A Joint Normal-Binary (Probit) Model for High-Dimensional Longitudinal Data

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Supplementary Materials

A Proofs of the conditional distribution of a subvector of the continuous response

Consider the most general case, where we have the conditional distribution of a subvector of the continuous response given a subvector of the binary responses and a subvector of the continuous responses. In Delporte et al. (2022), the expected value is equal to

$$\begin{aligned} E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{Y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} = \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} = \mathbf{1}] &= \frac{e^{-0.5G_i}}{(2\pi)^{\frac{n_b}{2}} f(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} = \mathbf{1})} \frac{\sqrt{|\boldsymbol{E}_i||\boldsymbol{T}_i|}}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_{\mathbf{i}}||\boldsymbol{E}^{bb}_i|}} \\ & \quad \Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}; \boldsymbol{F}_i; \boldsymbol{T}_i) \\ & \quad \Bigg\{ \Big((\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}_1)^a + \boldsymbol{E}^{ab}_i(\boldsymbol{E}^{bb}_i)^{-1}(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}_1)^b) \Big) \\ &+ \Big((\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}_{\mathbf{i}}{}^{-1})^a - \boldsymbol{E}^{ab}_i(\boldsymbol{E}^{bb}_i)^{-1}(\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}_{\mathbf{i}}{}^{-1})^b \Big) \\ &\times \Big(\boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p) \big] + \boldsymbol{F}_i \Big) \Bigg\}, \end{aligned} \tag{A.1}$$

.

This expression can be simplified by proving that

$$\begin{gathered} \frac{e^{-0.5G_i}}{(2\pi)^{\frac{n_b}{2}} f(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} = \mathbf{1})} \frac{\sqrt{|\boldsymbol{E}_i||\boldsymbol{T}_i|}}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_{\mathbf{i}}||\boldsymbol{E}^{bb}_i|}} \\ \Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}; \boldsymbol{F}_i; \boldsymbol{T}_i) = 1 \end{gathered} \tag{A.2}$$

First consider,

$$\begin{aligned}
\boldsymbol{T}_i^{-1} &= (\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b} + \boldsymbol{B}_i^{-1} - (\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}) \\
\boldsymbol{B}_i^{-1}\boldsymbol{H}_i &= -\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}
\end{aligned}$$

Further, define

$$\begin{aligned}
\boldsymbol{M}_a &= -\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{aa})^{-1} \\
\boldsymbol{M}_b &= -\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}
\end{aligned}$$

Since $\boldsymbol{\Sigma}_i = \sigma_i^2\boldsymbol{I}$ and the fact that $\widetilde{\boldsymbol{Z}}_{ci}$ and $\widetilde{\boldsymbol{Z}}_{bi}$ are design matrices

$$(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b\prime} = \boldsymbol{M}_b\boldsymbol{E}^{bb} + \boldsymbol{M}_a\boldsymbol{E}^{ab}$$

As a result,

$$\begin{aligned}
\boldsymbol{T}_i^{-1} &= \boldsymbol{M}_b\boldsymbol{E}^{bb}\boldsymbol{M}_b' + \boldsymbol{M}_b\boldsymbol{E}^{ab}\boldsymbol{M}_a' + \boldsymbol{M}_a\boldsymbol{E}^{ab}\boldsymbol{M}_b' + \boldsymbol{M}_a\boldsymbol{E}^{ab}(\boldsymbol{E}_i^{bb})^{-1}\boldsymbol{E}^{ba}\boldsymbol{M}^{a} + \boldsymbol{B}_i^{-1} - \\
&\quad (\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}) \\
&= \boldsymbol{M}_b\boldsymbol{E}^{bb}\boldsymbol{M}_b' + \boldsymbol{M}_b\boldsymbol{E}^{ab}\boldsymbol{M}_a' + \boldsymbol{M}_a\boldsymbol{E}^{ab}\boldsymbol{M}_b' + \boldsymbol{M}_a\boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{-1})^{bb}\boldsymbol{E}_i^{ba}\boldsymbol{M}_a' - \\
&\quad \boldsymbol{M}_a\boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{-1})^{ba}((\boldsymbol{E}_i^{-1})^{aa})^{-1}(\boldsymbol{E}_i^{-1})^{ab}\boldsymbol{E}_i^{ba}\boldsymbol{M}_a' + \boldsymbol{B}_i^{-1} - (\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}),
\end{aligned}$$

where we used the inverse of partitioned matrices,

$$(\boldsymbol{E}_i^{bb})^{-1} = (\boldsymbol{E}_i^{-1})^{bb} - (\boldsymbol{E}_i^{-1})^{ba}((\boldsymbol{E}_i^{-1})^{aa})^{-1}(\boldsymbol{E}_i^{-1})^{ab},$$

Further,

$$\begin{aligned}
\boldsymbol{T}_i^{-1} &= \boldsymbol{I} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}' + \boldsymbol{M}_a\Bigg(\boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}\boldsymbol{E}_i^{ba} - \boldsymbol{E}_i^{aa}\Bigg)\boldsymbol{M}_a' \\
&= \boldsymbol{I} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}' - \boldsymbol{M}_a\Bigg((\boldsymbol{E}_i^{-1})^{aa}\Bigg)^{-1}\boldsymbol{M}_a',
\end{aligned}$$

where we used $\boldsymbol{B}_i^{-1} = \boldsymbol{I} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'$ and $(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}) = [\boldsymbol{M}_b\ \ \boldsymbol{M}_a]$ and the inverse of a partioned matrix.

If we re-substitute $\boldsymbol{M}_a = -\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{aa}_i)^{-1}$

$$\begin{aligned}
\boldsymbol{T}_i^{-1} &= \boldsymbol{I} - \widetilde{\boldsymbol{Z}}_{bi}\left[\boldsymbol{K}_i + \boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{aa}_i)^{-1}\left((\boldsymbol{E}_i^{-1})^{aa}\right)^{-1}(\boldsymbol{\Sigma}^{aa}_i)^{-1}\widetilde{\boldsymbol{Z}}^{a}_{ci}\boldsymbol{K}_i\right]\widetilde{\boldsymbol{Z}}'_{bi} \\
&= \boldsymbol{I} - \widetilde{\boldsymbol{Z}}_{bi}\left[\boldsymbol{W}_i\right]\widetilde{\boldsymbol{Z}}'_{bi}.
\end{aligned}$$

Next,

$$\boldsymbol{W}_i^{-1} = \boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{aa}_i)^{-1}\left[(\boldsymbol{E}_i^{-1})^{aa} + (\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{aa}_i)^{-1})'\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{aa}_i)^{-1}\right]^{-1}(\boldsymbol{\Sigma}^{aa}_i)^{-1}\widetilde{\boldsymbol{Z}}^{a}_{ci}$$

and

$$\begin{aligned}
\boldsymbol{E}_i^{-1} &= \boldsymbol{\Sigma}_i^{-1} + \boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i - (\boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci})^{-1}\right)\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1} \\
&= \boldsymbol{\Sigma}_i^{-1} + \boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big(\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i + \\
&\quad \boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{D}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci})^{-1}\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i - (\boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci})^{-1}\Big)\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1} \\
&= \boldsymbol{\Sigma}_i^{-1} + \boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i\right)\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1},
\end{aligned}$$

where we used $\boldsymbol{B}_i = \boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi})^{-1}\widetilde{\boldsymbol{Z}}'_{bi}$ and $\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi} = \boldsymbol{K}_i^{-1} - \boldsymbol{D}^{-1} - \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}$. As a result,

$$\begin{aligned}
\boldsymbol{W}_i^{-1} &= \boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{aa}_i)^{-1}\Big[(\boldsymbol{\Sigma}^{aa}_i)^{-1} - (\boldsymbol{\Sigma}^{aa}_i)^{-1}\widetilde{\boldsymbol{Z}}^{a}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{aa}_i)^{-1} + \\
&\quad (\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{aa}_i)^{-1})'\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{aa}_i)^{-1}\Big]^{-1}(\boldsymbol{\Sigma}^{aa}_i)^{-1}\widetilde{\boldsymbol{Z}}^{a}_{ci} \\
&= \boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{aa}_i)^{-1}\left[(\boldsymbol{\Sigma}^{aa}_i)^{-1}\right]^{-1}(\boldsymbol{\Sigma}^{aa}_i)^{-1}\widetilde{\boldsymbol{Z}}^{a}_{ci} \\
&= \boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{aa}_i)^{-1}\widetilde{\boldsymbol{Z}}^{a}_{ci} \\
&= \boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} + \widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi} - \widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{aa}_i)^{-1}\widetilde{\boldsymbol{Z}}^{a}_{ci} \\
&= \boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi} + \widetilde{\boldsymbol{Z}}^{b}_{ci}(\boldsymbol{\Sigma}^{bb}_i)^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}
\end{aligned}$$

As a consequence,

$$\boldsymbol{T}_i^{-1} = \boldsymbol{I} - \widetilde{\boldsymbol{Z}}_{bi}\left[\boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi} + \widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}\right]^{-1}\widetilde{\boldsymbol{Z}}'_{bi}, \tag{A.3}$$

which equals $(\boldsymbol{B}_i^*)^{-1}$, the inverse of the $\boldsymbol{B}_i$ matrix of the joint density $f(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}})$.

Next, consider

$$\begin{aligned}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i &= -\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{T}_i\Big((\boldsymbol{H}'_i\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\ &\quad (\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}_i^{-1})^{b\prime}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{y}}^b_{ci} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b)\Big)\end{aligned} \tag{A.4}$$

Next,

$$\begin{aligned}(\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}_i^{-1})^{b'}(\boldsymbol{E}_i^{bb})^{-1} &= -\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}_i^{aa})^{-1}\boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1} \\ &= -\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}_i^{aa})^{-1} \\ &\quad \Big\{ - \widetilde{\boldsymbol{Z}}^{a}_{ci}\big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\big]^{-1}\widetilde{\boldsymbol{Z}}^{b'}_{ci} \\ &\quad \big[\boldsymbol{\Sigma}_i^{bb} - \widetilde{\boldsymbol{Z}}^{b}_{ci}(-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci})^{-1}\widetilde{\boldsymbol{Z}}^{b'}_{ci}\big]^{-1}\Big\} \\ &= -\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}_i^{aa})^{-1}\widetilde{\boldsymbol{Z}}^{a}_{ci}\big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\big]^{-1}\widetilde{\boldsymbol{Z}}^{b'}_{ci} \\ &\quad \big[(\boldsymbol{\Sigma}_i^{bb})^{-1} - (\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\big] \\ &= -\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}_i^{aa})^{-1}\widetilde{\boldsymbol{Z}}^{a}_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1},\end{aligned}$$

where $\boldsymbol{K}_i^* = (\boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi} + \widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci})^{-1}$ and we made the following substitutions

$$\begin{aligned}\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci} &= (\boldsymbol{K}_i^*)^{-1} + \widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}'_{bi} - \boldsymbol{D}^{-1}, \\ (\boldsymbol{K}_i)^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}(\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} &= -\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}'_{bi} - \boldsymbol{D}^{-1}.\end{aligned}$$

Next, consider

$$\boldsymbol{Z}^{a'}_{ci}(\boldsymbol{\Sigma}_i^{aa})^{-1}\widetilde{\boldsymbol{Z}}^{a}_{ci} = \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} - \widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}$$

As a consequence,

$$\begin{aligned}
(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b'}(\boldsymbol{E}_i^{bb})^{-1} &= -\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i(\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} - \widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b})\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1} \\
&= -\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1} - \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i(\boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} - \boldsymbol{D}^{-1} - (\boldsymbol{K}_i^*)^{-1} + \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} + \boldsymbol{D}^{-1})\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1} \\
&= -\widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{K}_i^*)\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}
\end{aligned}$$

Hence,

$$-\boldsymbol{T}_i(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b\prime}(\boldsymbol{E}_i^{bb})^{-1}\widetilde{\boldsymbol{y}}_{ci}^{b} = -\boldsymbol{H}_i^*\widetilde{\boldsymbol{y}}_{ci}^{b}, \tag{A.5}$$

where equals $\boldsymbol{H}_i^*$ equals the $\boldsymbol{H}_i$ matrix of the joint density $f(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}})$.

Next, we will rewrite the second part of (A.4)

$$
\begin{aligned}
\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i + \boldsymbol{H}_i^*\widetilde{\boldsymbol{y}}_{ci}^b &= -\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{T}_i\Big((\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad (\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b\prime}(\boldsymbol{E}_i^{bb})^{-1}(-(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b)\Big) \\
&= -\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{B}_i^*(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b\prime}(\boldsymbol{\Sigma}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&= -\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b\prime}(\boldsymbol{\Sigma}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&= -\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\Big(\boldsymbol{\Sigma}_i - \widetilde{\boldsymbol{Z}}_{ci}\big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\Big)\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b\prime}(\boldsymbol{\Sigma}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&= -\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\Big(\widetilde{\boldsymbol{Z}}_{ci}\big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\Big)\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b\prime}(\boldsymbol{\Sigma}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&= -\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\big[-\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} - \boldsymbol{D}^{-1}\big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b\prime}(\boldsymbol{\Sigma}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b,
\end{aligned}
$$

where we rewrote $\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} = \boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}$

Next,

$$
\begin{aligned}
(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b &= \boldsymbol{E}_i^{bb}(\boldsymbol{V}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \boldsymbol{E}_i^{ba}(\boldsymbol{V}_i^{-1})^{ab}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&\quad + \boldsymbol{E}_i^{bb}(\boldsymbol{V}_i^{-1})^{ba}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \boldsymbol{E}_i^{ba}(\boldsymbol{V}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a \\
&= \left(\boldsymbol{\Sigma}_i^{bb} - \widetilde{\boldsymbol{Z}}_{ci}^{b}\big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b'}\right)(\boldsymbol{V}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \\
&\quad \left(\boldsymbol{\Sigma}_i^{ba} - \widetilde{\boldsymbol{Z}}_{ci}^{b}\big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}\right)(\boldsymbol{V}_i^{-1})^{ab}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \\
&\quad \left(\boldsymbol{\Sigma}_i^{bb} - \widetilde{\boldsymbol{Z}}_{ci}^{b}\big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b'}\right)(\boldsymbol{V}_i^{-1})^{ba}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \\
&\quad \left(\boldsymbol{\Sigma}_i^{ba} - \widetilde{\boldsymbol{Z}}_{ci}^{b}\big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}\right)(\boldsymbol{V}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a
\end{aligned}
$$

As a consequence,

$$\begin{aligned}
\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i + \boldsymbol{H}_i^*\widetilde{\boldsymbol{y}}_{ci}^b \;=\;& -\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\big[-\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi} - \boldsymbol{D}^{-1}\big]^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{ab}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b - \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{ba}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{ba}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a \\
=\;& -\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{ab}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{ab}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b - \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{ba}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a
\end{aligned}$$

where we substituted $\widetilde{\boldsymbol{Z}}_{ci}^{b'}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b} = (\boldsymbol{K}_i^*)^{-1} - \widetilde{\boldsymbol{Z}}_{ci}^{b'}\widetilde{\boldsymbol{Z}}_{ci}^{b} - \boldsymbol{D}^{-1}$.

Next, consider

$$\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{ab}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a = \widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{a}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})$$

and

$$\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{ba}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b = \widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{b}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})$$

As a result,

$$\begin{aligned}
\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i + \boldsymbol{H}_i^*\widetilde{\boldsymbol{y}}_{ci}^b &= -\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\left[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right]^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\left[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^b\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\left[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^a\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^a\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&= -\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^a\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&= -\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi})^{-1}\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^a\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}((\boldsymbol{K}_i^*)^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi})^{-1}\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^a\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}
\end{aligned}$$

where we have rewritten $\boldsymbol{B}_i = I + \widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi})^{-1}\widetilde{\boldsymbol{Z}}_{bi}'$ and $\boldsymbol{B}_i^* = I + \widetilde{\boldsymbol{Z}}_{bi}((\boldsymbol{K}_i^*)^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi})^{-1}\widetilde{\boldsymbol{Z}}_{bi}'$.

