2.d : $R^k = R^{k+1}$

assume $(a, c) \in R^{k+1}$

and we also assume $(a, c) \in R^k$

$\because R^{k+1} = R^k \cup (R ; R^k)$

and $(a, c) \notin R^k$

$\therefore (a, c) \in (R ; R^k)$

$\therefore \exists (a, m_k) \in R \quad (m_k, c) \in R^k$

$\because (m_k, c) \notin R^i$ $i \leq k-1$ [if $(m_k, c) \in R^{k-1}$ and $(a, m_k) \in R$ we can get $(a, c) \in R^k$]

so $(m_k, c) \in R ; R^{k-2}$

$\therefore \exists (m_k, m_{k-1}) \in R \quad (m_{k-1}, c) \in R^{k-1}$

we still have $(m_{k-1}, c) \notin R^i \quad (i \leq k-2)$

[because, if $(m_{k-1}, c) \in R^{k-2}$, then $(m_k, c) \in R^{k-1}$]

and so on ...

arrange the result

(a, m_k) $\qquad (m_k, c) \notin R^i \quad i \le k-1$

(m_k, m_{k-1}) $\qquad (m_{k-1}, c) \notin R^i \quad i \le k-2$

(m_{k-1}, m_{k-2}) $\qquad (m_{k-2}, c) \notin R^i \quad i \le k-3$

$\vdots$

(m_3, m_2) $\qquad (m_2, c) \notin R^i \quad i \le 1$

(m_2, m_1) $\qquad (m_1, c) \notin R^i \quad i \le 1$

(m_1, m_0) $\qquad (m_0, c) \notin R^0$

and $m_0 \ne m_1 \ne m_2 \cdots \ne m_k$

$m_i \in S$

$|\{m_0, m_1, m_2 \cdots m_k\}| = K = |S|$

$\therefore a \in \{m_0, m_1 \cdots m_k\}$

$\therefore (a, c) \in R^k$

与假设矛盾（假设 $(a, c) \notin R^k$）

$\therefore (a, c) \in R^k$

此题得证

2、e. 证明 R^k is T

assume $(a,b),(b,c)\in R^k$

we assume $(a,c)\notin R^k$

① $(a,b),(b,c)\in R$

矛盾，$(a,c)\in R^k$

② $(a,b)\in R$ or $(b,c)\in R$

两者有一个成立

当 $(a,b)\in R$，显然矛盾

当 $(b,c)\in R$，$(a,b)\Rightarrow (a,m_k),(m_k,b)\in R^k$

$m_k\neq b$ 且 $(m_k,b)\notin R$ [if $(m_k,b)\in R$，$(a,c)\in R^k$]

$\therefore (m_k,b)\Rightarrow (m_k,m_{k-1})\in R$，$(m_{k-1},b)\in R^{k-2}$ [$(m_{k-1},b)\in R$] ……

以此类推，最终可得 $\{m_k, m_{k-1}, m_{k-2}, m_{k-3}, \dots m_1\}$ $1\le |S|\le k$

其中有 $m_i=a$

$\therefore (a,b)\in R$

与假设矛盾

$\therefore$ 此情况矛盾

③ $(a,b)\notin R$ and $(b,c)\notin R$

(a,m_k) $(m_k,b)\in R$,

(b,n_k) $(n_k,c)\in R$

$\because (a,b)\in R$ $\therefore m_k\neq b$, $n_k\neq c$

即 $(b,b)\notin R^i$ $(c,c)\notin R^i$

矛盾，因此可证 $(a,c)\in R^k$，T

2. f: $Q = (R \cup R^{\epsilon})$

① R: $Q_0 = \{(x,x) \mid x \in S\}$

$Q_0 \subseteq Q^R$

∴ R 成立

S. ① $(a,b) \in Q$

$\exists (b,a) \in Q$

due to $R \cup R^{\epsilon}$

② $(a,b) \notin Q$

∴ $\exists (a, m_k) \in Q$

$(m_k, b) \in Q^R$

∵ $Q \neq R \cup R^{\epsilon}$

∴ $(m_k, a) \in Q$.

③ if $(m_k, b) \in Q$ 得证

if $(m_k, b) \notin Q$, $\exists$

$\exists (m_k, m_{k-1}) \in Q$, $(m_{k-1}, b) \in Q^R$

重复步骤③

最终我们找到一个 $\in Q$ 的 (m_{iQ}, b)

这样有：$(a, m_k) \in R$

$(m_k, m_{k-1}) \in R$

$(m_{k-1}, m_{k-2}) \in R$

$\vdots$

$(m_i, b) \in R$.

$\because R \cup R^{c}$

$\therefore (m_k, a), (m_{k-1}, m_k), (m_{k-2}, m_{k-1}) \ldots (b, m_i) \in Q$

$\therefore (b, a) \in Q$

因此，S 成立

T: 由上一问可证

T 成立

Since 是 equavilance 关系