HW#2
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1. (a) In our case we have

$$a(u,v) = (u,v)_a = \int_\Omega \partial_x u \partial_x v + \partial_y u \partial_y v + uv \; dxdy$$

Then we show that

$$\begin{aligned} \|u-u_h\|_a^2 &= \int_\Omega (\partial_x (u-u_h))^2 + (\partial_y (u-u_h))^2 + (u-u_h)^2 \, dxdy \\ &= \sum_{j=1}^{2} \|\partial_{x_j} (u-u_h)\|_{L^2(\Omega)}^2 + \int_\Omega (u-u_h)^2 \, dxdy \\ &= |u-u_h|_{H^1(\Omega)}^2 + \|u-u_h\|_{L^2(\Omega)}^2 = \|u-u_h\|_{H^1(\Omega)}^2 . \end{aligned} \tag{1}$$

Then, using Céa's Lemma we observe

$$\|u-u_h\|_{H^1(\Omega)}^2 = \|u-u_h\|_a^2 \leq \|u-I_h u\|_a^2 \overset{(1)}{=} \|u-I_h u\|_{H^1(\Omega)}^2 \leq C_1^2 h^2 |u|_{H^2(\Omega)}^2$$

where in the last inequality we use the hypothesis. Taking square roots in the last inequality we have

$$\|u-u_h\|_{H^1(\Omega)} \leq C_1 h |u|_{H^2(\Omega)}.$$

So considering $C_2 = C_1$ we have the required inequality (where the constant C_2 is independet of h, u and u_h).

(b) Let $z \in H_0^1(\Omega)$ be the weak solution of the following boundary value problem

$$\begin{aligned} -\Delta z + z = u - u_h & \quad \text{in } \Omega \\ z = 0 \quad & \quad \text{in } \partial\Omega. \end{aligned}$$

The weak formulation of this problem is

$$\text{find } z \in H_0^1(\Omega) \text{ such that } (z,v)_a = (v, u-u_h) \quad \forall v \in H_0^1(\Omega).$$

If we consider $v = u - u_h \in H_0^1(\Omega)$ we have

$$\begin{aligned} (z, u-u_h)_a &= (u-u_h, z)_a \\ &= \int_\Omega \partial_x (u-u_h) \partial_x z + \partial_y (u-u_h) \partial_y z + (u-u_h) z \; dxdy \\ &= \int_\Omega ((u-u_h) z_x)_x + ((u-u_h) z_y)_y - (u-u_h)\Delta z + (u-u_h) z \; dxdy \\ &= \underbrace{\int_{\partial\Omega} -(u-u_h) z_y dx}_{=\text{ 0 because } u-u_h \in H_0^1} + \underbrace{\int_{\partial\Omega} (u-u_h) z_x dy}_{=\text{ 0 because } u-u_h \in H_0^1} + \int_\Omega (u-u_h) \underbrace{(-\Delta z + z)}_{=u-u_h} dxdy \\ &= \int_\Omega (u-u_h)^2 dxdy = \|u-u_h\|_{L^2(\Omega)}^2 \end{aligned}$$

where we used the Riemann-Green theorem.

If we denote by z_h the piecewise linear finite element approximation of z, using Galerking orthogonality we conclude that

$$\begin{aligned} \|u-u_h\|_{L^2(\Omega)}^2 &= (u-u_h, z-z_h)_a \\ &= (u-u_h, z-z_h)_a + (u-u_h, z_h)_a = (u-u_h, z-z_h)_a \end{aligned}$$

Applying the Cauchy-Schwarz inequality on the right-hand-side we get

$$\|u - u_h\|_{L^2(\Omega)}^2 \leq \|u - u_h\|_a \|z - z_h\|_a . \tag{2}$$

Now, by the Céa's Lemma

$$\|u - u_h\|_a \leq \|u - I_h u\|_a = \|u - I_h u\|_{H_0^1(\Omega)}^2 \leq C_1 h |u|_{H^2(\Omega)}$$

then, following with (2) we have

$$\|u - u_h\|_{L^2(\Omega)}^2 \leq C_1^2 h^2 |u|_{H^2(\Omega)} |z|_{H^2(\Omega)}. \tag{3}$$

