UCT

Polynomial Time

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Feasible Decision Problems

The question arises whether any time complexity class is a good match for our intuitive notion of “feasible computation” or “tractable problem.”

Note that whatever answer we give, we are in a similar situation as with the Church-Turing thesis: since we deal with intuitive notions there cannot be a formal proof – though one can try to amass ample evidence.

However, while there is fairly good agreement about what constitutes a computation, there is much less agreement about what constitutes a feasible computation. If one deals with very large data sets, anything beyond $n \log n$ is not really feasible.

(Deterministic) Polynomial Time

Recall our definition of polynomial time solvable problems:

$$\mathbb{P} = \mathrm{TIME}(n^{O(1)}) = \bigcup_{k \geq 0} \mathrm{TIME}(n^k).$$

Let’s put up a straw man: let’s declare $\mathbb{P}$ to be our stand-in for feasible computation.

But note that $\mathbb{P}$ is only about decision problems, in the world of actual algorithms there is no doubt that function problems are more important.

Types of Problems

Here is a fairly natural taxonomy of computational problems.

Decision Problems Return a Yes/No answer.

Counting Problems Determine the size of some object.

Function Problems Calculate a certain function.

Search Problems Select one particular solution.

How about all this other stuff?

Example: Vertex Cover

Problem: **Vertex Cover (decision)**
Instance: A ugraph G, a bound k.
Question: Does G have a vertex cover of size k?

Problem: **Vertex Cover (counting)**
Instance: A ugraph G.
Solution: Find the size of a minimal vertex cover of G.

Problem: **Vertex Cover (function)**
Instance: A ugraph G.
Solution: Find the lex-minimal vertex cover of G.

Problem: **Vertex Cover (search)**
Instance: A ugraph G.
Solution: Find a vertex cover of G of minimal size.

Connections

It should be clear that there are lots of connections between these different versions of the problem.

For example, if we can solve the counting/function/search version, then we also can handle the decision version.

The opposite direction is also true, more later.

Feasible Counting/Function/Search Problems

To deal with those, we need transducers as opposed to acceptors. Recall that in a transducers we attach an additional input/output tape to our Turing machines.

Definition

A function is polynomial time computable if it can be computed by a polynomial time Turing machine transducer.

The output for counting problems is just a number, but for function and search problems it can be some arbitrary combinatorial object (represented as a string).

Full Disclosure

Computing functions, as opposed to just solving decision problems, is beyond any doubt more important in the world of real algorithms. Interestingly, the concept of the class of polynomial time computable functions appeared a bit before modern complexity theory:

- von Neumann 1953
- Gödel 1956
- Cobham 1964
- Edmonds 1965

So why don't we bite the bullet and deal with feasible computable functions?

Because it's very hard. Decision problems make enough of a mess, so let's not worry about things that are even more complicated at this point.

Closure

We mention just one hugely important property of polynomial time computable functions: they can be composed sequentially without falling out of the class.

Lemma

Polynomial time functions are closed under composition.

Informally, if we have a fast algorithm to take A to B, and another fast algorithm to take B to C, then we also have a fast algorithm to take A to C.

Proof

Proof. The short version is: polynomials are closed under substitution.

More precisely, suppose $y = f(x)$ is computable in time at most $p(n)$ where $n = |x|$.

Then $|y| \leq p(n)$ and $z = g(y)$ is computable in time at most $q(p(n))$ for some polynomial q.

Hence we have a polynomial time bound for $z = g(f(x))$. □

In fact, if $p(n) = O(n^k)$ and $q = O(n^\ell)$, then the composition is $O(n^{k\ell})$ at most.

Polynomials are Great

Just to be clear: this result may seem trivial, but it depends crucially on our choice of polynomials as the resource bounds.

Closure fails for, say, exponential time $\mathrm{EXP}_1 = 2^{O(n)}$.

But similar claims do hold for

- linear time (easy),
- logarithmic space (hard, more later).

