Advection and diffusion in a porous medium

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Porous medium model

- What is a Porous medium?

Tissues can be treated as a porous medium as they are composed of dispersed cells separated by connective voids which allow for flow of nutrients.

Mass diffusion and advection in brain tissues

- Diffusion is an essential mechanism for delivering glucose and oxygen from the vascular system to brain cells.
- Advection is a transport mechanism of a substance or conserved property by a fluid due to the fluid's bulk motion.

The domain

We model a simplified situation in which the fluid is moving through a 2D thin fracture that could be seen as an oversimplified porous medium,
The domain is the form:

$$\Omega = \{\mathbf{x} = (x, y,) \in \mathbb{R}^2, 0 < x < L, 0 < y < H_0\}$$

where H_0 is the height of the channel and L is the characteristic lenght of the domain. The following assumption holds:

$$H_0 \ll L \rightarrow \epsilon = \frac{H_0}{L} \ll 1$$

Mass diffusion and advection in brain tissues

Boundary and initial condition

Boundary condition:

- No-slip boundary condition applies on the bed surface

$$u_y = 0 \qquad \text{on } y = 0, y = H_0$$

Concentration initial condition is:

$$c = c_0 \qquad \text{at } t = 0$$

where c_0 is a smooth function, indicating that in a typical cell of the porous medium the initial concentration is uniform.

Advection-diffusion equation

Outline

The transport of oxygen can be described by the advection-diffusion equation. It be derived from the continuity equation.

$$\frac{\delta c}{\delta t} + \nabla \cdot \mathbf{J} = 0$$

where c is the concentration of the species and $\mathbf{J}$ is the flux. Sources of flux:

- diffusive flux
- advective flux

The Fick's law

Diffusive flux

Fick's law describes concentration c of a compound in a material

$$\frac{\delta}{\delta t}\left(\iiint\limits_V c\delta V\right) = -\oint\!\!\!\oint\limits_S D\nabla c\delta S$$

Using the divergence theorem and recalling that V is an arbitrary volume we can rewrite the above equation as:

$$\frac{\delta c}{\delta t} = -D\nabla\cdot(\nabla C_f) = -D\nabla^2 C_f$$

where c is the concentration of the nutrients .

Advective flux

The advection is the transport mechanism of a species by a fluid due to the fluid's bulk motion.

$$\frac{\delta c}{\delta t} = \nabla \cdot (\mathbf{u} c)$$

where $\mathbf{u} = (u_x, u_y) = (u_x, 0)$ is the blood flow term (i.e the velocity vector field).

The uptake term

The uptake term represents the consumption of the species by the surrounding tissue. This can be modeled by the following equation:

$$F(c) = F_0 c$$

where F_0 is the surface reactivity coefficient.

The advection-diffusion equation

Final form

Conservation of c reads:

$$\frac{\delta c}{\delta t} + \nabla \cdot ((\mathbf{u}c) - D^* \nabla c) = F(c)$$

where D^* is the *effective diffusion coefficient* given by:

$$D^* = \frac{\phi D}{\tau^2}$$

where ϕ is the porosity and τ is the tortuosity of the capillaries

The blood flow model

The blood flow term could be derived starting from the Navier-Stokes equations for incompressible fluids:

$$\rho\frac{\delta \mathbf{u}}{\delta t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\rho = -\nabla P + \nu\nabla^2\mathbf{u} + F$$

$$\nabla\cdot\mathbf{u} = 0$$

The blood flow model

Asymptotic expansion

The velocity $\mathbf{u}$ and the pressure P can be written as asymptotic expansions in ϵ:

$$\mathbf{u}^\epsilon = \epsilon^2 \mathbf{u}^0 + \epsilon^3 \mathbf{u} + O(\epsilon^4)$$
$$P^\epsilon = P^0 + \epsilon P + \epsilon^2 P^2 + O(\epsilon^3)$$

