Undergraduate Complexity Theory
Lecture 17: Savitch's Theorem & NL

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1 Lecture Notes

Theorem 1.1 (Savitch's Theorem)**.** *For any function $f : \mathbb{N} \to \mathbb{R}^+$, where $f(n) \geq n$,*

$$\mathsf{NSPACE}(f(n)) \subseteq \mathsf{SPACE}(f^2(n))$$

idea of Savitch's Theorem: "Middle-first search".

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procedure PATH?(x, y, k)            ▷ whether there is a path from x to y within 2^k steps
    if k = 0 then
        return truth value of x = y
    else
        for w ∈ V do
            if PATH?(x, w, k − 1) && PATH?(w, y, k − 1) then
                return true
    return false
```

need to store $\log n$ stack variables (the depth of recursion), each with $O(\log n)$ space, so $O(\log^2 n)$ space in total. time complexity: $O(n^k)$.

Definition 1.2 (Nondeterminism-based definition of NL)**.** $\mathsf{NL} = \mathsf{NSPACE}(\log n)$.

Proposition 1.3. $\mathsf{ST\text{-}PATH} \in \mathsf{NL}$.

Proof. Nondeterministically choose each step, and maintain a counter for length. □

Theorem 1.4. $\mathsf{NL} \subseteq \mathsf{P}, \mathsf{NL} \subseteq \mathsf{SPACE}(\log^2 n)$.

Proof. see reading section. □

2 Reading

2.1 sipser 8.4 (The Classes L and NL)

> If M is a TM that has a separate read-only input tape and w is an input, a configuration of M on w is a setting of the state, the work tape, and the positions of the two tape heads. The input w is not a part of the configuration of M on w.

Thus total number of configurations of M on w is $|Q|nf(n)|\Gamma|^{f(n)}$, i.e. $n2^{O(f(n))}$, can extend Savitch's Theorem to $f(n) \geq \log n$.