COMP9020 Week 2
Binary Relations

- Textbook (R & W) - Ch. 3., Sec. 3.1, 3.4; Ch. 11, Sec. 11.1

Applications in Computer Science

Many relations that appear in CS fall into two broad categories:

Equivalence relations (generalizing "equality"):

- Programs that exhibit the same behaviour
- Logically equivalent statements
- The `.equals()` method in Java

Partial orders (generalizing "less than or equal to"):

- Object inheritance
- Simulation
- Requirement specifications
- The `.compareTo()` method in Java

Summary of topics

- Defining binary relations
- Properties of binary relations
- Equivalence relations, classes, and partitions
- Orderings

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Binary relations

A **binary relation between** S **and** T is a subset of $S \times T$: i.e. a set of ordered pairs.
Also: over S and T; from S to T; on S (if $S = T$).

Example (Special (Trivial) Relations)

Identity (diagonal, equality) $E = \{ (x, x) : x \in S \}$

Empty $\emptyset$

Universal $U = S \times S$

Defining binary relations: Set-based definitions

Defining a relation $R \subseteq S \times T$:

- Explicitly listing tuples: e.g. $\{(1,1),(2,3),(3,2)\}$
- Set comprehension: $\{(x,y) \in [1,3] \times [1,3] : 5 | xy - 1\}$
- Construction from other relations:
  $\{(1,1)\} \cup \{(2,3)\} \cup \{(2,3)\}^{\leftarrow}$

Defining binary relations: Matrix representation

Defining a relation $R \subseteq S \times T$:

Rows enumerated by elements of S, columns by elements of T:

Examples

- The relation $\{(1,1),(2,3),(3,2)\} \subseteq [1,3] \times [1,3]$:

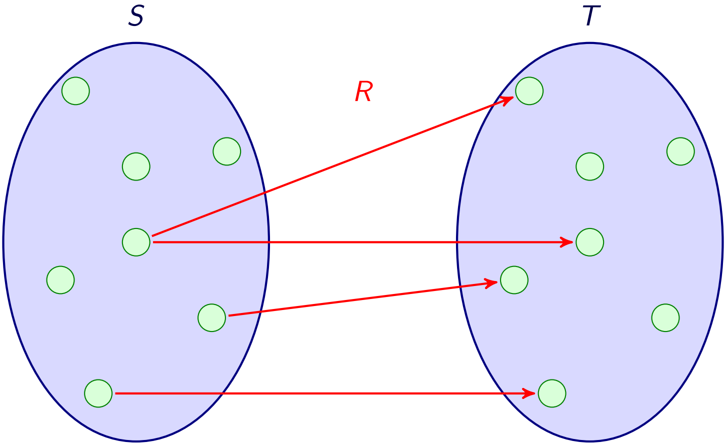
$$\begin{bmatrix} \bullet & \circ & \circ \\ \circ & \circ & \bullet \\ \circ & \bullet & \circ \end{bmatrix}$$

- The relation $\{(1,1),(1,2),(1,3),(1,4),(2,2),(3,2)\} \subseteq [1,3] \times [1,4]$:

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \circ & \circ \\ \circ & \bullet & \circ & \circ \end{bmatrix}$$

Defining binary relations: Graphical representation

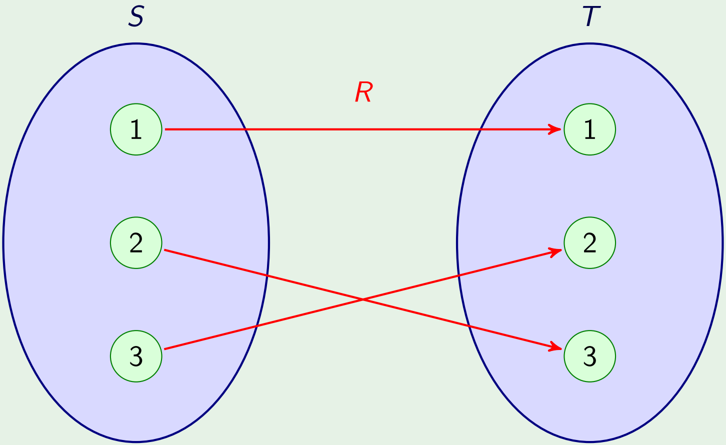
Defining a relation $R \subseteq S \times T$:



