

# Undergraduate Complexity Theory

## Lecture 17: Savitch's Theorem & NL

Marcythm

July 20, 2022

### 1 Lecture Notes

**Theorem 1.1** (Savitch's Theorem). *For any function  $f : \mathbb{N} \rightarrow \mathbb{R}^+$ , where  $f(n) \geq n$ ,*

$$\text{NSPACE}(f(n)) \subseteq \text{SPACE}(f^2(n))$$

idea of Savitch's Theorem: "Middle-first search".

---

```
procedure PATH?( $x, y, k$ )                                ▷ whether there is a path from  $x$  to  $y$  within  $2^k$  steps
  if  $k = 0$  then
    return truth value of  $x = y$ 
  else
    for  $w \in V$  do
      if PATH?( $x, w, k - 1$ ) && PATH?( $w, y, k - 1$ ) then
        return true
  return false
```

---

need to store  $\log n$  stack variables (the depth of recursion), each with  $O(\log n)$  space, so  $O(\log^2 n)$  space in total. time complexity:  $O(n^k)$ .

**Definition 1.2** (Nondeterminism-based definition of NL).  $\text{NL} = \text{NSPACE}(\log n)$ .

**Proposition 1.3.**  $\text{ST-PATH} \in \text{NL}$ .

*Proof.* Nondeterministically choose each step, and maintain a counter for length. □

**Theorem 1.4.**  $\text{NL} \subseteq \text{P}, \text{NL} \in \text{SPACE}(\log^2 n)$ .

*Proof.* see reading section. □

### 2 Reading

#### 2.1 sipser 8.4 (The Classes L and NL)

If  $M$  is a TM that has a separate read-only input tape and  $w$  is an input, a configuration of  $M$  on  $w$  is a setting of the state, the work tape, and the positions of the two tape heads. The input  $w$  is not a part of the configuration of  $M$  on  $w$ .

Thus total number of configurations of  $M$  on  $w$  is  $|Q|nf(n)|\Gamma|^{f(n)}$ , i.e.  $n2^{O(f(n))}$ , can extend Savitch's Theorem to  $f(n) \geq \log n$ .