To conclude the proof we will prove that for all $w \in H^2(\Omega) \cap H_0^1(\Omega)$

$$|w|_{H^2(\Omega)}^2 \leq \|\Delta w - w\|_{L^2(\Omega)}^2 . \tag{4}$$

In fact, if the previous inequality holds, for $w = z$ we have

$$|z|_{H^2(\Omega)} \leq \|\Delta z - z\|_{L^2(\Omega)} = \|-\Delta z + z\|_{L^2(\Omega)} = \|u - u_h\|_{L^2(\Omega)}$$

and we obtain from (3)

$$\|u - u_h\|_{L^2(\Omega)} \leq C_1^2 h^2 |u|_{H^2(\Omega)}$$

that completes the prove. So considering $C_3 = C_1^2$ we have the required inequality (where the constant C_3 is independet of h, u and u_h).

To prove (4), firstly we observe:

$$\begin{aligned}
\|\Delta w\|_{L^2(\Omega)}^2 &= \int_\Omega (\partial_{xx} w + \partial_{yy} w)^2 dxdy \\
&= \int_\Omega (\partial_{xx} w)^2 + 2\partial_{xx} w \partial_{yy} w + (\partial_{yy} w)^2 dxdy \\
&= \int_\Omega (\partial_{xx} w)^2 + 2(\partial_x \partial_y w)^2 + (\partial_{yy} w)^2 dxdy \\
&= |w|_{H^2(\Omega)}^2
\end{aligned}$$

where we used Integration by parts. Then, we follow

$$\begin{aligned}
|w|_{H^2(\Omega)}^2 = \|\Delta w\|_{L^2(\Omega)}^2 = \|\Delta w - w + w\|_{L^2(\Omega)}^2 &= \int_\Omega (\Delta w - w + w)^2 \, dxdy \\
&= \int_\Omega (\Delta w - w)^2 \, dxdy + \int_\Omega 2 (\Delta w - w) \, w + w^2 dxdy \\
&= \|\Delta w - w\|_{L^2(\Omega)}^2 + \int_\Omega 2w\Delta w - w^2 dxdy;
\end{aligned}$$

To finish the demostration, we only have to proof that the last integral is less or equal than zero:

$$\begin{aligned}
\int_\Omega 2w\Delta w - w^2 \; dxdy \overset{(-w^2 \leq 0)}{\leq} \int_\Omega 2w\Delta w \; dxdy &= 2\int_\Omega w\Delta w \; dxdy \\
&= 2\int_\Omega (w\partial_x w)_x + (w\partial_y w)_y - (\partial_x w)^2 - (\partial_y w)^2 \, dxdy \\
&\leq 2\int_\Omega (w\partial_x w)_x + (w\partial_y w)_y \\
&= 2\int_{\partial\Omega} \underbrace{w\partial_y w \; dx + w\partial_x w \; dy}_{= \, 0 \text{ because } w \in H_0^1} = 0.
\end{aligned}$$

where we used Riemann-Green Theorem again and the fact that $-(\partial_x w)^2 \leq 0$ and $-(\partial_y w)^2 \leq 0$.

2. (a) The equations for the $u(t) := X(t)$ and $v(t) := Y(t)$ are given by

$$u_t = A + u^2 v - (B+1)u$$
$$v_t = Bu - u^2 v.$$

To find the equilibria point of the system we must solve

$$\begin{cases} u_t = 0 \\ v_t = 0 \end{cases} \iff \begin{cases} A + u^2 v - (B+1)u & = 0 \\ Bu - u^2 v & = 0 \end{cases}$$

Considering $v = B/u$ in the second equation and replacing it in the first we arrive to the unique solution $P = (A, B/A)$.

