Øving 6

12.1

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a)

$$v(y) : \mathbb{R} \to \mathbb{R}, \quad y = g(x,t) = x + ct$$
$$\frac{\partial v}{\partial x} = \frac{dv}{dy} \cdot \frac{\partial g}{\partial x} = v'(x+ct), \quad \frac{\partial v}{\partial t} = \frac{dv}{dy} \cdot \frac{\partial g}{\partial t} = c \cdot v'(x+ct)$$
$$\frac{\partial^2 v}{\partial x^2} = \frac{d^2 v}{dy^2} \cdot \left(\frac{\partial g}{\partial x}\right)^2 = v''(x+ct), \quad \frac{\partial^2 v}{\partial t^2} = \frac{d^2 v}{dy^2} \cdot \left(\frac{\partial g}{\partial t}\right)^2 = c^2 v''(x+ct)$$

$$w(z) : \mathbb{R} \to \mathbb{R}, \quad z = h(x,t) = x - ct$$
$$\frac{\partial^2 w}{\partial x^2} = \frac{d^2 w}{dz^2} \cdot \left(\frac{\partial h}{\partial x}\right)^2 = w''(x-ct), \quad \frac{\partial^2 w}{\partial t^2} = \frac{d^2 w}{dz^2} \cdot \left(\frac{\partial h}{\partial t}\right)^2 = c^2 w''(x-ct)$$

$$u(x,t) = v(x+ct) + w(x-ct), \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$c^2 v''(x+ct) + c^2 w''(x-ct) = c^2 \left(v''(x+ct) + w''(x-ct)\right)$$
$$c^2 \left(v''(x+ct) + w''(x-ct)\right) = c^2 \left(v''(x+ct) + w''(x-ct)\right)$$
$$\blacksquare$$

d)

$$u(x,y) = v(x) + w(y), \qquad u_{xy} = 0$$
$$u_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(w'(y)\right) = 0$$
$$\blacksquare$$

$$u(x,y) = v(x)w(y), \qquad u u_{xy} = u_x u_y$$
$$u \cdot u_{xy} = u \cdot \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = v(x)w(y) \cdot \frac{\partial}{\partial x}\left(v(x)w'(y)\right) = v(x)w(y) \cdot v'(x)w'(y)$$
$$u_x u_y = \left(v'(x)w(y)\right) \cdot \left(v(x)w'(y)\right) = v(x)w(y) \cdot v'(x)w'(y)$$
$$\blacksquare$$

$$u(x,t) = v(x+2t) + w(x-2t), \qquad u_{tt} = 4u_{xx}$$
$$u_{tt} = 2^2 v''(x+2t) + (-2)^2 w''(x-2t) =$$
$$4(v''(x+2t) + w''(x-2t)) = 4u_{xx}$$
$$\blacksquare$$

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$$u(x,y) = a\ln\left(x^2+y^2\right)+b, \qquad u_{xx}+u_{yy}=0$$

$$u_x = \frac{2ax}{x^2+y^2}$$
$$u_{xx} = \frac{2a\left(x^2+y^2\right)-2ax\cdot 2x}{\left(x^2+y^2\right)^2} = \frac{2a\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}$$
$$u_{yy} = \frac{2a\left(x^2-x^2\right)}{\left(x^2+y^2\right)^2} = -\frac{2a\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}$$
$$\frac{2a\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2} - \frac{2a\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2} = 0$$

$$u\left(x,y \mid x^2+y^2=1\right) = a\ln 1 + b = 110 \Rightarrow b = 110$$
$$u\left(x,y \mid x^2+y^2=100\right) = a\ln 100 + 110 = 0 \Rightarrow a = \frac{-110}{2\ln 10}$$

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$$u_{yy}+6u_y+13u = 4e^{3y}$$

$$\lambda^2+6\lambda+13 = (\lambda+3-2i)(\lambda+3+2i)$$
$$u(x,y) = u_p(y)+u_h(x,y)$$
$$u_h = e^{-3y}\left(C_1(x)\cos 2y + C_2(x)\sin 2y\right)$$

$$u_p = Ae^{3y}, \quad u_p' = 3Ae^{3y}, \quad u_p'' = 9Ae^{3y}$$
$$9A+6\cdot 3A+13A = 4 \Rightarrow A = \frac{1}{10}$$

$$u(x,y) = \frac{1}{10}e^{3y} + e^{-3y}\left(C_1(x)\cos 2y + C_2(x)\sin 2y\right)$$

