# COMP9020 Week 2
# Binary Relations

- Textbook (R & W) - Ch. 3., Sec. 3.1, 3.4; Ch. 11, Sec. 11.1

# Applications in Computer Science

Many relations that appear in CS fall into two broad categories:

Equivalence relations (generalizing "equality"):

- Programs that exhibit the same behaviour
- Logically equivalent statements
- The `.equals()` method in Java

Partial orders (generalizing "less than or equal to"):

- Object inheritance
- Simulation
- Requirement specifications
- The `.compareTo()` method in Java

# Summary of topics

- Defining binary relations
- Properties of binary relations
- Equivalence relations, classes, and partitions
- Orderings

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# Binary relations

A **binary relation between** $S$ **and** $T$ is a subset of $S \times T$: i.e. a set of ordered pairs.
Also: over $S$ and $T$; from $S$ to $T$; on $S$ (if $S = T$).

**Example (Special (Trivial) Relations)**

**Identity** (diagonal, equality) $\quad E = \{ (x, x) : x \in S \}$

**Empty** $\emptyset$

**Universal** $U = S \times S$

# Defining binary relations: Set-based definitions

Defining a relation $R \subseteq S \times T$:

- Explicitly listing tuples: e.g. $\{(1,1),(2,3),(3,2)\}$
- Set comprehension: $\{(x,y) \in [1,3] \times [1,3] : 5|xy-1\}$
- Construction from other relations: $\{(1,1)\} \cup \{(2,3)\} \cup \{(2,3)\}^{\leftarrow}$

# Defining binary relations: Matrix representation

Defining a relation $R \subseteq S \times T$:

Rows enumerated by elements of $S$, columns by elements of $T$:

### Examples

- The relation $\{(1,1),(2,3),(3,2)\} \subseteq [1,3] \times [1,3]$:

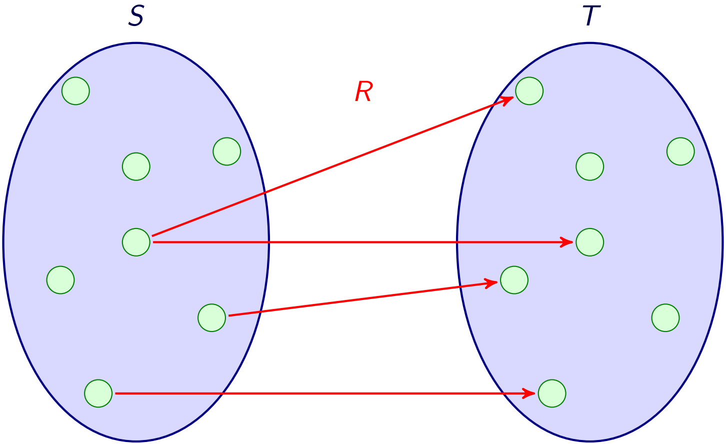
$$\begin{bmatrix} \bullet & \circ & \circ \\ \circ & \circ & \bullet \\ \circ & \bullet & \circ \end{bmatrix}$$

- The relation $\{(1,1),(1,2),(1,3),(1,4),(2,2),(3,2)\} \subseteq [1,3] \times [1,4]$:

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \circ & \circ \\ \circ & \bullet & \circ & \circ \end{bmatrix}$$

# Defining binary relations: Graphical representation

Defining a relation $R \subseteq S \times T$:



