A constrained monopod based on a rotating plane with anisotropic viscous friction

1 Introduction

We study methods for the experimental determination of the viscous friction parameters. During the experiments, test objects contact with moving and stationary planes. The parameters arising in special modes of motion give information about the parameters of the friction model.

2 The problem

Consider a motion of a rigid body along a horizontal plane that rotates at a constant angular velocity $\boldsymbol{\omega}$ about a fixed vertical axis. The body and the plane are in contact at a single point. Let's call the body a monopod. The motion of the monopod is constrained so that it can rotate freely about a certain vertical axis. It is fixed both in an absolute frame and in a coordinate system rigidly connected to the monopod.

The rotating plane acts on the monopod in the support point P with linear viscous friction

$$\boldsymbol{F}_{\text{fr}} = -c_1 v_1 \boldsymbol{e}_1 - c_2 v_2 \boldsymbol{e}_2. \tag{1}$$

There are concentric circles on the plane with a center O. Unit orthogonal vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are fixed in the rotating plane and depend on the point P: $\boldsymbol{e}_1 \parallel \overrightarrow{OP}$, $\boldsymbol{e}_2 \perp \overrightarrow{OP}$; c_1, c_2 are linear viscous friction coefficients for vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$; $v_1 = (\boldsymbol{v}^{\text{r}}, \boldsymbol{e}_1)$, $v_2 = (\boldsymbol{v}^{\text{r}}, \boldsymbol{e}_2)$, $\boldsymbol{v}^{\text{r}}$ is velocity of point P relative to the rotating plane.

Our goal is to build an algorithm for determining c_1, c_2. Introduce a fixed frame $OXYZ$ so that, the plane OXY coincides with the horizontal plane, and the axis OZ is directed vertically upward. A point A represents a projection onto the plane OXY of some point that lies on the rotation axis of the monopod. The axis OX passes through point A. Let r be a distance between the rotation axis of the plane OZ and the rotation axis of the monopod ($r = |\overrightarrow{OA}|$) and ρ be a distance between the rotation axis of the monopod and the supporting point P ($\rho = |\overrightarrow{AP}|$). $Axyz$ is a coordinate system fixed in the monopod. The axis Az is parallel to the axis OZ, the axes Ax and Ay lie in the plane OXY, and the axis Ax is directed along the vector $\overrightarrow{AP}$. I is a moment of inertia of the monopod about the axis Az. Let φ be a rotation angle of the monopod about the axis Az.

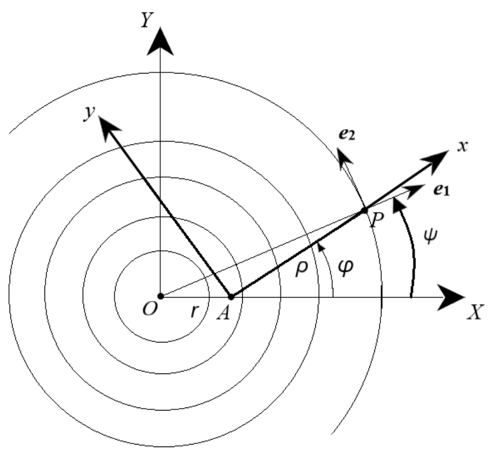

Figure 1: Projection of the system onto the plane OXY

The monopod's equation of motion follows from The Angular Momentum Theorem with respect point A about the axis Az

$$\ddot{\varphi} + a c_2 (\dot{\varphi} - \omega) + \omega r c_2 \frac{a}{\rho} \sin\varphi + \frac{a(c_1 - c_2) r^2 \cos^2\varphi}{r^2 + \rho^2 - 2r\rho \sin\varphi} \dot{\varphi} = 0, \quad a = \frac{\rho^2}{I}. \tag{2}$$

At $r < \rho$, there are no equilibrium positions, but a unique stable limit cycle exists. If the rotation axis of the monopod Az coincides with the rotation axis of the plane OZ ($r = 0$), the periodic motion is a uniform rotation of the monopod with the angular velocity equal to the angular velocity of the plane: $\varphi(t) = \omega t$.