Defining binary relations: Graphical representation

Example

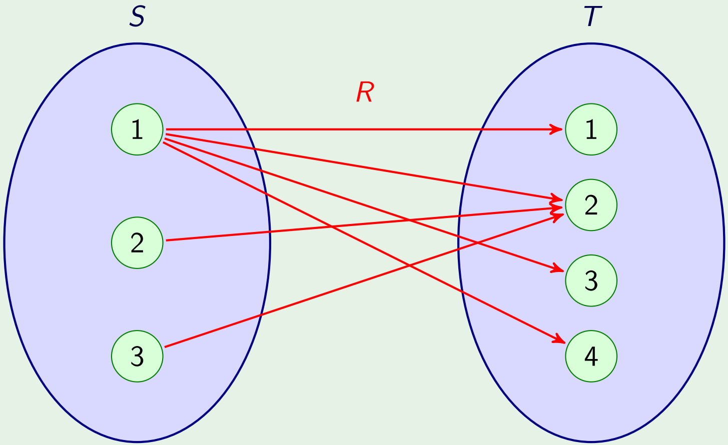
$R = \{(1,1),(2,3),(3,2)\} \subseteq [1,3] \times [1,3]$:



Defining binary relations: Graphical representation

Example

$\{(1,1),(1,2),(1,3),(1,4),(2,2),(3,2)\} \subseteq [1,3] \times [1,4]$:



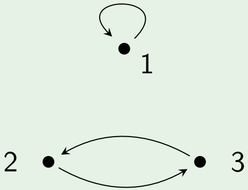
Defining binary relations: Graph representation

If $S = T$ we can define $R \subseteq S \times S$ as a **directed graph** (week 5).

- Nodes: Elements of S
- Edges: Elements of R

Example

$R = \{(1,1),(2,3),(3,2)\} \subseteq [1,3] \times [1,3]$:



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Properties of Binary Relations $R \subseteq S \times S$

Definition

(R)	reflexive	For all $x \in S$: $(x, x) \in R$
(AR)	antireflexive	For all $x \in S$: $(x, x) \notin R$
(S)	symmetric	For all $x, y \in S$: If $(x, y) \in R$ then $(y, x) \in R$
(AS)	antisymmetric	For all $x, y \in S$: If (x, y) and $(y, x) \in R$ then $x = y$
(T)	transitive	For all $x, y, z \in S$: If (x, y) and $(y, z) \in R$ then $(x, z) \in R$

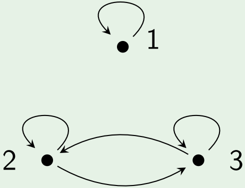
NB

- *Properties have to hold for all elements*
- *(S), (AS), (T) are conditional statements – they will hold if there is nothing which satisfies the 'if' part*

Relation properties: Examples

Examples

(R) Reflexivity: $(x, x) \in R$ for all x



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Relation properties: Examples

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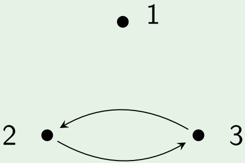

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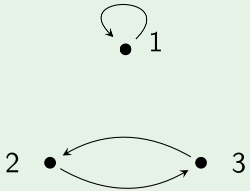
Relation properties: Examples

Examples

(R) Reflexivity: $(x, x) \in R$ for all x

(AR) Antireflexivity: $(x, x) \notin R$ for all x

(S) Symmetry: If $(x, y) \in R$ then $(y, x) \in R$ for all x, y



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Relation properties: Examples