Two Standard Objections

The notation $T_{\mathcal{M}}(n) = O(n^k)$ hides two constants:

$$\exists\, n_0, c\, \forall\, n \geq n_0\, (T_{\mathcal{M}}(n) \leq c \cdot n^k)$$

What if these constants are huge? Something like 1000^{1000}? Note that there are only around 10^{80} particles in the universe[†].

This would render the asymptotic bounds entirely useless for anything resembling feasible computation. It could even wreck physical computation entirely.

[†] Absurdly large n_0 appear with annoying frequency in algorithm analysis, say, fast integer and matrix multiplication.

Standard Answer

This objection has merit in principle, but in the RealWorld$^{\text{TM}}$ it is pointless: for practical problems it is a matter of experience that the constants are easy to determine and are (almost) always very reasonable.

In fact, we can even compute the constants by writing down the algorithms very carefully in a low-level language like C.

For practical algorithms this is somewhat of a pain, but not really terribly difficult (as long as we are fairly relaxed about the bounds).

Example: Quadratic

Consider a code fragment like

```
// P
  for i = 1 to n do
      for j = 1 to i do
          blahblahblah
```

Suppose `blahblahblah` is constant time, a bunch of comparisons and assignments, say.

Clearly, the running time of P is quadratic, $O(n^2)$.

Assembly

But if we wanted to, we could write P in assembly,

```
   0:   55                      push   ebp
   1:   89 e5                   mov    ebp,esp
   3:   83 e4 f0                and    esp,0xfffffff0
   6:   83 ec 10                sub    esp,0x10
   9:   c7 04 24 00 00 00 00    mov    DWORD PTR [esp],0x0
```

Now we can count the number of steps each execution of P takes, likewise for the control structures. The result might be $55n + O(1)$.

But it surely won't be $1000^{1000}n + O(1)$.

Digression: Knuth

D. Knuth (everybody bow three times in the direction of Stanford) actually thinks that pure asymptotic bounds are fairly cheesy.

We should never say “this algorithm has complexity $O(n)$”, we should explicitly figure out the coefficients of the leading term. Say

$$\ldots = 7 \cdot n + O(\log n)$$

This is easy to say for Knuth, who just develops the necessary math on the fly, but quite challenging for ordinary mortals.

The Bibles

D. E. Knuth
The Art of Computer Programming
Addison-Wesley, 1968–now

D. E. Knuth, O. Patashnik, R. Graham
Concrete Mathematics
Addison-Wesley, 1988

Aside

Knuth is very unhappy about people publishing algorithms without ever implementing them (one really should distinguish more carefully between computable function and algorithm).

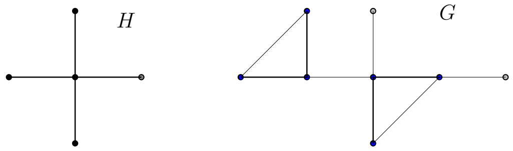
http://www.informit.com/articles/article.aspx

Truth in Advertising

The claim that we can always figure out the multiplicative constant is somewhat of a white lie. There are algorithms in graph theory, based on the Robertson-Seymour theorem for graph minors where these constants are utterly unknown.

Graph H is a minor of graph G if H is (isomorphic to) a graph obtained from G by

- edge deletions,
- edge contractions, and
- deletion of isolated vertices.



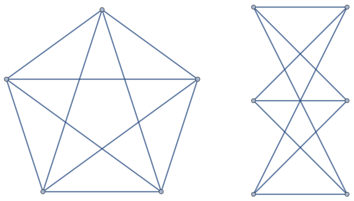
Closed Classes

A collection of finite graphs $\mathcal{G}$ is closed under minors if whenever $G \in \mathcal{G}$ and H is a minor of G, then $H \in \mathcal{G}$.

The classical example is the class of planar graphs: every minor of a planar graph is always planar. This produces a famous theorem:

Theorem (Kuratowski-Wagner)

A graph is planar if, and only if, it has no minor K_5 or $K_{3,3}$.



