COMP9020 Week 3
Functions and Big-O notation

- Textbook (R & W) - Ch. 1, Sec. 1.7; Ch. 4., Sec. 4.3

Applications of Functions and Big-O notation

- Functions, methods, procedures in programming
- Computer programs "are" functions
- Graphical transformations
- Algorithmic analysis

Summary of topics

- Functions recap
- Inverse functions
- Matrices
- Introduction to big-O notation

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Functions

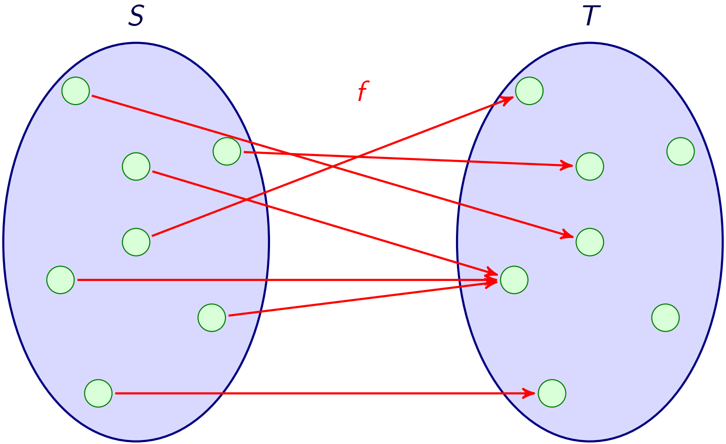
Definition

A **function**, $f : S \rightarrow T$, is a binary relation $f \subseteq S \times T$ such that for all $s \in S$ there is *exactly one* $t \in T$ such that $(s, t) \in f$.

We write $f(s)$ for the unique element related to s.

A **partial function** $f : S \nrightarrow T$ is a binary relation $f \subseteq S \times T$ such that for all $s \in S$ there is *at most one* $t \in T$ such that $(s, t) \in f$. That is, it is a function $f : S' \longrightarrow T$ for $S' \subseteq S$

Graphical representation



Functions

$f : S \longrightarrow T$ describes pairing of the sets: it means that f assigns to every element $s \in S$ a unique element $t \in T$. To emphasise where a specific element is sent, we can write $f : x \mapsto y$, which means the same as $f(x) = y$

		Symbol	
S	**domain** of f	$\mathrm{Dom}(f)$	(inputs)
T	**co-domain** of f	$\mathrm{Codom}(f)$	(*possible* outputs)
$f(S)$	**image** of f	$\mathrm{Im}(f)$	(*actual* outputs)
$= \{ f(x) : x \in \mathrm{Dom}(f) \}$			

Important!

The domain and co-domain are critical aspects of a function's definition.

$$f : \mathbb{N} \to \mathbb{Z} \quad \text{given by} \quad f(x) \mapsto x^2$$

and

$$g : \mathbb{N} \to \mathbb{N} \quad \text{given by} \quad g(x) \mapsto x^2$$

are different functions even though they have the same behaviour!

Composition of Functions

Composition of functions is described as

$$g \circ f : x \mapsto g(f(x)), \quad \text{requiring } \operatorname{Im}(f) \subseteq \operatorname{Dom}(g)$$

Composition is associative

$$h \circ (g \circ f) = (h \circ g) \circ f, \quad \text{can write } h \circ g \circ f$$

Composition of Functions

If a function maps a set into itself, i.e. when $\mathrm{Dom}(f) = \mathrm{Codom}(f)$ (and thus $\mathrm{Im}(f) \subseteq \mathrm{Dom}(f)$), the function can be composed with itself — **iterated**

$$f \circ f, f \circ f \circ f, \ldots, \quad \text{also written } f^2, f^3, \ldots$$

Identity function on S

$$\mathrm{Id}_S(x) = x, x \in S; \mathrm{Dom}(\mathrm{Id}_S) = \mathrm{Codom}(\mathrm{Id}_S) = \mathrm{Im}(\mathrm{Id}_S) = S$$