Next, the substitution of

$$\begin{aligned}
\boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} &= \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} + \boldsymbol{D}^{-1}, \\
(\boldsymbol{K}_i^*)^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} &= \widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b} + \boldsymbol{D}^{-1}, \\
\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} &= \boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} - \boldsymbol{D}^{-1}, \\
\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} &= (\boldsymbol{K}_i^*)^{-1} - \widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b} - \boldsymbol{D}^{-1},
\end{aligned}$$

results in

$$\begin{aligned}
\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i + \boldsymbol{H}_i^*\widetilde{\boldsymbol{y}}_{ci} &= -\widetilde{\boldsymbol{Z}}_{bi}(\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}'_{ci} + \boldsymbol{D}^{-1})^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{K}_i^* + \boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*)\widetilde{\boldsymbol{Z}}^{a'}_{ci}\left((\boldsymbol{\Sigma}_i + \widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{ci})^{-1}\right)^a\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&= -\widetilde{\boldsymbol{Z}}_{bi}(\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}'_{ci} + \boldsymbol{D}^{-1})^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{K}_i^* + \boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*)\widetilde{\boldsymbol{Z}}^{a'}_{ci}\left((\boldsymbol{\Sigma}_i + \widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{ci})^{-1}\right)^{ab}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{K}_i^* + \boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*)\widetilde{\boldsymbol{Z}}^{a'}_{ci}\left((\boldsymbol{\Sigma}_i + \widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{ci})^{-1}\right)^{aa}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}.
\end{aligned}$$

Next, we substitute

$$\begin{aligned}
\left((\boldsymbol{\Sigma}_i + \widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{ci})^{-1}\right)^{ab} &= -(\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}^{a}_{ci}(\boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}'_{ci})\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{bb}, \\
\left((\boldsymbol{\Sigma}_i + \widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{ci})^{-1}\right)^{aa} &= (\boldsymbol{\Sigma}_i^{-1})^{aa} - (\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}^{a}_{ci}(\boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}'_{ci})\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{aa}, \\
\boldsymbol{K}_i^* + \boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^* &= (\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}^{b}_{ci} + \boldsymbol{D}^{-1})^{-1}, \\
\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}^{b}_{ci} &= \widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}^{b}_{ci} + \boldsymbol{D}^{-1} - \boldsymbol{D}^{-1}.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i + \boldsymbol{H}_i^*\widetilde{\boldsymbol{y}}_{ci} &= -\widetilde{\boldsymbol{Z}}_{bi}(\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} + \boldsymbol{D}^{-1})^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}(\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}^{b}_{ci} + \boldsymbol{D}^{-1})^{-1}\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}^{a}_{ci}(\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} + \boldsymbol{D}^{-1})^{-1} \\
&\quad \widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}(\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}^{b}_{ci} + \boldsymbol{D}^{-1})^{-1}\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a - \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}(\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}^{b}_{ci} + \boldsymbol{D}^{-1})^{-1}\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} + \boldsymbol{D}^{-1})^{-1} \\
&\quad \widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a \\
&= -\widetilde{\boldsymbol{Z}}_{bi}(\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}^{b}_{ci} + \boldsymbol{D}^{-1})^{-1}\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b,
\end{aligned}$$

where we rewrote $\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a = \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}$.

Further,

$$
\begin{aligned}
\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i + \boldsymbol{H}_i^*\widetilde{\boldsymbol{y}}_{ci} &= -\Big(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^* - \widetilde{\boldsymbol{Z}}_{bi}\big((\boldsymbol{K}_i^*)^{-1}\big)^{-1} + \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}\big[(\boldsymbol{K}_i^*)^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\big]^{-1}\Big)\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&= -\Bigg(\bigg[\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\big[(\boldsymbol{K}_i^*)^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\big]^{-1}\widetilde{\boldsymbol{Z}}_{bi}'\bigg]\widetilde{\boldsymbol{Z}}_{bi} \\
&\quad \bigg[\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b} + \boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\bigg]^{-1}\Bigg)\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&= -\boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&= \boldsymbol{H}_i^*(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b.
\end{aligned} \tag{A.6}
$$

The combination of (A.3),(A.5) and (A.6) results in the following equality

$$
\Phi((\boldsymbol{X}_{bi}\boldsymbol{\beta})^b - \boldsymbol{H}_i^*(\widetilde{\boldsymbol{y}}_{ci}^b - (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b); \boldsymbol{B}_i^*) \;=\; \Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}, \boldsymbol{F}_i, \boldsymbol{T}_i). \tag{A.7}
$$

Next consider,

$$
\begin{aligned}
G_i &= \Big(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Big)'(\boldsymbol{E}_i^{bb})^{-1}\Big(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Big) - \boldsymbol{F}_i'\boldsymbol{T}_i^{-1}\boldsymbol{F}_i + \\
&\quad (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) - (\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) \\
&= -\Big(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\widetilde{\boldsymbol{y}}_{ci}^b - (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b) - (\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\Big)'\boldsymbol{T}_i \\
&\quad \Big(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\widetilde{\boldsymbol{y}}_{ci}^b - (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b) - (\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\Big) \\
&\quad +\Big(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Big)'(\boldsymbol{E}_i^{bb})^{-1}\Big(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Big) \\
&\quad +(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) - (\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}).
\end{aligned}
$$

Now consider the terms from the latter equation where $\widetilde{\boldsymbol{y}}^b_{ci}$ occurs twice:

$$
\begin{aligned}
&-\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}^*_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{bb}\widetilde{\boldsymbol{y}}^b_{ci}\right)'\boldsymbol{T}_i\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}^*_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{bb}\widetilde{\boldsymbol{y}}^b_{ci}\right)+\widetilde{\boldsymbol{y}}^{b'}_{ci}(\boldsymbol{E}^{bb}_i)^{-1}\widetilde{\boldsymbol{y}}^b_{ci}\\
&\quad= -\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}^*_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{bb}\widetilde{\boldsymbol{y}}^b_{ci}\right)'\boldsymbol{T}_i\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}^*_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{bb}\widetilde{\boldsymbol{y}}^b_{ci}\right)+\\
&\qquad\widetilde{\boldsymbol{y}}^{b'}_{ci}\left((\boldsymbol{E}^{-1}_i)^{bb}-(\boldsymbol{E}^{-1}_i)^{ba}((\boldsymbol{E}^{-1}_i)^{aa})^{-1}(\boldsymbol{E}^{-1}_i)^{ab}\right)\widetilde{\boldsymbol{y}}^b_{ci}\\
&\quad= \widetilde{\boldsymbol{y}}^{b'}_{ci}\Big\{-(\boldsymbol{\Sigma}^{-1}_i)^{bb}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}^*_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{B}^*_i\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{K}^*_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{bb}+(\boldsymbol{\Sigma}^{-1}_i)^{bb}-(\boldsymbol{\Sigma}^{-1}_i)^{bb}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{bb}-\\
&\qquad(\boldsymbol{\Sigma}^{-1}_i)^{bb}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{aa}\widetilde{\boldsymbol{Z}}^a_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{bb}+\\
&\qquad(\boldsymbol{\Sigma}^{-1}_i)^{bb}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{aa}\widetilde{\boldsymbol{Z}}^a_{ci}\left(-\boldsymbol{K}^{-1}_i+\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{aa}\widetilde{\boldsymbol{Z}}^a_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{aa}\widetilde{\boldsymbol{Z}}^a_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{bb}\Big\}\widetilde{\boldsymbol{y}}^b_{ci}\\
&\quad= \widetilde{\boldsymbol{y}}^{b'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{bb}\widetilde{\boldsymbol{y}}^b_{ci}+\widetilde{\boldsymbol{y}}^{b'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{bb}\widetilde{\boldsymbol{Z}}^b_{ci}\Big\{-\boldsymbol{K}^*_i\widetilde{\boldsymbol{Z}}'_{bi}\left(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\big((\boldsymbol{K}^*_i)^{-1}-\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\big)^{-1}\widetilde{\boldsymbol{Z}}'_{bi}\right)\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}^*_i-\boldsymbol{K}_i-\\
&\qquad\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{aa}\widetilde{\boldsymbol{Z}}^a_{ci}\boldsymbol{K}_i+\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{aa}\widetilde{\boldsymbol{Z}}^a_{ci}\left(-\boldsymbol{K}^{-1}_i+\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{aa}\widetilde{\boldsymbol{Z}}^a_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{aa}\widetilde{\boldsymbol{Z}}^a_{ci}\boldsymbol{K}_i\Big\}\\
&\qquad\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}^{-1}_i)^{bb}\widetilde{\boldsymbol{y}}^b_{ci},
\end{aligned}
$$

where we first used the general inverse of block matrices

$$(\boldsymbol{E}^{bb}_i)^{-1}=(\boldsymbol{E}^{-1}_i)^{bb}-(\boldsymbol{E}^{-1}_i)^{ba}((\boldsymbol{E}^{-1}_i)^{aa})^{-1}(\boldsymbol{E}^{-1}_i)^{ab}$$

and next substituted $\boldsymbol{B}^*_i=\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\big((\boldsymbol{K}^*_i)^{-1}-\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\big)^{-1}\widetilde{\boldsymbol{Z}}'_{bi}$.
Further, we repeatedly use $\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}=\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}-\boldsymbol{K}^{-1}_i+\boldsymbol{K}^{-1}_i$, which results in the

following

$$
\begin{aligned}
&-\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\right)'\boldsymbol{T}_i\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\right)+\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\widetilde{\boldsymbol{y}}_{ci}^{b} \\
&= \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}+\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b}\left\{\boldsymbol{K}_i^*-\left((\boldsymbol{K}_i^*)^{-1}-\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\right)^{-1}-\left(\boldsymbol{K}_i^{-1}-\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}_{ci}^{a}\right)^{-1}\right\} \\
&\quad\ \widetilde{\boldsymbol{Z}}_{ci}^{b}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b} \\
&= \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}+\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b}\left\{\boldsymbol{K}_i^*-\left((\boldsymbol{K}_i^*)^{-1}-\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\right)^{-1}-\boldsymbol{K}_i^*\right\}\widetilde{\boldsymbol{Z}}_{ci}^{b}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b} \\
&= \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}-\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b}\left(\boldsymbol{D}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b}\right)^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b} \\
&= \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{V}_i^*)^{-1}\widetilde{\boldsymbol{y}}_{ci}^{b},
\end{aligned}
$$

where $(\boldsymbol{V}_i^*)^{-1}$ equals the inverse of the $\boldsymbol{V}_i$ matrix of the joint density $f(\widetilde{\boldsymbol{y}}_{ci}^{b}, \widetilde{\boldsymbol{y}}_{bi})$.

Next, consider the terms where $\widetilde{\boldsymbol{y}}_{ci}^{b}$ occurs once, at the start of the term,

$$
\begin{aligned}
&\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b}+\\
&\qquad\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\right)'\boldsymbol{T}_i(\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}-\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b}\\
&=\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b}\\
&-\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
&+\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^{ba}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{a}-\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b}\\
&+\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b}\\
&=\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b}-\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b}\\
&-\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\right)^{b}\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}+\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^{b}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
&=\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b}-\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b}\\
&-\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\right)^{b}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}+\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^{b}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
&=\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b}-\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b}\\
&-\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left((\boldsymbol{V}_i-\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\boldsymbol{V}_i)\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\right)^{b}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
&+\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^{b}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
&=\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b}-\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b}\\
&=-\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{V}_i^*)^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b},
\end{aligned}
$$

where we have rewritten

$$
\begin{aligned}
(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b} \;&=\; \left(\boldsymbol{I}-\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^{ba}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{a}+\\
&\quad\left(\boldsymbol{I}-\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i)\right)^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b}
\end{aligned}
$$

Next, we will scrutinize the terms where $\widetilde{\boldsymbol{y}}^b_{ci}$ does not appear

$$
\begin{aligned}
&-\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}^*_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}^*_i\widetilde{\boldsymbol{Z}}^{b'}_{ci})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b-\\
&\qquad\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}^*_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)'\boldsymbol{T}_i(\boldsymbol{B}^*_i)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}-\\
&\qquad\left((\boldsymbol{B}^*_i)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\right)'\boldsymbol{T}_i\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}^*_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)-\\
&\qquad\left((\boldsymbol{B}^*_i)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\right)'\boldsymbol{T}_i\left((\boldsymbol{B}^*_i)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\right)+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})-\\
&\qquad(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})+(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b'}(\boldsymbol{E}^{bb}_i)^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})\\
=&-\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}^*_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}^*_i\widetilde{\boldsymbol{Z}}^{b'}_{ci})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b+\\
&+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}^{b'}_{ci}\boldsymbol{K}^*_i\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{H}'_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}^*_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\\
&-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{H}'_i(\boldsymbol{B}^*_i)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
&+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b'}(\boldsymbol{E}^{bb}_i)^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b'}(\boldsymbol{E}^{bb}_i)^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}'_i\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i\big)^{-1}\boldsymbol{H}_i\right)\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
&-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\left(\boldsymbol{V}_i\boldsymbol{H}'_i\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i\big)^{-1}\boldsymbol{H}_i\right)^{b'}(\boldsymbol{E}^{bb}_i)^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\\
&+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\left(\boldsymbol{V}_i\boldsymbol{H}'_i\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i\big)^{-1}\boldsymbol{H}_i\right)^{b'}(\boldsymbol{E}^{bb}_i)^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}'_i\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i\big)^{-1}\boldsymbol{H}_i\right)^{b}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
=&(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b'}(\boldsymbol{V}^*_i)^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\\
&-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{H}'_i(\boldsymbol{B}^*_i)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
&+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b'}(\boldsymbol{E}^{bb}_i)^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}'_i\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i\big)^{-1}\boldsymbol{H}_i\right)^{b}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
&-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\left(\boldsymbol{V}_i\boldsymbol{H}'_i\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i\big)^{-1}\boldsymbol{H}_i\right)^{b'}(\boldsymbol{E}^{bb}_i)^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\\
&+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\left(\boldsymbol{V}_i\boldsymbol{H}'_i\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i\big)^{-1}\boldsymbol{H}_i\right)^{b'}(\boldsymbol{E}^{bb}_i)^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}'_i\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i\big)^{-1}\boldsymbol{H}_i\right)^{b}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
=&(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b'}(\boldsymbol{V}^*_i)^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\\
&-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\Big[-\boldsymbol{V}_i^{-1}+\boldsymbol{H}'_i(\boldsymbol{B}^*_i)^{-1}\boldsymbol{H}_i+\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\Big]\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}+\\
&+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\left(\boldsymbol{V}_i\boldsymbol{H}'_i\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i\big)^{-1}\boldsymbol{H}_i\right)^{b'}(\boldsymbol{E}^{bb}_i)^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}'_i\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i\big)^{-1}\boldsymbol{H}_i\right)^{b}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta},
\end{aligned}
$$

where we implemented the results from the previous proofs.

