Undergraduate Complexity Theory
Lecture 10: Reductions

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1 Lecture Notes

Theorem 1.1. *If* $\mathsf{SAT} \in \mathsf{P}$*, then* $\mathsf{4COL} \in \mathsf{P}$*.*

idea: given M_{SAT} solving SAT in poly-time, can build algo M_{4COL} solving $\mathsf{4COL}$ in poly-time, using M_{SAT} as a subroutine. The whole idea is denoted as $\mathsf{4COL} \leq^P \mathsf{SAT}$, which means $\mathsf{4COL}$ is at most polynomial slower than SAT.

Definition 1.2. Let A, B be languages, A has a poly-time **mapping reduction** to B, written $A \leq_m^P B$, if exists poly-time algo $R : \{0,1\}^* \to \{0,1\}^*$, s.t. $\forall x : x \in A \iff R(x) \in B$.

Theorem 1.3. *If* $A \leq_m^P B$*,* $B \in \mathsf{P}$*, then* $A \in \mathsf{P}$*.*

Proof. Given poly-time M_B for B, reduction R, $M_B(R(x))$ decides $x \overset{?}{\in} A$ in poly($|x|$) time. □

Theorem 1.4. $\leq_m^P$ *is transitive: if* $A \leq_m^P B, B \leq_m^P C$*, then* $A \leq_m^P C$*.*

Definition 1.5. $\mathsf{4CHROMA} = \{\langle G \rangle : G\text{'s chromatic number is } 4\}$.

Claim 1.6. $\mathsf{4CHROMA} \leq_T^P \mathsf{SAT}$*, where* T *stands for Turing, can use the subroutine many times.*

Remark 1.7. Here can not use $\phi = \phi_4 \wedge \neg\phi_3$, since $\neg\phi_3$ is satisfiable can not imply ϕ_3 is not satisfiable.

Theorem 1.8. *If* $A \leq_T^P B$*,* $B \in \mathsf{P}$*, then* $A \in \mathsf{P}$*.*

Theorem 1.9. *If* $A \leq_m^P B$*,* $B \in \mathsf{NP}$*, then* $A \in \mathsf{NP}$*.*

Remark 1.10. This theorem seemingly false for $\leq_T^P$.

Theorem 1.11. $\mathsf{CIRCUIT\text{-}SAT} \leq_m^P \mathsf{FORMULA\text{-}SAT}$*.*

Theorem 1.12. $\mathsf{CIRCUIT\text{-}SAT} \leq_m^P \mathsf{3\text{-}SAT}$*.*

upcoming: Cook-Levin Theorem

Theorem 1.13 (Cook-Levin Theorem)**.** $\forall L \in \mathsf{NP} : L \leq_m^P \mathsf{CIRCUIT\text{-}SAT}$*.*

2 Reading

2.1 Sipser 7.4 (NP-completeness)

1. poly-time reductions (mapping reduction)
2. Definition of NP-complete