

**Due: Sunday, 24th November, 23:59**

Submission is through WebCMS/give and should be a single pdf file, maximum size 2Mb. Prose should be typed, not handwritten. Use of  $\text{\LaTeX}$  is encouraged, but not required.

Discussion of assignment material with others is permitted, but the work submitted *must* be your own in line with the University's plagiarism policy.

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**Problem 1**

(40 marks)

For this question, let  $F$  denote the set of well-formed formulas over a set  $\text{PROP}$  of propositional variables.

- (a) Show that the logical equivalence relation,  $\equiv$ , is an equivalence relation on  $F$ . (12 marks)
- (b) List four elements in  $[\perp]$ , the equivalence class of  $\perp$ . (4 marks)
- (c) For all  $\varphi, \varphi', \psi, \psi' \in F$  with  $\varphi \equiv \varphi'$  and  $\psi \equiv \psi'$ ; show that:
  - (i)  $\neg\varphi \equiv \neg\varphi'$  (4 marks)
  - (ii)  $\varphi \wedge \psi \equiv \varphi' \wedge \psi'$  (4 marks)
  - (iii)  $\varphi \vee \psi \equiv \varphi' \vee \psi'$  (4 marks)

Let us define  $F_{\equiv}$  to be the set of **equivalence classes** of  $F$  under  $\equiv$ . That is,

$$F_{\equiv} := \{[\varphi] : \varphi \in F\}.$$

Part (c) above shows that the following operations are *well-defined*<sup>1</sup> on  $F_{\equiv}$ :

- $[\varphi] \wedge [\psi]$  defined to be  $[\varphi \wedge \psi]$
- $[\varphi] \vee [\psi]$  defined to be  $[\varphi \vee \psi]$
- $[\varphi]'$  defined to be  $[\neg\varphi]$

- (d) Show that  $F_{\equiv}$  together with the operations defined above forms a Boolean Algebra. Note: you will have to give a suitable definition of a zero element and a one element in  $F_{\equiv}$ . (12 marks)

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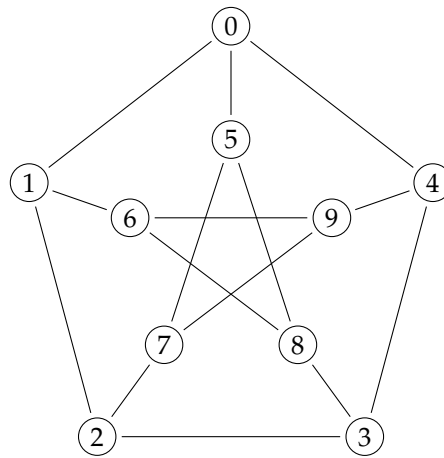
<sup>1</sup>well-defined means that the output is not dependent on different representations of the same input.

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**Problem 2**

(10 marks)

This is the Petersen graph:



- (a) Show that the Petersen graph contains a subdivision of  $K_5$ . (5 marks)
- (b) Show that the Petersen graph contains a subdivision of  $K_{3,3}$ . (5 marks)

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**Problem 3**

(10 marks)

Harry would like to take each of the following subjects: Defence against the Dark Arts; Potions; Herbology; Transfiguration; and Charms. Unfortunately some of the classes clash, meaning Harry cannot take them both. The list of clashes are:

- Defence against the Dark Arts clashes with Potions and Charms
- Potions also clashes with Herbology
- Herbology also clashes with Transfiguration, and
- Transfiguration also clashes with Charms.

Harry would like to know the **maximum** number of classes he can take.

- (a) Model this as a graph problem. Remember to:
- (i) Clearly define the vertices and edges of your graph. (4 marks)
- (ii) State the associated graph problem that you need to solve. (2 marks)
- (b) Give the solution to the graph problem corresponding to this scenario; and solve Harry's problem. (4 marks)

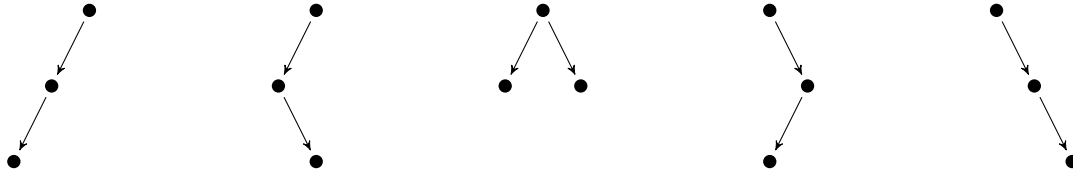
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**Problem 4**

(22 marks)

Recall from Assignment 2 the definition of a binary tree data structure: either an empty tree, or a node with two children that are trees.

