Undergraduate Complexity Theory
Lecture 18: NL-Completeness and Logspace Reductions

Marcythm

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1 Lecture Notes

Theorem 1.1. *For* $f(n) \geq \log n$, $\mathsf{NSPACE}(f(n)) \subseteq \mathsf{TIME}(2^{O(f(n))})$.

Theorem 1.2. *For* $f(n) \geq \log n$, $\mathsf{NSPACE}(f(n)) \subseteq \mathsf{SPACE}(f(n)^2)$.

Corollary 1.3. $\mathsf{NPSPACE} \subseteq \mathsf{PSPACE}$.

Corollary 1.4. $\mathsf{NPSPACE} = \mathsf{PSPACE}$.

Claim 1.5. $\mathsf{ST\text{-}PATH}$ *is "*NL*-complete".*

How to define sensible reduction for logspace? $\leq_m^P$ is a bad choice, since L is not closed under it.
Ideal: $A \in \mathsf{NL}, A \leq B, B \in \mathsf{L}$, then $A \in \mathsf{L}$.
idea: When using reductions, must be as weak as the weakest class you care about.
Need to define $\leq_m^L$, "log-space reduction", want:

1. closure: $A \leq_m^L B, B \in \mathsf{L} \implies A \in \mathsf{L}$, and $B \in \mathsf{NL} \implies A \in \mathsf{NL}$.
2. transitivity: $A \leq_m^L B, B \leq_m^L C \implies A \leq_m^L C$.

Definition 1.6. $A \leq_m^L B$ if $\exists R : \{0,1\}^* \to \{0,1\}^*$ computable (write-once output tape) in $O(\log n)$ space s.t. $\forall x : x \in A \iff R(x) \in B$.

model for space-bounded computation w/ output: read-only input tape, space-bounded ($O(\log n)$) r/w work tapes, write-once output tape.

Theorem 1.7. $\mathsf{ST\text{-}PATH}$ *is "*NL*-complete".*

Proof. $\mathsf{ST\text{-}PATH} \in \mathsf{NL}$.
Let $A \in \mathsf{NL}$, we need to show $A \leq_m^L \mathsf{ST\text{-}PATH}$. Say N is an $O(\log n)$-space nondeterministic TM deciding A. Claim: exists a deterministic $O(\log n)$-space-computable $R : \{0,1\}^* \to \{0,1\}^*$ that, given $x \in \Sigma^n$, outputs config graph $G_{N,x}$ and C_{start}, C_{acc}.
Thus, $x \in A \iff N(x)$ acc $\iff \exists$ path $C_{start} \to C_{acc}$ in $G_{N,x} \iff R(x) \in \mathsf{ST\text{-}PATH}$. □

Theorem 1.8. *If* $P, Q : \{0,1\}^* \to \{0,1\}^*$ *are computable in* $O(\log n)$ *space, then so is* $R(x) = Q(P(x))$.

core technique: recalculate $P(x)$ every time it is acquired by Q to save space.

Corollary 1.9 (closure)**.** $A \leq_m^L B, B \in \mathsf{L} \implies A \in \mathsf{L}$.

Corollary 1.10 (transitivity)**.** $A \leq_m^L B, B \leq_m^L C \implies A \leq_m^L C$.

Exercise 1.11. $A \leq_m^L, B \in \mathsf{NL} \implies A \in \mathsf{NL}$.

2 Reading

2.1 sipser 8.5 (NL completeness)

definition of log space reducibility
log space reducibility implies poly-time reducibility

Theorem 2.1. $A \leq_m^L B, B \in \mathsf{L} \implies A \in \mathsf{L}$.

Theorem 2.2. $\mathsf{ST\text{-}PATH}$ *is* NL*-complete.*

Corollary 2.3. $\mathsf{NL} \subseteq \mathsf{P}$.