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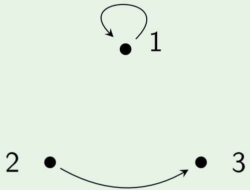
(AR) Antireflexivity: $(x, x) \notin R$ for all x

(S) Symmetry: If $(x, y) \in R$ then $(y, x) \in R$ for all x, y

(AS) Antisymmetry: $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$ for all x, y



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Relation properties: Examples

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(R) Reflexivity: $(x,x) \in R$ for all x

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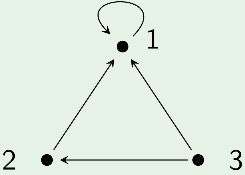
(S) Symmetry: If $(x,y) \in R$ then $(y,x) \in R$ for all x, y

(AS) Antisymmetry: $(x,y) \in R$ and $(y,x) \in R$ implies $x = y$ for all x, y

(T) Transitivity: $(x,y) \in R$ and $(y,z) \in R$ implies $(x,z) \in R$ for all x, y, z.



$$\begin{bmatrix} \bullet & \circ & \circ \\ \bullet & \circ & \circ \\ \bullet & \bullet & \circ \end{bmatrix}$$

Interaction of Properties

A relation *can* be both symmetric and antisymmetric. Namely, when R consists only of some pairs $(x, x), x \in S$.

A relation *cannot* be simultaneously reflexive and antireflexive (unless $S = \emptyset$).

NB

$$\left.\begin{array}{r} \textit{nonreflexive} \\ \textit{nonsymmetric} \end{array}\right\} \textit{ is not the same as } \left\{\begin{array}{l} \textit{antireflexive/irreflexive} \\ \textit{antisymmetric} \end{array}\right.$$

Exercises

Exercises

$\boxed{3.1.1}$ The following relations are on $S = \{1, 2, 3\}$.
Which of the properties (R), (AR), (S), (AS), (T) does each satisfy?

(a) $(m, n) \in R$ if $m + n = 3$?

(e) $(m, n) \in R$ if $\max\{m, n\} = 3$?

$\boxed{3.1.2(b)}$ $(m, n) \in R$ if $m < n$?

Exercises

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Which of the properties (R), (AR), (S), (AS), (T) does each satisfy?

(a) $(m, n) \in R$ if $m + n = 3$? (AR) and (S)

(e) $(m, n) \in R$ if $\max\{m, n\} = 3$? (S)

$\boxed{3.1.2(b)}$ $(m, n) \in R$ if $m < n$? (AR), (AS), (T)

Exercises

Exercises

Complete the following table of common relations (over $\mathbb{Z}$) and their properties:

	(R)	(AR)	(S)	(AS)	(T)
$=$					
$\leq$					
$<$					
$\emptyset$					
$\mathcal{U} = \mathbb{Z} \times \mathbb{Z}$					
$\mid$					
$\equiv \pmod 3$					

Exercises

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	(R)	(AR)	(S)	(AS)	(T)
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$\leq$	✓			✓	✓
$<$					
$\emptyset$					
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$<$		✓		✓	✓
$\emptyset$		✓	✓	✓	✓
$\mathcal{U} = \mathbb{Z} \times \mathbb{Z}$	✓		✓		✓
$\mid$	✓			✓	✓
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Exercises

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$<$		$\checkmark$		$\checkmark$	$\checkmark$
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$\mid$	$\checkmark$			$\checkmark$	$\checkmark$
$= \pmod 3$	$\checkmark$		$\checkmark$		$\checkmark$

Exercises

Exercises

$\boxed{3.1.10(a)}$ Give examples of relations with specified properties.
(AS), (T), not (R).

Exercises

Exercises

$\boxed{3.1.10(a)}$ Give examples of relations with specified properties.

(AS), (T), not (R).

