Segment Tree and Lazy Propagation

beOI Training

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Motivating problem

You are given an integer array A of size n ($n < 10^6$).
Given two integers a and b, can you give the sum of the entries in A between indices a and b ?

$$A[a] + \ldots + A[b-1]$$

Well that's easy, just iterate over the interval and sum!

Motivating problem

You are given an integer array A of size n ($n < 10^6$).
Given two integers a and b, can you give the sum of the entries in A between indices a and b ?

$$A[a] + \ldots + A[b-1]$$

100000 times?

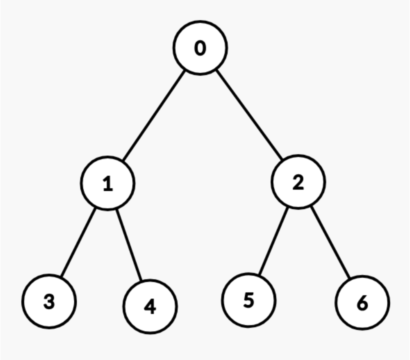
This is called the **range sum query** (RSQ) problem.

Naive solution

For each query, iterate over the corresponding range and sum the entries.
If k is the number of queries, time complexity is $\mathcal{O}(nk)$.
TLE

Array representation of a binary tree

- 0-based array, index $0 =$ root
- For each node of index p,
  - left child has index $2p + 1$
  - right child has index $2p + 2$



Segment Tree

Each node is responsible of one segment
Root represents the whole array $[0, n[$
Given a node representing segment $[l, r[$

- left child represents the segment's first half $\left[l, \frac{l+r}{2}\right[$
- right child represents the segment's second half $\left[\frac{l+r}{2}, r\right[$

The value of a node will be the **sum of the entries in segment** $[l, r[$.

Querying

When we query the sum in an interval, we look for **big segments that are contained within the query range**, and sum their values.
Recursively,

- segment is within query range $\Rightarrow$ return value of the node;
- segment and query range are disjoint $\Rightarrow$ do nothing;
- otherwise, return sum of both children.

Querying implementation

```
// p is array index of current node,
// [L,R[ is current segment,
// [i,j[ is search interval
// returns: position of the minimum element
int query(int p, int L, int R, int i, int j) {
    // inside  query range
    if (i <= L && R <= j) return st[p];
    // outside query range
    if (i >= R || L >= j) return 0;
    // sum the left and right subintervals
    return query(2*p+1, L, (L+R)/2, i, j)
           + query(2*p+2, (L+R)/2, R, i, j);
}

// Starting a query:
query(0, 0, n, i, j);
// CAREFUL with i, j: 0-indexed, inclusive-exclusive!
```

Querying complexity

At each level, at most 4 nodes are visited (see coach for proof).
There are exactly $\lceil \log_2 n \rceil$ levels.

$$\mathcal{O}(4 \times \lceil \log_2 n \rceil) = \mathcal{O}(\log n)$$

Overall complexity $\mathcal{O}(k \log n)$ is now reasonable!
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Building

Building the Segment Tree is also done recursively.
For each node,

- if no child, store current value;
- otherwise,
  - build left child;
  - build right child;
  - store sum of children.

Building implementation

```
void build(int p, int L, int R, vector<int> const& A) {
    if (L+1 == R) {
        // Single element in the segment
        st[p] = A[L];
    } else {
        // Build both children and then combine
        build(2*p+1, L, (L+R)/2, A);
        build(2*p+2, (L+R)/2, R, A);
        st[p] = st[2*p+1] + st[2*p+2];
    }
}

// Call with:
build(0, 0, n, A);
```

Building complexity

We visit every node once.
In general, the number of nodes is
$N + \frac{N}{2} + \frac{N}{4} + \cdots + 2 + 1 \approx 2N$, so time complexity is

$$\mathcal{O}(2 \times N) = \mathcal{O}(N)$$

This also proves memory is $\mathcal{O}(N)$ (in practice one always takes an array of $4 \times N$ for safety).

Segment Trees are extremely powerful!

We saw how to solve the range **sum** query problem.
But we can do much more than that!

- Range minimum query
- Range maximum query
- Range *insert any function here* query

One last operation

Suppose that, between queries, the array is being **updated**.
Naive solution: re-build the Segment Tree in $\mathcal{O}(N)$.
TLE
Segment Trees allow efficient **updating**!

Updating

To update p, we only need to update the segments that contain p.
Update the leaf to root path in $\mathcal{O}(\log N)$!