12.3

Generell løsning av bølgeligning:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = 1, \quad 0 \le x \le 1$$

$$u_n(x,t) = F(x)G(t)$$
$$F(x)G''(t) = F''(x)G(t)$$
$$\frac{G''(t)}{G(t)} = \frac{F''(x)}{F(x)} = k$$
$$F''(x) - kF(x) = 0, \quad G''(t) - kG(t) = 0, \quad k = -p^2$$
$$F''(x) + p^2F(x) = 0, \qquad F(x) = A\cos px + B\sin px$$
$$F(0) = A = 0, \qquad F(1) = B\sin p = 0$$
$$p = n\pi, \qquad n \in \mathbb{Z}$$
$$F(x) = sin(n\pi x)$$
$$G(t) = A_n \cos n\pi t + B_n \sin n\pi t$$
$$u_n(x,t) = (A_n \cos n\pi t + B_n \sin n\pi t)\sin n\pi x$$
$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (A_n \cos n\pi t + B_n \sin n\pi t)\sin n\pi x$$

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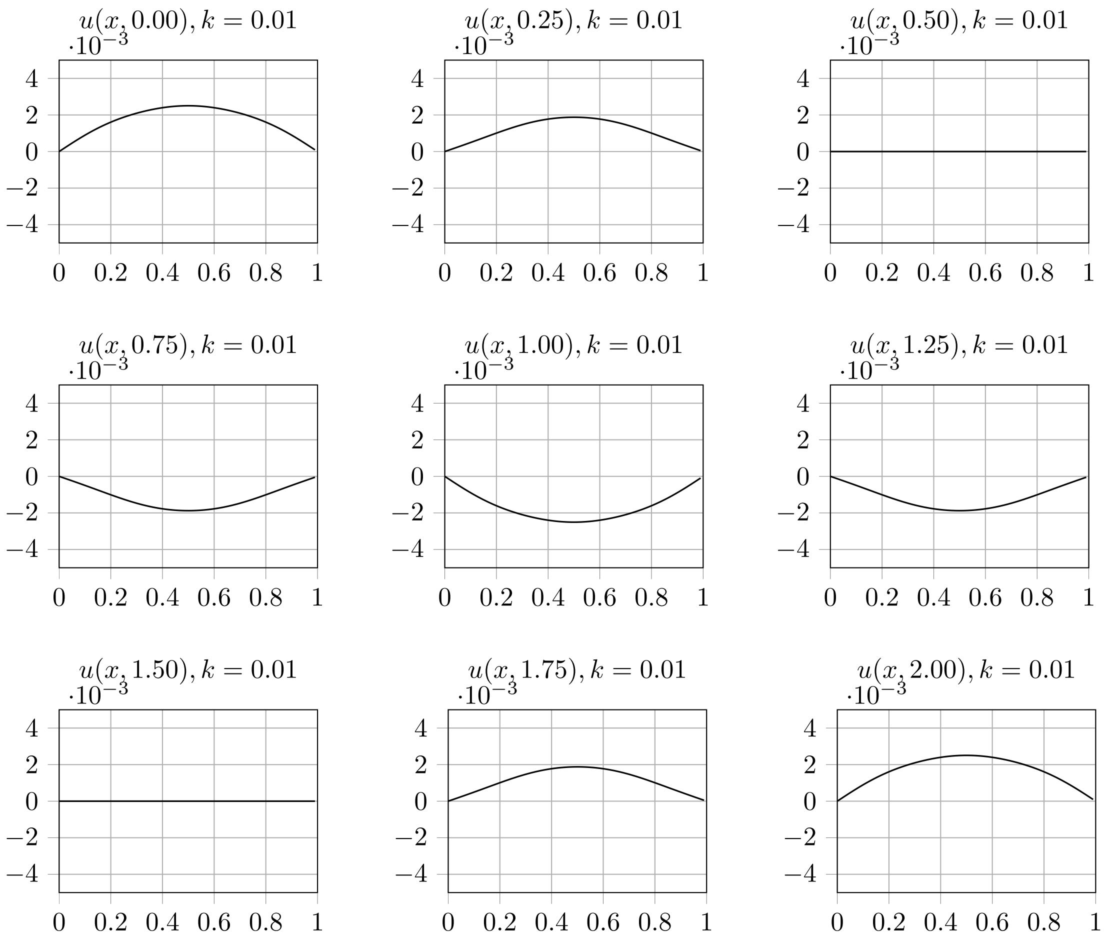
$$u(x,0) = kx(1-x), \qquad u_t(x,0) = 0$$