For $g : S \longrightarrow T \quad g \circ \mathrm{Id}_S = g, \ \mathrm{Id}_T \circ g = g$

Properties of Functions

Function is called **surjective** or **onto** if every element of the codomain is mapped to by at least one x in the domain, i.e.

$$\mathrm{Im}(f) = \mathrm{Codom}(f)$$

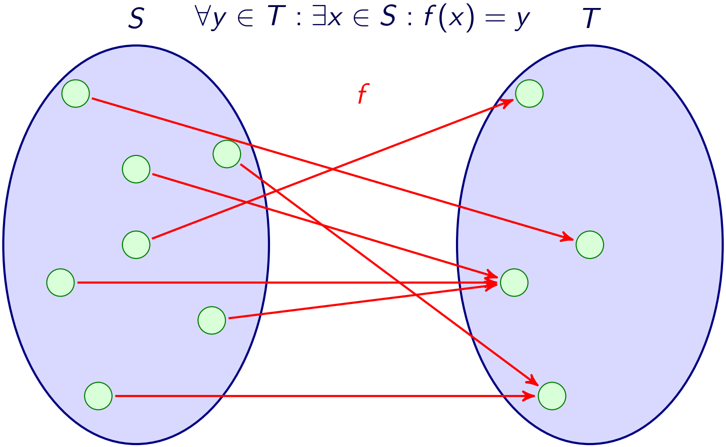
Examples (of functions that are surjective)

- $f : \mathbb{N} \longrightarrow \mathbb{N}$ with $f(x) \mapsto x$
- Floor, ceiling

Examples (of functions that are not surjective)

- $f : \mathbb{N} \longrightarrow \mathbb{N}$ with $f(x) \mapsto x^2$
- $f : \{a, \ldots, e\}^* \longrightarrow \{a, \ldots, e\}^*$ with $f(w) \mapsto awe$

Graphical representation: Surjective



Injective Functions

Function is called **injective** or **1–1** (**one-to-one**) if different x implies different $f(x)$, i.e.

$$\text{If } f(x) = f(y) \text{ then } x = y$$

Examples (of functions that are injective)

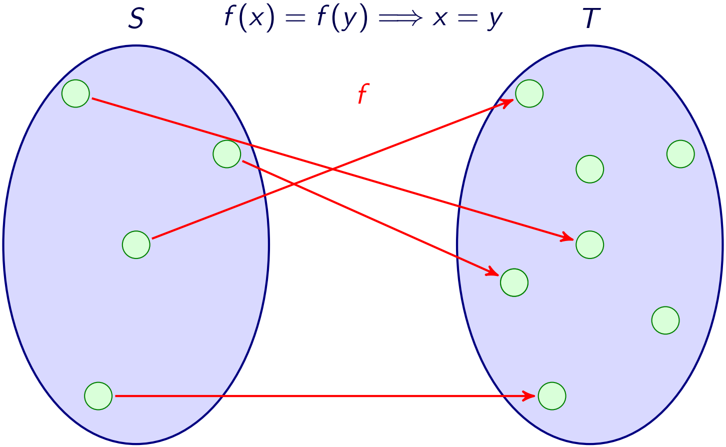
- $f : \mathbb{N} \longrightarrow \mathbb{N}$ with $f(x) \mapsto x$
- set complement (for a fixed universe)

Examples (of functions that are not injective)

- absolute value, floor, ceiling
- length of a word

Function is **bijective** if it is both surjective and injective.

Graphical representation: Injective



Functions on finite sets

NB

For a **finite** *set* S *and* $f : S \longrightarrow S$ *the properties*

1. *surjective, and*
2. *injective*

are equivalent.

Converse of a function

Question

$f^{\leftarrow}$ *is a relation; when is it a function?*

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?

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Inverse Functions

Definition

If $f^{\leftarrow}$ is a function then it is called the **inverse function**; denoted f^{-1}.

NB

f^{-1} only exists if f is a bijection.
$f^{\leftarrow}$ always exists.

f^{-1} is the procedure of "undoing" f.

