

Undergraduate Complexity Theory

Lecture 16: Space Complexity

Marcyhm

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1 Lecture Notes

Definition 1.1. *Space complexity* of a decider TM is a function $S : \mathbb{N} \rightarrow \mathbb{N}$ s.t.

$$S(n) = \max_{|x|=n} \{\# \text{ distinct tape cells accessed by } M(x)\}$$

another version: read-only input tape, work tapes, for sublinear spaces.

Remark 1.2. To simulate multitape TM with space S , only needs single-tape TM with space $O(S)$.

Definition 1.3. $\text{SPACE}(f(n)) = \{A : \text{exists a TM deciding } A \text{ with space complexity } O(f(n))\}$.

Fact 1.4. Using $O(\log n)$ space a TM on input x can compute (write in base 2) $n = |x|$.

Definition 1.5. $L = \text{SPACE}(\log n)$.

e.g. $A = \{0^n 1^n : n \in \mathbb{N}\} \in L$. PALINDROMES $\in L$

Intuitively, L is pseudocode with constant (i.e. $O(1)$) # of variables, which are ints ranging from 0 to $\text{poly}(n)$. no array/memory allocation, read-only input lookups, basic arith ops on vars.

ST-PATH = $\{\langle G, s, t \rangle : \text{exists path } s \rightarrow t \text{ in } G\}$, in L ? BFS in $\Theta(n)$ space, DFS the same.

Theorem 1.6 (Savitch's Theorem '70). $\text{ST-PATH} \in \text{SPACE}(\log^2 n)$.

Theorem 1.7 (Reingold '04). *Undirected* ST-PATH $\in L$.

CIRCUIT-EVAL = $\{\langle C, x \rangle : C(x) = 1\} \in P$, doable in linear space, believed not in sublinear space.

3SAT: $O(n)$ space.

Definition 1.8. $\text{PSPACE} = \bigcup_{k \in \mathbb{N}} \text{SPACE}(n^k)$.

Exercise 1.9. $\text{NP} \subseteq \text{PSPACE}$.

Fact 1.10. $\text{TIME}(f(n)) \subseteq \text{SPACE}(f(n))$. e.g. $P \subseteq \text{PSPACE}$.

Theorem 1.11. For $f(n) \geq \log n$, $\text{SPACE}(f(n)) \subseteq \text{TIME}(2^{O(f(n))})$.

Corollary 1.12. $L \subseteq P$.

Corollary 1.13. $\text{PSPACE} \subseteq \text{EXP}$.

current hierarchy: $L \subseteq P \subseteq \{\text{NP} | \text{coNP}\} \subseteq \text{PSPACE} \subseteq \text{EXP}$.

interesting: find a problem in P but not in L ; in PSPACE but not in $\text{NP} \cup \text{coNP}$.

already knows $P \neq \text{EXP}$ by T.H.T, so whether $P \neq \text{PSPACE}$ or $\text{PSPACE} \neq \text{EXP}$?

Definition 1.14. $f(n)$ is *space-constructible* iff $f(n) \geq \log n$ and can compute $f(n)$ in $O(f(n))$ space.

Theorem 1.15 (Space Hierarchy Theorem). Let $f(n)$ be a space-constructible function, then exists language $A \in \text{SPACE}(f(n))$ that $\notin \text{SPACE}(g(n))$ for any $g(n) = o(f(n))$.

2 Reading

2.1 sipser 8.0 (Space Complexity)

1. definition of SPACE and NSPACE.
2. Space appears to be more powerful than time because space can be reused, whereas time cannot.

2.2 sipser 8.1 (Savitch's Theorem)

Theorem 2.1 (Savitch's Theorem). *For any function $f : \mathbb{N} \rightarrow \mathbb{R}^+$, where $f(n) \geq n$,*

$$\text{NSPACE}(f(n)) \subseteq \text{SPACE}(f^2(n))$$

2.3 sipser 8.2 (The Class PSPACE)

definition of PSPACE, so far $P \subseteq NP \subseteq PSPACE = \text{NPSPACE} \subseteq \text{EXP}$.