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \sin n\pi x = 0 \Rightarrow B_n = 0$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin n\pi x = kx(1-x)$$

$$A_n = 2\int_0^1 kx(1-x)\sin n\pi x \; dx = \begin{cases} \frac{8k}{\pi^3 n^3} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$u(x,t) = \frac{8k}{\pi^3} \sum_{n \text{ odd}}^{\infty} \frac{1}{n^3} \cos n\pi t \; \sin n\pi x$$



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$$u(x,0) = \begin{cases} 0 & x < \frac{1}{4} \\ x - \frac{1}{4} & \frac{1}{4} < x < \frac{1}{2} \\ \frac{3}{4} - x & \frac{1}{2} < x < \frac{3}{4} \\ 0 & x > \frac{3}{4} \end{cases}, \qquad u_t(x,0) = 0$$

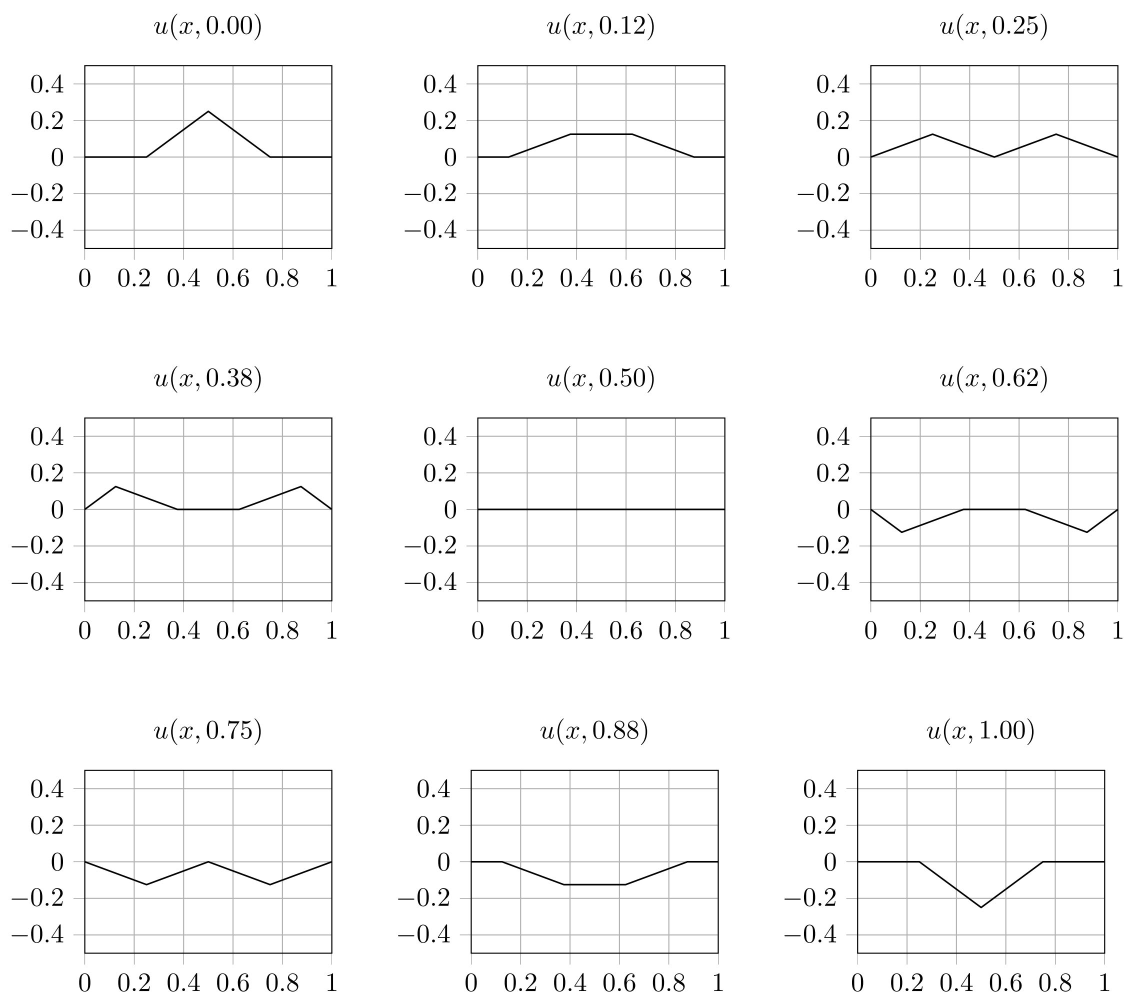
$$B_n = 0$$

