COMP9020 Week 3 Recap and Administrivia

Administrivia

- Assignment 1 due at **23:59 this Sunday**
- Challenge questions posted
- Quiz 5

Week 3 Recap

Relations and functions:

- Binary relations:
  - Equivalence relations, equivalence classes
  - Partial orders, Hasse diagram, lub, glb
  - Total orders, topological sorting, lexicographic, lenlex
- Functions:
  - Injective, surjective, bijective
  - Inverse function
  - Matrices

Equivalence classes: Proof example

Fact

Let $R \subseteq S \times S$ *be an equivalence relation. Then for all* $s, t \in S$*:*

$$[s] = [t] \quad \textit{if and only if} \quad (s, t) \in R.$$

Equivalence classes: Proof example

Proof

Suppose $[s] = [t]$. Recall $[s] = \{x \in S : (s, x) \in R\}$. We will show that $(s, t) \in R$.

Because R is reflexive, $(t, t) \in R$.

Therefore $t \in [t]$.

Because $[t] = [s]$, it follows that $t \in [s]$.

But then $(s, t) \in R$ by the definition of $[s]$.

Equivalence classes: Proof example

Proof

Now suppose $(s, t) \in R$. We will show $[s] = [t]$ by showing $[s] \subseteq [t]$ and $[t] \subseteq [s]$.

Take any $x \in [s]$.

By the definition of $[s]$, $(s, x) \in R$.

Since R is symmetric $(x, s) \in R$.

Since R is transitive and $(s, t) \in R$ we have that $(x, t) \in R$.

Since R is symmetric $(t, x) \in R$.

Therefore, $x \in [t]$.

Therefore $[s] \subseteq [t]$.

Equivalence classes: Proof example

Proof

Now suppose $(s, t) \in R$. We will show $[s] = [t]$ by showing $[s] \subseteq [t]$ and $[t] \subseteq [s]$.

Take any $x \in [t]$.

By the definition of $[t]$, $(t, x) \in R$.

Since R is transitive and $(s, t) \in R$ we have that $(s, x) \in R$.

Therefore $x \in [s]$.

Therefore $[t] \subseteq [s]$. $\square$

glb and lub

Definition

Let $R \subseteq S \times S$ be a partial order and let $A \subseteq S$.

- x is an **upper bound** for A if $(a, x) \in R$ for all $a \in A$
- x is a **lower bound** for A if $(x, a) \in R$ for all $a \in A$
- The **set of upper bounds** for A is defined as $\mathrm{ub}(A) = \{x : (a, x) \in R \text{ for all } a \in A\}$
- The **set of lower bounds** for A is defined as $\mathrm{lb}(A) = \{x : (x, a) \in R \text{ for all } a \in A\}$
- The **least upper bound** of A, $\mathrm{lub}(A)$, is the minimum of $\mathrm{ub}(A)$ (if it exists)
- The **greatest lower bound** of A, $\mathrm{glb}(A)$ is the maximum of $\mathrm{lb}(A)$ (if it exists)

NB

Greatest and Least correspond to maximum and minimum resp.

glb and lub

To show x is glb$(()A)$ you need to show:

- $(x, a) \in R$ for all $a \in A$: x is a lower bound
- If $(y, a) \in R$ for all $a \in A$ then $(y, x) \in R$: x is the greatest of all lower bounds

Example

Pow(X) ordered by $\subseteq$.

- $\text{glb}(A, B) = A \cap B$
- $\text{lub}(A, B) = A \cup B$

glb and lub examples

Examples

- $\mathbb{F}(\mathbb{N})$ — all finite subsets, has no *arbitrary* lub property; glb exists, it is the intersection, hence always finite;
- $\mathbb{I}(\mathbb{N})$ — all infinite subsets, may not have an arbitrary glb; lub exists, it is the union, which is always infinite.

Need to know for this course

- Binary relations:
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- Functions:
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  - Inverse function
  - Matrices