Some examples over $\mathbb{N}$, $\mathrm{Pow}(\mathbb{N})$:

- strict order of numbers $x < y$
- simple (weak) order, but with some pairs (x, x) removed from R
- being a prime divisor
  $(p, n) \in R$ iff p is prime and $p|n$
  - not reflexive: $(1, 1) \notin R, (4, 4) \notin R, (6, 6) \notin R$
  - transitivity is meaningful only for the pairs $(p, p), (p, n)$, $p|n$ for p prime

Exercises

Exercises

$\boxed{3.1.10(b)}$ Give examples of relations with specified properties.

(S), not (R), not (T).

Exercises

Exercises

$\boxed{3.1.10(b)}$ Give examples of relations with specified properties.

(S), not (R), not (T).

Simplest example - inequality

Exercises

$\boxed{\text{3.6.10 (supp)}}$

R is a relation on $\mathbb{N} \times \mathbb{N}$, i.e. it is a subset of $\mathbb{N}^2 \times \mathbb{N}^2$
$(m, n)\, R\, (p, q)$ if $m = p \pmod 3$ or $n = q \pmod 5$.
(a) Is R reflexive?

(b) Is R symmetric?

(c) Is R transitive?

Exercises

Exercises

$\boxed{\text{3.6.10 (supp)}}$

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$(m, n)\, R\, (p, q)$ if $m = p \pmod 3$ or $n = q \pmod 5$.

(a) Is R reflexive?

Yes: $m = m \pmod 3$ (and $n = n \pmod 5$) so $(m, n) R (m, n)$.

(b) Is R symmetric?

(c) Is R transitive?

Exercises

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(a) Is R reflexive?

Yes: $m = m \pmod 3$ (and $n = n \pmod 5$) so $(m, n)R(m, n)$.

(b) Is R symmetric?

Yes: by symmetry of $. = . \pmod n$.

(c) Is R transitive?

Exercises

Exercises

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$(m, n)\, R\, (p, q)$ if $m = p \pmod 3$ or $n = q \pmod 5$.
(a) Is R reflexive?
Yes: $m = m \pmod 3$ (and $n = n \pmod 5$) so $(m, n)R(m, n)$.
(b) Is R symmetric?
Yes: by symmetry of $. = . \pmod n$.
(c) Is R transitive? No: Consider $(1, 1)$, $(1, 4)$ and $(2, 4)$.

Summary of topics

- Defining binary relations
- Properties of binary relations
- Equivalence relations, classes, and partitions
- Orderings

Equivalence relations

Equivalence relations capture a general notion of "equality". They are relations which are:

- Reflexive (R): Every object should be "equal" to itself
- Symmetric (S): If x is "equal" to y, then y should be "equal" to x
- Transitive (T): If x is "equal" to y and y is "equal" to z, then x should be "equal" to z.

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Definition

A binary relation $R \subseteq S \times S$ is *equivalence relation* if it satisfies (R), (S), (T).

Example

Partition of $\mathbb{Z}$ into classes of numbers with the same remainder on division by p; it is particularly important for p prime

$$\mathbb{Z}(p) = \mathbb{Z}_p = \{0, 1, \ldots, p - 1\}$$

One can define all four arithmetic operations (with the usual properties) on $\mathbb{Z}_p$ for a prime p; division has to be restricted when p is not prime.

NB

$(\mathbb{Z}_p, +, \cdot, 0, 1)$ *are fundamental algebraic structures known as* **rings**. *These structures are very important in coding theory and cryptography.*

Equivalence Classes and Partitions

Suppose $R \subseteq S \times S$ is an equivalence relation

The **equivalence class** $[s]$ (w.r.t. R) of an element $s \in S$ is

$$[s] = \{t : t \in S \text{ and } sRt\}$$

Fact

$s \; R \; t$ if and only if $[s] = [t]$.

Equivalence classes: Proof example

Proof

Suppose $[s] = [t]$. Recall $[s] = \{x \in S : (s, x) \in R\}$. We will show that $(s, t) \in R$.

Because R is reflexive, $(t, t) \in R$.

Therefore $t \in [t]$.

Because $[t] = [s]$, it follows that $t \in [s]$.

But then $(s, t) \in R$ by the definition of $[s]$.