Next consider,

$$\begin{aligned}\Big(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\Big)^b &= \widetilde{\boldsymbol{Z}}_{ci}^b\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}')^{-1}\boldsymbol{H}_i\\ (\boldsymbol{E}_i^{bb})^{-1} &= (\boldsymbol{\Sigma}_i^{-1})^{bb}-(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^b\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\end{aligned}$$

As a consequence,

$$\begin{aligned}&\Big(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\Big)^{b'}(\boldsymbol{E}_i^{bb})^{-1}\Big(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\Big)^b\\
=&\Big(\widetilde{\boldsymbol{Z}}_{ci}^b\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}')^{-1}\boldsymbol{H}_i\Big)'\Big((\boldsymbol{\Sigma}_i^{-1})^{bb}-(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^b\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\Big)\\
&\Big(\widetilde{\boldsymbol{Z}}_{ci}^b\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}')^{-1}\boldsymbol{H}_i\Big)\\
=&\boldsymbol{H}_i'\Big(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\Big)^{-1}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\Big((\boldsymbol{K}_i^*)^{-1}-\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}-\boldsymbol{D}^{-1}\Big)\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\Big(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\Big)^{-1}\boldsymbol{H}_i\\
&-\boldsymbol{H}_i'\Big(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\Big)^{-1}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\Big((\boldsymbol{K}_i^*)^{-1}-\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}-\boldsymbol{D}^{-1}\Big)\boldsymbol{K}_i*\\
&\Big((\boldsymbol{K}_i^*)^{-1}-\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}-\boldsymbol{D}^{-1}\Big)\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\Big(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\Big)^{-1}\boldsymbol{H}_i\\
=&\boldsymbol{H}_i'\boldsymbol{H}_i-\boldsymbol{H}_i'\Big(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\Big)^{-1}\boldsymbol{H}_i-\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{H}_i\\
=&\boldsymbol{H}_i'\boldsymbol{H}_i-\boldsymbol{H}_i'\Big(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\Big)^{-1}\boldsymbol{H}_i-\boldsymbol{H}_i'\Big(\widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{K}_i^*)^{-1}\widetilde{\boldsymbol{Z}}_{bi}'+\boldsymbol{I}-\boldsymbol{I}\Big)\boldsymbol{H}_i\\
=&-\boldsymbol{H}_i'\Big(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\Big)^{-1}\boldsymbol{H}_i+\boldsymbol{H}_i'(\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\\
=&-\boldsymbol{H}_i'\boldsymbol{H}_i+\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\Big(\boldsymbol{D}^{-1}+\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\Big)^{-1}\boldsymbol{H}_i+\boldsymbol{H}_i'(\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\\
=&\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\Big(\boldsymbol{D}^{-1}+\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}'\Big)^{-1}\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{H}_i+\\
&\boldsymbol{H}_i'(\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i,\end{aligned}$$

Further,

$$
\begin{aligned}
&\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^{b}-\boldsymbol{H}_i'(\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\\
=&-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\\
&+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{H}_i\\
&-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{H}_i-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\\
&+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{H}_i\\
&+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\\
&-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\\
&-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{H}_i\\
&+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{H}_i+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{H}_i\\
&+\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}-\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}-\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\\
&-\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\\
&+\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{H}_i\\
&-\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{H}_i-\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{H}_i-\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{H}_i\\
=&\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}\\
&-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}\\
&-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}+\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}\\
&-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}-\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}\\
&+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}\\
=&-\boldsymbol{V}_i^{-1}+\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1},
\end{aligned}
$$

where we rewrote the following matrices

$$\begin{aligned}
\boldsymbol{V}_i^{-1} &= \boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1} \\
&\quad -\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\left(\boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\right)^{-1}\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1} \\
\boldsymbol{E}_i &= \boldsymbol{\Sigma}_i - \widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}'_{ci}
\end{aligned}$$

As a result,

$$\begin{aligned}
&-\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b - \\
&\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)'\boldsymbol{T}_i(\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\left((\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\right)'\boldsymbol{T}_i\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right) - \\
&\left((\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\right)'\boldsymbol{T}_i\left((\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\right) + (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) - \\
&(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) + (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b\prime}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) \\
&= (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b\prime}(\boldsymbol{V}_i^*)^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b
\end{aligned}$$

Hence,

$$G_i = \left(\widetilde{\boldsymbol{y}}_{ci}^{b'} - (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)'(\boldsymbol{V}_i^*)^{-1}\left(\widetilde{\boldsymbol{y}}_{ci}^{b'} - (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right) \tag{A.8}$$

Next, consider

$$
\begin{aligned}
\frac{|\boldsymbol{E}_i||\boldsymbol{T}_i||\boldsymbol{V}_i^*|}{|\boldsymbol{V}_i||\boldsymbol{B}_i||\boldsymbol{E}_i^{bb}|} &= \frac{|(\boldsymbol{E}_i^{bb})^{-1}||\boldsymbol{V}_i^*|}{|\boldsymbol{V}_i||\boldsymbol{E}_i^{-1}||\boldsymbol{B}_i||\boldsymbol{T}_i^{-1}|} \qquad (A.9)\\
&= \frac{|(\boldsymbol{\Sigma}_i^{bb})^{-1} - (\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}||\boldsymbol{\Sigma}_i^{bb} + \widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}|}{|\boldsymbol{V}_i||\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i + \boldsymbol{V}_i^{-1}||\boldsymbol{B}_i||\boldsymbol{T}_i^{-1}|}\\
&= \frac{|\boldsymbol{I} - (\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}^{b'}_{ci} + (\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci} - (\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}^*\widetilde{\boldsymbol{Z}}^{b}{}'_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}|}{|\boldsymbol{V}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i + \boldsymbol{I}_n||\boldsymbol{B}_i||\boldsymbol{T}_i^{-1}|}\\
&= |\boldsymbol{I} - (\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}^{b'}_{ci} +\\
&\quad \boldsymbol{\Sigma}_i^{bb}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci} + \boldsymbol{\Sigma}_i^{bb}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}^*\left((\boldsymbol{K}^*)^{-1} - \widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi} - \boldsymbol{D}^{-1}\right)\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}|\\
&\quad \frac{1}{|\boldsymbol{I}_p + \boldsymbol{B}_i^{-1}\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'||\boldsymbol{B}_i||\boldsymbol{T}_i^{-1}|}\\
&= \frac{|\boldsymbol{I} + (\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}|}{|\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'||\boldsymbol{T}_i^{-1}|}\\
&= \frac{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|}{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{bi}||\boldsymbol{T}_i^{-1}|}\\
&= \frac{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|}{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{bi}||\boldsymbol{I} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|}\\
&= \frac{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|}{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{bi} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|}\\
&= \frac{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|}{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{bi} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\left((\boldsymbol{K}_i^*)^{-1} - \boldsymbol{D}^{-1} - \widetilde{\boldsymbol{Z}}^{b}{}'_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci}\right)\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|}\\
&= \frac{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|}{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|},\\
&= 1,
\end{aligned}
$$

where the Sylvester identity det(I+AB)=det(I+BA) is repeatedly used.

When we combine the results of (A.7), (A.8) and (A.9) in (A.1), the expected value simplifies to

$$\begin{aligned} E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}^{\boldsymbol{a}}|\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}^{\boldsymbol{b}}=\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}},\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}}=\mathbf{1}] &= \Big((\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}_1)^a \\ &\quad +\boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}}-(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}_1)^b)\Big) \\ &+ \Big((\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_{\mathbf{i}}{}^{-1})^a-\boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_{\mathbf{i}}{}^{-1})^b\Big) \\ &\times \Big(\boldsymbol{T}_i\big[-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)\big]+\boldsymbol{F}_i)\Big), \end{aligned}$$

In addition, if we consider the special case of (A.1), where we only condition on the binary response, the expected value simplifies to

$$\begin{aligned} E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{Y}}_{\boldsymbol{bi}}=\mathbf{1}] = &\quad \boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}+\boldsymbol{H}_i'\boldsymbol{B}_{\mathbf{i}}{}^{-1}\boldsymbol{F}_i) \\ &+\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_{\mathbf{i}}{}^{-1}\boldsymbol{T}_i\big[-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)\big]. \end{aligned}$$

B Prediction and confidence intervals for conditional expected values

2.1 Conditional distribution of the the continuous response given the binary responses

The prediction interval of the conditional expected value is composed of the second central moment and the standard errors of the transformed parameters. More specifically, the 95% prediction interval can be computed with the following general formula

$$\Bigg[E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}}=\mathbf{1}]-1.96\sqrt{E\left[\left(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}-E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}}=\mathbf{1}]\right)^2\right]+\frac{\partial G(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}'}Var(\hat{\boldsymbol{\beta}})\frac{\partial G(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}}},$$
$$E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}}=\mathbf{1}]+1.96\sqrt{E\left[\left(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}-E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}}=\mathbf{1}]\right)^2\right]+\frac{\partial G(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}'}Var(\hat{\boldsymbol{\beta}})\frac{\partial G(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}}}\Bigg].$$

The second central moment is derived by Delporte et al. (2022) but can be simplified using (A.2). The expression is as follows

$$\begin{aligned}
&E\left[\left(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}-E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}}=\mathbf{1}]\right)\left(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}-E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}}=\mathbf{1}]\right)'\right] \\
&= \boldsymbol{E}_i+\boldsymbol{E}_{\boldsymbol{i}}\boldsymbol{V}_{\boldsymbol{i}}^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_{\boldsymbol{i}}^{-1}\boldsymbol{E}_{\boldsymbol{i}} \\
&\quad +\boldsymbol{E}_i\boldsymbol{H}'\boldsymbol{B}_i^{-1}\left(\boldsymbol{N}+\boldsymbol{J}\boldsymbol{J}_i'\right)\boldsymbol{B}_{\boldsymbol{i}}^{-1}\boldsymbol{H}\boldsymbol{E}_i+ \\
&\quad \boldsymbol{E}_i\boldsymbol{V}_{\boldsymbol{i}}^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\boldsymbol{J}'\boldsymbol{B}_i^{-1}\boldsymbol{H}\boldsymbol{E}_i+\boldsymbol{E}_i\boldsymbol{H}'\boldsymbol{B}_i^{-1}\boldsymbol{J}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_{\boldsymbol{i}}^{-1}\boldsymbol{E}_i \\
&\quad -E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}}=\mathbf{1})E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}}=\mathbf{1})',
\end{aligned} \tag{B.1}$$

where $\boldsymbol{J}$ is the expected value of the truncated multivariate normal density, and $\boldsymbol{N}$ is the second central moment of the latter density. They are defined as follows:

$$
\begin{aligned}
\boldsymbol{J} &= \boldsymbol{T_i}\big[-F_1(a_1) \quad -F_2(a_2) \quad ... \quad -F_{\widetilde{p}_i}(a_{\widetilde{p}_i})\big] + \boldsymbol{F}_i,\\
\boldsymbol{a} &= \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta},\\
\varphi(\boldsymbol{x}) &= \begin{cases} \frac{\phi(\boldsymbol{x},\boldsymbol{F}_i,\boldsymbol{T_i})}{\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta}+\boldsymbol{H}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta},\boldsymbol{F}_i,\boldsymbol{T_i})}, & \text{for } \boldsymbol{x} \le \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta},\\ 0, & \text{otherwise.}\end{cases},\\
F_i(x_i) &= \int_{-\infty}^{a_1}\dots\int_{-\infty}^{a_{i-1}}\int_{-\infty}^{a_{i+1}}\dots\int_{-\infty}^{a_{\widetilde{p}_i}}\varphi(x_1,...x_{i-1},x,x_{i+1},...x_{\widetilde{p}_i})dx_{\widetilde{p}_i},..dx_{i+1}dx_{i-1}...dx_1,\\
N_{i,j} &= T_{i\,i,j} + \sum_{k=1}^{\widetilde{p}_i} T_{i\,i,k}\frac{-T_{i\,j,k}a_kF_k(a_k)}{T_{i\,k,k}} + \sum_{k=1}^{\widetilde{p}_i} T_{i\,i,k}\sum_{q\neq k}\left(T_{i\,j,q} - \frac{T_{i\,k,q}T_{i\,j,k}}{T_{i\,k,k}}\right)\\
&\quad \cdot - F_{k,q}(a_k,a_q) - J_iJ_k,\\
F_{k,q}(x,y) &= \int_{-\infty}^{a_1}...\int_{-\infty}^{a_{k-1}}\int_{-\infty}^{a_{k+1}}..\int_{-\infty}^{a_{q-1}}\int_{-\infty}^{a_{q+1}}...\int_{-\infty}^{a_{\widetilde{n}_i}}\phi(x,y,\boldsymbol{x}_{-k,-q})d\boldsymbol{x}_{-k,-q}.
\end{aligned}
$$

The derivative of the expected value with respect to a coefficient $\boldsymbol{\beta}_{c2}$ of a predictor X_{c2} of the continuous response vector equals

$$
\begin{aligned}
\frac{\partial E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}}=\boldsymbol{1}]}{\partial\boldsymbol{\beta}_{c2}} &= \boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}'_{c2i} + \boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}_i^{-1}\boldsymbol{T}_i\boldsymbol{B}_i^{-1}\boldsymbol{H}_i\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}'_{bi}\\
&\quad + \boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}_i^{-1}\frac{\boldsymbol{\nu} - \boldsymbol{\lambda}\boldsymbol{T}_i\big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p)\big]}{\Phi(o,\boldsymbol{T}_i)}
\end{aligned}
\tag{B.2}
$$

with

$$
\begin{aligned}
\boldsymbol{\lambda} &= \sum_{k=1}^{\widetilde{p}_i}(\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{c2i} - \boldsymbol{T}_i\cdot\boldsymbol{B}_i^{-1}\boldsymbol{H}_i\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{c2i})_k\phi\big[(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i)_k, \boldsymbol{T}_{i,kk}\big]\\
&\quad \times\Phi\big[(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i)_{-k}; \boldsymbol{T}_{i,-k|k}\big],\\
\boldsymbol{\nu} &= \sum_{k=1}^{\widetilde{p}_i}(\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{c2i} - \boldsymbol{T}_i\cdot\boldsymbol{B}_i^{-1}\boldsymbol{H}_i\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{c2i})_k g_k(o_k),\\
g_k(x_k) &= \int_{-\infty}^{o_1}\dots\int_{-\infty}^{o_{i-1}}\int_{-\infty}^{o_{i+1}}\dots\int_{-\infty}^{o_{\widetilde{p}_i}}[x_1..x_{k-1}o_kx_{k+1}..x_{\widetilde{p}_i}]'\phi([x_1..x_{k-1}o_kx_{k+1}..x_{\widetilde{p}_i}]',\boldsymbol{T}_i)d\boldsymbol{x}_{-k},\\
\boldsymbol{o} &= \widetilde{\boldsymbol{X}}_{2i}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i
\end{aligned}
$$

Further, $\boldsymbol{T}_i$ is partitioned as

$$\boldsymbol{T}_i = \begin{bmatrix} \boldsymbol{T}_{11}^{(k)} & \boldsymbol{T}_{c2}^{(k)} \\ \boldsymbol{T}_{2c}^{(k)} & T_{kk} \end{bmatrix},$$

and $\boldsymbol{T}_{-k|k}$ is defined as

$$\boldsymbol{T}_{-k|k} = \boldsymbol{T}_{11}^{(k)} - \boldsymbol{T}_{c2}^{(k)} T_{kk}^{-1} \boldsymbol{T}_{2c}^{(k)}.$$

Next, for a coefficient $\boldsymbol{\beta}_{b2}$ of a predictor X_{b2} of the binary response vector the derivative is the following

$$\frac{E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} = \boldsymbol{1}]}{\partial \boldsymbol{\beta}_{b2}} = \boldsymbol{E}_i \boldsymbol{H}_i' \boldsymbol{B}_i^{-1} \frac{\xi - \Omega \cdot \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p) \big]}{\Phi(\boldsymbol{o}, \boldsymbol{T}_i)},$$

where

$$\begin{aligned}
\Omega &= \sum_{k=1}^{\widetilde{p}} \widetilde{\boldsymbol{X}}'_{b2ik} \phi \big[(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i)_k, \boldsymbol{T}_{i,kk} \big] \Phi \big[(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i)_{-k}; \boldsymbol{T}_{i,-k|k} \big], \\
\xi &= \sum_{k=1}^{\widetilde{p}_i} \widetilde{\boldsymbol{X}}'_{b2ik} g_k(o_k).
\end{aligned}$$

The derivative of the expected value with respect to an arbitrary component of $\boldsymbol{D}_{lm}$, denoted by τ equals

$$\begin{aligned}
\frac{\partial E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} = \boldsymbol{1}]}{\partial \tau} = \; & \boldsymbol{E}_i^* \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{E}_i (\boldsymbol{V}_i^{-1} \boldsymbol{V}_i^* \boldsymbol{V}_i^{-1}) \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
& \boldsymbol{E}_i^* \boldsymbol{H}' \boldsymbol{B}_i^{-1} (\boldsymbol{F}_i + \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p) \big]) + \\
& \boldsymbol{E}_i \boldsymbol{H}_i^{*'} \boldsymbol{B}_i^{-1} (\boldsymbol{F}_i + \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p) \big]) - \\
& \boldsymbol{E}_i \boldsymbol{H}_i (\boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1}) (\boldsymbol{F}_i + \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p) \big]) + \\
& \boldsymbol{E}_i \boldsymbol{H}_i \boldsymbol{B}_i^{-1} (\boldsymbol{tr}^* + \boldsymbol{F}_i^*)
\end{aligned}$$