# Defining binary relations: Graphical representation

**Example**

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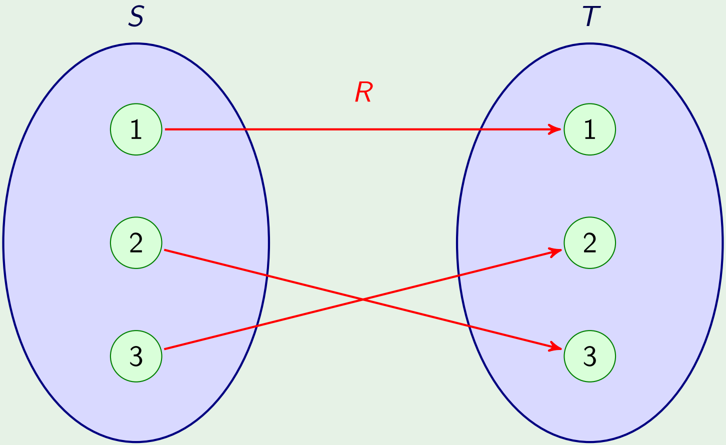
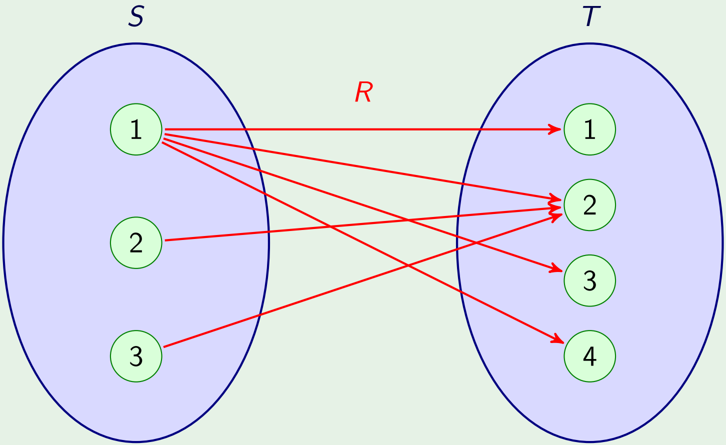


# Defining binary relations: Graphical representation

**Example**

$\{(1,1),(1,2),(1,3),(1,4),(2,2),(3,2)\} \subseteq [1,3] \times [1,4]$:



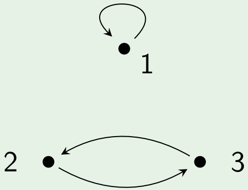
# Defining binary relations: Graph representation

If $S = T$ we can define $R \subseteq S \times S$ as a **directed graph** (week 5).

- Nodes: Elements of $S$
- Edges: Elements of $R$

### Example

$R = \{(1,1),(2,3),(3,2)\} \subseteq [1,3] \times [1,3]$:



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# Properties of Binary Relations $R \subseteq S \times S$

## Definition

| | | |
|---|---|---|
| (R) | reflexive | For all $x \in S$: $(x, x) \in R$ |
| (AR) | antireflexive | For all $x \in S$: $(x, x) \notin R$ |
| (S) | symmetric | For all $x, y \in S$: If $(x, y) \in R$ then $(y, x) \in R$ |
| (AS) | antisymmetric | For all $x, y \in S$: If $(x, y)$ and $(y, x) \in R$ then $x = y$ |
| (T) | transitive | For all $x, y, z \in S$: If $(x, y)$ and $(y, z) \in R$ then $(x, z) \in R$ |

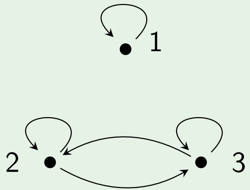
## NB

- *Properties have to hold for all elements*
- *(S), (AS), (T) are conditional statements – they will hold if there is nothing which satisfies the 'if' part*

# Relation properties: Examples

## Examples

**(R)** Reflexivity: $(x, x) \in R$ for all $x$

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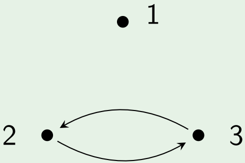


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# Relation properties: Examples

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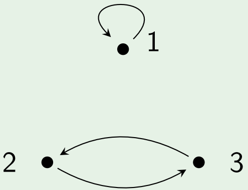
**(R)** Reflexivity: $(x, x) \in R$ for all $x$

**(AR)** Antireflexivity: $(x, x) \notin R$ for all $x$

**(S)** Symmetry: If $(x, y) \in R$ then $(y, x) \in R$ for all $x$, $y$

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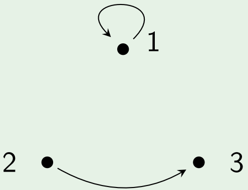
**(AS)** Antisymmetry: $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$ for all $x$, $y$

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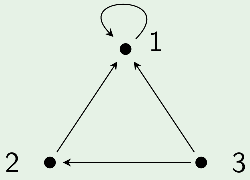
**(AS)** Antisymmetry: $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$ for all $x$, $y$

**(T)** Transitivity: $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ for all $x$, $y$, $z$.

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# Interaction of Properties

A relation *can* be both symmetric and antisymmetric. Namely, when $R$ consists only of some pairs $(x, x), x \in S$.
A relation *cannot* be simultaneously reflexive and antireflexive (unless $S = \emptyset$).

