Innlevering 4

Oppgave 1

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_0^{\infty} \frac{\theta^4}{6} x^3 e^{tx\theta x} dx = \frac{\theta^4}{6} \int_0^{\infty} x^3 e^{-(\theta-t)x} dx = (*)$$

$$y = (\theta - t)x \qquad x = \frac{y}{\theta - t} \qquad dx = \frac{dy}{\theta - t}$$

$$(*) = \frac{\theta^4}{6} \int_0^{\infty} \frac{y^3}{(\theta-t)^3} e^{-y} \frac{dy}{\theta - t} = \frac{1}{6} \frac{\theta^4}{(\theta-t)^4} \int_0^{\infty} y^3 e^{-y} dy$$

$$= \frac{1}{6} \frac{1}{\left(1 - \frac{t}{\theta}\right)^4} \Gamma(3) = \left(1 - \frac{t}{\theta}\right)^{-4}$$

$$E(X) = M_X'(0) = \frac{4}{\theta} \left(1 - \frac{t}{\theta}\right)^{-5} \Bigg|_0 = \frac{4}{\theta}$$

Oppgave 2

a)

$$\int_{\theta}^{\infty} ce^{-(x-\theta)} dx = -c \left[e^{-(x-\theta)}\right]_{\theta}^{\infty} = c = 1$$

$$P(X > \theta + 1) = \int_{\theta+1}^{\infty} e^{-(x-\theta)} dx = -\left[e^{-(x-\theta)}\right]_{\theta+1}^{\infty} = e^{-1} = 0.368$$

b)

$$M_Y(t) = M_X(2t) \cdot e^{3t} \qquad M_X(t) = \frac{1}{1-t} \cdot e^{\theta}$$

Siden X er eksponentialfordelt og forskjøvet θ til høyre

$$M_Y(X) = \frac{1}{1-2t} \cdot e^{3t+2\theta t} \Rightarrow f(y) = e^{-(2y-3-2\theta)} \quad y > 3 + 2\theta$$

$$P(Y > \theta + 1) = 1 \Leftarrow \theta + 1 < 3 + 2\theta$$

c)

$$P(W > w) = P(\cap_{i=1}^{10} X_i > w) = \prod_{i=1}^{10} P(X_i > w) = P(X > w)^{10} = 1 - G(w)$$
$$P(X > w) = 1 - F(w) = \int_w^\infty e^{-(x-\theta)dx} = e^{-(w-\theta)}$$
$$G(w) = 1 - \left(e^{-(w-\theta)}\right)^{10} = 1 - e^{-10(w-\theta)}$$
$$g(w) = \frac{d}{dx}G(w) = 10e^{-10(w-\theta)}$$
$$P(W > \theta + 1) = 1 - G(\theta + 1) = e^{-10(\theta+1-\theta)} = e^{-10} = 0.0000454$$

Oppgave 3

$$Var(\hat{\mu}) = 0.1^2 = 0.01$$
$$Var(\tilde{\mu}) = \frac{1}{4} \cdot 0.2^2 + \frac{1}{4} \cdot 0.1^2 = \frac{0.04}{4} + \frac{0.01}{4} = 0.0125$$
$$Var(\mu^*) = \frac{0.04}{25} + \frac{16}{25} \cdot 0.01 = 0.008$$

En god estimator er forventningsrett og har lav varians

μ^* har lavest varians og er derfor å foretrekke

Oppgave 4

a)

$$P(X > 2) = 1 - P(X < 2) = 1 - F(2) = 1 - (1 - e^{-0.04 \cdot 2^2}) = e^{-0.16} = 0.8521$$

$$P(X > 5 | X > 2) = \frac{P(X > 5 \cup X > 2)}{P(X > 2)} = \frac{P(X > 5)}{e^{-0.16}} =$$
$$(1 - F(5)) \cdot e^{0.16} = e^{-0.04 \cdot 5^2} \cdot e^{0.16} = e^{0.16-1} = e^{-0.84} = 0.4317$$