Properties of the inverse

Fact

If $f : S \to T$ and f^{-1} exists then:

$$f^{-1} \circ f = Id_S \quad \text{and} \quad f \circ f^{-1} = Id_T.$$

Conversely, if $f : S \to T$ and $g : T \to S$ and

$$g \circ f = Id_S \quad \text{and} \quad f \circ g = Id_T$$

then f^{-1} exists and is equal to g.

Exercises

Exercises

$\boxed{1.7.5}$ f and g are 'shift' functions $\mathbb{N} \longrightarrow \mathbb{N}$ defined by $f(n) = n + 1$, and $g(n) = \max(0, n - 1)$

(c) Is f injective? surjective?

(d) Is g injective? surjective?

(e) Do f and g commute, i.e. $\forall n \, ((f \circ g)(n) = (g \circ f)(n))$?

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Exercises

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$\boxed{1.7.6}$ $\Sigma = \{a, b, c\}$

(c) Is length : $\Sigma^* \longrightarrow \mathbb{N}$ surjective?

(d) length$^{\leftarrow}(2) \stackrel{?}{=}$

$\boxed{1.7.12}$ Verify that $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$ defined by $f(x, y) = (x + y, x - y)$ is invertible.

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Matrices

An $\mathbf{m} \times \mathbf{n}$ **matrix** is a rectangular array with m horizontal rows and n vertical columns.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

NB

Matrices are important objects in Computer Science, e.g. for

- *optimisation*
- *graphics and computer vision*
- *cryptography*
- *information retrieval and web search*
- *machine learning*

Matrices

NB

A matrix represents a **linear transformation***: a function that maps vectors to vectors in a "nice" way.*
Matrix multiplication corresponds to the composition of transformations.

Basic Matrix Operations

The **transpose** $\mathbf{A}^\mathsf{T}$ of an $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ is the $n \times m$ matrix whose entry in the ith row and jth column is a_{ji}.

Example

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix} \qquad \mathbf{A}^\mathsf{T} = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix}$$

NB

A matrix $\mathbf{M}$ *is called symmetric if* $\mathbf{M}^\mathsf{T} = \mathbf{M}$

The **sum** of two $m \times n$ matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ is the $m \times n$ matrix whose entry in the ith row and jth column is $a_{ij} + b_{ij}$.

Example

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 5 & 3 \\ 2 & 3 & -2 & 1 \\ 4 & -2 & 0 & 2 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 3 & -1 & 5 & 7 \\ 5 & 5 & -3 & 3 \\ 8 & -2 & 1 & 5 \end{bmatrix}$$

Fact

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ *and* $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

Given $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ and $c \in \mathbb{R}$, the **scalar product** $c\mathbf{A}$ is the $m \times n$ matrix whose entry in the ith row and jth column is $c \cdot a_{ij}$.

Example

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix} \qquad 2\mathbf{A} = \begin{bmatrix} 4 & -2 & 0 & 8 \\ 6 & 4 & -2 & 4 \\ 8 & 0 & 2 & 6 \end{bmatrix}$$

The **product** of an $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ and an $n \times p$ matrix $\mathbf{B} = [b_{jk}]$ is the $m \times p$ matrix $\mathbf{C} = [c_{ik}]$ defined by

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} \qquad \text{for } 1 \le i \le m \text{ and } 1 \le k \le p$$

Example

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

NB

The rows of $\mathbf{A}$ *must have the same number of entries as the columns of* $\mathbf{B}$.

