# Undergraduate Complexity Theory
# Lecture 16: Space Complexity

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## 1 Lecture Notes

**Definition 1.1.** *Space complexity* of a decider TM is a function $S : \mathbb{N} \to \mathbb{N}$ s.t.

$$S(n) = \max_{|x|=n}\{\# \text{ distinct tape cells accessed by } M(x)\}$$

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another version: read-only input tape, work tapes, for sublinear spaces.

**Remark 1.2.** To simulate multitape TM with space $S$, only needs single-tape TM with space $O(S)$.

**Definition 1.3.** $\mathsf{SPACE}(f(n)) = \{A : \text{exists a TM deciding } A \text{ with space complexity } O(f(n))\}$.

**Fact 1.4.** *Using $O(\log n)$ space a TM on input $x$ can compute (write in base 2) $n = |x|$.*

**Definition 1.5.** $\mathsf{L} = \mathsf{SPACE}(\log n)$.

e.g. $A = \{0^n 1^n : n \in \mathbb{N}\} \in \mathsf{L}$. $\mathsf{PALINDROMES} \in \mathsf{L}$
Intuitively, $\mathsf{L}$ is pesudocode with constant (i.e. $O(1)$) # of variables, which are ints ranging from 0 to poly$(n)$. no array/memory allocation, read-only input lookups, basic arith ops on vars.
$\mathsf{ST\text{-}PATH} = \{\langle G, s, t\rangle : \text{exists path } s \to t \text{ in G}\}$, in $\mathsf{L}$? BFS in $\Theta(n)$ space, DFS the same.

**Theorem 1.6** (Savitch's Theorem '70)**.** $\mathsf{ST\text{-}PATH} \in \mathsf{SPACE}(\log^2 n)$.

**Theorem 1.7** (Reingold '04)**.** *Undirected* $\mathsf{ST\text{-}PATH} \in \mathsf{L}$.

$\mathsf{CIRCUIT\text{-}EVAL} = \{\langle C, x\rangle : C(x) = 1\} \in \mathsf{P}$, doable in linear space, believed not in sublinear space.
$\mathsf{3SAT}$: $O(n)$ space.

**Definition 1.8.** $\mathsf{PSPACE} = \bigcup_{k\in\mathbb{N}} \mathsf{SPACE}(n^k)$.

**Exercise 1.9.** $\mathsf{NP} \subseteq \mathsf{PSPACE}$.

**Fact 1.10.** $\mathsf{TIME}(f(n)) \subseteq \mathsf{SPACE}(f(n))$. *e.g.* $\mathsf{P} \subseteq \mathsf{PSPACE}$.

**Theorem 1.11.** *For $f(n) \geq \log n$,* $\mathsf{SPACE}(f(n)) \subseteq \mathsf{TIME}(2^{O(f(n))})$.

**Corollary 1.12.** $\mathsf{L} \subseteq \mathsf{P}$.

**Corollary 1.13.** $\mathsf{PSPACE} \subseteq \mathsf{EXP}$.

current hierarchy: $\mathsf{L} \subseteq \mathsf{P} \subseteq \{\mathsf{NP}|\mathsf{coNP}\} \subseteq \mathsf{PSPACE} \subseteq \mathsf{EXP}$.
interesting: find a problem in $\mathsf{P}$ but not in $\mathsf{L}$; in $\mathsf{PSPACE}$ but not in $\mathsf{NP} \cup \mathsf{coNP}$.
already knows $\mathsf{P} \neq \mathsf{EXP}$ by T.H.T, so whether $\mathsf{P} \neq \mathsf{PSPACE}$ or $\mathsf{PSPACE} \neq \mathsf{EXP}$?

**Definition 1.14.** $f(n)$ is *space-constructible* iff $f(n) \geq \log n$ and can compute $f(n)$ in $O(f(n))$ space.

**Theorem 1.15** (Space Hierarchy Theorem)**.** *Let $f(n)$ be a space-constructible function, then exists language $A \in \mathsf{SPACE}(f(n))$ thats $\notin \mathsf{SPACE}(g(n))$ for any $g(n) = o(f(n))$.*

# 2 Reading

## 2.1 sipser 8.0 (Space Complexity)

1. definition of SPACE and NSPACE.
2. Space appears to be more powerful than time because space can be reused, whereas time cannot.

## 2.2 sipser 8.1 (Savitch's Theorem)

**Theorem 2.1** (Savitch's Theorem)**.** *For any function $f : \mathbb{N} \to \mathbb{R}^+$, where $f(n) \geq n$,*

$$\mathsf{NSPACE}(f(n)) \subseteq \mathsf{SPACE}(f^2(n))$$

## 2.3 sipser 8.2 (The Class PSPACE)

definition of PSPACE, so far $\mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE} = \mathsf{NPSPACE} \subseteq \mathsf{EXP}$.