**NB**

$$\left.\begin{array}{r}\textit{nonreflexive}\\ \textit{nonsymmetric}\end{array}\right\}\ \textit{is not the same as}\ \left\{\begin{array}{l}\textit{antireflexive/irreflexive}\\ \textit{antisymmetric}\end{array}\right.$$

# Exercises

## Exercises

$\boxed{3.1.1}$ The following relations are on $S = \{1, 2, 3\}$.
Which of the properties (R), (AR), (S), (AS), (T) does each satisfy?

(a) $(m, n) \in R$ if $m + n = 3$?

(e) $(m, n) \in R$ if $\max\{m, n\} = 3$?

$\boxed{3.1.2(b)}$ $(m, n) \in R$ if $m < n$?

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# Exercises

## Exercises

Complete the following table of common relations (over $\mathbb{Z}$) and their properties:

| | $(R)$ | $(AR)$ | $(S)$ | $(AS)$ | $(T)$ |
|---|---|---|---|---|---|
| $=$ | | | | | |
| $\leq$ | | | | | |
| $<$ | | | | | |
| $\emptyset$ | | | | | |
| $\mathcal{U} = \mathbb{Z} \times \mathbb{Z}$ | | | | | |
| $\mid$ | | | | | |
| $\equiv \pmod 3$ | | | | | |

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| $\mid$ | ? | | | | |
| $= \pmod 3$ | ? | | | | |

# Exercises

**Exercises**

$\boxed{3.1.10(a)}$ Give examples of relations with specified properties.

(AS), (T), not (R).

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$\boxed{3.1.10(b)}$ Give examples of relations with specified properties.

(S), not (R), not (T).

# Exercises

**Exercises**

$\boxed{\text{3.1.10(b)}}$ Give examples of relations with specified properties.
(S), not (R), not (T).
?

# Exercises

**Exercises**

$\boxed{\text{3.6.10 (supp)}}$

$R$ is a relation on $\mathbb{N} \times \mathbb{N}$, i.e. it is a subset of $\mathbb{N}^2 \times \mathbb{N}^2$

$(m, n)\, R\, (p, q)$ if $m = p \pmod 3$ or $n = q \pmod 5$.

(a) Is $R$ reflexive?

(b) Is $R$ symmetric?

(c) Is $R$ transitive?

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- Defining binary relations
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# Equivalence relations

Equivalence relations capture a general notion of "equality". They are relations which are:

- Reflexive (R): Every object should be "equal" to itself
- Symmetric (S): If $x$ is "equal" to $y$, then $y$ should be "equal" to $x$
- Transitive (T): If $x$ is "equal" to $y$ and $y$ is "equal" to $z$, then $x$ should be "equal" to $z$.

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**Definition**

A binary relation $R \subseteq S \times S$ is *equivalence relation* if it satisfies (R), (S), (T).

### Example

Partition of $\mathbb{Z}$ into classes of numbers with the same remainder on division by $p$; it is particularly important for $p$ prime

$$\mathbb{Z}(p) = \mathbb{Z}_p = \{0, 1, \ldots, p-1\}$$

One can define all four arithmetic operations (with the usual properties) on $\mathbb{Z}_p$ for a prime $p$; division has to be restricted when $p$ is not prime.

### NB

*$(\mathbb{Z}_p, +, \cdot, 0, 1)$ are fundamental algebraic structures known as* **rings**. *These structures are very important in coding theory and cryptography.*

# Equivalence Classes and Partitions

Suppose $R \subseteq S \times S$ is an equivalence relation
The **equivalence class** $[s]$ (w.r.t. $R$) of an element $s \in S$ is

$$[s] = \{t : t \in S \text{ and } sRt\}$$

**Fact**

*$s\ R\ t$ if and only if $[s] = [t]$.*

# Partitions

## Definition

A **partition** of a set $S$ is a collection of sets $S_1, \ldots, S_k$ such that

- $S_i$ and $S_j$ are disjoint (for $i \neq j$)
- $S = S_1 \cup S_2 \cup \cdots \cup S_k = \bigcup_{i=1}^{k} S_i$

The collection of all equivalence classes $\{[s] : s \in S\}$ forms a partition of $S$

In the opposite direction, a partition of a set defines the equivalence relation on that set. If $S = S_1 \cup \cdots \cup S_k$, then we can define $\sim \subseteq S \times S$ as:

$$s \sim t \text{ exactly when } s \text{ and } t \text{ belong to the same } S_i.$$

# Exercises

**Exercises**

$\boxed{\text{3.6.6 (supp)}}$ Show that $m \sim n$ iff $m^2 = n^2 \pmod 5$ is an equivalence on $S = \{1, \ldots, 7\}$. Find all the equivalence classes.