To allow for a convenient solution for a general case, the following expression was evaluated numerically

$$\boldsymbol{tr}^* = \frac{\partial \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p) \big]}{\partial \tau}$$

In addition,

$$\begin{aligned}
\boldsymbol{D}^*_{lm} &= \frac{\partial \boldsymbol{D}}{\partial \tau}\\
\boldsymbol{B}^*_i &= \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}_{bi}\ (\boldsymbol{K}_i \boldsymbol{D}^{-1} \boldsymbol{D}^*_{lm} \boldsymbol{D}^{-1} \boldsymbol{K}_i) \widetilde{\boldsymbol{Z}}^{'}_{bi} \boldsymbol{B}_i\\
\boldsymbol{V}^*_i &= \widetilde{\boldsymbol{Z}}_{ci} \boldsymbol{D}^*_{lm} \widetilde{\boldsymbol{Z}}^{'}_{ci}\\
\boldsymbol{H}^*_i &= -\boldsymbol{B}^*_i \widetilde{\boldsymbol{Z}}_{bi} \boldsymbol{K}_i \widetilde{\boldsymbol{Z}}^{'}_{ci} \boldsymbol{\Sigma}^{-1}_i - \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}^{'}_{bi} (\boldsymbol{K}_i \boldsymbol{D}^{-1} \boldsymbol{D}^*_{lm} \boldsymbol{D}^{-1} \boldsymbol{K}_i) \widetilde{\boldsymbol{Z}}_{ci} \boldsymbol{\Sigma}^{-1}_i\\
\boldsymbol{E}^*_i &= -\boldsymbol{E}_i \Big[- \boldsymbol{V}^{-1}_i \boldsymbol{V}^*_i \boldsymbol{V}^{-1}_i + \boldsymbol{H}^{*'} \boldsymbol{B}^{-1}_i \boldsymbol{H}_i + \boldsymbol{H}'_i \Big(- \widetilde{\boldsymbol{Z}}_{bi}\ (\boldsymbol{K}_i \boldsymbol{D}^{-1} \boldsymbol{D}^*_{lm} \boldsymbol{D}^{-1} \boldsymbol{K}_i) \widetilde{\boldsymbol{Z}}^{'}_{bi} \Big) \boldsymbol{H}_i +\\
&\quad \boldsymbol{H}'_i \boldsymbol{B}^{-1}_i \boldsymbol{H}^*_i \Big] \boldsymbol{E}_i\\
\boldsymbol{T}^*_i &= -\boldsymbol{T}_i \Big[- \widetilde{\boldsymbol{Z}}_{bi} (\boldsymbol{K}_i \boldsymbol{D}^{-1} \boldsymbol{D}^*_{lm} \boldsymbol{D}^{-1} \boldsymbol{K}_i) \widetilde{\boldsymbol{Z}}^{'}_{bi} - (\boldsymbol{H}^{*'}_i \boldsymbol{B}^{-1}_i)' \boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}^{-1}_i +\\
&\quad (\boldsymbol{H}'_i \boldsymbol{B}^{-1}_i \boldsymbol{B}^*_i \boldsymbol{B}^{-1}_i)' \boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}^{-1}_i - (\boldsymbol{H}'_i \boldsymbol{B}^{-1}_i)' \boldsymbol{E}^*_i \boldsymbol{H}'_i \boldsymbol{B}^{-1}_i - (\boldsymbol{H}'_i \boldsymbol{B}^{-1}_i)' \boldsymbol{E}_i \boldsymbol{H}^{*'}_i \boldsymbol{B}^{-1}_i -\\
&\quad (\boldsymbol{H}'_i \boldsymbol{B}^{-1}_i)' \boldsymbol{E}_i (-\boldsymbol{H}'_i \boldsymbol{B}^{-1}_i \boldsymbol{B}^*_i \boldsymbol{B}^{-1}_i) \Big] \boldsymbol{T}_i\\
\boldsymbol{F}^*_i &= \boldsymbol{T}^*_i (\boldsymbol{H}'_i \boldsymbol{B}^{-1})' \boldsymbol{E}_i \boldsymbol{V}^{-1}_i \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta} + \boldsymbol{T}_i (\boldsymbol{H}^{*'}_i \boldsymbol{B}^{-1})' \boldsymbol{E}_i \boldsymbol{V}^{-1}_i \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta} +\\
&\quad \boldsymbol{T}_i (-\boldsymbol{H}'_i \boldsymbol{B}^{-1}_i \boldsymbol{B}^*_i \boldsymbol{B}^{-1}_i)' \boldsymbol{E}_i \boldsymbol{V}^{-1}_i \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta} + \boldsymbol{T}_i (\boldsymbol{H}'_i \boldsymbol{B}^{-1})' \boldsymbol{E}^*_i \boldsymbol{V}^{-1}_i \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta} +\\
&\quad \boldsymbol{T}_i (\boldsymbol{H}'_i \boldsymbol{B}^{-1})' \boldsymbol{E}_i (-\boldsymbol{V}^{-1}_i \boldsymbol{V}^*_i \boldsymbol{V}^{-1}_i) \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta}
\end{aligned}$$

Lastly, the derivative of the expected value with respect to σ^2_c equals

$$\begin{aligned}
\frac{\partial E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} | \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} = \boldsymbol{1}]}{\partial \sigma^2_c} = \quad & \boldsymbol{E}^*_i \boldsymbol{V}^{-1}_i \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta} - \boldsymbol{E}_i (\boldsymbol{V}^{-1}_i \boldsymbol{S}^*_c \boldsymbol{V}^{-1}_i) \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta} +\\
& \boldsymbol{E}^*_i \boldsymbol{H}' \boldsymbol{B}^{-1}_i (\boldsymbol{F}_i + \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p) \big]) +\\
& \boldsymbol{E}_i \boldsymbol{H}^{*'}_i \boldsymbol{B}^{-1}_i (\boldsymbol{F}_i + \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p) \big]) -\\
& \boldsymbol{E}_i \boldsymbol{H}_i (\boldsymbol{B}^{-1}_i \boldsymbol{B}^*_i \boldsymbol{B}^{-1}_i) (\boldsymbol{F}_i + \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p) \big]) +\\
& \boldsymbol{E}_i \boldsymbol{H}_i \boldsymbol{B}^{-1}_i (\boldsymbol{F}^*_i + \boldsymbol{tr}^*)
\end{aligned}$$

To allow for a convenient solution for a general case, the following expressions was evaluated numerically

$$\boldsymbol{tr}^* = \frac{\partial \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p) \big]}{\partial \sigma^2_c}$$

In addition,

$$
\begin{aligned}
\boldsymbol{S}_c^* &= \frac{\partial \boldsymbol{\Sigma}_i}{\partial \sigma_c^2} \\
\boldsymbol{K}_i^* &= \boldsymbol{K}_i \widetilde{\boldsymbol{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \boldsymbol{S}_c^* \boldsymbol{\Sigma}_i^{-1} \widetilde{\boldsymbol{Z}}_{ci} \boldsymbol{K}_i \\
\boldsymbol{B}_i^* &= \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}_{bi} \boldsymbol{K}_i^* \widetilde{\boldsymbol{Z}}_{bi}' \boldsymbol{B}_i \\
\boldsymbol{H}_i^* &= -\boldsymbol{B}_i^* \widetilde{\boldsymbol{Z}}_{bi} \boldsymbol{K}_i \widetilde{\boldsymbol{Z}}_{ci} \boldsymbol{\Sigma}_i^{-1} - \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}_{bi}' \boldsymbol{K}_i^* \widetilde{\boldsymbol{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} + \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}_{bi}' \boldsymbol{K}_i \widetilde{\boldsymbol{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \boldsymbol{S}_c^* \boldsymbol{\Sigma}_i^{-1} \\
\boldsymbol{E}_i^* &= -\boldsymbol{E}_i \Big[- \boldsymbol{V}_i^{-1} \boldsymbol{S}_c^* \boldsymbol{V}_i^{-1} + \boldsymbol{H}^{*'} \boldsymbol{B}_i^{-1} \boldsymbol{H}_i - \boldsymbol{H}' \boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1} \boldsymbol{H}_i + \boldsymbol{H}_i' \boldsymbol{B}_i^{-1} \boldsymbol{H}_i^* \Big] \boldsymbol{E}_i \\
\boldsymbol{T}_i^* &= -\boldsymbol{T}_i \bigg[\bigg(- \boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1} - (\boldsymbol{H}_i^{*'} \boldsymbol{B}_i^{-1})' \boldsymbol{E}_i \boldsymbol{H}_i' \boldsymbol{B}_i^{-1} - \\
&\quad (-\boldsymbol{H}_i' \boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1})' \boldsymbol{E}_i \boldsymbol{H}_i' \boldsymbol{B}_i^{-1} - (\boldsymbol{H}_i' \boldsymbol{B}_i^{-1})' \boldsymbol{E}_i^* \boldsymbol{H}_i' \boldsymbol{B}_i^{-1} - (\boldsymbol{H}_i' \boldsymbol{B}_i^{-1})' \boldsymbol{E}_i \boldsymbol{H}_i^{*'} \boldsymbol{B}_i^{-1} - \\
&\quad (\boldsymbol{H}_i' \boldsymbol{B}_i^{-1})' \boldsymbol{E}_i (-\boldsymbol{H}_i' \boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1}) \bigg] \boldsymbol{T}_i \\
\boldsymbol{F}_i^* &= \boldsymbol{T}_i^* (\boldsymbol{H}_i' \boldsymbol{B}^{-1})' \boldsymbol{E}_i \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta} + \boldsymbol{T}_i (\boldsymbol{H}_i^{*'} \boldsymbol{B}^{-1})' \boldsymbol{E}_i \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta} + \\
&\quad \boldsymbol{T}_i (-\boldsymbol{H}_i' \boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1})' \boldsymbol{E}_i \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta} + \boldsymbol{T}_i (\boldsymbol{H}_i' \boldsymbol{B}^{-1})' \boldsymbol{E}_i^* \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta} + \\
&\quad \boldsymbol{T}_i (\boldsymbol{H}_i' \boldsymbol{B}^{-1})' \boldsymbol{E}_i (-\boldsymbol{V}_i^{-1} \boldsymbol{S}_c^* \boldsymbol{V}_i^{-1}) \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta}
\end{aligned}
$$

2.2 Conditional distribution of a subvector of the continuous response given a subvector of the binary responses and a subvector of the continuous responses

The second central moment is calculated in Delporte et al. (2022) and can be simplified using (A.2). This results in the following equation

$$
\begin{aligned}
&E\left[\left(\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}} - E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} = \boldsymbol{1}]\right)\left(\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}} - E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} = \boldsymbol{1}]\right)'\right] \\
&= \boldsymbol{E}^{aa}_i - \boldsymbol{E}^{ab}_i(\boldsymbol{E}^{bb}_i)^{-1}\boldsymbol{E}^{ba}_i + (\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}^{-1}_i)^a\Big(\boldsymbol{N} + \boldsymbol{J}\boldsymbol{J}'\Big)(\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}^{-1}_i)^{a'} \\
&\quad + (\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}^{-1}_i)^a\boldsymbol{J}((\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}^{-1}_i\boldsymbol{E}_i)^{a'} + (\boldsymbol{E}_i\boldsymbol{V}^{-1}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a\boldsymbol{J}'(\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}^{-1}_i)^{a'} + \\
&\quad (\boldsymbol{E}_i\boldsymbol{V}^{-1}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a(\boldsymbol{E}_i\boldsymbol{V}^{-1}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{a'} \\
&\quad + \Bigg\{(\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}^{-1}_i)^a\boldsymbol{J}\left(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i\boldsymbol{V}^{-1}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)' - (\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}^{-1}_i)^a\Big(\boldsymbol{N} + \boldsymbol{J}\boldsymbol{J}'\Big)(\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}^{-1}_i)^{b'} \\
&\quad + (\boldsymbol{E}_i\boldsymbol{V}^{-1}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a\left(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i\boldsymbol{V}^{-1}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)' - (\boldsymbol{E}_i\boldsymbol{V}^{-1}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a\boldsymbol{J}'(\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}^{-1}_i)^{b'}\Bigg\}(\boldsymbol{E}^{bb}_i)^{-1}\boldsymbol{E}^{ba}_i \\
&\quad + \boldsymbol{E}^{ab}_i(\boldsymbol{E}^{bb}_i)^{-1}\Bigg\{\left(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i\boldsymbol{V}^{-1}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)((\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i\boldsymbol{E}_i)^{a'} - (\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}^{-1}_i)^b\boldsymbol{J}((\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i\boldsymbol{E}_i)^a \\
&\quad + \left(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i\boldsymbol{V}^{-1}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)\boldsymbol{J}'(\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}^{-1}_i)^{a'} - (\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}^{-1}_i)^b\Big(\boldsymbol{N} + \boldsymbol{J}\boldsymbol{J}'\Big)(\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}^{-1}_i)^{a'}\Bigg\} \\
&\quad + \boldsymbol{E}^{ab}_i(\boldsymbol{E}^{bb}_i)^{-1}\Bigg\{(\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}^{-1}_i)^b\Big(\boldsymbol{N} + \boldsymbol{J}\boldsymbol{J}')\Big)(\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}^{-1}_i)^{b'} \\
&\quad - (\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}^{-1}_i)^b\boldsymbol{J}\left(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i\boldsymbol{V}^{-1}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)' \\
&\quad - \left(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i\boldsymbol{V}^{-1}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)\boldsymbol{J}'(\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}^{-1}_i)^{b'} + \left(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i\boldsymbol{V}^{-1}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right) \\
&\quad \left(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i\boldsymbol{V}^{-1}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b)\right)'\Bigg\}(\boldsymbol{E}^{bb}_i)^{-1}\boldsymbol{E}^{ba}_i - E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} = \boldsymbol{1}]E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} = \boldsymbol{1}]',
\end{aligned}
$$

with $\boldsymbol{J}$ as the expected value of the truncated multivariate normal density, and $\boldsymbol{N}$ is the second central moment of the latter density. They are defined in B.1.
The standard errors of the transformed parameters are derived by Delporte et al. (2022) with the delta method, but are here simplified with (A.2). The derivative of the expected value with respect to $\boldsymbol{\beta}_{c2}$, an arbitrary coefficient of a predictor of the

continuous response vector X_{c2} is the following:

$$\begin{aligned}\frac{\partial E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}},\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}}]}{\partial\boldsymbol{\beta}_{c2}} &= (\boldsymbol{E}_i\boldsymbol{V}_i^{-1})^a\widetilde{\boldsymbol{X}}_{c2i} - \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1})^b\widetilde{\boldsymbol{X}}_{c2i}\\ &+ \Big((\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^a - \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^b\Big)(\boldsymbol{\nu}+\boldsymbol{\delta}_i),\end{aligned}$$

with

$$\begin{aligned}\boldsymbol{\delta}_i &= \boldsymbol{T}_i\Big(-(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{c2i})^b + (\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{c2i})\Big)\\ \boldsymbol{\nu} &= \frac{\sum_{k=1}^{\widetilde{p}_i}(\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{12i}-\boldsymbol{\delta}_i)_k g_k(o_k) - \boldsymbol{\Theta}\boldsymbol{T}_i\big[-F_1(o_1)\quad -F_2(o_2)\quad \ldots\quad -F_p(o_p)\big]}{\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta}+\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta},\boldsymbol{F}_i,\boldsymbol{T}_i)},\\ g_k(\boldsymbol{x}_k) &= \int_{-\infty}^{o_1}\cdots\int_{-\infty}^{o_{i-1}}\int_{-\infty}^{o_{i+1}}\cdots\int_{-\infty}^{o_{\widetilde{p}_i}}[x_1..x_{k-1}o_kx_{k+1}..x_{\widetilde{p}_i}]'\phi([x_1..x_{k-1}o_kx_{k+1}..x_{\widetilde{p}_i}]',\boldsymbol{T}_i)d\boldsymbol{x}_{-k}\\ \boldsymbol{\Theta} &= \sum_{k=1}^{\widetilde{p}}(\boldsymbol{H}_{ik}X_{c2i}-\boldsymbol{\delta}_{ik})\phi(X_{bi}\boldsymbol{\beta}+\boldsymbol{H}_iX_{ci}\boldsymbol{\beta}-\boldsymbol{F}_i)_k,\boldsymbol{T}_{kk})\Phi\big[(X_{bi}\boldsymbol{\beta}+\boldsymbol{H}_iX_{ci}\boldsymbol{\beta}-\boldsymbol{F}_i)_{-k},\boldsymbol{T}_{-k|k}\big],\end{aligned}$$

where $\boldsymbol{T}_{-k|k}$ is defined in (B.2).

