15-455: UCT **K. Sutner**

Assignment 1 Due: **Friday 01/28/2022**.

1. Sequence Numbers (20)

Background

A variadic function $f : \mathbb{N}^* \to \mathbb{N}$ is called a coding function if there are "inverse" functions $g : \mathbb{N} \to \mathbb{N}$ and $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$g(f(a_1, a_2, \dots, a_n)) = n,$$
$$h(f(a_1, a_2, \dots, a_n), i\,) = a_i, \quad 1 \le i \le n$$

for all sequences $a_1, a_2, \dots, a_n$. Thus g determines the length of the sequence and h decodes it back into its elements. Moreover, h and g are supposed to be easily computable but let's ignore that for the time being. Now consider the pairing function π defined by

$$\pi(x, y) = \binom{x + y + 1}{2} + x + 1$$

and define f as follows:

$$f(\mathsf{nil}) = 0$$
$$f(a) = \pi(0, a)$$
$$f(a_1, \dots, a_n) = \pi(f(a_2, \dots, a_n), a_1)$$

Task

A. Show that π is injective.

B. Show that f is a coding function. Make sure to explain what the appropriate decoding functions g and h are.

C. What would happen if we replaced $\pi(x, y)$ by $\pi(x, y) - 1$? How could you fix the issue?

2. Write-First Turing Machines (40)

Background

It is customary to define Turing machines via a transition function of the form

$$\delta : Q \times \Gamma \to \Gamma \times \Delta \times Q$$

Here Q is the set of states, Γ the tape alphabet including a blank symbol, and $\Delta = \{-1, 0, +1\}$ indicates movement of the head. An instruction $\delta(p, a) = (b, d, q)$ is interpreted as follows: if the machines is in state p and reads symbol a on the tape, it will write symbol b, move the head by d and go into state q. So the inner loop looks like this:

$$\text{read} — \text{write} — \text{move} — \text{goto}$$

Every action after the read depends on the symbol on the tape. This seems fairly natural, but there are other possibilities. For example, the machine could use a basic cycle

$$\text{write} — \text{move} — \text{read} — \text{goto}$$

The corresponding transition function has the format

$$\gamma : Q \to \Gamma \times \Delta \times (\Gamma \to Q)$$

Suppose the machine is in state p and $\gamma(p) = (b, d, f)$. Then the machine first writes b and moves the head according to d. Lastly, it reads the current tape symbol c (in a possibly new position) and transitions into state $f(c)$. For the sake of clarity we refer to these machines as write-first Turing machines (WFTM); their traditional counterparts will be called read-first Turing machines (RFTM).

Task

A. Give a precise definition of what it means for a write-first Turing machine to compute a function.

B. Show that every write-first Turing machine can be simulated by a read-first machine.

C. Show that every read-first Turing machine can be simulated by a write-first machine.

D. How do the machines compare in size?

Comment

For simplicity we assume here that δ is total, so you will have to find a way to redefine halting.

3. Minimal Machines (40)

Background

All models of computation can be associated with a natural size function. This is particularly obvious for machine-based models: the machine is just a finite data structure, and has a canonical size. For example, we could define the size of a Turing machine M to be the product $|Q||\Sigma|$, or the number of bits needed to specify its transition function. Or we could think of the index $\widehat{M}$ as a natural number, and use that number.

Fix one such measure, and call M minimal if no smaller machine is equivalent to M. Here equivalent means that $\forall\, z\, (M(z) \simeq M'(z))$: the computations may unfold in a different way, but the final result has to be the same for all inputs.

Task

A. Explain intuitively why minimality of TMs should not be semidecidable. You might want to start with decidability.

B. Assume that minimality of TMs is semidecidable. Show that there is an effective enumeration (N_e) of all minimal TMs.

C. Show that minimality of TMs fails to be semidecidable using the recursion theorem and part (A).

Comment

This can be proven without the recursion theorem, but the argument is much easier using the theorem.