Equivalence classes: Proof example

Proof

Now suppose $(s, t) \in R$. We will show $[s] = [t]$ by showing $[s] \subseteq [t]$ and $[t] \subseteq [s]$.

Take any $x \in [s]$.

By the definition of $[s]$, $(s, x) \in R$.

Since R is symmetric $(x, s) \in R$.

Since R is transitive and $(s, t) \in R$ we have that $(x, t) \in R$.

Since R is symmetric $(t, x) \in R$.

Therefore, $x \in [t]$.

Therefore $[s] \subseteq [t]$.

Equivalence classes: Proof example

Proof

Now suppose $(s,t) \in R$. We will show $[s] = [t]$ by showing $[s] \subseteq [t]$ and $[t] \subseteq [s]$.

Take any $x \in [t]$.

By the definition of $[t]$, $(t,x) \in R$.

Since R is transitive and $(s,t) \in R$ we have that $(s,x) \in R$.

Therefore $x \in [s]$.

Therefore $[t] \subseteq [s]$. $\square$

Partitions

Definition

A **partition** of a set S is a collection of sets $S_1, \ldots, S_k$ such that

- S_i and S_j are disjoint (for $i \neq j$)
- $S = S_1 \cup S_2 \cup \cdots \cup S_k = \bigcup_{i=1}^{k} S_i$

The collection of all equivalence classes $\{[s] : s \in S\}$ forms a partition of S

In the opposite direction, a partition of a set defines the equivalence relation on that set. If $S = S_1 \cup \cdots \cup S_k$, then we can define $\sim \subseteq S \times S$ as:

$$s \sim t \text{ exactly when } s \text{ and } t \text{ belong to the same } S_i.$$

Exercises

Exercises

$\boxed{\text{3.6.6 (supp)}}$ Show that $m \sim n$ iff $m^2 = n^2 \pmod 5$ is an equivalence on $S = \{1, \dots, 7\}$. Find all the equivalence classes.

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$\boxed{\text{3.6.6 (supp)}}$ Show that $m \sim n$ iff $m^2 = n^2 \pmod 5$ is an equivalence on $S = \{1, \ldots, 7\}$. Find all the equivalence classes.

(a) It just means that $m = n \pmod 5$ or $m = -n \pmod 5$, e.g. $1 = -4 \pmod 5$. This satisfies (R), (S), (T).

(b) We have
$[1] = \{1, 4, 6\}$
$[2] = \{2, 3, 7\}$
$[5] = \{5\}$

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Partial Order

A **partial order** $\preceq$ on S satisfies (R), (AS), (T).
We call $(S, \preceq)$ a **poset** — partially ordered set

Examples

Posets:

- $(\mathbb{Z}, \leq)$
- $(\mathrm{Pow}(X), \subseteq)$ for some set X
- $(\mathbb{N}, |)$

Not posets:

- $(\mathbb{Z}, <)$
- $(\mathbb{Z}, |)$

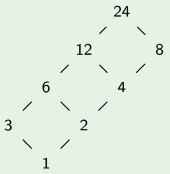
Hasse diagram

Every finite poset $(S, \preceq)$ can be represented with a **Hasse diagram**:

- Nodes are elements of S
- An edge is drawn *upward* from x to y if $x \prec y$ and there is no z such that $x \prec z \prec y$

Example

Hasse diagram for positive divisors of 24 ordered by $|$:



Ordering Concepts

Definition

Let $(S, \preceq)$ be a poset.