# Exercises

**Exercises**

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# Partial Order

A **partial order** $\preceq$ on $S$ satisfies (R), (AS), (T).
We call $(S, \preceq)$ a **poset** — partially ordered set

## Examples

Posets:

- $(\mathbb{Z}, \leq)$
- $(\mathrm{Pow}(X), \subseteq)$ for some set $X$
- $(\mathbb{N}, |)$

Not posets:

- $(\mathbb{Z}, <)$
- $(\mathbb{Z}, |)$

# Hasse diagram

Every finite poset $(S, \preceq)$ can be represented with a **Hasse diagram**:

- Nodes are elements of $S$
- An edge is drawn *upward* from $x$ to $y$ if $x \prec y$ and there is no $z$ such that $x \prec z \prec y$

## Example

Hasse diagram for positive divisors of 24 ordered by $|$:

$$\begin{tikzcd}
 & & & 24 & \\
 & & 12 \arrow[ur, dash] & & 8 \arrow[ul, dash] \\
 & 6 \arrow[ur, dash] & & 4 \arrow[ul, dash] \arrow[ur, dash] & \\
3 \arrow[ur, dash] & & 2 \arrow[ul, dash] \arrow[ur, dash] & & \\
 & 1 \arrow[ul, dash] \arrow[ur, dash] & & &
\end{tikzcd}$$

# Ordering Concepts

## Definition

- **Minimal** and **maximal** elements (they always exist in every finite poset)
- **Minimum** and **maximum** — unique minimal and maximal element (might not exist)
- **lub** (least upper bound) and **glb** (greatest lower bound) of a subset $A \subseteq S$ of elements
  $\mathrm{lub}(A)$ — minimum of $\{x \in S : x \succeq a \text{ for all } a \in A\}$
  $\mathrm{glb}(A)$ — maximum of $\{x \in S : x \preceq a \text{ for all } a \in A\}$
- **Lattice** — poset where $\mathrm{lub}(x, y)$ and $\mathrm{glb}(x, y)$ exist for every pair of elements $x, y$.

# Examples

## Examples

- $\mathsf{Pow}(\{a, b, c\})$ with the order $\subseteq$
  $\emptyset$ is minimum; $\{a, b, c\}$ is maximum
- $\mathsf{Pow}(\{a, b, c\}) \setminus \{\{a, b, c\}\}$ (proper subsets of $\{a, b, c\}$)
  Each two-element subset $\{a, b\}, \{a, c\}, \{b, c\}$ is maximal.
  - But there is no maximum
- $\{1, 2, 3, 4, 6, 8, 12, 24\}$ partially ordered by divisibility is a lattice
  - e.g. $\mathsf{lub}(\{4, 6\}) = 12$; $\mathsf{glb}(\{4, 6\}) = 2$
- $\{1, 2, 3\}$ partially ordered by divisibility is not a lattice
  - $\{2, 3\}$ has no lub
- $\{2, 3, 6\}$ partially ordered by divisibility
  - $\{2, 3\}$ has no glb
- $\{1, 2, 3, 12, 18, 36\}$ partially ordered by divisibility
  - $\{2, 3\}$ has no lub ($12, 18$ are minimal upper bounds)

## NB

*An infinite lattice need not have a lub (or no glb) for an arbitrary infinite subset of its elements, in particular no such bound may exist for* **all** *its elements.*

## Examples

- $\mathbb{Z}$ — neither lub nor glb;
- $\mathbb{F}(\mathbb{N})$ — all finite subsets, has no *arbitrary* lub property; glb exists, it is the intersection, hence always finite;
- $\mathbb{I}(\mathbb{N})$ — all infinite subsets, may not have an arbitrary glb; lub exists, it is the union, which is always infinite.

# Exercises

## Exercises

$\boxed{11.1.5}$ Consider poset $(\mathbb{R}, \leq)$

(a) Is this a lattice?

(b) Give an example of a non-empty subset of $\mathbb{R}$ that has no upper bound.