The product of a $1 \times n$ *matrix and an* $n \times 1$ *matrix is usually called the* **inner product** *of two* **n-dimensional vectors.**

Example

Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$$

Calculate $\mathbf{AB}$, $\mathbf{BA}$

Example

Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$$

Calculate **AB**, **BA**

$$\mathbf{AB} = \begin{bmatrix} -10 & 5 \\ -20 & 10 \end{bmatrix} \qquad \mathbf{BA} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

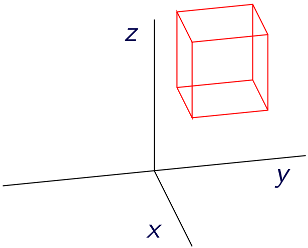
NB

In general, $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$

Example: Computer Graphics

Rotating an object w.r.t. the x axis by degree α:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} 5 & 5 & 7 & 7 & 5 & 7 & 5 & 7 \\ 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 \\ 9 & 7 & 7 & 9 & 7 & 7 & 9 & 9 \end{bmatrix}$$



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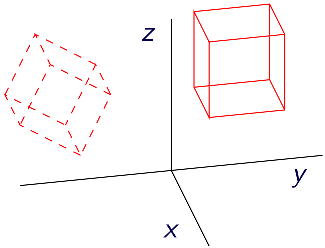

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Motivation

Want to compare algorithms – particularly ones that can solve *arbitrarily large* instances.

We would like to be able to talk about the resources (running time, memory, energy consumption) required by a program/algorithm as a function $f(n)$ of the size n of its input.

Example

How long does a given sorting algorithm take to run on a list of n elements?

Problem 1: the exact running time may depend on

- compiler optimisations
- processor speed
- cache size

Each of these may affect the resource usage by up to a *linear* factor, making it hard to state a general claim about running times.

Problem 2: Many algorithms that arise in practice have resource usage that can be expressed only as a rather complicated function. E.g.

$$f(n) = 20n^2 + 2n\log(n) + (n-100)\log(n)^2 + \frac{1}{2^n}\log(\log(n))$$

The main contribution to the value of the function for "large" input sizes n is the term of the *highest order*:

$$20n^2$$

We would like to be able to *ignore the terms of lower order*

$$2n\log(n) + (n-100)\log(n)^2 + \frac{1}{2^n}\log(\log(n))$$

Order of Growth

Example

Consider two algorithms, one with running time $f_1(n) = \frac{1}{10}n^2$, the other with running time $f_2 = 10n \log n$ (measured in milliseconds).

Input size	$f_1(n)$	$f_2(n)$
100	0.01s	2s
1000	1s	30s
10000	1m40s	6m40s
100000	2h47m	1h23m
1000000	11d14h	16h40h
10000000	3y3m	8d2h

Order of growth provides a way to abstract away from these two problems, and focus on what is essential to the size of the function, by saying that "the (complicated) function f is of *roughly the same size* (for large input) as the (simple) function g"

NB

Asymptotic analysis is about how costs **scale** *as the input increases.*

"Big-Oh" Asymptotic Upper Bounds

Definition

Let $f, g : \mathbb{N} \to \mathbb{R}$. We say that g is *asymptotically less than* f (or: f **is an upper bound of** g) if there exists $n_0 \in \mathbb{N}$ and a real constant $c > 0$ such that for all $n \geq n_0$,

$$g(n) \leq c \cdot f(n)$$

Write $O(f(n))$ for the class of all functions g that are asymptotically less than f.

Example

$$g(n) = 3n + 1 \quad \rightarrow \quad g(n) \leq 4n\,, \text{ for all } n \geq 1$$

Therefore, $3n + 1 \in O(n)$

Example

$$\frac{1}{10}n^2 \in O(n^2) \qquad 10n\log n \in O(n\log n) \qquad O(n\log n) \subsetneq O(n^2)$$

The traditional notation has been

$$g(n) = O(f(n))$$

instead of $g(n) \in O(f(n))$.

It allows one to use $O(f(n))$ or similar expressions as part of an equation; of course these 'equations' express only an approximate equality. Thus,

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + O(n)$$

means

"There exists a function $f(n) \in O(n)$ such that $T(n) = 2T(\frac{n}{2}) + f(n)$."