- **Minimal** element: x such that there is no y with $y \preceq x$
- **Maximal** element: x such that there is no y with $x \preceq y$
- **Minimum (least)** element: x such that $x \preceq y$ for all $y \in S$
- **Maximum (greatest)** element: x such that $y \preceq x$ for all $y \in S$

NB

- *There may be multiple minimal/maximal elements.*
- *Minimum/maximum elements are the unique minimal/maximal elements if they exist.*
- *Minimal/maximal elements always exist in finite posets, but not necessarily in infinite posets.*

Examples

Examples

- $\mathrm{Pow}(\{a, b, c\})$ with the order $\subseteq$
  $\emptyset$ is minimum; $\{a, b, c\}$ is maximum
- $\mathrm{Pow}(\{a, b, c\}) \setminus \{\{a, b, c\}\}$ (proper subsets of $\{a, b, c\}$)
  Each two-element subset $\{a, b\}, \{a, c\}, \{b, c\}$ is maximal.
  - But there is no maximum

Ordering Concepts

Definition

Let $(S, \preceq)$ be a poset.

- x is an **upper bound** for A if $a \preceq x$ for all $a \in A$
- x is a **lower bound** for A if $x \preceq a$ for all $a \in A$
- The **set of upper bounds** for A is defined as $\text{ub}(A) = \{x : a \preceq x \text{ for all } a \in A\}$
- The **set of lower bounds** for A is defined as $\text{lb}(A) = \{x : x \preceq a \text{ for all } a \in A\}$
- The **least upper bound** of A, $\text{lub}(A)$, is the minimum of $\text{ub}(A)$ (if it exists)
- The **greatest lower bound** of A, $\text{glb}(A)$ is the maximum of $\text{lb}(A)$ (if it exists)

glb and lub

To show x is $\mathrm{glb}(A)$ you need to show:

- x is a lower bound: $x \preceq a$ for all $a \in A$.
- x is the greatest of all lower bounds: If $y \preceq a$ for all $a \in A$ then $y \preceq x$.

Example

$\mathrm{Pow}(X)$ ordered by $\subseteq$.

- $\mathrm{glb}(A, B) = A \cap B$
- $\mathrm{lub}(A, B) = A \cup B$

Ordering Concepts

Definition

Let $(S, \preceq)$ be a poset.

- $(S, \preceq)$ is a **lattice** if $\mathrm{lub}(x, y)$ and $\mathrm{glb}(x, y)$ exist for every pair of elements $x, y \in S$.
- $(S, \preceq)$ is a **complete lattice** if $\mathrm{lub}(A)$ and $\mathrm{glb}(A)$ exist for every subset $A \subseteq S$.

NB

A finite lattice is always a complete lattice.

Examples

Examples

- $\{1,2,3,4,6,8,12,24\}$ partially ordered by divisibility is a lattice
  - e.g. $\text{lub}(\{4,6\}) = 12$; $\text{glb}(\{4,6\}) = 2$
- $\{1,2,3\}$ partially ordered by divisibility is not a lattice
  - $\{2,3\}$ has no lub
- $\{2,3,6\}$ partially ordered by divisibility
  - $\{2,3\}$ has no glb
- $\{1,2,3,12,18,36\}$ partially ordered by divisibility
  - $\{2,3\}$ has no lub ($12, 18$ are minimal upper bounds)

NB

An infinite lattice need not have a lub (or no glb) for an arbitrary infinite subset of its elements, in particular no such bound may exist for **all** *its elements.*

Examples

- $(\mathbb{Z}, \leq)$: neither $\mathsf{lub}(\mathbb{Z})$ nor $\mathsf{glb}(\mathbb{Z})$ exist
- $(\mathcal{F}(\mathbb{N}), \subseteq)$ [all finite subsets of $\mathbb{N}$]: lub exists for pairs of elements but not generally for (infinite) sets of elements. glb exists for any set of elements: intersection of a set of finite sets is finite.
- $(\mathcal{I}(\mathbb{N}), \subseteq)$ [all infinite subsets of $\mathbb{N}$]: glb does not exist for some pairs of elements (e.g. odds and evens). lub exists for any set of elements: union of a set of infinte sets is always infinte.

Exercises

Exercises

$\boxed{11.1.5}$ Consider poset $(\mathbb{R}, \leq)$

(a) Is this a lattice?

(b) Give an example of a non-empty subset of $\mathbb{R}$ that has no upper bound.