(c) Find $\text{lub}(\{\, x \in \mathbb{R} : x < 73 \,\})$

(d) Find $\text{lub}(\{\, x \in \mathbb{R} : x \leq 73 \,\})$

(e) Find $\text{lub}(\{\, x : x^2 < 73 \,\})$

(f) Find $\text{glb}(\{\, x : x^2 < 73 \,\})$

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(f) Find $\mathrm{glb}(\{\ x : x^2 < 73\ \})$ ?

# Total orders

## Definition

A **total order** is a partial order that also satisfies:

(L) *Linearity* (any two elements are comparable):

$$\text{For all } x, y \text{ either: } x \leq y \text{ or } y \leq x \text{ (or both if } x = y)$$

## NB

*On a finite set all total orders are "isomorphic"*
*On an infinite set there is quite a variety of possibilities.*

# Examples

**Examples**

- $\mathbb{Z}$ with $\leq$:
  (no minimum/maximum element)
- $\mathbb{Z}$ with $\{(x, y) : x < 0 \leq y$ or $|x| \leq |y|\}$:
  (no maximum element, minimum element is -1)
- $\mathbb{Z}$ with $\{(x, y) : x < 0 \leq y$ or $x \geq y\}$:
  (minimum element -1, maximum element 0)

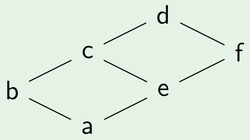
# Ordering of a Poset — Topological Sort

## Definition

For a poset $(S, \preceq)$ any total order $\leq$ that is consistent with $\preceq$ (if $a \preceq b$ then $a \leq b$) is called a **topological sort**.

## Example

Consider




The following all are topological sorts:

$a \leq b \leq e \leq c \leq f \leq d$

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# Well-Ordered Sets

## Definition

A *well-ordered set* is a poset where every subset has a least element.

## NB

*The greatest element is not required.*

## Examples

- $\mathbb{N} = \{0, 1, \ldots\}$
- $\mathbb{N}_1 \dot{\cup} \mathbb{N}_2 \dot{\cup} \mathbb{N}_3 \dot{\cup} \ldots$, where each $\mathbb{N}_i \simeq \mathbb{N}$ and $\mathbb{N}_1 < \mathbb{N}_2 < \mathbb{N}_3 \cdots$

## NB

*Well-ordered sets are an important mathematical tool to prove termination of programs.*

# Combining Orders

**Product order** — can combine any partial orders. In general, it is only a *partial order*, even if combining total orders.
For $s, s' \in S$ and $t, t' \in T$ define

$$(s, t) \preceq (s', t') \quad \text{if } s \preceq s' \text{ and } t \preceq t'$$

# Practical Orderings

They are, effectively, *total* orders on the *product* of ordered sets.

- **Lexicographic order** — defined on all of $\Sigma^*$. It extends a total order already assumed to exist on $\Sigma$.
- **Lenlex** — the order on (potentially) the entire $\Sigma^*$, where the elements are ordered first by length.
  $\Sigma^{(1)} \prec \Sigma^{(2)} \prec \Sigma^{(3)} \prec \cdots$, then lexicographically within each $\Sigma^{(k)}$. In practice it is applied only to the finite subsets of $\Sigma^*$.
- **Filing order** — lexicographic order confined to the strings of the same length.
  It defines total orders on $\Sigma^i$, separately for each $i$.

# Example

## Example

$\boxed{11.2.5}$ Let $\mathbb{B} = \{0, 1\}$ with the usual order $0 < 1$. List the elements $101, 010, 11, 000, 10, 0010, 1000$ of $\mathbb{B}^*$ in the

(a) Lexicographic order

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$\boxed{11.2.8}$ When are the lexicographic order and *lenlex* on $\Sigma^*$ the same?

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$000, 0010, 010, 10, 1000, 101, 11$

(b) Lenlex order

$10, 11, 000, 010, 101, 0010, 1000$

$\boxed{11.2.8}$ When are the lexicographic order and *lenlex* on $\Sigma^*$ the same?

Only when $|\Sigma| = 1$.

# Exercises

## Exercises

$\boxed{11.6.6}$ True or false?

(a) If a set $\Sigma$ is totally ordered, then the corresponding lexicographic partial order on $\Sigma^*$ also must be totally ordered.

(b) If a set $\Sigma$ is totally ordered, then the corresponding lenlex order on $\Sigma^*$ also must be totally ordered.

(c) Every finite partially ordered set has a Hasse diagram.

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