Undergraduate Complexity Theory
Lecture 19: From P-Completeness to PSPACE-Completeness

Marcythm

July 21, 2022

1 Lecture Notes

today: A P-complete problem (under $\leq_m^L$) C: $C \in \mathsf{L}(\mathsf{NL}, \mathsf{NC}) \iff \mathsf{P} \subseteq \mathsf{L}(\mathsf{NL}, \mathsf{NC})$.
Informally, $\mathsf{NC} = \{L : L$ solvable in polylog(n) time, given poly(n) processors$\}$.

Fact 1.1. P*-complete language:* Horn-SAT*,* Linear-Programming*,* CIRCUIT-EVAL

Fact 1.2. *Empirically, whenever* $A \leq_m^P B$*, it seems that* $A \leq_m^L B$ *too.*

Theorem 1.3. $\forall A \in \mathsf{NP} : A \leq_m^L \mathsf{CIRCUIT\text{-}SAT} \leq_m^L \mathsf{3\text{-}SAT}$.

Theorem 1.4. *Cook-Levin Theorem is true even under* $\leq_m^L$.

sketch of proof: similar to Cook-Levin, each gadget can be constructed in logspace.

Corollary 1.5. CIRCUIT-EVAL *is* P*-complete.*

Definition 1.6. Q-SAT, or TQBF, where Q stands for quantified: Totally Quantified Boolean Formula.

$$\mathsf{TQBF} = \{\text{true sentences of form } Q_1 x_1 Q_2 x_2 \ldots \phi(x) \text{ where } Q_i \in \{\forall, \exists\}\}$$

FORMULA-SAT is a variant where all Q_is are $\exists$.

Proposition 1.7. $\mathsf{TQBF} \in \mathsf{PSPACE}$.

use recursive algo with linear space and exp time.

Theorem 1.8. TQBF *is* PSPACE*-hard.*

Proof. similar to Cook-Levin, but use quantifiers to reduce size (on timestamp) from exp to poly. □

Corollary 1.9. TQBF *is* PSPACE*-complete.*

2 Reading

2.1 sipser 8.3 (PSPACE completeness)

definition of PSPACE-complete: poly-time reduction.

Remark 2.1. Whenever we define complete problems for a complexity class, the reduction model must be more limited than the model used for defining the class itself.

definition of TQBF: prenex normal form (all quantifiers appears in the beginning)
formula game, generalized geography, and reduction from GG to TQBF.