The derivative of the expected value with respect to a coefficient β_{b2} of a predictor $\boldsymbol{X}_{b2}$ of the binary response vector equals

$$\begin{aligned}\frac{\partial E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}},\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}}=\boldsymbol{1}]}{\partial\boldsymbol{\beta}_{b2}} &= \Big((\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^a - \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^b\Big)\\ &\quad\frac{\zeta-\Omega\boldsymbol{T}_i\big[-F_1(o_1)\quad -F_2(o_2)\quad \ldots\quad -F_p(o_p)\big]}{\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta}+\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta},\boldsymbol{F}_i,\boldsymbol{T}_i)},\end{aligned}$$

with

$$\begin{aligned}\zeta &= \sum_{k=1}^{\widetilde{p}_i}\widetilde{X}'_{b2ik}g_k(o_k),\\ \Omega &= \sum_{k=1}^{\widetilde{p}}X_{b2ik}\phi(\boldsymbol{X}_{bi}\boldsymbol{\beta}+\boldsymbol{H}_i\boldsymbol{X}_{ci}\boldsymbol{\beta}-\boldsymbol{F}_i)_k,T_{kk})\Phi\big[\phi(\boldsymbol{X}_{bi}\boldsymbol{\beta}+\boldsymbol{H}_i\boldsymbol{X}_{ci}\boldsymbol{\beta}-\boldsymbol{F}_i)_{-k},\boldsymbol{T}_{-k|k}\big].\end{aligned}$$

The derivative of the expected value with respect to an arbitrary component of $\boldsymbol{D}_{lm}$, denoted by τ equals

$$
\begin{aligned}
\frac{\partial E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{Y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} = \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} = \mathbf{1}]}{\partial \tau} \quad = \quad & (\boldsymbol{E}^*_i \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{E}_i \boldsymbol{V}_i^{-1} \boldsymbol{V}^*_i \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \\
& \big[(\boldsymbol{E}^*_i)^{ab}(\boldsymbol{E}^{bb}_i)^{-1} - \boldsymbol{E}^{ab}_i((\boldsymbol{E}^{bb}_i)^{-1}(\boldsymbol{E}^*_i)^{bb}(\boldsymbol{E}^{bb}_i)^{-1})\big] \\
& \big[(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b)\big] \\
+ \quad & \boldsymbol{E}^{ab}_i (\boldsymbol{E}^{bb}_i)^{-1} \big[- \boldsymbol{E}^*_i \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \boldsymbol{E}_i \boldsymbol{V}_i^{-1} \boldsymbol{V}^*_i \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})\big]^b \\
+ \quad & \Bigg((\boldsymbol{E}^*_i \boldsymbol{H}'_i \boldsymbol{B}_{\mathbf{i}}{}^{-1} + \boldsymbol{E}_i \boldsymbol{H}^{*'}_i \boldsymbol{B}_{\mathbf{i}}{}^{-1} - \boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1} \boldsymbol{B}^*_i \boldsymbol{B}_i^{-1})^a \\
- \quad & \Bigg[(\boldsymbol{E}^*_i)^{ab}(\boldsymbol{E}^{bb}_i)^{-1} - \boldsymbol{E}^{ab}_i((\boldsymbol{E}^{bb}_i)^{-1}(\boldsymbol{E}^*_i)^{bb}(\boldsymbol{E}^{bb}_i)^{-1}) \Bigg] (\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_{\mathbf{i}}{}^{-1})^b \\
- \quad & \boldsymbol{E}^{ab}_i (\boldsymbol{E}^{bb}_i)^{-1} (\boldsymbol{E}^*_i \boldsymbol{H}'_i \boldsymbol{B}_{\mathbf{i}}{}^{-1} + \boldsymbol{E}_i \boldsymbol{H}^{*'}_i \boldsymbol{B}_{\mathbf{i}}{}^{-1} - \boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1} \boldsymbol{B}^*_i \boldsymbol{B}_i^{-1})^b \Bigg) \\
\times \quad & \Big(\boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p)\big] + \boldsymbol{F}_i \Big) \\
+ \quad & \Big((\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_{\mathbf{i}}{}^{-1})^a - \boldsymbol{E}^{ab}_i (\boldsymbol{E}^{bb}_i)^{-1} (\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_{\mathbf{i}}{}^{-1})^b \Big) \boldsymbol{tr}^*,
\end{aligned}
$$

To allow for a convenient solution for a general case, the following expressions was evaluated numerically

$$
\boldsymbol{tr}^* \quad = \quad \frac{\partial \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p)\big] + \boldsymbol{F}_i}{\partial \tau}
$$

In addition,

$$
\begin{aligned}
\boldsymbol{D}^*_{lm} &= \frac{\partial \boldsymbol{D_{lm}}}{\partial \tau} \\
\boldsymbol{B}^*_i &= \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}_{bi}\ (\boldsymbol{K}_i \boldsymbol{D}^{-1} \boldsymbol{D}^*_{lm} \boldsymbol{D}^{-1} \boldsymbol{K}_i) \widetilde{\boldsymbol{Z}}'_{bi} \boldsymbol{B}_i \\
\boldsymbol{V}^*_i &= \widetilde{\boldsymbol{Z}}_{ci} \boldsymbol{D}^*_{lm} \widetilde{\boldsymbol{Z}}'_{ci} \\
\boldsymbol{H}^*_i &= -\boldsymbol{B}^*_i \widetilde{\boldsymbol{Z}}_{bi} \boldsymbol{K}_i \widetilde{\boldsymbol{Z}}'_{ci} \boldsymbol{\Sigma}_i^{-1} - \boldsymbol{B}^*_i \widetilde{\boldsymbol{Z}}'_{bi} (\boldsymbol{K}_i \boldsymbol{D}^{-1} \boldsymbol{D}^*_{lm} \boldsymbol{D}^{-1} \boldsymbol{K}_i) \widetilde{\boldsymbol{Z}}_{ci} \boldsymbol{\Sigma}_i^{-1} \\
\boldsymbol{E}^*_i &= -\boldsymbol{E}_i \Big[- \boldsymbol{V}_i^{-1} \boldsymbol{V}^*_i \boldsymbol{V}_i^{-1} + \boldsymbol{H}^{*'} \boldsymbol{B}_i^{-1} \boldsymbol{H}_i + \boldsymbol{H}'_i \Big(- \widetilde{\boldsymbol{Z}}'_{bi}\ (\boldsymbol{K}_i \boldsymbol{D}^{-1} \boldsymbol{D}^*_{lm} \boldsymbol{D}^{-1} \boldsymbol{K}_i) \widetilde{\boldsymbol{Z}}_{bi} \Big) \boldsymbol{H}_i + \\
&\quad \boldsymbol{H}'_i \boldsymbol{B}_i^{-1} \boldsymbol{H}^{*'}_i \Big] \boldsymbol{E}_i \\
\boldsymbol{T}^*_i &= -\boldsymbol{T}_i \Big[\Big(\boldsymbol{E}^*_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1} + \boldsymbol{E}_i \boldsymbol{H}^{*'}_i \boldsymbol{B}_i^{-1} - \boldsymbol{E}_i \boldsymbol{H}_i (\boldsymbol{B}_i^{-1} \boldsymbol{B}^*_i \boldsymbol{B}_i^{-1}) \Big)^{b'} (\boldsymbol{E}_i^{bb})^{-1} (\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1})^b - \\
&\quad (\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1})^{b'} (\boldsymbol{E}_i^{bb})^{-1} (\boldsymbol{E}^*_i)^{bb} (\boldsymbol{E}_i^{bb})^{-1} (\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1})^b + \\
&\quad (\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1})^{b'} (\boldsymbol{E}_i^{bb})^{-1} \Big(\boldsymbol{E}^*_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1} + \boldsymbol{E}_i \boldsymbol{H}^{*'}_i \boldsymbol{B}_i^{-1} - \boldsymbol{E}_i \boldsymbol{H}_i (\boldsymbol{B}_i^{-1} \boldsymbol{B}^*_i \boldsymbol{B}_i^{-1}) \Big)^b - \\
&\quad (\boldsymbol{B}_i^{-1} \boldsymbol{B}^*_i \boldsymbol{B}_i^{-1}) - \big(\boldsymbol{H}^{*'}_i \boldsymbol{B}_i^{-1} - \boldsymbol{H}'_i (\boldsymbol{B}_i^{-1} \boldsymbol{B}^*_i \boldsymbol{B}_i^{-1}) \big)' \boldsymbol{E}_i (\boldsymbol{H}'_i \boldsymbol{B}_i^{-1}) - (\boldsymbol{H}'_i \boldsymbol{B}_i^{-1})' \boldsymbol{E}^*_i (\boldsymbol{H}'_i \boldsymbol{B}_i^{-1}) - \\
&\quad (\boldsymbol{H}'_i \boldsymbol{B}_i^{-1})' \boldsymbol{E}^*_i \big(\boldsymbol{H}^{*'}_i \boldsymbol{B}_i^{-1} - \boldsymbol{H}'_i (\boldsymbol{B}_i^{-1} \boldsymbol{B}^*_i \boldsymbol{B}_i^{-1}) \big) \Big] \boldsymbol{T}_i
\end{aligned}
$$

Finally, the derivative of the expected value with respect to σ^2 equals

$$
\begin{aligned}
\frac{\partial E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}} | \widetilde{\boldsymbol{Y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} = \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} = \boldsymbol{1}]}{\partial \sigma^2} &= (\boldsymbol{E}^*_i \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta} - \boldsymbol{E}_i \boldsymbol{V}_i^{-1} \boldsymbol{S}^*_c \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta})^a + \\
&\quad \big[(\boldsymbol{E}^*_i)^{ab} (\boldsymbol{E}_i^{bb})^{-1} - \boldsymbol{E}_i^{ab} ((\boldsymbol{E}_i^{bb})^{-1} (\boldsymbol{E}^*_i)^{bb} (\boldsymbol{E}_i^{bb})^{-1}) \big] \\
&\quad \big[(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta})^b) \big] \\
&+ \boldsymbol{E}_i^{ab} (\boldsymbol{E}_i^{bb})^{-1} \big[- \boldsymbol{E}^*_i \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta} + \boldsymbol{E}_i \boldsymbol{V}_i^{-1} \boldsymbol{S}^*_c \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta}) \big]^b \\
&+ \Big((\boldsymbol{E}^*_i \boldsymbol{H}'_i \boldsymbol{B}_{\mathbf{i}}{}^{-1} + \boldsymbol{E}_i \boldsymbol{H}^{*'}_i \boldsymbol{B}_{\mathbf{i}}{}^{-1} - \boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1} \boldsymbol{B}^*_i \boldsymbol{B}_i^{-1})^a \\
&- \big[(\boldsymbol{E}^*_i)^{ab} (\boldsymbol{E}_i^{bb})^{-1} - \boldsymbol{E}_i^{ab} ((\boldsymbol{E}_i^{bb})^{-1} (\boldsymbol{E}^*_i)^{bb} (\boldsymbol{E}_i^{bb})^{-1}) \big] (\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_{\mathbf{i}}{}^{-1})^b \\
&- \boldsymbol{E}_i^{ab} (\boldsymbol{E}_i^{bb})^{-1} (\boldsymbol{E}^*_i \boldsymbol{H}'_i \boldsymbol{B}_{\mathbf{i}}{}^{-1} + \boldsymbol{E}_i \boldsymbol{H}^{*'}_i \boldsymbol{B}_{\mathbf{i}}{}^{-1} - \boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1} \boldsymbol{B}^*_i \boldsymbol{B}_i^{-1})^b \Big) \\
&\times \Big(\boldsymbol{T}_i \big[- F_1(o_1) \quad - F_2(o_2) \quad \ldots \quad - F_p(o_p) \big] + \boldsymbol{F}_i \Big) \\
&+ \Big((\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_{\mathbf{i}}{}^{-1})^a - \boldsymbol{E}_i^{ab} (\boldsymbol{E}_i^{bb})^{-1} (\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_{\mathbf{i}}{}^{-1})^b \Big) \boldsymbol{tr}^*,
\end{aligned}
$$

To allow for a convenient solution for a general case, the following expression was evaluated numerically

$$\boldsymbol{tr}^* = \frac{\partial \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p) \big] + \boldsymbol{F}_i}{\partial \sigma^2}$$

In addition,

$$
\begin{aligned}
\boldsymbol{S}_c^* &= \frac{\partial \boldsymbol{\Sigma}_i}{\partial \sigma_c^2} \\
\boldsymbol{K}_i^* &= \boldsymbol{K}_i \widetilde{\boldsymbol{Z}}_{ci}^{'} \boldsymbol{\Sigma}_i^{-1} \boldsymbol{S}_c^* \boldsymbol{\Sigma}_i^{-1} \widetilde{\boldsymbol{Z}}_{ci} \boldsymbol{K}_i \\
\boldsymbol{B}_i^* &= \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}_{bi} \boldsymbol{K}_i^* \widetilde{\boldsymbol{Z}}_{bi}^{'} \boldsymbol{B}_i \\
\boldsymbol{H}_i^* &= -\boldsymbol{B}_i^* \widetilde{\boldsymbol{Z}}_{bi} \boldsymbol{K}_i \widetilde{\boldsymbol{Z}}_{ci} \boldsymbol{\Sigma}_i^{-1} - \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}_{bi}^{'} \boldsymbol{K}_i^* \widetilde{\boldsymbol{Z}}_{ci}^{'} \boldsymbol{\Sigma}_i^{-1} + \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}_{bi}^{'} \boldsymbol{K}_i \widetilde{\boldsymbol{Z}}_{ci}^{'} \boldsymbol{\Sigma}_i^{-1} \boldsymbol{S}_c^* \boldsymbol{\Sigma}_i^{-1} \\
\boldsymbol{E}_i^* &= -\boldsymbol{E}_i \left[-\boldsymbol{V}_i^{-1} \boldsymbol{S}_c^* \boldsymbol{V}_i^{-1} + \boldsymbol{H}^{*'} \boldsymbol{B}_i^{-1} \boldsymbol{H}_i - \boldsymbol{H}^{'} \boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1} \boldsymbol{H}_i + \boldsymbol{H}_i^{\prime} \boldsymbol{B}_i^{-1} \boldsymbol{H}_i^* \right] \boldsymbol{E}_i \\
\boldsymbol{T}_i^* &= -\boldsymbol{T}_i \Bigg[\Bigg(\boldsymbol{E}_i^* \boldsymbol{H}_i^{\prime} \boldsymbol{B}_i^{-1} + \boldsymbol{E}_i \boldsymbol{H}_i^{*'} \boldsymbol{B}_i^{-1} - \boldsymbol{E}_i \boldsymbol{H}_i (\boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1}) \Bigg)^{b'} (\boldsymbol{E}_i^{bb})^{-1} (\boldsymbol{E}_i \boldsymbol{H}_i^{\prime} \boldsymbol{B}_i^{-1})^{b} - \\
&\quad (\boldsymbol{E}_i \boldsymbol{H}_i^{\prime} \boldsymbol{B}_i^{-1})^{b'} (\boldsymbol{E}_i^{bb})^{-1} (\boldsymbol{E}_i^*)^{bb} (\boldsymbol{E}_i^{bb})^{-1} (\boldsymbol{E}_i \boldsymbol{H}_i^{\prime} \boldsymbol{B}_i^{-1})^{b} + \\
&\quad (\boldsymbol{E}_i \boldsymbol{H}_i^{\prime} \boldsymbol{B}_i^{-1})^{b'} (\boldsymbol{E}_i^{bb})^{-1} \Bigg(\boldsymbol{E}_i^* \boldsymbol{H}_i^{\prime} \boldsymbol{B}_i^{-1} + \boldsymbol{E}_i \boldsymbol{H}_i^{*'} \boldsymbol{B}_i^{-1} - \boldsymbol{E}_i \boldsymbol{H}_i (\boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1}) \Bigg)^{b} - \\
&\quad (\boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1}) - \big(\boldsymbol{H}_i^{*'} \boldsymbol{B}_i^{-1} - \boldsymbol{H}_i^{'} (\boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1}) \big)^{\prime} \boldsymbol{E}_i \boldsymbol{H}_i^{\prime} \boldsymbol{B}_i^{-1} - (\boldsymbol{H}_i^{\prime} \boldsymbol{B}_i^{-1})^{\prime} \boldsymbol{E}_i^* (\boldsymbol{H}_i^{\prime} \boldsymbol{B}_i^{-1}) - \\
&\quad (\boldsymbol{H}_i^{\prime} \boldsymbol{B}_i^{-1})^{\prime} \boldsymbol{E}_i^* \big(\boldsymbol{H}_i^{*'} \boldsymbol{B}_i^{-1} \boldsymbol{H}_i^{'} (\boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1}) \big) \Bigg] \boldsymbol{T}_i
\end{aligned}
$$