Wagner's Conjecture

In 1937, K. Wagner conjectured that a similar theorem holds for every minor closed class $\mathcal{C}$ of finite graphs:

Conjecture

Suppose $\mathcal{C}$ is minor closed. There are finitely many obstruction graphs $H_1, H_2, \ldots, H_r$ such that G is in $\mathcal{C}$ iff G does not have H_i as a minor for all $i = 1, \ldots, r$.

In other words, all anti-chains of graphs (wrt the minor order) are finite.

Note that this yields a decision algorithm: we just check if some given graph G has one of the forbidden minors.

Robertson-Seymour

Wagner's conjecture gave rise to the following amazing theorem[†]:

Theorem (Robertson, Seymour)

Every minor-closed family of graphs has a finite obstruction set.

This was proven in a series of 23 papers from 1984 to 2004, an incredible tour de force. The total proof is hundreds of pages long.

The proof has been examined very carefully and is probably correct, but putting it through a theorem prover would not hurt.

[†] This is very closely connected to Higman's theorem about the lack of infinite antichains in the subsequence order.

LaLaLand to Computational Universe

The proof of the Robertson-Seymour theorem is brutally non-constructive: the finite obstruction set exists in some set-theoretic universe, but we have no way of constructing it.

Worse, often we don't even know its (finite) cardinality.

Does it have any computational content? Well, for planar graphs is certainly does: we could check planarity by looking for minors K_3 and $K_{3,3}$. Needless to say, there are much better planarity testing algorithms.

Theorem To Algorithm

Suppose that graph H is fixed.

Problem:	**Minor Query**
Instance:	A ugraph G.
Question:	Is H a minor of G?

There is an algorithm that tests whether H is a minor of G in $O(n^2)$ steps, $n = |G|$.

Note that H has to be fixed, otherwise we could check whether G contains a cycle of length $|G|$ in quadratic time (aka Hamiltonian cycle).

Doom and Disaster

But then we can check all H from a finite list of obstruction graphs in quadratic time.

Hence we can check membership in any minor closed class $\mathcal{G}$ in quadratic time.

Alas, there is a glitch: the finite obstruction list is obtained non-constructively; it exists somewhere, and we can prove its existence using sufficiently strong set theory, but we cannot in general determine its elements—and, in fact, not even its cardinality. So we get a quadratic time algorithm, but we cannot bound the multiplicative constant.

Ouch.

Objection Two

What if $T_{\mathcal{M}}(n) = O(n^{1000})$?

This is polynomial time, but practically useless. By the time hierarchy theorem, we know that problems exist that can be solved in time $O(n^{1000})$ but essentially not in less time.

But note that these problems are constructed by diagonalization techniques, and are thus entirely artificial; they do not correspond to decent RealWorld$^{\text{TM}}$ problems.

Weird Empirical Fact

If a natural problem is in $\mathbb{P}$ at all, then it can actually be solved in time $O(n^{10})$ – for some small value of 10.

Take this with ample amounts of salt, but experience has shown so far that there simply are no natural problems where the best known algorithm has running time $O(n^{1000})$.

Alas, no one has any idea why this low-degree principle appears to be true. Note the qualifier "natural". Everyone understands intuitively what this means, but it seems very difficult to give a formal definition.

Close Call

In 2002, Agrawal, Kayal and Saxena published a paper that shows that primality testing is in polynomial time.

Amazingly, the algorithm uses little more than high school arithmetic.

The original algorithm had time complexity $\widetilde{O}(n^{12})$, but has since been improved to $\widetilde{O}(n^6)$.

Alas, the AKS algorithm seems useless in the RealWorld$^{\text{TM}}$, probabilistic algorithms are much superior[†].

[†] An indication that $\mathbb{P}$ is probably not quite the right definition of feasible computation.