$$A_n = 2\left(\int_{1/4}^{1/2}\left(x - \frac{1}{4}\right)\sin n\pi x\ dx - \int_{1/2}^{3/2}\left(\frac{3}{4} - x\right)\sin n\pi x\ dx\right) =$$

$$\frac{2}{\pi^2 n^2}\left(2\sin\frac{1}{2}\pi n - \sin\frac{1}{4}\pi n - \sin\frac{3}{4}\pi n\right) =$$

$$\frac{4}{\pi^2 n^2}\begin{cases}\sin\frac{1}{2}\pi n - \sin\frac{1}{4}\pi n & n \text{ odd} \\ 0 & n \text{ even}\end{cases}$$

$$u(x,t) = \frac{4}{\pi^2}\sum_{n \text{ odd}}^{\infty}\frac{1}{n^2}\left(\sin\frac{1}{2}\pi n - \sin\frac{1}{4}\pi n\right)\cos n\pi t\ \sin n\pi x$$



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$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt) \sin nx$$

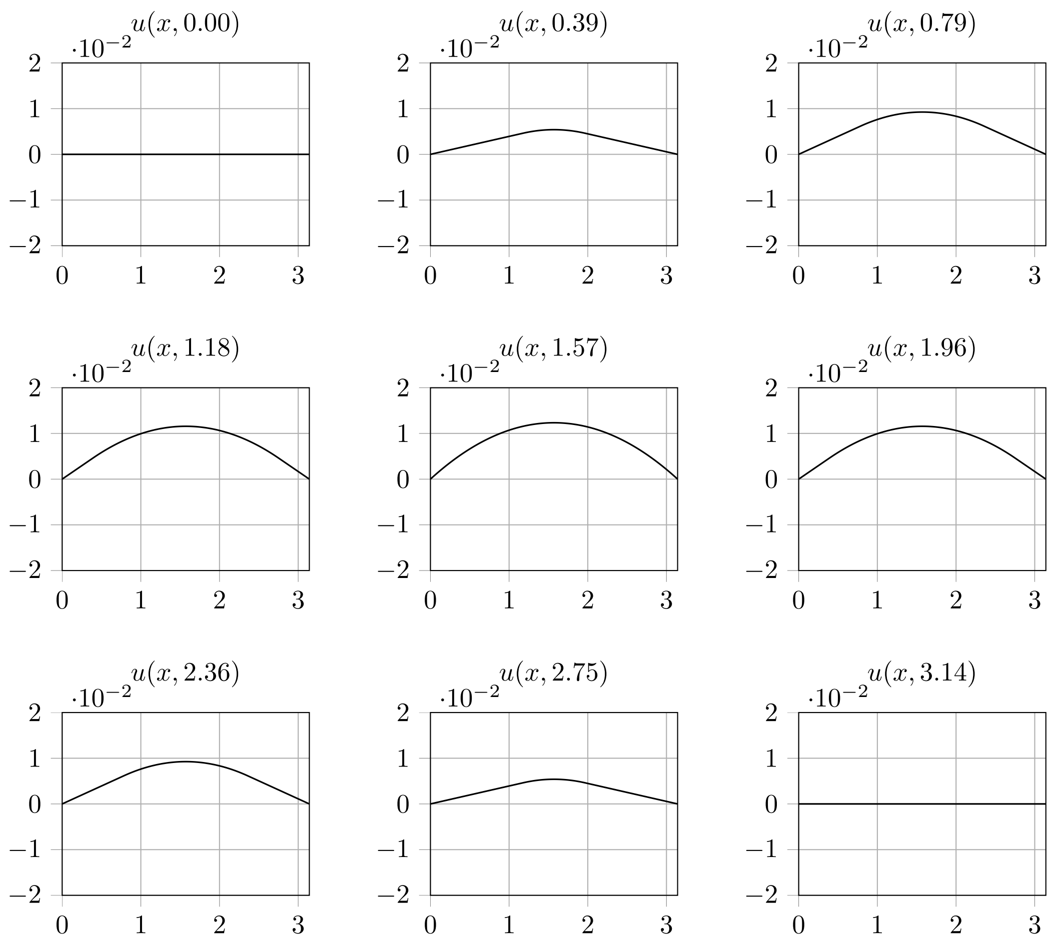
$$u_t(x,t) = \sum_{n=1}^{\infty} (-nA_n \sin nt + nB_n \cos nt) \sin nx$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx = 0 \Rightarrow A_n = 0$$