C Confidence intervals for conditional probabilities

We will first derive the confidence interval of (2.9). First, the logit transformation is applied to the probability to transform it to the continuous scale:

$$z = \operatorname{logit}\Big(f(\widetilde{\boldsymbol{Y}}_{\boldsymbol{bi}} = \mathbf{1} | \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}) \Big) = \operatorname{logit}\Big(\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i) \Big).$$

Next the derivative of z with respect to a coefficient β_{c2} of a predictor of the continuous response vector $\mathbf{X}_{c2}$ is derived in order to obtain the transformed standard errors on the continuous scale

$$\frac{\partial z}{\partial \beta_{c2}} = -\frac{\sum_{k=1}^{\widetilde{p}_i} \boldsymbol{H}_{ik} \boldsymbol{X}_{c2i} \phi\big[(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_k; B_{kk}\big] \Phi\big[(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_{-k}; \mathbf{B}_{-k|k}\big]}{\Big(\Phi\big[\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\big]\Big)^2 - \Phi\big[\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\big]}, \tag{C.1}$$

where kk denotes the element on row k and column k, k denotes the row k of the matrix or element k of the vector. In addition, $\mathbf{B}_i$ is partitioned as follows

$$\mathbf{B}_i = \begin{bmatrix} \mathbf{B}_{11}^{(k)} & \mathbf{B}_{12}^{(k)} \\ \mathbf{B}_{21}^{(k)} & B_{kk} \end{bmatrix}.$$

Next, $\mathbf{B}_{-k|k}$ is defined as

$$\mathbf{B}_{-k|k} = \mathbf{B}_{11}^{(k)} - \mathbf{B}_{12}^{(k)} B_{kk}^{-1} \mathbf{B}_{21}^{(k)}, \tag{C.2}$$

which has been retrieved from Poddar (2016), in their Appendix A.

Next, the gradient of a coefficient β_{b2} of one of the predictors of the binary response vector X_{b2} is defined as

$$\frac{\partial z}{\partial \beta_{b2}} = -\frac{\sum_{k=1}^{\widetilde{p}_i} X_{b2ik} \phi\big[(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_k; B_{kk}\big] \Phi\big[(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_{-k}; \boldsymbol{B}_{-k|k}\big]}{\Big(\Phi\big[\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\big]\Big)^2 - \Phi\big[\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\big]}. \tag{C.3}$$

To allow for a convenient solution for a general case, the parts of the two following gradients expressions were evaluated numerically. The first expression shows the gradient of the residual variance σ_{c1}^2 of a continuous response Y_{c1}, and the second expression contains the gradient of τ, an arbitrary component of the variance-covariance matrix of the random effects $\boldsymbol{D}$

$$\begin{aligned}
\frac{\partial z}{\partial \sigma_{c1}^2} &= \frac{\partial \Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i)}{\partial \sigma_{c1}^2} \frac{-1}{\left(\Phi\left[\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\right]\right)^2 - \Phi\left[\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\right]}, \\
\frac{\partial z}{\partial \tau} &= \frac{\partial \Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i)}{\partial \tau} \frac{-1}{\left(\Phi\left[\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\right]\right)^2 - \Phi\left[\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\right]}.
\end{aligned}$$

Hence, the 95% confidence interval can be constructed as

$$\operatorname{expit}\left\{z \pm 1.96\sqrt{\left(\frac{\partial z}{\partial \boldsymbol{\theta}}\right)' \operatorname{Var}(\hat{\boldsymbol{\theta}}) \left(\frac{\partial z}{\partial \boldsymbol{\theta}}\right)}\right\}, \tag{C.4}$$

where $\boldsymbol{\theta}$ signals the vector of estimated parameters.

We will now derive the confidence interval of (2.10). First, let z be a logit transformation of the latter conditional probability. The gradient of a coefficient β_{c2} of a predictor of the continuous responses $\mathbf{X}_{c2}$ is derived:

$$\begin{aligned}
\frac{\partial z}{\partial \beta_{c2}} &= \left\{\sum_{k=1}^{\widetilde{p}_i} \boldsymbol{H}_{ik}\boldsymbol{X}_{c2i}\phi\left[(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\mathbf{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}))_k; B_{kk}\right]\Phi\left[(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_{-k}; \mathbf{B}_{-k|k}\right]\right. \\
&\times \Phi(\widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\mathbf{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i) \\
&- \sum_{k=1}^{\widetilde{p}^b_i} \boldsymbol{H}^b_{ik}\boldsymbol{X}^b_{c2i}\phi\left[(\widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\mathbf{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}))_k; B^{bb}_{kk}\right]\Phi\left[(\widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\mathbf{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}))_{-k}; \mathbf{B}^{bb}_{-k|k}\right] \\
&\left.\times \Phi\left[\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\mathbf{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i\right]\right\} \\
&\times \frac{-\left(\Phi\left[\widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\mathbf{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i\right]\right)^{-2}}{\left(\frac{\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\mathbf{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i)}{\Phi(\widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\mathbf{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i)}\right)^2 - \left(\frac{\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\mathbf{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i)}{\Phi(\widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\mathbf{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i)}\right)},
\end{aligned} \tag{C.5}$$

where $\mathbf{B}_{-k|k}$ is defined in (C.1).

Next, the gradient of a coefficient β_{b2} of one of the predictors of the binary responses

$\boldsymbol{X}_{b2}$ is defined as

$$\begin{aligned}
\frac{\partial z}{\partial \beta_{b2}} &= \Bigg\{ \sum_{k=1}^{\widetilde{p}_i} X_{b2ik}\phi\big[(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}))_k; B_{kk}\big]\Phi\big[(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_{-k}; \mathbf{B}_{-k|k}\big] \qquad \text{(C.6)}\\
&\times\ \Phi(\widetilde{\boldsymbol{X}}^{b}_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i)\\
&-\ \sum_{k=1}^{\widetilde{p}^b_i} X_{b2ik}\phi\big[(\widetilde{\boldsymbol{X}}^{b}_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}))_k; B^{bb}_{kk}\big]\Phi\big[(\widetilde{X}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}))_{-k}; \mathbf{B}^{bb}_{-k|k}\big]\\
&\times\ \Phi\big[\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i\big]\Bigg\}\\
&\times\ \frac{-\left(\Phi\big[\widetilde{\boldsymbol{X}}^{b}_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i\big]\right)^{-2}}{\left(\frac{\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); B_i)}{\Phi(\widetilde{\boldsymbol{X}}^{b}_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i)}\right)^2 - \left(\frac{\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); B_i)}{\Phi(\widetilde{\boldsymbol{X}}^{b}_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); B^{bb}_i)}\right)}.
\end{aligned}$$

To allow for a convenient solution for a general case, the following expressions were evaluated numerically

$$\begin{aligned}
st^* &= \frac{\partial\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_{\mathbf{i}})}{\partial\sigma^2_{c1}},\\
sn^* &= \frac{\partial\Phi(\widetilde{\boldsymbol{X}}^{b}_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i)}{\partial\sigma^2_{c1}},\\
dt^* &= \frac{\partial\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_{\mathbf{i}})}{\partial\tau},\\
dn^* &= \frac{\partial\Phi(\widetilde{\boldsymbol{X}}^{b}_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i)}{\partial\tau}.
\end{aligned}$$

The gradients with respect to the residual error σ^2_{c1} and an arbitrary component of the variance-covariance matrix of the random effects τ, equal

$$\begin{aligned}
\frac{\partial z}{\partial \sigma^2_{c1}} &= \Big\{ st^*\Phi(\widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i) - \\
&\quad sn^*\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i) \Big\} \\
&\times \frac{-\left(\Phi\big[\widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i\big]\right)^{-2}}{\left(\frac{\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta}-\boldsymbol{H}_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}-\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta});\boldsymbol{B}_i)}{\Phi(\widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta}-\boldsymbol{H}^b_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}-\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta});\boldsymbol{B}^{bb}_i)}\right)^2 - \left(\frac{\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta}-\boldsymbol{H}_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}-\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta});\boldsymbol{B}_i)}{\Phi(\widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta}-\boldsymbol{H}^b_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}-\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta});\boldsymbol{B}^{bb}_i)}\right)}, \qquad \text{(C.7)}\\
\frac{\partial z}{\partial \tau} &= \Big\{ dt^*\Phi(\widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i) - \\
&\quad dn^*\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i) \Big\} \\
&\times \frac{-\left(\Phi\big[\widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i\big]\right)^{-2}}{\left(\frac{\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta}-\boldsymbol{H}_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}-\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta});\boldsymbol{B}_i)}{\Phi(\widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta}-\boldsymbol{H}^b_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}-\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta});\boldsymbol{B}^{bb}_i)}\right)^2 - \left(\frac{\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta}-\boldsymbol{H}_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}-\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta});\boldsymbol{B}_i)}{\Phi(\widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta}-\boldsymbol{H}^b_i(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}-\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta});\boldsymbol{B}^{bb}_i)}\right)}. \qquad \text{(C.8)}
\end{aligned}$$

Again, the confidence interval with appropriate bounds can be constructed with (C.4).

D The manifest correlation function

4.1 Proof

The formula of the manifest correlation as described in Delporte et al. (2022) is the following:

$$\rho_{Y_{lij},Y_{mik}} = \frac{\left(\frac{1}{|\boldsymbol{D}_{lm}|^{1/2}|\boldsymbol{M}_i|^{1/2}L_i^{1/2}} - 1\right)\boldsymbol{x}'_{lij}\boldsymbol{\beta}\Phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta}) + \frac{1}{|\boldsymbol{D}_{lm}|^{1/2}|\boldsymbol{M}_i|^{1/2}L_i}\boldsymbol{z}'_{lij}\boldsymbol{M}_i^{-1}\boldsymbol{z}'_{mik}\phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta})}{\sqrt{\left(\boldsymbol{z}'_{lij}\boldsymbol{D}_{lm}\boldsymbol{z}_{lij} + \Sigma_{lij}\right)\Phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta})(1-\Phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta}))}},$$

where $\boldsymbol{D}_{lm}$ denotes the submatrix of $\boldsymbol{D}$ relating to the variances and covariances of the random effects of both responses i and j and

$$\begin{aligned}
\boldsymbol{M}_i &= \boldsymbol{D}_{lm}{}^{-1} + \boldsymbol{z}_{mik}\boldsymbol{z}'_{mik} \\
L_i &= I - \boldsymbol{z}'_{mik}\boldsymbol{M}_i^{-1}\boldsymbol{z}_{mik}.
\end{aligned}$$

Now, consider

$$\begin{aligned}
|\boldsymbol{M}_i| \times |\boldsymbol{D}| \times L_i &= 1 \\
|\boldsymbol{M}_i| \times |\boldsymbol{D}| &= L_i^{-1} \\
|\boldsymbol{M}_i \times \boldsymbol{D}| &= (1 - \boldsymbol{z}'_{mik}\boldsymbol{M}_i^{-1}\boldsymbol{z}_{mik})^{-1} \\
|(\boldsymbol{D}^{-1} + \boldsymbol{z}_{mik}\boldsymbol{z}'_{mik}) \times \boldsymbol{D}| &= 1 - \boldsymbol{z}'_{mik}(-\boldsymbol{M}_i + \boldsymbol{z}_{mik}\boldsymbol{z}'_{mik})^{-1}\boldsymbol{z}_{mik} \\
|I + \boldsymbol{D}\boldsymbol{z}_{mik}\boldsymbol{z}'_{mik}| &= 1 - \boldsymbol{z}'_{mik}(-\boldsymbol{M}_i + \boldsymbol{z}_{mik}\boldsymbol{z}'_{mik})^{-1}\boldsymbol{z}_{mik} \\
|I + \boldsymbol{D}\boldsymbol{z}_{mik}\boldsymbol{z}'_{mik}| &= 1 - \boldsymbol{z}'_{mik}(-\boldsymbol{D}^{-1} - \boldsymbol{z}_{mik}\boldsymbol{z}'_{mik} + \boldsymbol{z}_{mik}\boldsymbol{z}'_{mik})^{-1}\boldsymbol{z}_{mik} \\
|I + \boldsymbol{D}\boldsymbol{z}_{mik}\boldsymbol{z}'_{mik}| &= 1 + \boldsymbol{z}'_{mik}\boldsymbol{D}\boldsymbol{z}_{mik}
\end{aligned}$$

As a consequence, the expression simplifies to

$$\rho_{Y_{lij},Y_{mik}} = \frac{\frac{1}{L_i^{1/2}}\boldsymbol{z}'_{lij}\boldsymbol{M}_i^{-1}\boldsymbol{z}_{mik}\phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta})}{\sqrt{\left(\boldsymbol{z}'_{lij}\boldsymbol{D}_{lm}\boldsymbol{z}_{lij} + \Sigma_{lij}\right)\Phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta})(1-\Phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta}))}},$$

4.2 Standard errors

We extended the methodology of Delporte et al. (2022) by calculating the formula of the standard errors of (4.1). The Fisher Z transformation is first applied to (4.1) in order to transform the probability which takes values on the unit interval to a quantity which takes values on the entire real line. Now, the delta method (Oehlert, 1992) has to be applied to calculate the standard errors, since the estimates of the joint model are first entered in formula (4.1) and then transformed with the Fisher z transformation. Hence, the standard error of the Fisher transformed correlation z equals

$$SE(z) = \sqrt{\frac{\partial z}{\partial \boldsymbol{\theta'}} \mathrm{Var}(\hat{\boldsymbol{\theta}}) \frac{\partial z}{\partial \boldsymbol{\theta}}}, \tag{D.1}$$

where $\boldsymbol{\theta}$ indicates the parameter vector.