Let us consider $f(u,v) = A + u^2 v - (B+1)u$ and $g(u,v) = Bu - u^2 v$. To prove that P is unstable we start by computing the Jacobian matrix:

$$J(u,v) = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix} = \begin{bmatrix} 2uv - (B+1) & u^2 \\ B - 2uv & -u^2 \end{bmatrix}$$

so, we have

$$J(P) = J(A, B/A) = \begin{bmatrix} B-1 & A^2 \\ -B & -A^2 \end{bmatrix}.$$

The eigenvalues of $J(P)$ are given by

$$|J(P) - \lambda I| = 0 \Leftrightarrow \lambda^2 + \lambda(A^2 - B + 1) + A^2 = 0$$

By Routh-Hurwitz Criteria [2] for second-order polynomials , P is unstable if and only if

$$A^2 - B + 1 < 0$$
$$A^2 > 0$$

So P is unstable if and only if $B > A^2 + 1$.

(b) Matlab file **Ex2b.m**.

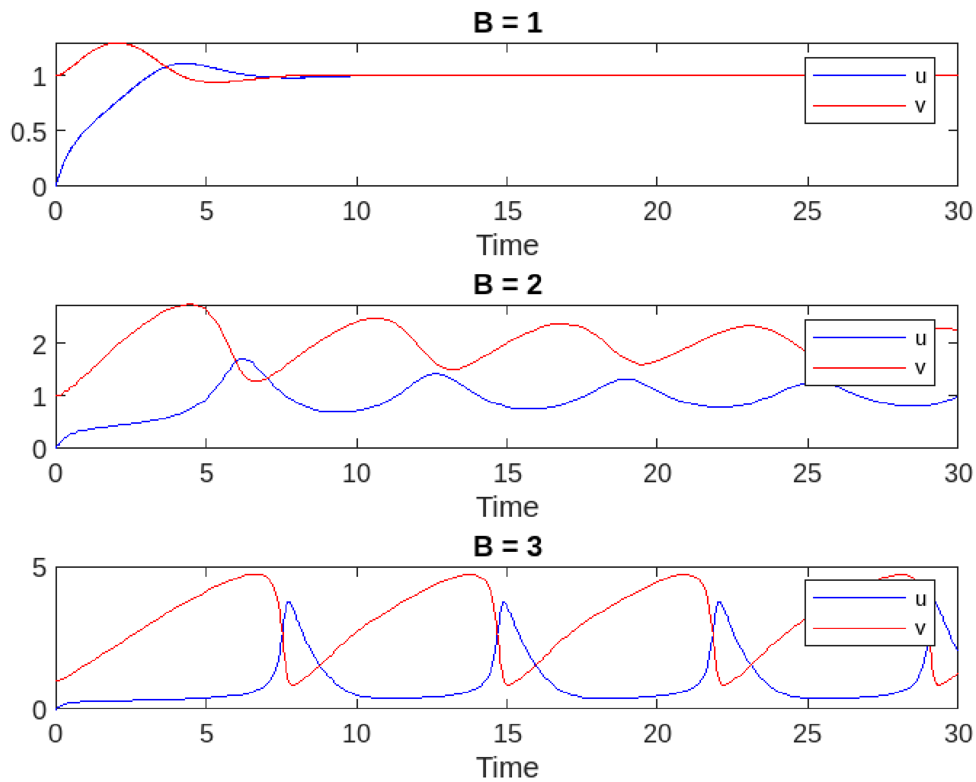

Figura 1: Numerical solutions for A = 1 and: (i) B = 1; (ii) B = 2; (iii) B = 3

(c) Matlab file **Ex2c.m**.

