{\pi n^2}\left(2\sin n\frac{\pi}{2} - \sin n\pi\right) = \begin{cases} \frac{4}{\pi n^2}(-1)^{\frac{n-1}{2}} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$f_N(x) = \sum_{n \text{ odd}}^{N} \frac{4}{\pi n^2}(-1)^{\frac{n-1}{2}} \sin nx$$

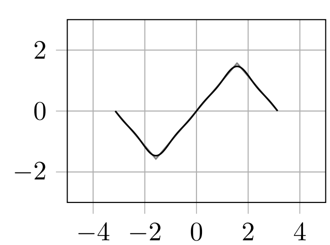
$F_1(x)$



$F_3(x)$



$F_5(x)$



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Fourier sine series for $\sin x = \sin x$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin x \; dx = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \; dx = \frac{2}{\pi\left(n^2-1\right)}\left[\cos x \cos nx + n \sin x \sin nx\right]_0^{\pi} =$$

$$\frac{2}{\pi\left(n^2-1\right)}\left(-\cos n\pi - 1\right) = \begin{cases} \frac{4}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$f_N(x) = \frac{2}{\pi} + \sum_{n \text{ even}}^{N} \frac{4}{\pi\left(1-n^2\right)} \cos nx$$