$\frac{\partial z}{\partial \boldsymbol{\beta}_{m2}}$ for a coefficient of an arbitrary predictor for the binary response $\boldsymbol{X}_{m2}$ equals

$$\begin{aligned}\frac{\partial z}{\partial \boldsymbol{\beta}_{m2}} &= \frac{-1}{\rho^2-1}\frac{1}{\nu^2}\Bigg\{\nu\frac{-X_{m2ik}\boldsymbol{x}'_{mik}\boldsymbol{\beta}L_i\phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta})}{L_i^{1/2}}\boldsymbol{z}'_{lij}\boldsymbol{M}_i^{-1}\boldsymbol{z}_{mik} - \\ &\quad \frac{1}{L_i^{1/2}}\boldsymbol{z}'_{lij}\boldsymbol{M}_i^{-1}\boldsymbol{z}_{mik}\phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta}) \\ &\quad \frac{1}{2\nu}\left[\boldsymbol{z}'_{lij}\boldsymbol{D}_{lm}\boldsymbol{z}_{lij} + \Sigma_{lij}\right]X_{m2ik}\sqrt{L}\phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta})(1-2\Phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta}))\Bigg\}\end{aligned}$$

where ρ equals the non-transformed correlation between Y_{lij} and Y_{mik} and

$$\nu = \sqrt{\left(\boldsymbol{z}'_{lij}\boldsymbol{D}_{lm}\boldsymbol{z}_{lij} + \Sigma_{lij}\right)\Phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta})(1-\Phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta}))}.$$

Next, the derivative of σ^2_{lij} equals

$$\frac{\partial z}{\partial \sigma^2_{lij}} = \frac{\Phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta})(1-\Phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta}))}{\rho^2-1}\left\{\frac{1}{L_i^{1/2}}\boldsymbol{z}'_{lij}\boldsymbol{M}_i^{-1}\boldsymbol{z}_{mik}\phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta})\right\}\frac{1}{2\nu^3}.$$

The derivative of an arbitrary component of $\boldsymbol{D}_{lm}$, denoted by τ equals

$$\frac{\partial z}{\partial \tau} = \frac{-1}{\rho^2-1}\left[\frac{\nu t_1 - t_2\frac{1}{L_i^{1/2}}\boldsymbol{z}'_{lij}\boldsymbol{M}_i^{-1}\boldsymbol{z}_{mik}\phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta})}{\nu^2}\right],$$

where

$$
\begin{aligned}
\boldsymbol{D}^* &= \frac{\partial \boldsymbol{D}_{lm}}{\partial \tau}, \\
\boldsymbol{M}^* &= -\boldsymbol{D}_{lm}^{-1}\boldsymbol{D}^*\boldsymbol{D}_{lm}^{-1}, \\
L_i^* &= -\boldsymbol{z}'_{mik}\boldsymbol{M}_i^{-1}\boldsymbol{D}_{lm}^{-1}\boldsymbol{D}^*\boldsymbol{D}_{lm}^{-1}\boldsymbol{M}_i^{-1}\boldsymbol{z}_{mik} \\
\zeta &= \frac{L_i^*\boldsymbol{x}'_{mik}\boldsymbol{\beta}}{2\sqrt{L_i}}\phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta}) \\
t_1 &= -\frac{1}{2}L_i^{-3/2}L_i^*\boldsymbol{z}'_{lij}\boldsymbol{M}_i^{-1}\boldsymbol{z}_{mik}\phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta}) - \frac{1}{L_i^{1/2}}\boldsymbol{z}'_{lik}\boldsymbol{M}_i^{-1}\boldsymbol{M}^*\boldsymbol{M}_i^{-1}\boldsymbol{z}_{mik}\phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta}) - \\
&\quad \frac{1}{2L_i^{1/2}}\boldsymbol{z}'_{lij}\boldsymbol{M}_i^{-1}\boldsymbol{z}_{mik}L_i^*\big(\boldsymbol{x}'_{mik}\boldsymbol{\beta}\big)^2\phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta}), \\
t_2 &= \frac{1}{2\nu}\Big\{\boldsymbol{z}'_{lij}\boldsymbol{D}^*\boldsymbol{z}_{lij}\Phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta})(1-\Phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta}) + \\
&\quad \zeta\big(\boldsymbol{z}'_{lij}\boldsymbol{D}_{lm}\boldsymbol{z}_{lij} + \Sigma_{lij}\big)(1-2\Phi(L_i^{1/2}\boldsymbol{x}'_{mik}\boldsymbol{\beta}))\Big\}.
\end{aligned}
$$

E Pairwise modelling of independent subsamples

The computational complexity of our models in the case studies is reduced by the use of multiple pseudo-likelihood methods. Firstly, the pairwise method of Fieuws and Verbeke (2006) is implemented. This method estimates the parameters in the joint model by fitting a bivariate model for each pair of the responses. As a result, some parameters are estimated only once (e.g., the covariances between the random effects of different responses), while other parameters are estimated multiple times (e.g., the variance of th random intercept of a response). To obtain a single estimate for the parameters that are estimated multiple times, an appropriately weighted average of the estimated values can be used. More details and the calculation of the standard errors can be found in Fieuws and Verbeke (2006).

Secondly, the partitioned samples method of Molenberghs et al. (2011) can be of use to reduce computational complexity. In this method, the dataset is divided into random subsamples and each sample is analysed separately. This results in multiple estimates for each parameter, which can be transformed into a single estimate by calculating an appropriate weighted average.

Thirdly, the latter two methods can be combined to reduce the computational complexity even more. Hence, we divide the data in subsamples and proceed to analyse each sample with the pairwise method. As a result, we have estimates for each pair of responses for each subsample. Again, we can combine those estimates by calculating the appropriately weighted average to obtain a single estimate for each parameter. More details can be found in Ivanova et al. (2017).

F Case study: Covid-19

Table 10: Parameter estimates (standard errors) of perceived infectability and germ aversion.

Effect	Infectability		Germ aversion	
Intercept	4.536	(0.199)	4.428	(0.026)
Time 1	-0.026	(0.074)	-0.022	(0.007)
Time 2	0.019	(0.083)	0.068	(0.009)
Time 3	-0.145	(0.095)	0.006	(0.011)
Time 4	-0.126	(0.111)	-0.198	(0.015)
Gender: Male	-0.225	(0.069)	-0.343	(0.004)
Large city	0.135	(0.111)	0.072	(0.008)
Suburbs	-0.006	(0.089)	-0.054	(0.006)
Small city	0.112	(0.079)	0.096	(0.005)
Countryside	-0.109	(0.153)	-0.115	(0.015)
Age	-0.002	(0.003)	0.015	(<0.001)
Student	-0.100	(0.146)	0.01	(0.016)
Permanent disability	0.498	(0.211)	0.451	(0.029)
Children: No	0.080	(0.075)	0.121	(0.004)
No parents <60 alive	0.154	(0.077)	0.125	(0.004)
Perceived income	-0.177	(0.029)	-0.121	(0.001)
Time 1 x Gender: Male	0.001	(0.045)	0.072	(0.002)
Time 2 x Gender: Male	-0.020	(0.048)	-0.123	(0.003)
Time 3 x Gender: Male	0.025	(0.057)	-0.089	(0.004)
Time 4 x Gender: Male	-0.014	(0.061)	-0.071	(0.004)
Time 1 x Age	0.000	(0.001)	0.004	(<0.001)
Time 2 x Age	-0.004	(0.002)	0.003	(<0.001)
Time 3 x Age	-0.001	(0.002)	0.003	(<0.001)
Time 4 x Age	-0.001	(0.002)	0.005	(<0.001)

Table 11: Parameter estimates (standard errors) of quality newspaper, social media, internet.

Effect	Quality paper		Social media		Internet	
Intercept	-1.805	(0.476)	0.445	(0.308)	0.505	(0.282)
Time 1	-0.624	(0.313)	-0.658	(0.262)	-0.149	(0.245)
Time 2	-0.682	(0.337)	-0.914	(0.294)	-0.521	(0.262)
Time 3	-0.467	(0.38)	-0.84	(0.339)	-0.632	(0.300)
Time 4	-0.56	(0.456)	-1.257	(0.391)	-0.901	(0.308)
Gender: Male	-0.158	(0.178)	-0.04	(0.122)	0.084	(0.115)
Large city	0.871	(0.259)	0.523	(0.180)	0.211	(0.157)
Suburbs	0.285	(0.218)	0.184	(0.136)	0.108	(0.124)
Small city	0.127	(0.200)	0.135	(0.127)	0.066	(0.112)
Countryside	-0.047	(0.339)	-0.239	(0.192)	0.089	(0.167)
Age	0.005	(0.007)	-0.012	(0.005)	-0.002	(0.005)
Student	0.88	(0.360)	0.259	(0.241)	-0.165	(0.224)
Permanent disability	-0.574	(0.496)	-0.047	(0.280)	-0.097	(0.261)
Children: No	0.162	(0.171)	-0.031	(0.112)	0.147	(0.100)
No parents <60 alive	0.216	(0.205)	0.021	(0.126)	0.010	(0.112)
Perceived income	0.18	(0.068)	0.086	(0.044)	-0.053	(0.038)
Time 1 x Gender: Male	0.138	(0.177)	0.024	(0.145)	-0.045	(0.141)
Time 2 x Gender: Male	0.51	(0.187)	0.225	(0.158)	0.016	(0.147)
Time 3 x Gender: Male	0.731	(0.216)	0.255	(0.187)	0.156	(0.167)
Time 4 x Gender: Male	0.705	(0.245)	0.269	(0.216)	0.134	(0.178)
Time 1 x Age	0.008	(0.006)	0.007	(0.005)	-0.001	(0.005)
Time 2 x Age	0.001	(0.006)	0.007	(0.005)	0.002	(0.005)
Time 3 x Age	-0.004	(0.007)	0.003	(0.006)	0.005	(0.006)
Time 4 x Age	-0.002	(0.008)	0.007	(0.007)	0.007	(0.006)

Table 12: Manifest correlations between perceived infectability and usage of social Media of quality newspapers.

	Wave(SNS)				
Wave(infect)	0	1	2	3	4
0	.041[.040;.042]	.033[.032;.034]	.028[.027;.029]	.024[.022;.025]	.018[.016;.020]
1	.036[.035;.037]	.029[.028;.030]	.024[.023;.025]	.020[.019;.021]	.015[.013;.017]
2	.031[.030;.032]	.024[.023;.025]	.02.0.019;.021]	.016[.015;.018]	.012[.010;.014]
3	.025[.024;.026]	.02.0.019;.021]	.016[.015;.017]	.013[.011;.014]	.009[.007;.011]
4	.020[.019;.021]	.015[.014;.016]	.012[.011;.013]	.009[.008;.011]	.006[.004;.008]

Table 13: Manifest correlations between perceived infectability and internet usage .

	Wave(internet)				
Wave(infect)	0	1	2	3	4
0	.037[.036;.038]	.040[.039;.041]	.043[.042;.044]	.05.0.049;.052]	.053[.051;.055]
1	.032[.031;.033]	.036[.035;.037]	.041[.04.0.042]	.049[.048;.050]	.053[.051;.054]
2	.026[.025;.028]	.032[.031;.033]	.038[.037;.039]	.047[.046;.048]	.052[.05.0.054]
3	.021[.02.0.022]	.028[.027;.029]	.035[.034;.036]	.045[.043;.046]	.051[.049;.053]
4	.015[.014;.017]	.024[.022;.025]	.032[.03.0.033]	.042[.041;.044]	.049[.047;.051]

Table 14: Manifest correlations between germ aversion and quality newspaper usage.

	Wave(paper)				
Wave(germ)	0	1	2	3	4
0	-.009[-.010;-.008]	.004[.003;.005]	.015[.014;.016]	.025[.024;.027]	.035[.033;.037]
1	-.010[-.011;-.009]	.002[.001;.002]	.012[.011;.013]	.021[.020;.023]	.030[.028;.032]
2	-.011[-.012;-.010]	-.001[-.002;.000]	.009[.008;.010]	.017[.016;.018]	.025[.023;.026]
3	-.012[-.013;-.011]	-.003[-.004;-.002]	.005[.004;.007]	.013[.011;.014]	.019[.018;.021]
4	-.013[-.014;-.011]	-.005[-.006;-.004]	.002[.001;.004]	.008[.007;.010]	.014[.012;.016]

Table 15: Manifest correlations between germ aversion and usage of social media of quality newspapers.

	Wave(SNS)				
Wave(germ)	0	1	2	3	4
0	.023[.022;.024]	.019[.018;.020]	.017[.016;.018]	.015[.014;.017]	.012[.01.0.015]
1	.012[.011;.013]	.011[.010;.012]	.011[.010;.012]	.011[.010;.013]	.010[.008;.012]
2	.001[.000;0.002]	.003[.002;.004]	.005[.004;.006]	.007[.006;.009]	.008[.006;.010]
3	-.010[-.011;-.009]	-.005[-.006;-.004]	.000[-.001;.001]	.003[.002;.005]	.006[.004;.008]
4	-.020[-.021;-.019]	-.012[-.013;-.011]	-.006[-.007;-.004]	.000[-.002;.001]	.004[.002;.006]

Table 16: Manifest correlations between germ aversion and internet usage

	Wave(internet)				
Wave(germ)	0	1	2	3	4
0	.037[.036;.038]	.030[.029;.031]	.024[.023;.025]	.021[.019;.022]	.015[.013;.017]
1	.032[.031;.033]	.028[.027;.029]	.026[.025;.027]	.025[.024;.026]	.022[.020;.024]
2	.027[.026;.028]	.026[.025;.027]	.026[.026;.027]	.029[.028;.030]	.029[.027;.030]
3	.021[.02.0.022]	.024[.023;.025]	.027[.026;.028]	.032[.031;.034]	.034[.032;.036]
4	.016[.015;.017]	.022[.02.0.023]	.027[.026;.028]	.035[.034;.037]	.040[.038;.042]

G Vaccination data

7.1 SAS Code

Obs	SurveyID	wave	bereid_of_vac	bereid_of_vac_a	CO_Outcome_Pos	CO_Infection	gender	age_group	region
1	1.00	1	0	0	12.50	7.50	2	2	Flanders
2	1.00	2	1	1	-1.00	-3.50	2	2	Flanders
3	1.00	3	1	1	26.50	-13.50	2	2	Flanders
4	1.00	4	1	1	-7.50	-12.50	2	2	Flanders
5	1.00	5	1	1	-4.50	4.00	2	2	Flanders
6	2.00	1	1	1	8.00	31.00	2	2	Flanders
7	2.00	2	.	.	.	.	2	2	Flanders
8	2.00	3	.	.	.	.	2	2	Flanders
9	2.00	4	1	1	3.00	7.00	2	2	Flanders
10	2.00	5	1	1	-5.00	-10.00	2	2	Flanders

Figure 6: First ten obeservations in the dataset

Figure 6 shows the first ten observations in the dataset. In order to analyze the data, each measurement of each response of each individual had to be on a separate record. This was done using the following SAS code

```
data v.final2;
length distvar $11;
length response $11;
length linkvar $11;
length var $11;
set v.final2;
time=wave-1;
response = bereid_of_vac;
var='bereid';
distvar    = "Binary";
linkvar =  "PROBIT";
output;
response = bereid_of_vac_a;
var='bereidA';
distvar    = "Binary";
linkvar =  "PROBIT";
```

```
output;
response = co_infection;
var='CI';
distvar    = "Normal";
linkvar = "IDEN";
output;
response = CO_Outcome_Neg;
var='CON';
distvar    = "Normal";
linkvar = "IDEN";
output;
run;
```

In addition, we created separate variables to use later for the random intercept and randoms slope for each variable. Since the variances of the random effects were expected to be small, the values were divided by ten.

```
data v.final2;
set v.final2;
if var= "CI" then do;
ci_int=1;
ci_t=time/10;
end;
else do;
ci_int=0;
ci_t=0;
end;
if var= "bereidA" then do;
bereidA_int=1;
bereidA_t=time/10;
end;
else do;
bereidA_int=0;
bereidA_t=0;
end;
if var= "bereid" then do;
bereid_int=1;
bereid_t=time/10;
```

```
end;
else do;
bereid_int=0;
bereid_t=0;
end;
if var= "CON" then do;
con_int=1;
con_t=time/10;
end;
else do;
con_int=0;
con_t=0;
end;
run;
```

We then took a random sample, and analyzed each pair of responses separately on each sample. For the bivariate analysis of a pair of binary responses or a pair of mixed response types, the analysis was performed in `SAS PROC GLIMMIX`. The following code provides an example

```
proc glimmix data=vac2.s5 method=quad(qpoints=5) initglm asycov GRADIENT
   HESSIAN SUBGRADIENT=vac2.g_cib5;
class var citimeclss(ref='1') surveyid distvar linkvar age_group(ref='4')
   region(ref='Flanders') ;
nloptions maxfunc=10000 maxiter=10000 technique=newrap;
model response(ref='1') = var
citimeclss bereid_int*wave_1 bereid_int*time
var*gender_1 var*age_group var*region
gender_1*citimeclss age_group*citimeclss region*citimeclss
gender_1*bereid_int*wave_1 age_group*bereid_int*wave_1
   region*bereid_int*wave_1
gender_1*bereid_int*time age_group*bereid_int*time region*bereid_int*time
/noint s dist=byobs(distvar) link=byobs(linkvar) solution;
random ci_int ci_t bereid_int bereid_t/type=un subject=surveyID ;
ods output hessian=vac2.h_cib5 parameterestimates=vac2.parms_cib5
   CovParms=vac2.r_cib5;
where ci_int=1 or bereid_int=1;
run;
```