$$f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}(1 - e^{-\alpha x^2}) = 2\alpha x e^{-\alpha x^2}$$

b)

$$l(\alpha) = f(x_1, x_2, ..., x_n; \alpha) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} 2\alpha x_i e^{-\alpha x_i^2} = 2^n \alpha^n e^{-\alpha \sum_{i=1}^{n} x_i^2} \prod_{i=1}^{n} x_i$$

$$L(\alpha) = \ln l(\alpha) = n \ln 2 + n \ln \alpha + \sum_{i=1}^{n} \ln x_i - \alpha \sum_{i=1}^{n} x_i^2$$

$$\frac{dL}{d\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} x_i^2 = 0 \alpha = \frac{n}{\sum_{i=1}^{n} x_i^2} \neq= \frac{n-1}{\sum_{i=1}^{n} x_i^2}$$

$$\hat{\mu} = \frac{\sqrt{\pi}}{2\sqrt{\hat{\alpha}}}$$

$$\hat{\alpha} = \frac{5}{3^2 + 4.5^2 + 5^2 + 7^2 + 6.5^2 + 5^2} = 0.02933$$

$$\hat{\mu} = \frac{\sqrt{\pi}}{2\sqrt{(0.02933)}} = 5.175$$

c)

$$Y = u(X) = X^2 \qquad X = u^{-1}(Y) = w(Y) = \sqrt{Y}$$

$$g(y) = f(w(y)) \cdot \left| \frac{1}{2\sqrt{y}} \right| = \cancel{2}\alpha \cancel{\sqrt{y}} e^{-\alpha y} \frac{1}{\cancel{2\sqrt{y}}} = \alpha e^{-\alpha y}$$

$$E(\alpha) = E(\frac{n-1}{\sum_{i=1}^{n} x_i^2}) = (n-1)E(\frac{1}{\sum_{i=1}^{n} x_i^2}) = (n-1)\frac{1}{(n-1)E(X^2)} = \frac{1}{1/\alpha} = \alpha$$

Oppgave 5

a)

$$P(X > 1000) = 1 - F(1000) = 0.0228$$
$$P(500 < X < 1000) = F(1000) - F(500) = 0.9759$$

b)

$$f(x) = \frac{3^x}{x!} \cdot e^{-3}$$
$$P(X=0) = f(0) = \frac{3^0}{0!} \cdot e^{-3} = e^{-3} = 0.04979$$

$$P(X > 3 | X > 0) = \frac{P(X > 3 \cap X > 0)}{P(X > 0)} = \frac{P(X > 3)}{P(X > 0)} =$$
$$\frac{1 - (f(0) + f(1) + f(2) + f(3))}{1 - f(0)} = \frac{1 - (e^{-3} + 3e^{-3} + 4.5e^{-3} + 4.5e^{-3})}{1 - e^{-3}} = 0.3713$$

c)

$$P(X > 0) = 0.5(1 - F(0)) = 0.5 - 0.5e^{-4} = 0.4908$$

$$E(X) = 0.5 \cdot 0 + 0.5 \cdot 4 = 2$$

d)

$$l(\mu, \theta) = f(x_1, x_2, ..., x_20; \mu, \theta) = \prod_{i=1}^{20} f(x_i) =$$
$$\prod_{i=1}^{8} \theta + (1-\theta)e^{-\mu} \cdot \prod_{i=1}^{12} (1-\theta)\frac{\mu^{x_i}}{x_i!} e^{-\mu} =$$
$$\left(\theta + (1-\theta)e^{-\mu}\right)^8 \cdot \prod_{i=1}^{12} (1-\theta)\frac{\mu^{x_i}}{x_i!} e^{-\mu} =$$
$$\left(\theta + (1-\theta)e^{-\mu}\right)^8 \cdot (1-\theta)^{12} \cdot e^{-12\mu} \cdot \mu^{\sum_{i=1}^{12} x_i} \cdot \prod_{i=1}^{12} \frac{1}{x!}$$
$$L(\mu, \theta) = \ln l(\mu, \theta) = 8 \ln\left(\theta + (1-\theta)e^{\mu}\right) + 12 \ln(1-\theta) - 12\mu + \ln \mu \sum_{i=1}^{12} x_i - \sum_{i=1}^{12} \ln x_i!$$