(c) Find $\text{lub}(\{ x \in \mathbb{R} : x < 73 \})$

(d) Find $\text{lub}(\{ x \in \mathbb{R} : x \leq 73 \})$

(e) Find $\text{lub}(\{ x : x^2 < 73 \})$

(f) Find $\text{glb}(\{ x : x^2 < 73 \})$

Exercises

Exercises

$\boxed{11.1.5}$ Consider poset $(\mathbb{R}, \leq)$

(a) Is this a lattice? Yes

(b) Give an example of a non-empty subset of $\mathbb{R}$ that has no upper bound. $\mathbb{R}_{>0} = \{ r \in \mathbb{R} : r > 0 \}$

(c) Find $\text{lub}(\{ x \in \mathbb{R} : x < 73 \})$ 73

(d) Find $\text{lub}(\{ x \in \mathbb{R} : x \leq 73 \})$ 73

(e) Find $\text{lub}(\{ x : x^2 < 73 \})$ $\sqrt{73}$

(f) Find $\text{glb}(\{ x : x^2 < 73 \})$ $-\sqrt{73}$

Total orders

Definition

A **total order** is a partial order that also satisfies:

(L) *Linearity* (any two elements are comparable):

$$\text{For all } x, y \text{ either: } x \leq y \text{ or } y \leq x \text{ (or both if } x = y)$$

NB

On a finite set all total orders are "isomorphic"
On an infinite set there is quite a variety of possibilities.

Examples

Examples

- $\mathbb{Z}$ with $\leq$:
  (no minimum/maximum element)
- $\mathbb{Z}$ with $\{(x, y) : x < 0 \leq y \text{ or } |x| \leq |y|\}$:
  (no maximum element, minimum element is -1)
- $\mathbb{Z}$ with $\{(x, y) : x < 0 \leq y \text{ or } x \geq y\}$:
  (minimum element -1, maximum element 0)

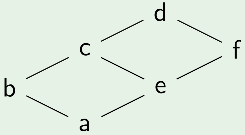
Ordering of a Poset — Topological Sort

Definition

For a poset $(S, \preceq)$ any total order $\leq$ that is consistent with $\preceq$ (if $a \preceq b$ then $a \leq b$) is called a **topological sort**.

Example

Consider



The following all are topological sorts:

$a \leq b \leq e \leq c \leq f \leq d$

$a \leq e \leq b \leq f \leq c \leq d$

$a \leq e \leq f \leq b \leq c \leq d$

Well-Ordered Sets

Definition

A *well-ordered set* is a poset where every subset has a least element.

NB

The greatest element is not required.

Examples

- $\mathbb{N} = \{0, 1, \ldots\}$
- $\mathbb{N}_1 \dot{\cup} \mathbb{N}_2 \dot{\cup} \mathbb{N}_3 \dot{\cup} \ldots$, where each $\mathbb{N}_i \simeq \mathbb{N}$ and $\mathbb{N}_1 < \mathbb{N}_2 < \mathbb{N}_3 \cdots$

NB

Well-ordered sets are an important mathematical tool to prove termination of programs.

Combining Orders

Product order — can combine any partial orders. In general, it is only a *partial order*, even if combining total orders.
For $s, s' \in S$ and $t, t' \in T$ define

$$(s, t) \preceq (s', t') \quad \text{if } s \preceq s' \text{ and } t \preceq t'$$

Practical Orderings

They are, effectively, *total* orders on the *product* of ordered sets.

- **Lexicographic order** — defined on all of Σ^*. It extends a total order already assumed to exist on Σ.
- **Lenlex** — the order on (potentially) the entire Σ^*, where the elements are ordered first by length.
  $\Sigma^{(1)} \prec \Sigma^{(2)} \prec \Sigma^{(3)} \prec \cdots$, then lexicographically within each $\Sigma^{(k)}$. In practice it is applied only to the finite subsets of Σ^*.
- **Filing order** — lexicographic order confined to the strings of the same length.
  It defines total orders on Σ^i, separately for each i.