For a pair of continuous responses, the starting values for the parameter estimates were obtained via univariate linear mixed models in `SAS PROC MIXED`. Next, the responses were jointly analyzed via `SAS PROC NLMIXED`

```
proc nlmixed data=v.s5 qpoints=5 maxiter=10000
maxfunc=100000 technique=newrap hess subgrad=v.g_cicon5;
/*starting values for the parameter estimates*/
parms
/*CON*/
beta201 =  1.0742
beta202 =  -1.6482
beta203 =  -2.2175
beta204 =  0.01856
beta205 =  0.9816
beta206 =  3.1154
beta207 =  -4.7174
beta208 =  -1.5571
beta209 =  -3.7211
beta210 =  -5.3718
beta211 =  2.7919
beta212 =  3.298
beta213 =  1.0026
beta214 =  -2.2369
beta215 =  -4.1239
beta216 =  -3.7082
beta217 =  -3.4485
beta218 =  4.852
beta219 =  0.4306
beta220 =  5.381
beta221 =  8.4258
beta222 =  1.4438
beta223 =  7.1814
beta224 =  -0.5756
beta225 =  6.4732
beta226 =  6.1074
beta227 =  3.3694
beta228 =  8.8807
beta229 =  -0.3548
beta230 =  2.0902
```

```
beta231 =  6.9178
beta232 =  1.0865
beta233 =  2.7396
beta234 =  -0.2869
beta235 =  2.2958
beta236 =  3.8862
beta237 =  -2.4378
beta238 =  0.8178
beta239 =  -0.4347
beta240 =  -1.4864
beta241 =  3.6942
beta242 =  1.3872
beta243 =  0.664
beta244 =  -0.2542
beta245 =  2.2136

/*ci*/
beta11  =  9.0346
beta12  =  -1.0924
beta13  =  -2.6435
beta14  =  3.9815
beta15  =  -0.09557
beta16  =  0.02026
beta21  =  12.6359
beta22  =  2.264
beta23  =  0.2143
beta24  =  0.6565
beta25  =  4.5365
beta26  =  1.35
beta27  =  -0.2102
beta32  =  1.4685
beta33  =  0.8375
beta34  =  -2.3893
beta35  =  -1.7486
beta36  =  -9.9476
beta37  =  -4.1605
beta38  =  2.7575
beta39  =  2.1868
```

```
beta40  =  -2.1077
beta41  =  -9.0811
beta42  =  -5.3943
beta43  =  1.2656
beta44  =  -0.3119
beta45  =  1.5218
beta46  =  -7.7712
beta47  =  -6.8263
beta48  =  -1.4169
beta49  =  -3.7714
beta50  =  -1.7791
beta51  =  -5.0436
beta52  =  -1.9424
beta53  =  0.4107
beta54  =  -0.5557
beta55  =  -5.3334
beta56  =  -0.1553
beta57  =  0.6125
beta58  =  2.0198
beta59  =  1.2917
beta60  =  -0.9959
beta61  =  0.4817
beta62  =  1.5869
beta63  =  4.8991


/*initial random effects estimates*/
ri_d=141.39
ris_d=-103.56
rs_d=465.55
rii_ds=41.2709
rsi_ds=-51.4533
ri_s=153.86
ris_ds=-108.01
rss_ds=510.96
ris_s=-115.37
rs_s=777.01
res_d=211.42
```

```
res_s=166.11;



if var='CI' then do;
mean= u1+ u2*ci_t + beta11
+ beta12 * wave_2
+ beta13 * wave_3
+ beta14 * wave_4
+ beta15 * wave_5
+ beta16 * gender_1
+ beta21 * age_group_1
+ beta22 * age_group_2
+ beta23 * age_group_3
+ beta24 * age_group_5
+ beta25 * age_group_6
+ beta26 * region_brussels
+ beta27 * region_wallonia
+ beta32 * gender_1*wave_2
+ beta33 * gender_1*wave_3
+ beta34 * gender_1*wave_4
+ beta35 * gender_1*wave_5
+ beta36 * age_group_1*wave_2
+ beta37 * age_group_2*wave_2
+ beta38 * age_group_3*wave_2
+ beta39 * age_group_5*wave_2
+ beta40 * age_group_6*wave_2
+ beta41 * age_group_1*wave_3
+ beta42 * age_group_2*wave_3
+ beta43 * age_group_3*wave_3
+ beta44 * age_group_5*wave_3
+ beta45 * age_group_6*wave_3
+ beta46 * age_group_1*wave_4
+ beta47 * age_group_2*wave_4
+ beta48 * age_group_3*wave_4
+ beta49 * age_group_5*wave_4
+ beta50 * age_group_6*wave_4
+ beta51 * age_group_1*wave_5
```

```
+  beta52   *  age_group_2*wave_5
+  beta53   *  age_group_3*wave_5
+  beta54   *  age_group_5*wave_5
+  beta55   *  age_group_6*wave_5
+  beta56   *  region_brussels*wave_2
+  beta57   *  region_wallonia*wave_2
+  beta58   *  region_brussels*wave_3
+  beta59   *  region_wallonia*wave_3
+  beta60   *  region_brussels*wave_4
+  beta61   *  region_wallonia*wave_4
+  beta62   *  region_brussels*wave_5
+  beta63   *  region_wallonia*wave_5;
dens1 = -0.5*log(3.14159265358) - log(sqrt(res_d))
-0.5*(response-mean)**2/(res_d);
ll = dens1;
end;

if var='CON' then do;
mean= u5+ u6*con_t + beta201
+  beta202 *  wave_2
+  beta203 *  wave_3
+  beta204 *  wave_4
+  beta205 *  wave_5
+  beta206 *  gender_1
+  beta207 *  age_group_1
+  beta208 *  age_group_2
+  beta209 *  age_group_3
+  beta210 *  age_group_5
+  beta211 *  age_group_6
+  beta212 *  region_brussels
+  beta213 *  region_wallonia
+  beta214 *  gender_1*wave_2
+  beta215 *  gender_1*wave_3
+  beta216 *  gender_1*wave_4
+  beta217 *  gender_1*wave_5
+  beta218 *  age_group_1*wave_2
+  beta219 *  age_group_2*wave_2
+  beta220 *  age_group_3*wave_2
```

```
+ beta221 * age_group_5*wave_2
+ beta222 * age_group_6*wave_2
+ beta223 * age_group_1*wave_3
+ beta224 * age_group_2*wave_3
+ beta225 * age_group_3*wave_3
+ beta226 * age_group_5*wave_3
+ beta227 * age_group_6*wave_3
+ beta228 * age_group_1*wave_4
+ beta229 * age_group_2*wave_4
+ beta230 * age_group_3*wave_4
+ beta231 * age_group_5*wave_4
+ beta232 * age_group_6*wave_4
+ beta233 * age_group_1*wave_5
+ beta234 * age_group_2*wave_5
+ beta235 * age_group_3*wave_5
+ beta236 * age_group_5*wave_5
+ beta237 * age_group_6*wave_5
+ beta238 * region_brussels*wave_2
+ beta239 * region_wallonia*wave_2
+ beta240 * region_brussels*wave_3
+ beta241 * region_wallonia*wave_3
+ beta242 * region_brussels*wave_4
+ beta243 * region_wallonia*wave_4
+ beta244 * region_brussels*wave_5
+ beta245 * region_wallonia*wave_5;
dens = -0.5*log(3.14159265358) - log(sqrt(res_s))
-0.5*(response-mean)**2/(res_s);
ll = dens;
end;

model response~general(ll);
random u1 u2 u5 u6 ~normal([0,0,0,0],[ri_d, ris_d ,rs_d, rii_ds, rsi_ds,
   ri_s, ris_ds, rss_ds, ris_s, rs_s]) subject=surveyid;
ods output hessian=v.h_cicon5 parameterestimates=v.parms_cicon5;
where var='CON' or var='CI';
run;
```

7.2 Tables

Table 18: *Parameter estimates (standard errors) of comparative optimism of infection and comparative optimism of severe outcomes.*

Effect	Infection		Severe outcomes	
Intercept	10.409	(1.773)	-0.250	(1.558)
January	-0.319	(2.234)	-0.213	(1.925)
February	-1.619	(2.120)	-0.886	(1.872)
March	0.928	(2.270)	0.832	(2.043)
May	0.591	(2.261)	0.564	(2.078)
Male	-1.130	(1.549)	0.422	(1.303)
$I(\text{Age} \leq 24)$	2.679	(3.538)	0.422	(1.303)
$I(25 \leq \text{Age} \leq 34)$	0.616	(2.382)	1.558	(1.888)
$I(35 \leq \text{Age} \leq 44)$	0.147	(2.396)	-0.637	(2.037)
$I(55 \leq \text{Age} \leq 64)$	-0.176	(2.412)	-0.284	(2.127)
$I(65 \leq \text{Age})$	2.294	(2.301)	3.741	(2.064)
Brussels capital region	-1.402	(3.001)	0.550	(2.265)
Walloon region	1.006	(1.759)	1.131	(1.417)
January×Male	0.527	(1.941)	-0.294	(1.614)
February×Male	0.385	(1.906)	-0.059	(1.619)
March×Male	0.458	(1.947)	-1.089	(1.692)
May×Male	0.340	(1.998)	-0.258	(1.766)
January×$I(\text{Age} \leq 24)$	-0.931	(4.539)	-2.316	(3.596)
January×$I(25 \leq \text{Age} \leq 34)$	-1.842	(2.986)	-0.903	(2.440)
January×$I(35 \leq \text{Age} \leq 44)$	0.497	(3.007)	1.641	(2.477)
January×$I(55 \leq \text{Age} \leq 64)$	0.895	(2.975)	0.827	(2.569)
January×$I(65 \leq \text{Age})$	-0.786	(2.861)	-0.075	(2.461)
February×$I(\text{Age} \leq 24)$	-0.941	(4.795)	1.220	(3.923)

February×I(25 ≤Age≤ 34)	-0.754	(2.979)	-0.930	(2.487)
February×I(35 ≤Age≤ 44)	-0.235	(2.940)	1.879	(2.486)
February×I(55 ≤Age≤ 64)	1.739	(2.869)	1.607	(2.566)
February×I(65 ≤Age)	-0.757	(2.793)	0.002	(2.417)
March×I(Age≤ 24)	-2.209	(4.912)	-0.191	(3.597)
March×I(25 ≤Age≤ 34)	-2.911	(3.244)	0.053	(2.619)
March×I(35 ≤Age≤ 44)	-1.410	(2.960)	0.597	(2.610)
March×I(55 ≤Age≤ 64)	-0.390	(2.982)	0.083	(2.720)
March×I(65 ≤Age)	-2.396	(2.834)	-1.769	(2.624)
May×I(Age≤ 24)	-1.098	(4.999)	0.389	(3.978)
May×I(25 ≤Age≤ 34)	-1.529	(3.207)	-0.323	(2.638)
May×I(35 ≤Age≤ 44)	-1.093	(3.024)	0.378	(2.736)
May×I(55 ≤Age≤ 64)	0.408	(3.009)	0.292	(2.859)
May×I(55 ≤Age≤ 64)	-4.566	(2.902)	-2.258	(2.646)
January×Brussels capital region	0.301	(3.535)	1.497	(2.876)
January×Walloon region	-1.682	(2.262)	0.339	(1.812)
February×Brussels capital region	0.581	(3.410)	1.872	(2.727)
February×Walloon region	-0.235	(2.240)	0.225	(1.840)
March×Brussels capital region	-1.105	(3.697)	2.662	(2.919)
March×Walloon region	-2.210	(2.249)	0.594	(1.890)
May×Brussels capital region	0.876	(3.780)	2.734	(3.098)
May×Walloon region	-0.160	(2.320)	1.020	(1.967)

Table 17: Parameter estimates(standard errors) of own vaccination hesitancy and perceived vaccination hesitancy of peers.

Effect	Own vaccination hesitancy		Vaccination hesitancy peers	
Intercept	2.427	(0.461)	0.150	(0.252)
December	-1.946	(0.489)	0.366	(0.311)
Time	-3.981	(0.503)	-1.227	(0.152)
Male	0.042	(0.370)	-0.073	(0.218)
I(Age$\leq$ 24)	-0.238	(0.801)	0.198	(0.446)
I(25 $\leq$Age$\leq$ 34)	-0.576	(0.590)	0.140	(0.318)
I(35 $\leq$Age$\leq$ 44)	-0.035	(0.553)	0.123	(0.334)
I(55 $\leq$Age$\leq$ 64)	0.528	(0.625)	-0.021	(0.361)
I(65 $\leq$Age)	0.957	(0.643)	-0.377	(0.366)
Brussels capital region	-1.011	(0.599)	0.329	(0.368)
Walloon region	-0.801	(0.423)	0.319	(0.233)
December$\times$Male	0.498	(0.413)	0.309	(0.265)
Time$\times$Male	-0.419	(0.346)	-0.098	(0.129)
December$\times$I(Age$\leq$ 24)	-0.244	(0.893)	-0.567	(0.530)
December$\times$I(25 $\leq$Age$\leq$ 34)	0.157	(0.644)	-0.533	(0.395)
December$\times$I(35 $\leq$Age$\leq$ 44)	-0.207	(0.610)	-0.343	(0.409)
December$\times$I(55 $\leq$Age$\leq$ 64)	-0.088	(0.669)	0.245	(0.444)
December$\times$I(65 $\leq$Age)	0.044	(0.708)	0.842	(0.435)
Time$\times$I(Age$\leq$ 24)	0.951	(0.698)	0.368	(0.258)
Time$\times$I(25 $\leq$Age$\leq$ 34)	1.193	(0.567)	0.367	(0.189)
Time$\times$I(35 $\leq$Age$\leq$ 44)	0.469	(0.548)	0.185	(0.193)
Time$\times$I(55 $\leq$Age$\leq$ 64)	-1.110	(0.672)	-0.324	(0.226)
Time$\times$I(65 $\leq$Age)	-2.303	(0.639)	-0.401	(0.221)
December$\times$Brussels capital region	0.335	(0.678)	-0.628	(0.450)
December$\times$Walloon region	-0.184	(0.459)	-0.871	(0.285)
Time$\times$Brussels capital region	1.886	(0.528)	0.468	(0.209)
Time$\times$Walloon region	2.052	(0.426)	0.579	(0.139)

7.3 Comparative optimism and vaccination intention peers

Table 19: Manifest correlations between vaccine intention peers and comparative optimism of infection.

	Wave(Intention)				
Wave(optimism)	Intention 1	Intention 2	Intention 3	Intention 4	Intention 5
1	.016[-.011;.043]	-.006[-.032;.019]	-.017[-.044;.011]	-.019[-.048;.009]	-.021[-.049;.008]
2	.015[-.011;.040]	-.005[-.029;.019]	-.014[-.039;.011]	-.017[-.042;.009]	-.018[-.044;.008]
3	.013[-.012;.038]	-.004[-.027;.020]	-.012[-.035;.012]	-.014[-.038;.010]	-.015[-.039;.009]
4	.011[-.014;.036]	-.002[-.026;.021]	-.009[-.033;.015]	-.011[-.035;.014]	-.012[-.036;.013]
5	.009[-.017;.035]	-.001[-.026;.024]	-.006[-.032;.020]	-.007[-.034;.019]	-.008[-.035;.019]

Table 20: Manifest correlations between vaccine intention peers and comparative optimism of severe outcomes.

	Wave(Intention)				
Wave(optimism)	Intention 1	Intention 2	Intention 3	Intention 4	Intention 5
1	.011[-.015;.038]	-.002[-.028;.024]	-.009[-.037;.018]	-.011[-.039;.017]	-.012[-.041;.016]
2	.011[-.015;.036]	.003[-.021;.027]	-.003[-.028;.022]	-.005[-.030;.021]	-.006[-.031;.020]
3	.010[-.016;.035]	.008[-.016;.032]	.004[-.020;.028]	.002[-.022;.026]	.001[-.023;.026]
4	.008[-.018;.034]	.013[-.011;.037]	.010[-.014;.035]	.009[-.016;.034]	.008[-.018;.033]
5	.007[-.020;.034]	.017[-.008;.043]	.016[-.011;.042]	.014[-.013;.041]	.014[-.014;.041]