$$\frac{\partial L}{\partial \theta} = \frac{8}{\theta + (1-\theta)e^{\mu}} \cdot (1 - e^{\mu}) - \frac{1}{1-\theta} = 0$$

Fikk ikke tid til å formulere denne helt ferdig, men gleder meg til lf :)

Oppgave 6

a)

$$E(\hat{a}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\cdot n \cdot E(X) = E(X)$$
$$Var(\hat{a}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\cdot n \cdot Var(X) = \frac{1}{n}Var(X)$$

$\hat{a}$ er normalfordelt fordi det er en lineærkombinasjon av normalfordelte stokastiske variabler

b)

$$l(a,\sigma_G^2) = \prod_{i=1}^{n} f_1(X_i)\cdot\prod_{i=1}^{m} f_2(y_i) = \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma_G}e^{-\frac{(X_i-a)^2}{2\sigma_G^2}}\cdot\prod_{i=1}^{m}\frac{4}{\sqrt{2\pi}\sigma_G}e^{-\frac{8(Y_i-a)^2}{\sigma_G^2}}$$
$$\ln l = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma_G^2 - \sum_{i=1}^{n}\frac{(X_i-a)^2}{2\sigma_G^2} + \ln 4 - \frac{m}{2}\ln 2\pi - \frac{m}{2}\ln\sigma_G^2 - \sum_{i=1}^{m}\frac{8(Y_i-a)^2}{\sigma_G^2} =$$
$$\ln 4 - \frac{n+m}{2}\ln 2\pi - \frac{n+m}{2}\ln\sigma_G^2 - \frac{1}{2\sigma_G^2}\sum_{i=1}^{n}(X_i-a)^2 + \frac{8}{\sigma_G^2}\sum_{i=1}^{m}(Y_i-a)^2$$
$$\frac{\partial}{\partial a}\ln l = \frac{1}{\sigma_G^2}\sum_{i=1}^{n}(X_i-a) + \frac{16}{\sigma_G^2}\sum_{i=1}^{m}(Y_i-a) = 0$$
$$n\bar{X} - na + 16m\bar{Y} - 16ma = 0 \qquad \hat{a} = \frac{n\bar{X}+16m\bar{Y}}{n+16m}$$

$$\frac{\partial}{\partial\,\sigma_G^2}\ln l = -\frac{n+m}{2\sigma_G^2} + \frac{1}{2\left(\sigma_G^2\right)^2}\sum_{i=1}^{n}(X_i-a)^2 + \frac{8}{\left(\sigma_G^2\right)^2}\sum_{i=1}^{m}(Y_i-a)^2 = 0$$
$$(n+m)\sigma_G^2 = \frac{1}{2}\sum_{i=1}^{n}(X_i-a)^2 + 8\sum_{i=1}^{m}(Y_i-a)^2$$
$$\sigma_G^2 = \frac{\frac{1}{2}\sum_{i=1}^{n}(X_i-a)^2 + 8\sum_{i=1}^{m}(Y_i-a)^2}{n+m}$$

$$\text{Setter inn } a = \hat{a} = \frac{n\bar{X}+16m\bar{Y}}{n+16m}$$

$$\hat{\sigma}_G^2 = \frac{\frac{1}{2}\sum_{i=1}^{n}(X_i - \frac{n\bar{X}+16m\bar{Y}}{n+16m})^2 + 8\sum_{i=1}^{m}(Y_i - \frac{n\bar{X}+16m\bar{Y}}{n+16m})^2}{n+m}$$