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Scaling Down

“Small” classes like $\mathbb{P}$ or EXP don’t play well with our reductions from CRT. For example

$$A \in \mathrm{EXP} \qquad \text{implies} \qquad A \leq_T \emptyset$$

The problem is that the oracle TM can directly solve A, it has no need to query the oracle. And, $\leq_T$ can eviscerate much more than just EXP.

We need to limit the compute power of the reduction mechanism.

Polynomial Time Turing Reductions

Perhaps the most obvious attempt would be to use Turing reductions, but with the additional constraint that the Turing machine must have polynomial running time. This is called a polynomial time Turing reduction.

Notation: $A \leq_T^p B$.

Note that this actually makes sense for any kind of problem, not just decision problems. Also, it really captures nicely the idea that problem A is easier than B since we could use a fast algorithm for B to construct a fast algorithm for A.

Free Lunch

Just to be clear: in a computation with oracle B we only charge for the steps taken by the ordinary part of the Turing machine. For example, it costs to write a query for the oracle on the tape.

But the answer provided by the oracle appears in just one step, we do not care about the internals of the oracle.

Checking

Proposition

Polynomial time Turing reducibility is a preorder (reflexive and transitive).

For transitivity, this works since polynomials are closed under substitution. Hence we can from equivalence classes as usual, the polynomial time Turing degrees. We won’t pursue this idea here.

Compatibility

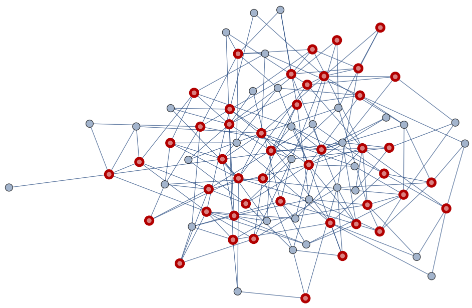
Proposition

$B \in \mathbb{P}$ and $A \leq_T^p B$ implies $A \in \mathbb{P}$.

Here we can simply replace the oracle by a real polynomial algorithm.

Since polynomials are closed under substitution, this will produce a polynomial time algorithm.

Recall: Vertex Cover



Finding minimal vertex covers is quite messy, even for moderate size graphs. In fact, even verifying a solution is hard.

Comparisons

Recall the 4 versions: decision, counting, function and search.

Intuitively, the function version is the hardest: once we have a lex-minimal cover, all other problems are trivial. Clearly, they are all polynomial Turing reducible to the function version.

The counting and decision versions are also Turing reducible to the search version.

Lastly, the decision version is Turing reducible to counting.

Decision $\leq_T^p$ Counting $\leq_T^p$ Search $\leq_T^p$ Function

The Other Way

Proposition

The function version of Vertex Cover is polynomial time Turing reducible to the decision version.

This requires a little argument. Let n be the number of vertices in G, say, $V = \{v_1, \ldots, v_n\}$ in lexicographic order.

- First, do a binary search to find the size k_0 of a minimal cover, using the oracle for the decision problem.
- Then find the least vertex v such that $G - v$ has a cover of size $k - 1$. Place v into C and remove it from G.
- Recurse.

This is polynomial time, even using Turing machines.

Justification

The last proposition is important: it justifies our narrow focus on decision problems: up to a polynomial factor, they are no worse than the fancier, more applicable versions. They all live in the same polynomial time Turing degree.

Of course, this does not help much when we are trying to find super-efficient algorithms. Typical example: Tarjan's strongly connected component algorithm, a form of DFS on steroids.

But, if all we want is to make sure that things can be done in polynomial time, the decision version is often good enough. Ditto for showing that things cannot happen in polynomial time.

Still No Good

Unfortunately, while polynomial Turing reductions are compatible with $\mathbb{P}$, they are still too brutish. To wit:

$$A \in \mathbb{P} \qquad \text{implies} \qquad A \leq_T^p \emptyset$$

The problem again is that the oracle TM can directly solve A, it has no need to query the oracle.

