COM2031 Discrete Structures
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Solutions

1.(15p) Prove that if $2n^2 + 3n$ is even integer, then n is even integer.

Solution: *(Proof by Contraposition)* $p \to q \equiv q' \to p'$,

$q' \to p'$ *: if n is not even integer, then* $2n^2 + 3n$ *is not even integer.*

assume n is not even integer

$$n = 2k + 1, \exists k \in Z$$
$$2n^2 + 3n = 2(2k + 1)^2 + 3(2k + 1)$$
$$2n^2 + 3n = 2(2k + 1)^2 + 6k + 3$$
$$2n^2 + 3n = 2(2k + 1)^2 + 6k + 2 + 1$$
$$2n^2 + 3n = 2[(2k + 1)^2 + 3k + 1] + 1 = 2m + 1, \exists m \in Z$$

2.(15p) Solve the recurrence relation $a_n = 3a_{n-1} + 10a_{n-2}$ with $a_0 = 1$ and $a_1 = 8$.

Solution :

$$r^2 - 3r - 10 = 0 \text{ (characteristic equation)}$$
$$(r - 5)(r + 2) = 0$$
$$r_1 = 5, r_2 = -2, \text{ then } \; a_n = c_1 5^n + c_2(-2)^n$$

$$a_0 = c_1 + c_2 = 1$$
$$a_1 = c_1 5 + c_2(-2) = 8 \;, \quad \text{then } \; a_n = \frac{10}{7}5^n - \frac{3}{7}(-2)^n$$

3.(20p) What value is returned by the following algorithm (in terms of n)? What is its basic operation? How many times is the basic operation executed? Give the worst-case running time of the algorithm using Big Oh notation.

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Bloktopia (n)
input : n is a positive integer
r ← 0
for i = 1 to n
    for j = 1  to i
        for k = i to i + j
            r ← r + 3
return r
```

Solution :

basic operation - r ← r + 3

$$T(n) = \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=i}^{i+j} 1 = \sum_{i=1}^{n}\sum_{j=1}^{i}(j + 1) = \sum_{i=1}^{n}(i^2 + 3i)/2 = \frac{1}{6}n(n + 1)(n + 5) = O(n^3)$$

the algorithm returns $3.T(n)$

4.(15p) Fourteen people are to be seated around two circular tables, one with 8 chairs and the other with 6 chairs. How many different seating arrangements are possible?

Solution :

$$\binom{14}{8}7!\,5!$$

5.(20p) Let $A = \{a, b, c\}$, $B = \{1, 2, \dots, n, n+1\}$ and $S = \{\, f\colon A \to B \mid f(a) < f(c) \text{ and } f(b) < f(c)\}$.

a) For $n \geq i \geq 1$, let $X_i = \{\, f\colon A \to B \mid f \in S \text{ and } f(c) = i+1\}$ what is the cardinality of X_i (in terms of i)? ($|X_i| = ?$)

b) Let $Y = \{\, f\colon A \to B \mid f \in S \text{ and } f(a) < f(b)\}$. What is the cardinality of Y(in terms of n)? ($|Y| = ?$)

Solution :

a) $X_i = \{\{(a, f(a)), (b, f(b)), (c, i+1)\} \mid f(a) < i+1 \text{ and } f(b) < i+1\}$

there are i different options for $f(a)$ and i different options for $f(b)$, thus $|X_i| = i^2$

b) $Y = \{\{(a, f(a)), (b, f(b)), (c, f(c))\} \mid f(a) < f(b) < f(c)\}$

Y consists of the functions $\{(a,1),(b,2),(c,3)\}, \{(a,1),(b,2),(c,4)\}, \dots,$

$\{(a, n-1), (b, n), (c, n+1)\}$

thus $|Y| = \sum_{i=1}^{n-1} i(i+1)/2$

6.(15p) Let $A = \{a, b, c, d, e, f, g\}$. How many subsets of the set A with 4 elements contains a, but does not contain e?

Solution:

$\{a, b, c, d, \cancel{e}, f, g\}$ $\quad$ $\{a, \_, \_, \_\}$ $\quad$ $\binom{5}{3}$