Alternative definition

Fact

$$f(n) \in O(g(n)) \quad \textit{if and only if} \quad \lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty.$$

Examples

$$5n^2 + 3n + 2 = O(n^2)$$

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$$n^3 + 2^{100} n^2 + 2n + 2^{2^{100}} = O(n^3)$$

Examples

$$5n^2 + 3n + 2 = O(n^2)$$

$$n^3 + 2^{100} n^2 + 2n + 2^{2^{100}} = O(n^3)$$

Generally, for constants $a_k \ldots a_0$,

$$a_k n^k + a_{k-1} n^{k-1} + \ldots + a_0 = O(n^k)$$

“Big-Omega” Asymptotic Lower Bounds

Definition

Let $f, g : \mathbb{N} \to \mathbb{R}$. We say that g is *asymptotically greater than* f (or: f **is an lower bound of** g) if there exists $n_0 \in \mathbb{N}$ and a real constant $c > 0$ such that for all $n \geq n_0$,

$$g(n) \geq c \cdot f(n)$$

Write $\Omega(f(n))$ for the class of all functions g that are asymptotically greater than f.

Example

$$g(n) = 3n + 1 \quad \to \quad g(n) \geq 3n\,, \text{ for all } n \geq 1$$

Therefore, $3n + 1 \in \Omega(n)$

“Big-Theta” Notation

Definition

Two functions f, g have the *same order of growth* if they scale up in the same way:
There exists $n_0 \in \mathbb{N}$ and real constants $c > 0, d > 0$ such that for all $n \geq n_0$,

$$c \cdot f(n) \leq g(n) \leq d \cdot f(n)$$

Write $\Theta(f(n))$ for the class of all functions g that have the same order of growth as f.

If $g \in O(f)$ (or $\Omega(f)$) we say that f is an *upper bound* (*lower bound*) on the order of growth of g; if $g \in \Theta(f)$ we call it a **tight bound**.

Observe that, somewhat symmetrically

$$g \in \Theta(f) \iff f \in \Theta(g)$$

We obviously have

$$\Theta(f(n)) \subseteq O(f(n)) \qquad \text{and} \qquad \Theta(f(n)) \subseteq \Omega(f(n)),$$

in fact

$$\Theta(f(n)) = O(f(n)) \cap \Omega(f(n)).$$

At the same time the 'Big-Oh' is *not* a symmetric relation

$$g \in O(f) \not\Rightarrow f \in O(g),$$

but

$$g \in O(f) \Leftrightarrow f \in \Omega(g)$$

More Examples

- All logarithms $\log_b x$ have the same order, irrespective of the value of b

$$O(\log_2 n) = O(\log_3 n) = \ldots = O(\log_{10} n) = \ldots$$

- Exponentials r^n, s^n to different bases $r < s$ have different orders, e.g. there is no $c > 0$ such that $3^n < c \cdot 2^n$ for all n

$$O(r^n) \subsetneq O(s^n) \subsetneq O(t^n) \ldots \quad \text{for} \quad r < s < t \ldots$$

- Similarly for polynomials

$$O(n^k) \subsetneq O(n^l) \subsetneq O(n^m) \ldots \quad \text{for} \quad k < l < m \ldots$$

Here are some of the most common functions occurring in the analysis of the performance of programs (algorithm complexity):

$1,\ \log\log n,\ \log n,\ \sqrt{n}, \sqrt{n}(\log n)^k, \sqrt{n}(\log n)^2, \ldots$
$n,\ n\log\log n,\ n\log n,\ n^{1.5},\ n^2,\ n^3, \ldots$
$2^n,\ 2^n\log n,\ n2^n,\ 3^n,\ \ldots$
$n!,\ n^n,\ n^{2n}, \ldots,\ n^{n^2},\ n^{2^n}, \ldots$

Notation: $O(1) \equiv$ const, although technically it could be any function that varies between two constants c and d.

Exercises

Exercises

$\boxed{4.3.5}$ True or false?

(a) $2^{n+1} = O(2^n)$

(b) $(n+1)^2 = O(n^2)$

(c) $2^{2n} = O(2^n)$

(d) $(200n)^2 = O(n^2)$

$\boxed{4.3.6}$ True or false?

(b) $\log(n^{73}) = O(\log n)$

(c) $\log n^n = O(\log n)$

(d) $(\sqrt{n}+1)^4 = O(n^2)$

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