Undergraduate Complexity Theory
Lecture 23: The Polynomial Hierarchy

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1 Lecture Notes

motivation:
recall: NP and coNP

Observation 1.1. $\mathsf{P} = \mathsf{NP} \implies \mathsf{NP} = \mathsf{coNP}$ (contra: $\mathsf{NP} \neq \mathsf{coNP} \implies \mathsf{P} \neq \mathsf{NP}$)

$\mathsf{ECLIQUE} = \{\langle G, k\rangle : G$ is a graph with a largest clique of exactly size $k\}$
$\mathsf{SMALLEST\text{-}CIRCUIT} = \{\langle C\rangle : C$ is the smallest circuit computing the function C_f that C computes$\}$

Notation 1.2. $Q^n u$ means $Qu, |u| \leq m$ where $Q \in \{\exists, \forall\}$.

Definition 1.3. $L \in \Sigma_2$ if $\exists$ poly-time TM V, $\exists$ polynomial p s.t.

$$x \in L \iff \exists^{p(x)} u_1 \forall^{p(x)} u_2 : V(x, u_1, u_2) = 1$$

Definition 1.4. $L \in \Pi_2$ if $\exists$ poly-time TM V, $\exists$ polynomial p s.t.

$$x \in L \iff \forall^{p(x)} u_1 \exists^{p(x)} u_2 : V(x, u_1, u_2) = 1$$

Observation 1.5. $\Pi_2 = \mathsf{co}\text{-}\Sigma_2$.

Claim 1.6. $\mathsf{ECLIQUE} \in \Sigma_2$.

Exercise 1.7. $\mathsf{SMALLEST\text{-}CIRCUIT} \in \Pi_2$.

Notation 1.8. $\Sigma_0 = \Pi_0 = \mathsf{P}, \Sigma_1 = \mathsf{NP} = \exists\mathsf{P}, \Pi_1 = \mathsf{coNP} = \forall\mathsf{P}, \ldots, \Sigma_i = \exists\Pi_{i-1}, \Pi_i = \forall\Sigma_{i-1}$.

Observation 1.9. $\Sigma_i \subseteq \Sigma_{i+1} \cap \Pi_{i+1}$, $\Pi_i \subseteq \Pi_{i+1} \cap \Sigma_{i+1}$.

Definition 1.10 (Polynomial Hierarchy)**.** $\mathsf{PH} = \bigcup_{k\in\mathbb{N}} \Sigma_k = \bigcup_{k\in\mathbb{N}} \Pi_k$ contains those languages can be described by constant number of quantifiers.

Claim 1.11. $\mathsf{PH} \subseteq \mathsf{PSPACE}$.

Theorem 1.12. $\mathsf{P} = \mathsf{NP} \implies \mathsf{P} = \mathsf{PH}$.

Theorem 1.13. $\Sigma_i = \Pi_i \implies \Sigma_i = \Pi_i = \mathsf{PH}$. *(hierarchy "collapses" to the ith level)*

Complete problems for $\Sigma_i, \Pi_i, \mathsf{PH}$: Let $\varphi(y_1, y_2, \ldots, y_i)$ be a boolean formula where each y_j is a vector (or sequence) of boolean variables.

Definition 1.14. $\Sigma_i\text{-}\mathsf{SAT} = \{\langle\varphi(y_1, y_2, \ldots, y_i)\rangle : \exists z_1 \forall z_2 \cdots \varphi(z_1, z_2, \ldots, z_i) = 1\}$

Exercise 1.15. Σ_i-SAT *is* Σ_i*-complete.*

the same for Π_i.

Claim 1.16. *If* $\exists L$ *s.t.* L *is* PH*-complete, then* $\exists i$ *s.t.* $\mathsf{PH} = \Sigma_i$.

2 Reading

2.1 sipser 10.3 (Alternation)

definition of Alternating Turing Machine, ATIME, ASPACE, AP, $\mathsf{APSPACE}$, AL

Theorem 2.1. *For $f(n) \geq n$, we have $\mathsf{ATIME}(f(n)) \subseteq \mathsf{SPACE}(f(n)) \subseteq \mathsf{ATIME}(f^2(n))$.*
For $f(n) \geq \log n$, we have $\mathsf{ASPACE}(f(n)) = \mathsf{TIME}(2^{O(f(n))})$.

Corollary 2.2. $\mathsf{AL} = \mathsf{P}, \mathsf{AP} = \mathsf{PSPACE}, \mathsf{APSPACE} = \mathsf{EXP}$.

definition of PH.