How about the other reductions (many-one in particular) that we considered in the CRT part?

Weak Reductions

Many-one reducibility is based on a computable function translating from one problem to another. We'll just have to constrain the function a bit.

Definition

$A \subseteq \Sigma^\star$ is polynomial time (many-one) reducible to $B \subseteq \Sigma^\star$ if there is a polynomial time computable function f such that

$$x \in A \iff f(x) \in B.$$

Notation: $A \leq_m^p B$.

Properties

Proposition

Polynomial time many-one reducibility is a preorder.

Proof.

Reflexivity is trivial. For transitivity, consider a polynomial time reduction f from A to B and g from B to C.

Obviously $h(x) = g(f(x))$ is a reduction from A to C.

h is still polynomial time computable since polynomials are closed under substitution.

□

Better Mousetrap?

Proposition

$\leq_m^p$ is compatible wrto $\mathbb{P}$: if B is in $\mathbb{P}$ and $A \leq_m^p B$, then A is also in $\mathbb{P}$.

This is clear: we can replace the oracle B by a polynomial time algorithm.

But is $\leq_m^p$ really a useful reduction for $\mathbb{P}$?

We could use the mediating function to "reduce" a problem in $\mathrm{TIME}(n^{1000})$ to $\mathrm{TIME}(n)$.

Again: we really want the reductions to be lightweight, the oracle should be the place where the heavy lifting takes place.

Log-Space Reductions

As it turns out, a better reduction results from restricting f even more: we'll insist that f can be computed in logarithmic space. More details about space complexity later, for the time being use your intuition.

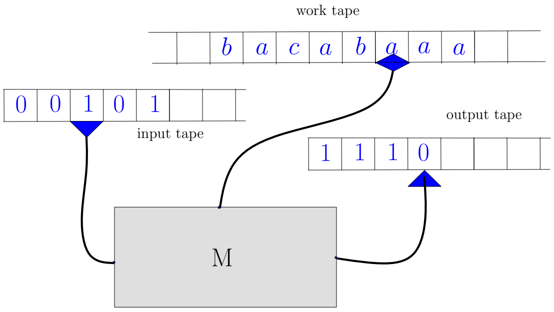
Definition

$A \subseteq \Sigma^\star$ is log-space reducible to $B \subseteq \Sigma^\star$ if there is a log-space computable function f such that

$$x \in A \iff f(x) \in B.$$

Notation: $A \leq_\ell B$.

Recall: Transducer



Only the work tape counts. We want it to be logarithmic in the size of the input.

But Why?

Clearly $A \leq_\ell B$ implies $A \leq_m^p B$.

But the mediating function f is now much more constrained. Think about graph problems where $[n]$ is the vertex set.

We cannot remember an arbitrary subset $S \subseteq [n]$.

Unfortunately, this wrecks many graph algorithms such as DFS that require linear memory and nicely forces the oracle to do most of the work.

Example: Neighborhood Traversals

Here is a typical code fragment in a graph algorithm:

```
foreach  v in V  do
foreach  (v,u) in E do
           blahblahblah
```

This idiom appears in lots of graph algorithms: look at the neighborhoods of all vertices.

Good news: it can be handled in logarithmic space: all we need is a few vertices in $[n]$ (these are written in binary), we can traverse all the edges (v, u) in any standard representation of a graph (this works fine for both adjacency matrices or adjacency lists, though they may produce different running times).

Structuring Polynomial Time

Unlike polynomial-time reductions, log-space reductions can be used to study the fine-structure of $\mathbb{P}$.

Definition ($\mathbb{P}$-Completeness)

A language B is $\mathbb{P}$-hard if for every language A in $\mathbb{P}$: $A \leq_\ell B$.
It is $\mathbb{P}$-complete if it is $\mathbb{P}$-hard and in $\mathbb{P}$.