Updating implementation

```
// i is the node that is to be updated
// x is the new value
void update(int p, int L, int R, int i, int x) {
    if (L+1 == R) {
        // Single element in the segment
        st[p] = x;
    } else {
        // Build both children and then combine
        if (i < (L+R)/2) {
            update(2*p+1, L, (L+R)/2, i, x);
        } else {
            update(2*p+2, (L+R)/2, R, i, x);
        }
        st[p] = st[2*p+1] + st[2*p+2];
    }
}

// Call with:
update(0, 0, n, i, x)
// Careful: i is 0-indexed
```

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Motivating problem

In the Range Sum Query (RSQ) problem, we add one operation: range update.
We want to update a range (e.g. increment every value in range by dx) efficiently.

Naive solution

At each range update query, re-build tree in $\mathcal{O}(N)$.
TLE

Let's be lazy!

Key idea behind lazy Segment Tree: don't update everything at once; put a flag on segments that need to be updated, and leave it for another traversal.

Propagation

Keep an array `lazy` that stores for each segment by how much each value needs to be incremented.
Every time we visit a node p (in `query` or `update`) where `lazy[p] != 0`,

- increment current segment by `lazy[p]` times size of segment;
- if node is not leaf,
  - increment `lazy[2*p+1]` by `lazy[p]`
  - increment `lazy[2*p+2]` by `lazy[p]`
- reset `lazy[p]`.

That is called **propagation**.
Obviously, complexity is $\mathcal{O}(1)$.

Propagation implementation

```
void propagate(int p, int L, int R) {
    if (lazy[p] != 0) {
        st[p] += (R-L)*lazy[p];

        if (L+1 != R) {
            lazy[2*p+1] += lazy[p];
            lazy[2*p+2] += lazy[p];
        }

        lazy[p] = 0;
    }
}
```

Querying

We do exactly the same, but we propagate at each node!
Complexity $\mathcal{O}(\log N)$.

Querying implementation

```
int query(int p, int L, int R, int i, int j) {
    // This line is new:
    propagate(p, L, R);

    if (i <= L && R <= j) return st[p];
    if (i >= R || L >= j) return 0;

    return query(2*p+1, L, (L+R)/2, i, j)
         + query(2*p+2, (L+R)/2, R, i, j);
}
```

Updating

For each node,

- propagate
- if outside of range, return
- if inside of range, set the lazy flag, and return
- otherwise
  - update left child
  - update right child
- merge both children (add them up)

Complexity $\mathcal{O}(\log N)$.

Updating implementation

```
// i, j: update range
// dx: by how much to increment
void update(int p, int L, int R, int i, int j, int dx) {
    // inside  update range
    if (i <= L && R <= j) {
        lazy[p] += dx;
        propagate(p, L, R);
        return;
    }
    // outside update range
    if (i >= R || L >= j) return 0;

    update(2*p+1, L, (L+R)/2, i, j, dx);
    update(2*p+2, (L+R)/2, R, i, j, dx);

    st[p] = st[2*p+1] + st[2*p+2];
}
```

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Segment Trees are extremely powerful! (part 2)

Many combinations of queries and updates can be solved with segment trees!

- Range minimum query with $+=$ dx updates: minimum position does not change in the interval.
- Range sum query with $*=$ product updates: sum over interval is multiplied by product.
- And so on...

Iterative segment trees

- Shorter and more efficient than the recursive segment trees we have seen so far.
- My opinion: more difficult to remember, **not worth it** for IOI.
- See: `https://codeforces.com/blog/entry/18051`

Fenwick trees

- Faster to write, but less flexible: operation must be invertible (e.g. RSQ but not RMQ).
- Range updates are possible but complicated.
- See: 09-fenwick-trees.

"Dynamic" segment trees

- What if we cannot fit the entire range in memory?
- For example: Range Sum Query on array of size 10^9, initialized with 0, but updated later.
  - Memory usage: 4 bytes per integer $\times$ $4n$ of storage.
  - $4 \times 4 \times 10^9 \approx 16$ Gb.
  - Memory Limit Exceeded
- Key idea $\Rightarrow$ Use an `unordered_map` instead of a `vector`, build tree as needed.

2D segment trees

- Key idea $\Rightarrow$ Inside each node of an outer segment tree, store an inner segment tree.
- Allows for queries in $\mathcal{O}\left(\log n \times \log n\right)$.
- Very complicated to implement.
- See also: Game from IOI 2013.
- (A quad tree does not work, worst cast complexity is $\mathcal{O}(n)$.)