Undergraduate Complexity Theory
Lecture 11: NP-Completeness and the Cook-Levin Theorem

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1 Lecture Notes

Recap: poly-time mapping reduction $\leq_m^P$, 3COL $\leq_m^P$ 4COL $\leq_m^P$ SAT $\leq_m^P$ CIRCUIT-SAT $\leq_m^P$ 3SAT. Actually 3SAT $\leq_m^P$ 3COL (which is hw5 p4), and this will be shown in the next lecture.

Theorem 1.1 (Cook-Levin Theorem)**.** $\forall L \in \mathsf{NP}, L \leq_m^P \mathsf{SAT}(\mathsf{3SAT}, \mathsf{3COL})$.

Corollary 1.2. *If* $\mathsf{SAT}(\mathsf{3SAT}, \mathsf{3COL}) \in \mathsf{P}$, $\mathsf{P} = \mathsf{NP}$.

Definition 1.3. Language S is called NP-hard, if $\forall L \in \mathsf{NP}, L \leq_m^P S$. (i.e. as hard as all problems in NP).

Remark 1.4. S is NP-hard does not imply $S \in \mathsf{NP}$.

Definition 1.5. Language S is NP-complete iff S is NP-hard and $S \in \mathsf{NP}$.

Theorem 1.6. *If S is* NP*-complete,* $S \in \mathsf{P} \iff \mathsf{P} = \mathsf{NP}$.

Theorem 1.7. *Suppose algo A is in time $T(n) \geq n$ on length-n inputs, then there exists circuit C of size $O(T(n)^2)$ outputs the same answer as A. (Furthermore, the circuit can be built in poly-time if $T(n) = kn^k$.)*

Proof. Denote $S_i^t \in \Gamma \cup Q$ as the symbol on location i of the TM's configuration at time t.

Claim 1.8. *S_i^k only depends on $S_{i-1}^{k-1}, S_{i-1}^k, S_{i-1}^{k+1}, S_{i-1}^{k+2}$. To be precise,*

$$S_i^k = \begin{cases} S_{i-1}^k, & S_{i-1}^{k-1} \notin \Gamma \\ f_\delta(S_{i-1}^{k-1}, S_{i-1}^k), & S_{i-1}^{k-1} \in \Gamma \end{cases}$$

Thus every symbol in configuration of any time can be written in boolean formula of symbols in previous configuration. Apply this recursively, we can construct the final output (i.e. the symbol in the final ($t = T(n)$) configuration) within boolean formula of $T(n)^2$ symbols and $T(n)^2$ size, one $T(n)$ for each step and another for each space. $\square$

2 Reading

2.1 Sipser 9.3 (Circuit Complexity)

1. Definition of boolean circuit, circuit family, size/depth complexity
2. Alternative proof of Cook-Levin Theorem, using boolean circuit (actually the one presented on class)

   Theorem 2.1. *For $t(n) \geq n$, if $A \in \mathsf{TIME}(t(n))$ then A has circuit complexity $O(t^2(n))$.*
3. CIRCUIT-SAT/3SAT is NP-complete.

2.2 Sipser 7.4 (NP-completeness)

1. Proof of Cook-Levin Theorem, using SAT directly.