Let  $T(n)$  denote the number of binary trees with  $n$  nodes. For example  $T(3) = 5$  because there are five binary trees with three nodes:



- (a) Using the recursive definition of a binary tree structure, or otherwise, derive a recurrence equation for  $T(n)$ . (8 marks)

A **full binary tree** is a non-empty binary tree where every node has either two non-empty children (i.e. is a fully-internal node) or two empty children (i.e. is a leaf).

- (b) Using observations from Assignment 2, or otherwise, explain why a full binary tree must have an odd number of nodes. (4 marks)
- (c) Let  $B(n)$  denote the number of full binary trees with  $n$  nodes. Derive an expression for  $B(n)$ , involving of  $T(n')$  where  $n' \leq n$ . *Hint: Relate the internal nodes of a full binary tree to  $T(n)$ .* (6 marks)

A well-formed formula is in **Negated normal form** if it consists of just  $\wedge$ ,  $\vee$ , and literals (i.e. propositional variables or negations of propositional variables). That is, a formula that results after two steps of the process for transforming a formula into a logically equivalent one. For example,  $p \vee (\neg q \wedge \neg r)$  is in negated normal form; but  $p \vee \neg(q \vee r)$  is not.

Let  $F(n)$  denote the number of well-formed, negated normal form formulas<sup>2</sup> there are that use precisely  $n$  propositional variables exactly one time each. So  $F(1) = 2$ ,  $F(2) = 16$ , and  $F(4) = 15360$ .

- (d) Using your answer for part (c), give an expression for  $F(n)$ . (4 marks)

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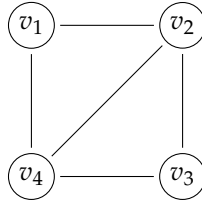
<sup>2</sup>Note: we do not assume  $\wedge$  and  $\vee$  are associative

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**Problem 5**

(18 marks)

Consider the following graph:



and consider the following process:

- Initially, start at  $v_1$ .
- At each time step, choose one of the vertices adjacent to your current location uniformly at random, and move there.

Let  $p_1(n)$ ,  $p_2(n)$ ,  $p_3(n)$ ,  $p_4(n)$  be the probability your location after  $n$  time steps is  $v_1$ ,  $v_2$ ,  $v_3$ , or  $v_4$  respectively. So  $p_1(0) = 1$  and  $p_2(0) = p_3(0) = p_4(0) = 0$ .

- (a) Express  $p_1(n+1)$ ,  $p_2(n+1)$ ,  $p_3(n+1)$ , and  $p_4(n+1)$  in terms of  $p_1(n)$ ,  $p_2(n)$ ,  $p_3(n)$ , and  $p_4(n)$ . (6 marks)
- (b) As  $n$  gets larger, each  $p_i(n)$  converges to a single value (called the steady state) which can be determined by setting  $p_i(n+1) = p_i(n)$  in the above equations. Determine the steady state probabilities for all vertices. (8 marks)
- (c) The distance between any two vertices is the length of the shortest path between them. What is your expected distance from  $v_1$  in the steady state? (4 marks)

## Advice on how to do the assignment

All submitted work must be done individually without consulting someone else's solutions in accordance with the University's "Academic Dishonesty and Plagiarism" policies.

- Assignments are to be submitted via WebCMS (or give) as a single pdf file.
- Be careful with giving multiple or alternative answers. If you give multiple answers, then we will give you marks only for your worst answer, as this indicates how well you understood the question.
- Some of the questions are very easy (with the help of the lecture notes or book). You can use the material presented in the lecture or book (without proving it). You do not need to write more than necessary (see comment above).
- When giving answers to questions, we always would like you to prove/explain/motivate your answers.
- If you use further resources (books, scientific papers, the internet,...) to formulate your answers, then add references to your sources.