We start by defining a mesh $Q_h^{\Delta t} \subset [0,1] \times [0,T]$, where $h = \frac{1}{N+1}$ and $\Delta t = \frac{T}{M}$ by

$$Q_h^{\Delta t} := \{(x_i, t^m) \,:\, x_i = ih\,,\, i = 0, ..., N+1\,;\, t^m = mT\,,\, m = 0, ..., M\}\,.$$

We use a Forward Finite Difference Euler scheme. Discretizing the equations, we obtain, for $1 \le i \le N\,, 0 \le n \le M-1$, the following

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = A + (U_i^n)^2\, V_i^n - (B+1)\, U_i^n + \alpha\, \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2}\,,$$

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} = B\, U_i^n - (U_i^n)^2\, V_i^n + \alpha\, \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{h^2}\,.$$

The boundary conditions are

$$\begin{cases} U_0^n = U_{N+1}^n = 1 \quad , \quad V_0^n = V_{N+1}^n = 3 \\ U_i^0 = 1 + \sin(2\pi x_i) \quad , \quad V_i^0 = 3 \end{cases} ,\, 0 \le i \le N+1\,,\, 0 \le n \le M\,.$$

Rearranging terms, we get formulas for U^{n+1} and V^{n+1} in terms of U^n and V^n, which can be written in matrix form by slightly modifying the matrix used when the boundary conditions are homogeneous. Then we can calculate the solution by iteratively calculating the values for U and V.

Examining the plots for U and V below, we can observe that the system has unstable dynamics, meaning that it appears to oscillate indefinitely. This is in agreement with 2.a) because the values of A and B verify the inequality $B > A^2 + 1$.

The solution obtained, for $T = 30$ and $r \approx 11.1630$ was

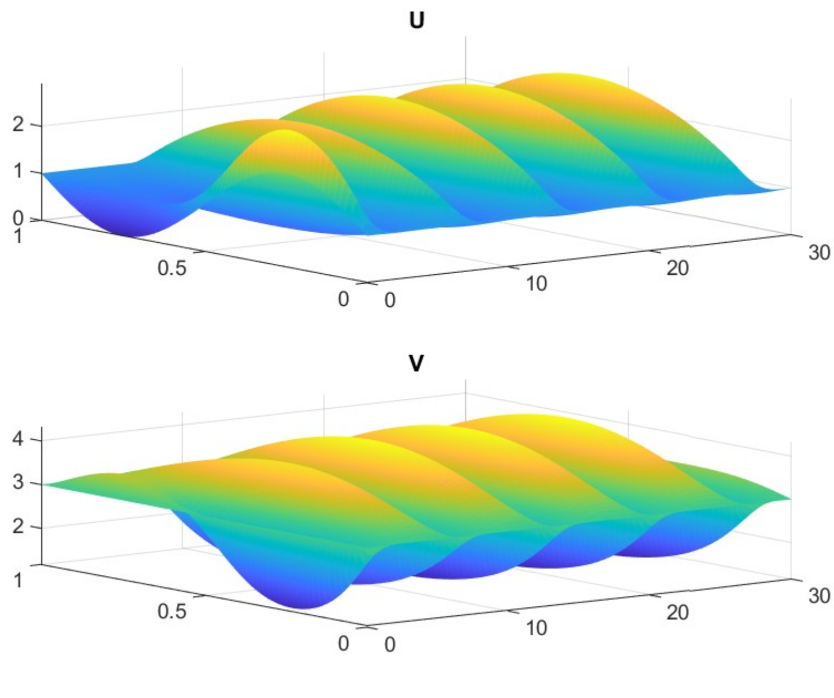

Figura 2: Numerical solutions for $A = 1, \quad B = 3 \quad$ and $\quad \alpha = \frac{1}{50}$.

Referências

[1] A. Araújo, Lecture Notes on Numerical Methods for Partial Differential Equations, Universidade de Coimbra, 2020.

[2] Mircea Ivanescu, Mechanical Engineer's Handbook, 2001
`https://www.sciencedirect.com/topics/engineering/routh-hurwitz-criterion`