Example

Example

$\boxed{11.2.5}$ Let $\mathbb{B} = \{0, 1\}$ with the usual order $0 < 1$. List the elements $101, 010, 11, 000, 10, 0010, 1000$ of $\mathbb{B}^*$ in the

(a) Lexicographic order

(b) Lenlex order

$\boxed{11.2.8}$ When are the lexicographic order and *lenlex* on Σ^* the same?

Example

Example

$\boxed{11.2.5}$ Let $\mathbb{B} = \{0, 1\}$ with the usual order $0 < 1$. List the elements $101, 010, 11, 000, 10, 0010, 1000$ of $\mathbb{B}^*$ in the

(a) Lexicographic order

$000, 0010, 010, 10, 1000, 101, 11$

(b) Lenlex order

$10, 11, 000, 010, 101, 0010, 1000$

$\boxed{11.2.8}$ When are the lexicographic order and *lenlex* on Σ^* the same?

Only when $|\Sigma| = 1$.

Exercises

Exercises

$\boxed{11.6.6}$ True or false?

(a) If a set Σ is totally ordered, then the corresponding lexicographic partial order on Σ^* also must be totally ordered.

(b) If a set Σ is totally ordered, then the corresponding lenlex order on Σ^* also must be totally ordered.

(c) Every finite partially ordered set has a Hasse diagram.

Exercises

Exercises

$\boxed{11.6.6}$ True or false?

(a) If a set Σ is totally ordered, then the corresponding lexicographic partial order on Σ^* also must be totally ordered. True

(b) If a set Σ is totally ordered, then the corresponding lenlex order on Σ^* also must be totally ordered.

(c) Every finite partially ordered set has a Hasse diagram.

Exercises

Exercises

$\boxed{11.6.6}$ True or false?

(a) If a set Σ is totally ordered, then the corresponding lexicographic partial order on Σ^* also must be totally ordered.
True

(b) If a set Σ is totally ordered, then the corresponding lenlex order on Σ^* also must be totally ordered.
True

(c) Every finite partially ordered set has a Hasse diagram.

Exercises

Exercises

11.6.6 True or false?

(a) If a set Σ is totally ordered, then the corresponding lexicographic partial order on Σ^* also must be totally ordered.
True

(b) If a set Σ is totally ordered, then the corresponding lenlex order on Σ^* also must be totally ordered.
True

(c) Every finite partially ordered set has a Hasse diagram.
True

Exercises

Exercises

$\boxed{11.6.6}$ True or false?

(d) Every finite partially ordered set has a topological sorting.

(e) Every finite partially ordered set has a minimum element.

(f) Every finite totally ordered set has a maximum element.

(g) An infinite partially ordered set cannot have a maximum element.

Exercises

Exercises

11.6.6 True or false?

- (d) Every finite partially ordered set has a topological sorting. True
- (e) Every finite partially ordered set has a minimum element.
- (f) Every finite totally ordered set has a maximum element.
- (g) An infinite partially ordered set cannot have a maximum element.

Exercises

Exercises

$\boxed{11.6.6}$ True or false?

(d) Every finite partially ordered set has a topological sorting.
True

(e) Every finite partially ordered set has a minimum element.
False

(f) Every finite totally ordered set has a maximum element.

(g) An infinite partially ordered set cannot have a maximum element.

Exercises

Exercises

$\boxed{11.6.6}$ True or false?

(d) Every finite partially ordered set has a topological sorting.
True

(e) Every finite partially ordered set has a minimum element.
False

(f) Every finite totally ordered set has a maximum element.
True

(g) An infinite partially ordered set cannot have a maximum element.

Exercises

Exercises

11.6.6 True or false?

(d) Every finite partially ordered set has a topological sorting.
True

(e) Every finite partially ordered set has a minimum element.
False

(f) Every finite totally ordered set has a maximum element.
True

(g) An infinite partially ordered set cannot have a maximum element.
False