Building a $\mathbb{P}$-hard set is easy: take an enumeration $(\mathcal{M}_e)$ of all polynomial time TMs, and build the analogue to the Halting set:

$$K_p = \{\, e\#x \mid \mathcal{M}_e(x) \text{ accepts} \,\}$$

Of course, there is no reason why K_p should be in $\mathbb{P}$.

More Precisely . . .

To obtain $(\mathcal{M}_e)$, we start with an arbitrary effective enumeration of all Turing machines and filter out the good machines. Let's ignore the problem that we cannot check whether the machines are total.

We would still need to run a poly-time tester that checks that $\mathcal{M}_e$ runs in polynomial time.

Unfortunately, this tester does not exist either: it is undecidable whether a machine runs in polynomial time. Intuitively we need to check

$$\exists\, k\, \forall\, n, x\, \big(|x| = n \Rightarrow T_{\mathcal{M}}(x) \leq n^k + k\big)$$

This looks completely hopeless, something like Σ_2.

Workaround

But we can construct a new machine $\mathcal{M}_e$ as follows. As always, assume that an enumeration (N_e) of all machines has infinitely many repetitions.

- attach a clock to N_e, and
- stop the computation after at most $n^e + e$ steps;
- return the same output as $N_e(x)$ if the computation has converged,
- otherwise return 0 (or some other default value).

Note that $\mathcal{M}_e$ is total by definition.

Furthermore, if some total machine $\mathcal{M}$ runs in polynomial time, then there is an index e such that $\mathcal{M}(x) \simeq \mathcal{M}_e(x)$ for all x.

If $\mathcal{M}$ is partial, we get the same behavior on the support.

Circuit Value Problem

Problem:	**Circuit Value Problem (CVP)**
Instance:	A Boolean circuit C, input value x.
Question:	Check if C evaluates to true on input x.

Obviously CVP is solvable in polynomial time (even linear time given a halfway reasonable representation).

There are several versions of CVP, here is a particularly simple one: compute the value of the "last" variable X_m given

$$X_0 = 0, \qquad X_1 = 1$$
$$X_i = X_{L_i} \diamondsuit_i X_{R_i}, \quad i = 2, \ldots, m$$

where $\diamondsuit_i = \wedge, \vee$ and $L_i, R_i < i$.

CVP is Hard

Theorem (Ladner 1975)

The Circuit Value Problem is $\mathbb{P}$-complete.

Sketch of proof. [†] For hardness consider any language A accepted by a polynomial time Turing machine $\mathcal{M}$.

We can encode the computation of the Turing machine $\mathcal{M}$ on x as a polynomial size circuit (polynomial in $n = |x|$): use lots of Boolean variables to express tape contents, head position and state.

Constructing this circuit only requires "local" memory; for example we need $O(\log n)$ bits to store a position on the tape.

The circuit evaluates to true iff the machine accepts. □

[†] We will provide much more detail in the proof of the Cook-Levin theorem.

Alternating Breadth First Search

A long list of $\mathbb{P}$-complete problems in known, though they tend to be a bit strange.

For example, consider an undirected graph $G = \langle V, E \rangle$ where E is partitioned as $E = E_0 \cup E_1$.

Question: Given a start vertex s and a target t, does t get visited along an edge in E_0 if the breadth-first search can only use edges in E_0 at even and E_1 at odd levels?

Theorem

ABFS is $\mathbb{P}$-complete (wrto logspace reductions).

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Complexity and Algorithmic Analysis

In any standard algorithm text you will find a statement along the lines of

Proposition

DFS runs in linear time.

What does that mean in our framework?

First, the size of the input, a ugraph $G = \langle V, E \rangle$ is usually given as $n + e$ where $n = |V|$ and $e = |E|$. This is justified by the standard adjacency list representation, and uniform cost function: in the RealWorld$^{\text{TM}}$, no one deals with graphs larger than $2^{64} \approx 1.85 \times 10^{19}$, much less 2^{128}.

If we were to implement DFS on a Turing machine, there would obviously be some slow-down. But still, there is a low-degree polynomial p so that the Turing machine runs in time $p(n + e)$.