$$u_t(x,0) = u(x,0) = \sum_{n=1}^{\infty} nB_n \sin nx \Rightarrow B_n = \int_0^{\pi} u_t(x,0) \sin nx \; dx =$$

$$0.01 \frac{2}{\pi n} \left(\int_0^{\frac{\pi}{2}} x \sin nx \; dx + \int_{-\frac{\pi}{2}}^{\pi} (\pi - x) \sin nx \; dx \right) =$$

$$0.01 \frac{4}{\pi n^3} \begin{cases} (-1)^{\frac{n+1}{2}} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \Rightarrow u(x,t) = \frac{0.04}{\pi} \sum_{n \text{ odd}}^{\infty} (-1)^{\frac{n+1}{2}} \frac{1}{n^3} \sin nt \sin nx$$



Denne

a)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2u, \qquad 0 < x < \pi, \quad t > 0$$
$$u_x(0,t) = 0, \quad u_x(\pi,t) = 0$$

$$u(x,t) = F(x)G(t)$$
$$\frac{\partial u}{\partial t} = F(x)G'(t), \quad \frac{\partial^2 u}{\partial x^2} = F''(x)G(t)$$
$$F(x)G'(t) = F''(x)G(t) + 2F(x)G(t)$$
$$\frac{G'(t)}{G(t)} = \frac{F''(x)}{F(x)} + 2 = k$$
$$F''(x) + (2-k)F(x) = 0 \qquad G'(t) - kG(t) = 0$$

$$2 - k = p^2, \quad F''(x) + p^2 F(x) = 0$$
$$F(x) = A\cos px + B\sin px$$
$$F'(x) = -Ap\sin px + Bp\cos px$$
$$F'(0) = Bp = 0 \Rightarrow B = 0$$
$$F'(\pi) = -Ap\sin p\pi = 0 \Rightarrow p = n, \quad A = 1$$
$$F(x) = \cos nx$$

$$G'(t) + \left(n^2 - 2\right) G(t) = 0$$
$$G(t) = Ce^{\left(2-n^2\right)t}$$

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{\left(2-n^2\right)t} \cos nx$$

b)

$$u(x,0) = \sum_{n=0}^{\infty} c_n \cos nx = (\cos x + 1)^2 = \frac{1}{2}\cos 2x + 2\cos x + \frac{3}{2}$$

$$u(x,t) = \frac{1}{2}e^{-2t}\cos 2x + 2e^{t}\cos x + \frac{3}{2}$$