15-455: UCT	**K. Sutner**
Assignment 2	Due: **Feb 04**, 2022.

1. Diophantine Solutions (20)

Background

According to a famous theorem by Matiyasevic, it is undecidable whether a multivariate polynomial with integer coefficients $P(\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{x}]$ has a solution over the integers. The same is true if we look of solutions of $\mathbb{N}^k$, for simplicity we'll use the second version.

Write $\#\mathsf{sol}(P)$ for the number of solutions of $P(\boldsymbol{x}) = 0$ over $\mathbb{N}^k$. If follows from Matiyasevic's theorem that "$\#\mathsf{sol}(P) = 0$" is undecidable.

Task

A. We are given a polynomial $P(\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{x}]$. Show that one can easily construct a polynomial Q such that the number of solutions of P over $\mathbb{Z}$ is the same as the number of solutions of Q over $\mathbb{N}$.

B. Given an arbitrary $k \in \mathbb{N}$, show that it is undecidable whether $\#\mathsf{sol}(P) = k$.

C. Show that it is undecidable whether $\#\mathsf{sol}(P) = \infty$.

Comment

For part (B), given a polynomial Q, construct a polynomial Q' such that $\#\mathsf{sol}(Q') = \#\mathsf{sol}(Q) + 1$.

This is a good example of a reduction: the very difficult part here is to show Matiyasevic's theorem; from there to asking specific cardinality questions is a fairly small step.

2. Graphs of Computable Functions (30)

Background

We have defined the complexity of sets in terms of the computability of their characteristic and semi-characteristic functions. One can also go in the opposite direction.

For any partial function $f : \mathbb{N} \nrightarrow \mathbb{N}$, we can define its graph to be the set

$$\mathsf{Gr}(f) = \{ (x, y) \mid f(x) \simeq y \} \subseteq \mathbb{N} \times \mathbb{N}$$

By an initial segment we mean the sets $\{ z \mid z < x \} \subseteq \mathbb{N}$ and $\mathbb{N}$ itself. The principal function of A is the unique order-preserving bijection between some initial segment and A. So the initial segment has the same cardinality as A.

Task

A. Show that a partial function $f : \mathbb{N} \nrightarrow \mathbb{N}$ is computable iff its graph is semidecidable.

B. What can you say about the graph of a total computable function?

C. Show that for any semidecidable set W and any partial computable function f the image $f(W) = \{ f(x) \mid f(x)\downarrow \wedge x \in W \}$ of W under f is again semidecidable.

D. Show that a set is decidable iff its principal function is computable.

E. Show that for any partial computable function f there is a partial computable function g such that for all x in the domain of f: $f(g(f(x))) \simeq f(x)$. If f were injective we could let $g = f^{-1}$, but the claim is that this works in general.

3. The Jump (25)

Background

Oracle Turing machines can be used to generalize Halting in a fairly natural manner. First define the jump A' of a set A as follows:

$$A' = \{\, e \mid \{e\}^A(e) \downarrow \}$$

For $A = \emptyset$ this is just the ordinary Halting set. But the double jump $\emptyset''$ should be even more complicated, never mind $\emptyset'''$ and so forth. In fact, one can show that the n-fold jump of $\emptyset$ is Σ_n-complete, but we won't go there.

Task

A. Show that A' is A-semidecidable but not A-decidable. Hence $A <_T A'$.

B. Show that B is A-semidecidable if, and only if, $B \leq_m A'$.

4. Classifying Index Sets (25)

Background

Consider the index sets

$$\mathsf{ONE} = \{\, e \mid |W_e| = 1 \,\}$$
$$\mathsf{EXT} = \{\, e \mid \{e\} \text{ is extendible to a total computable function} \,\}$$

Here a partial function $f : \mathbb{N} \nrightarrow \mathbb{N}$ is called extendible if there is a total function $F : \mathbb{N} \to \mathbb{N}$ such that $F \restriction D = f$ where $D \subseteq \mathbb{N}$ is the support of f.

Task

A. Find the location of ONE in the arithmetical hierarchy.

B. Find the location of EXT in the arithmetical hierarchy.

Lower bounds are not required, Extra Credit if you can prove a completeness result. But make sure your upper bounds are tight, a "solution" $\mathsf{EXT} \in \Sigma_{42}$ is useless.

Comment

It is known that $\mathsf{EXT} \neq \mathbb{N}$, there are partial computable functions that cannot be extended to a total computable function–which is really too bad, since otherwise we could just get rid of pesky partial functions.