The blood flow model

Asymptotic expansion

For a slow steady flow we ca neglect the left-hand term side in the momentum equation. Substituting the expansions and collecting equal power of ϵ yield:

$$\Big[-\nu\frac{\delta^2\mathbf{u}^0}{\delta y^2}+\nabla_{x'}p^0+\frac{\delta P^1}{\delta y}\mathbf{k}\Big]+\epsilon\Big[-\nu\frac{\delta^2\mathbf{u}^1}{\delta y^2}+\nabla_{x'}p^1+\frac{\delta P^2}{\delta y}\mathbf{k}\Big]+$$
$$+\epsilon^2\Big[-\nu\frac{\delta^2\mathbf{u}^2}{\delta y^2}-\nu\nabla_{x'}\mathbf{u}^{\mathbf{0}}+\nabla_{x'}p^2+\frac{\delta P^3}{\delta y}\mathbf{k}\Big]+\cdots=F$$

The blood flow model

Leading order solution

Keeping only the first term we arrive at:

$$-\nu \frac{\delta^2 \mathbf{u}^0}{\delta y^2} + \nabla_{x'} p^0 + \frac{\delta P^1}{\delta y} \mathbf{k} = F$$

Regarding that $\mathbf{u^0} = 0$ for $y = 0$ the solution is in the form:

$$\mathbf{u}^0 = \frac{1}{2\nu} y(h(x') - y)\mathbf{w}(x')$$

The blood flow model

Leading order solution

The standard Darcy's law:

$$\mathbf{w} + \nabla_{x'} P^0 = F$$

Darcy model ignores the boundary effects on the flow.

The blood flow model

Lower order terms

The equation for the first and the second order are:

$$O(\epsilon) \rightarrow -\nu \frac{\delta^2 \mathbf{u}^1}{\delta y^2} + \nabla_{x'} p^1 + \frac{\delta P^2}{\delta y}\mathbf{k} = 0$$

$$\nabla \cdot \mathbf{u^0} + \frac{\delta u_2^1}{\delta y} = 0$$

$$O(\epsilon^2) \rightarrow -\nu \frac{\delta^2 \mathbf{u}^2}{\delta y^2} - \nu \nabla_{x'} \mathbf{u^0} + \nabla_{x'} p^2 + \frac{\delta P^3}{\delta y}\mathbf{k} = 0$$

$$\nabla \cdot \mathbf{u^1} + \frac{\delta u_2^2}{\delta y} = 0$$

The blood flow model

Lower order terms

Order ϵ solution:

$$O(\epsilon) \to \begin{cases} u_3^1 & = - \int_0^y (div_{x'} \mathbf{u}^0(x', z) \delta z) \\ \mathbf{u}^1 & = u_3^1 \mathbf{k} \\ p^2 & = \nu \frac{\delta u_3^1}{\delta y} \end{cases}$$

The blood flow model

Lower order terms

Applying the previous solution to the order ϵ^2 system leads to:

$$O(\epsilon^2) \rightarrow \begin{cases} \mu B^{\epsilon}\mathbf{u}^{\epsilon} + \nabla_{x'} p^0 - \nu\nabla^2\mathbf{u} = F \\ div_{x'}\mathbf{v}^{\epsilon} = 0 \text{ in the domain } O \\ \mathbf{u}^{\epsilon} = 0 \text{ on } \delta O \end{cases}$$

Which is the Darcy-Brinkman law for viscous fluid with no-slip B.C.

The blood flow model

Comparison between the models

Remarks

- Diffusion and advection are both important to model the transport of nutrient through tissues.
- The uptake term takes into account the loss of nutrients and the chemical reactions.
- The Brinkmann model is more accurate and efficient than Darcy model.

Future work

- Model the uptake term considering the chemical effects
- Combine the diffusion-advection equation with the Brinkman model to find an expression for the concentration
- 3-D model and simulation

THANKS!