Consider the case when r is small: $r = \varepsilon\rho$, ε is a dimensionless parameter and $\varepsilon \ll 1$. Denote $T(\varepsilon)$ by the period of the periodic solution and perform the substitution of time $\tau = \dfrac{t}{T(\varepsilon)}$. All periodic solutions have the same unit period at this substitution of time.

For $\varepsilon = 0$ we have $\varphi(t, 0) = \omega t$ and $T(0) = \dfrac{2\pi}{\omega}$, so we present the periodic solution and the period in the following form

$$\varphi(\tau, \varepsilon) = 2\pi\tau + g_1(\tau)\varepsilon + g_2(\tau)\varepsilon^2 + O(\varepsilon^3), \quad \omega T(\varepsilon) = s(\varepsilon) = 2\pi + s_1\varepsilon + s_2\varepsilon^2 + O(\varepsilon^3).$$

Here $g_1(\tau), g_2(\tau)$ are periodic functions with the unit period. We substitute the solution $\varphi(\tau, \varepsilon)$ and the period $T(\varepsilon)$ in the equation written in the new time τ

$$\varphi'' + ac_2T(\varepsilon)(\varphi' - T(\varepsilon)\omega) + \varepsilon ac_2\omega T(\varepsilon)^2 \sin\varphi + \frac{a(c_1 - c_2)\varepsilon^2\cos^2\varphi}{1 - 2\varepsilon\sin\varphi + \varepsilon^2} T(\varepsilon)\varphi' = 0$$

and get two equations for determining $g_1(\tau)$, $g_2(\tau)$, s_1, and s_2. We successively find these functions and coefficients provided that $g_1(\tau)$, $g_2(\tau)$ are periodic. As a result, we obtain up to ε^2

$$\omega T(\varepsilon) = s(\varepsilon) = 2\pi + s_2\varepsilon^2, \qquad \frac{s_2(x)}{\pi} = \frac{c_1}{c_2} - \frac{1}{1 + c_2^2 x} = \delta - \frac{1}{1 + k_2^2}, \quad x = \left(\frac{a}{\omega}\right)^2. \tag{3}$$

We can use the formulas (3) to determine c_1 and c_2. Note, that the coefficient s_2 is hyperbola on x. We write two representations for s_2: on the left it is for a real experiment, on the right - for a numerical one.

The problem is to find the fitting hyperbola of the following form $f(x) = B - \dfrac{1}{1 + Ax}$. To obtain experimental points, we integrate the dimensionless equation of motion at fixed parameters ε, δ, k_2

$$\varphi'' + k_2(\varphi' - 1) + \varepsilon k_2 \sin\varphi + \frac{(\delta - 1)k_2\varepsilon^2\cos^2\varphi}{1 - 2\varepsilon\sin\varphi + \varepsilon^2}\varphi' = 0,$$

where $'$ is the derivative with respect to dimensionless time $\widetilde{\tau} = \omega t$.

Numerical simulation implements the real experiment. At the fixed angular velocity value of the plane ω, the distance between the rotation axes of the plane and the monopod is changed (k_2 corresponds to ω, ε corresponds to r in the numerical experiment). At each fixed distance, we measure the period of the periodic motion. The limit cycle is stable, so after some time, for any initial conditions, the motion will become stationary. As a result, we get a set of values r and T in real experiment. Using the formulas $\varepsilon = \dfrac{r}{\rho}$ and $s = \omega T$, we calculate the set of points ε and s and find the coefficient s_2 for the fitting parabola. Thus, by changing the angular velocity value ω in the experiment, we get different values s_2.

3 Conclusion

In our work, we have proved the existence of the unique periodic solution if the distance between the rotation axes of the plane and the monopod is small enough. We have found the dependence of the period on this distance and obtained the approximate formula for the friction coefficients. We have demonstrated the method for determining the viscous friction coefficients by the periodic motion of the monopod. Numerical simulation has shown that the method is good at determining the friction coefficients in some cases.