Work sheet week 5

lørdag 27. januar 2018 18.02

C.1

$$A\boldsymbol{x} = \lambda \boldsymbol{x} \rightarrow \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1 \cdot \begin{bmatrix} a \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ a \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \begin{bmatrix} ax_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ ax_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix}$$

$$ax_1 + x_2 = \lambda x_1$$
$$x_1 + ax_2 = \lambda x_2$$
$$\lambda = \frac{ax_1 + x_2}{x_1} = \frac{ax_2 + x_1}{x_2}$$

This only works for $x_1 = \pm x_2$ as if they would have any other relation $x_1 = cx_2$ the equality would not hold:

$$x_1 = cx_2$$
$$\frac{acx_2 + x_2}{cx_2} = \frac{ax_2 + cx_2}{x_2}$$
$$a + \frac{1}{c} = a + c$$

And we see this only holds with $c = \pm 1$

The value of λ can then be easily found. We get two different values depending on if $x_1 = x_2$ or $x_1 = -x_2$:

$$x_1 = x_2 \Rightarrow \lambda = \frac{ax_1 + x_1}{x_1} = a + 1$$
$$x_1 = -x_2 \Rightarrow \lambda = \frac{ax_1 - x_1}{x_1} = a - 1$$

This can then be summarized as:

$$\left. \begin{matrix} x_1 = \pm x_2 \\ \lambda = a + \dfrac{x_1}{x_2} \end{matrix} \right\} \Rightarrow A\boldsymbol{x} = \lambda \boldsymbol{x}$$

C.2

For the set $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ to span $\mathbb{C}^3$, the determinant of the matrix $[\boldsymbol{u} \quad \boldsymbol{v} \quad \boldsymbol{w}]$ has to be not equal to zero (as this would imply that theese vectors does not lie in the same plane).

$$\boldsymbol{u} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$|\boldsymbol{u} \quad \boldsymbol{v} \quad \boldsymbol{w}| = \begin{vmatrix} z_1 & i & 1 \\ z_2 & i & 2 \\ z_3 & i & 3 \end{vmatrix} = z_1 \begin{vmatrix} i & 2 \\ i & 3 \end{vmatrix} + z_2 \begin{vmatrix} i & 3 \\ i & 1 \end{vmatrix} + z_3 \begin{vmatrix} i & 1 \\ i & 2 \end{vmatrix}$$

$= z_1(3i - 2i) + z_2(i - 3i) + z_3(2i \ - i)$

$= i\, z_1 - 2i\, z_2 + i\, z_3 = i(z_1 - 2z_2 + z_3)$

$i(z_1 - 2z_2 + z_3) \neq 0$

$z_1 - 2z_2 + z_3 \neq 0$

We then see that all vectors $\boldsymbol{u}$ such that $\boldsymbol{u} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \neq 0$, that is all vectors $\boldsymbol{u}$ that does not lie in the plane

$z_1 - 2z_2 + z_3 = 0$ (which is the plane spanned by the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$) makes the set $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ span $\mathbb{C}^3$

C.3

If $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution then this also holds for $\boldsymbol{b} = \boldsymbol{0}$ such that $A\boldsymbol{x} = \boldsymbol{0}.$ Since $\boldsymbol{x} = \boldsymbol{0}$ will always be a solution to this equation and the solution is supposed to be unique, then $\boldsymbol{0}$ can be the only solution. Furthermore it can be shown that the equation $A\boldsymbol{x} = \boldsymbol{b}$ can only have one solution if the equation $A\boldsymbol{x} = \boldsymbol{0}$ Has only the trivial solution.

Suppose that the matrix $A = [\boldsymbol{u} \quad \boldsymbol{v}] = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$ then A represents the linear transform from having $\hat{\imath}$ and $\hat{\jmath}$ as basis vectors to having $\boldsymbol{u}$ and $\boldsymbol{v}$. For the equation $A\boldsymbol{x} = \boldsymbol{0}$ to have nontrivial solutions, $\boldsymbol{u}$ and $\boldsymbol{v}$ has to be paralell (otherwise no linear combination with non zero coefficients will send you back to the origin). This means that the set $\{\boldsymbol{u} \quad \boldsymbol{v}\}$ does not span the entire plane ($\mathbb{R}^2$ or $\mathbb{C}^2$ depending on if the equations exist in the complex plane or not) and there exist values for $\boldsymbol{b}$ that is not contained in the set spanned by $\{\boldsymbol{u} \quad \boldsymbol{v}\}$ which means that if the equation $A\boldsymbol{x} = \boldsymbol{0}$ has non trivial solutions then $A\boldsymbol{x} = \boldsymbol{b}$ does not have solutions for all $\boldsymbol{b}$, let alone a unique one.