ME 193B / 292B: Feedback Control of Legged Robots
HW #5

Problem 1. Three-Link Walker: Control

In this problem, you will implement your first walking controller. We will use the model of the three-link walker similar to the one you've developed in HW#1 (The three link model considered here can be found in pg. 67 of [1]). Consider the following outputs to be controlled

$$y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} := \begin{bmatrix} \theta_3 - \theta_3^d \\ \theta_2 + \theta_1 \end{bmatrix}, \tag{1}$$

where θ_3^d is set to a constant value and θ_1, θ_2 and θ_3 correspond to the absolute angles of the links. This output corresponds to a simple walking behavior that enforces the torso angle to remain constant while commanding the swing leg to behave as the mirror image of the stance leg, $\theta_1 = -\theta_2$. Additionally, assume Leg-1 is the stance leg and the moment of inertias for the links is zero. The coefficient of static friction $\mu = 0.8$. A starter code is provided to you that computes the dynamics of the system in stance phase: D, C, G and B matrices, the contact forces that enforce the holonomic constraints at the stance foot, as well as the impact map Δ. These are computed in terms of the generalized coordinates $q := \begin{bmatrix} x & y & q_1 & q_2 & q_3 \end{bmatrix}^T$, where x, y is the position of the hip and q_1, q_2, q_3 correspond to the relative angles of the links.

(a) The dynamics of the system can be computed as

$$D(q)\ddot{q} + C(q,\dot{q}) + G(q) = B(q) + J_{st}^T F_{st}, \tag{2}$$

where J_{st} is the Jacobian of the stance foot position and F_{st} is the stance foot contact force. Using the expressions for D, C, G, B and F_{st}, computed in the started code and with the state defined as $s := \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$, compute the expressions for the vectors $f(s)$ and $g(s)$ such that $\dot{s} = f(s) + g(s)u$.
Note: This is unlike the dynamics $\dot{s} = f(s) + g_1(s)u + g_2(s)F_{st}$ that we formulated in class, and shows yet another way to formulate state-space systems for dynamics with constraints.

(b) Using the given transformation matrices T and d in the starter code, compute the outputs y in terms of q_1, q_2 and q_3.

(c) Using the above expressions, compute the Lie Derivatives of the output: $L_f y$, $L_g y$, $L_f^2 y$ and $L_g L_f y$.

(d) Derive the relabeling matrix $R \in \mathbb{R}^{5\times 5}$ that maps the pre-impact configuration to the post-impact configuration, i.e.

$$q^+ = Rq^- \tag{3}$$
$$\dot{q}^+ = R\Delta_{\dot{q}}\left(q^-, \dot{q}^-\right), \tag{4}$$

where $\Delta_{\dot{q}}(q^-, \dot{q}^-)$ was computed in Problem 2, HW#2. (Note that in HW#2, we did not consider the relabeling matrix to compute the post-impact velocities).

Problem 2. Three-Link Walker: Simulation

(a) We will now consider a slightly different form of controller than what was presented in class. This controller also does input-output linearization, but the convergence properties are faster than exponential, resulting in finite-time convergence. Consider the controller,

$$u = L_g L_f y^{-1}\left(-L_f^2 y + v\right) \tag{5}$$

$$v = \begin{bmatrix} \frac{1}{\epsilon^2}\psi_a(y_1, \epsilon\dot{y}_1) \\ \frac{1}{\epsilon^2}\psi_a(y_2, \epsilon\dot{y}_2) \end{bmatrix}, \tag{6}$$

$$\psi_a(x_1, x_2) := -\text{sign}(x_2)\,|x_2|^a - \text{sign}(\phi_a(x_1, x_2))\,|\phi_a(x_1, x_2)|^{\frac{a}{2-a}}, \tag{7}$$

$$\phi_a(x_1, x_2) := x_1 + \frac{1}{2-a}\text{sign}(x_2)\,|x_2|^{2-a}, \tag{8}$$

With the above controller, simulate the system for 10 walking steps (using `ode45` in MATLAB). Use the initial condition $x_0 = \begin{bmatrix} -0.3827 \\ 0.9239 \\ 2.2253 \\ 3.0107 \\ 0.5236 \\ 0.8653 \\ 0.3584 \\ -1.0957 \\ -2.3078 \\ 2.0323 \end{bmatrix}$ and with $\theta_3^d = \pi/6$. Assume impact to occur when $\theta_1 = \pi/8$ (i.e. absolute angle of leg-1 (stance leg) is equal to $\pi/8$ radians) and detect -ve to +ve crossings.

Provide plots of:

i. θ_1 vs $\dot{\theta}_1$,

ii. u_1 and u_2 vs time (on the same plot).

iii. F_{st} vs time (both components of F_{st} on the same plot).

For the controller, choose $a = 0.9$ and $\epsilon = 0.1$. The control input here enforces the outputs to converge to the origin in finite-time (See pg. 168 of [1] for more details). Use the animation script provided to visualize your simulation.

(b) We will now check for the friction constraints at the stance foot. Specifically, we want F_{st} to satisfy the following conditions:

i. Vertical (second) component of F_{st} must be greater than 0, i.e. the ground cannot pull the robot and,

ii. The ratio of the horizontal (first) and vertical components of F_{st} must be less than the coefficient of static friction, i.e. we do not want the robot to slip.

Using the data from your simulation, check if the above conditions are met. Provide plots demonstrating this.

Instructions

1. You may submit either a typeset or handwritten solution. In either case, submit a **PDF** version of your solutions.

2. Start each problem on a separate page and box answers to each sub-question.
3. You may choose to use a symbolic math package such as the `Symbolic Math Toolbox` (`https://www.mathworks.com/help/symbolic/index.html`) in `MATLAB` or `Mathematica`.
4. Do include all your code, if any.
5. Please submit a single pdf of your HW. (If typset on a computer, please save to pdf. If handwritten, please scan to pdf.)
6. **Honor Code.** You are to do your own work. Discussing the homework with a friend is fine. Sharing results or MATLAB code is not.

References

[1] Eric R. Westervelt, Jessy W. Grizzle, Christine Chevallereau, Jun Ho Choi, and Benjamin Morris. Feedback Control of Dynamic Bipedal Robot Locomotion. *Crc Press*, 2007.