Skriftlig innlevering 4

Oppgave 1

$$A = \iint_S 1 \cdot dS$$

$$z = f(x, y) = \sqrt{x^2 + y^2}$$

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} dA = \sqrt{2} dxdy$$

$$A = \iint_S 1 \cdot dS = \iint_S \sqrt{2} dxdy = \sqrt{2} \iint_S dxdy$$

Dette tilsvarer $\sqrt{2}$ ganger arealet av projeksjonen av S ned i xy-planet:

$$x^2 + y^2 = z^2$$

$$x + 2z = 3 \Rightarrow z = \frac{3 - x}{2} \Rightarrow z^2 = \frac{(3 - x)^2}{4}$$

$$x^2 + y^2 = \frac{(3 - x)^2}{4} \Rightarrow 4x^2 + 4y^2 = (3 - x)^2 \Rightarrow 4x^2 + 4y^2 = 9 - 6x + x^2 \Rightarrow$$

$$3x^2 + 6x + 4y^2 = 9 \Rightarrow 3\left(x^2 + 2x + 1 - 1\right) + 4y^2 = 9 \Rightarrow 3\left((x + 1)^2 - 1\right) + 4y^2 = 9 \Rightarrow$$

$$3(x + 1)^2 - 3 + 4y^2 = 9 \Rightarrow 3(x + 1)^2 + 4y^2 = 12 \Rightarrow \frac{(x + 1)^2}{4} + \frac{y^2}{3} = 1$$

Dette gjenkjenner vi som elipsen med sentrum (-1, 0) og halvakser $a = 2, b = \sqrt{3}$ med areal $A = ab\pi = 2\pi\sqrt{3}$ og det totale arealet av kjegleflaten blir da $\sqrt{2} \cdot 2\pi\sqrt{3} = 2\pi\sqrt{6}$.

Oppgave 2

Greens teorem:

$$\oint_C \left(xy + \ln\left(x^2 + 1\right)\right) dx + \left(4x + e^{y^2} + 3\arctan y\right) dy =$$

$$\iint_D \left(\frac{\partial}{\partial x}\left(4x + e^{y^2} + 3\arctan y\right) - \frac{\partial}{\partial y}\left(xy + \ln\left(x^2 + 1\right)\right)\right) dxdy =$$

$$\iint_D (4 - x)\, dxdy$$

Innfører polarkoordinater:

$$\int_0^\pi \int_0^1 (4 - r\cos\theta)\, rdrd\theta = \int_0^\pi \int_0^1 \left(4r - r^2\cos\theta\right) drd\theta = \int_0^\pi \left[2r^2 - \frac{1}{3}r^3\cos\theta\right]_0^1 d\theta =$$

$$\int_0^\pi \left(2 - \frac{1}{3}\cos\theta\right) d\theta = \left[2\theta - \frac{1}{3}\sin\theta\right]_0^\pi = 2\pi$$

Oppgave 3

Divergensteoremet:

$$\oiint_{\partial T} \mathbf{F} \cdot \hat{\mathbf{N}} dS = \iiint_T \nabla \cdot \mathbf{F} dV = \iiint_T (3 - 2 + 8y)\, dV = \iiint_T (1 + 8y)\, dV$$

Finner grenser for T:

$$\begin{gathered} 0 \leq z \leq x \\ y^2 \leq x \leq 9 \\ -3 \leq y \leq 3 \end{gathered}$$

Evaluerer så trippelintegralet:

$$\iiint_T (1 + 8y)\, dV = \int_{-3}^{3} \int_{y^2}^{9} \int_{0}^{x} (1 + 8y)\, dz dx dy = \int_{-3}^{3} \int_{y^2}^{9} (1 + 8y) \cdot [z]_0^x \, dx dy =$$
$$\int_{-3}^{3} \int_{y^2}^{9} (1 + 8y)\, x dx dy = \frac{1}{2} \int_{-3}^{3} (1 + 8y) \cdot \left[x^2\right]_{y^2}^{9} dy = \frac{1}{2} \int_{-3}^{3} (1 + 8y) \left(9^2 - y^4\right) dy =$$
$$\frac{1}{2} \int_{-3}^{3} \left(9^2 + 8 \cdot 9^2 y - y^4 - 8y^5\right) dy = \frac{1}{2} \left[9^2 y + 4 \cdot 9^2 y^2 - \frac{1}{5} y^5 - \frac{4}{3} y^6\right]_{-3}^{3} = \frac{972}{5}$$

Oppgave 4

a)

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} (x + y) + \frac{\partial}{\partial y} (4x - y) + \frac{\partial}{\partial z} \left(z^2 + xy\right) = 1 - 1 + 2z = 2z$$
$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathbf{F}_1 & \mathbf{F}_2 & \mathbf{F}_3 \end{vmatrix} = \hat{\mathbf{i}} \cdot (x - 0) + \hat{\mathbf{j}} \cdot (0 - y) + \hat{\mathbf{k}} \cdot (4 - 1) = (x, -y, 3) \neq \vec{\mathbf{0}}$$
$$\Rightarrow \mathbf{F} \text{ ikke konservativt}$$

b)

Skjæringskurve:

$$z = \sqrt{10}$$
$$z = \sqrt{x^2 + y^2 + 1}$$
$$\sqrt{x^2 + y^2 + 1} = \sqrt{10} \Rightarrow x^2 + y^2 + 1 = 10 \Rightarrow x^2 + y^2 = 3^2$$

Dette gjenkjenner vi som en sirkel parallelt med xy-planet med sentrum i $\left(0, 0, \sqrt{10}\right)$ og radius lik 3. Mulig parametrisering er:

$$\mathbf{r}(\theta) = \left(3\cos\theta, 3\sin\theta, \sqrt{10}\right)$$
$$0 \leq \theta \leq 2\pi$$

Evaluerer så linjeintegralet:

$$d\mathbf{r} = 3\left(-\sin\theta, \cos\theta, 0\right) d\theta$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left(3\cos\theta + 3\sin\theta, 4 \cdot 3\cos\theta - 3\sin\theta, \sqrt{10}^2 + 9\cos\theta\sin\theta\right) \cdot 3\left(-\sin\theta, \cos\theta, 0\right) d\theta =$$

$$9\int_0^{2\pi} \left(-\left(\cos\theta + \sin\theta\right) \cdot \sin\theta + \left(4\cos\theta - \sin\theta\right) \cdot \cos\theta + 0\right) d\theta = 9\int_0^{2\pi} \left(5\cos^2\theta - \sin 2\theta - 1\right) d\theta =$$

$$9\int_0^{2\pi} \left(\frac{5}{2}\left(\cos 2\theta + 1\right) - \sin 2\theta - 1\right) d\theta = 9\int_0^{2\pi} \left(\frac{5}{2}\cos 2\theta - \sin 2\theta + \frac{3}{2}\right) d\theta =$$

$$9\left[\frac{5}{4}\sin 2\theta + \frac{1}{2}\cos 2\theta + \frac{3}{2}\theta\right]_0^{2\pi} = 27\pi$$