Easy versus Hard

It is great fun to figure out how small p can be made, but ultimately irrelevant: we want lower bounds, in particular statements like

Problem such-and-such **cannot** be solved in polynomial time.

A polynomial increase doesn't matter in this case: we might as well think about the computation in a luxurious RAM or a C program. If the Turing machine does not run in polynomial time, these won't either.

This makes it much easier to reason about the problem.

Towards The Abyss: Path Existence

Given a digraph and two vertices s and t, we can ask whether there is a path from s to t.

The brute-force approach would enumerate all simple paths starting at s, and check if any one of them hits t. Alas, this is exponential.

But we can simply run DFS (or BFS) starting at s, and get the answer in linear time.

Shortest Path

Given a digraph whose edges are labeled by a positive cost, one can ask: what is the shortest path from s to t?

Again, brute-force would require enumeration of all simple paths and require exponential time.

But there are better (and more complicated) algorithms such as Dijkstra or Bellman-Ford that work in polynomial time.

Sanity Check

Just this once, let us make sure that a Turing machine could handle, say, arrays in polynomial time, and even at fairly low degree.

We want a reasonable Turing implementation of **arrays** of integers: a mutating data structure to store a (fixed-length) sequence of integers. The key operations are read and write:

```
x := A[i];        \\ read
A[i] := x;        \\ write
```

To simplify matters, we will use three tapes on the TM to represent a single array: one for the actual array, one for the index i, and one for the value x.

TM Arrays

The array tape looks like

$$\#a_0\#a_1\#\dots\#a_{n-1}\#$$

where the numbers a_i are written in binary, and $\#$ is a special separator symbol.

For a read operation, we use the index i to find the ith separator, searching from the left. Then we copy the following binary number to the value tape.

For a write operation, we again search for the ith block. We then copy x over from the other tape, shifting the remainder of the tape contents according to the number of bits in x (compared to the previous number of bits).

Alternatively, we could determine the number k of bits of the largest number ever stored, and reserve k bits in each block right from the start.

Done!

Of course, this wrecks the crucial property of an array, constant time read/write access. But: It only takes $O(nk)$ steps to simulate the ops.

More generally, when we implement some graph algorithm such as Dijkstra's shortest path on a Turing machine, the whole computation slows down, but only by a polynomial amount. In particular, if our pseudo-code runs in polynomial time, then the TM also runs in polynomial time. So we actually have a proof (sketch) that the shortest path problem is in $\mathbb{P}$.

Again: we are losing all fine-grained information, but this is fine if one is mostly interested in separating feasible from infeasible. The key accomplishment of Dijkstra's algorithm is to avoid an exponential search, the TM still reflects this.

Disaster Strikes: Long Paths

Given a ugraph, we can ask: is there a simple path that has length at least k?

Certainly brute-force enumeration of all simple paths would dismantle this problem, and would require exponential time.

But this time no one knows a computational shortcut, in particular no polynomial time algorithm for this problem is known.

The problem gets nasty when $k = n - 1$: we are asking for a simple path using all vertices. This seems to be fundamentally different from anything like a shortest path.

Exponential Search

As we will see, there are lots of combinatorial problems like the last one that are easy to solve with an exponential search, everything else is polynomial.

Naturally, one would like to establish a lower bound, say, every algorithm requires 2^n steps (at least for infinitely many inputs).

Alas, these lower bounds seem be extremely elusive, to the point of constituting a major open problem in math, not just the theory of algorithms.

Old Example

We already talked about one other such problem: Vertex Cover.

We are given a ugraph G with n vertices and a bound $k \leq n$, and want to know whether G admits a VC of size k.

Obviously we can perform a brute force check on all subsets of size k, but there are $\binom{n}{k}$ such sets—and that number is **not** polynomial in n for variable k (though everything is fine for fixed k).

So, on the face of it, it is far from clear that VC can be handled in polynomial time.