A joint normal-binary(probit) model

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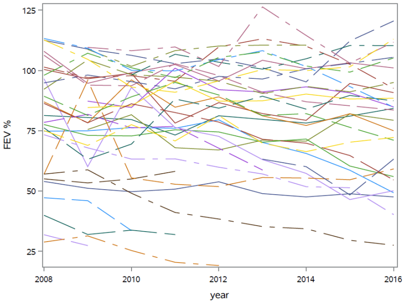
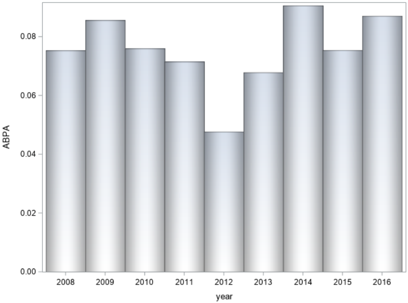
1 Introduction

- Repeated measurement of multiple responses
- Joint analysis outcomes





2 Existing methodology

Longitudinal continuous response

$$\begin{aligned} Y_{ij}|\boldsymbol{b}_i &= \boldsymbol{x}'_{ij}\beta + \boldsymbol{z}'_{ij}\boldsymbol{b}_i, + \epsilon_{ij} \\ \boldsymbol{b_i} &\sim N(\mathbf{0}, D) \\ \epsilon_i &\sim N(\mathbf{0}, \sigma_i^2 I_{n_i}) \end{aligned}$$

Longitudinal binary response

$$\begin{aligned} \Phi^{-1}(P(\boldsymbol{Y}_{ij} = 1)) &= \boldsymbol{x}_{ij}\boldsymbol{\beta} + \boldsymbol{z}'_{ij}\boldsymbol{b}_i \\ \boldsymbol{b_i} &\sim N(\mathbf{0}, D) \end{aligned}$$

3 Joint model

- Model both responses with a mixed model
- Let the random effects correlate

3 Joint model

$$
\begin{aligned}
f(\boldsymbol{y}_{1i}, \boldsymbol{y}_{2i} = 1) &= \Big(\int_{-\infty}^{+\infty}\Big)^q \int_{t=-\infty}^{t=X_{2i}\beta+Z_{2i}\boldsymbol{b}_i} \frac{1}{(2\pi)^{\frac{(q+n_i+p_i)}{2}}|D|^{1/2}|\Sigma_i|^{1/2}} \\
&\quad \times \mathsf{exp}\Big\{-\frac{1}{2}\big[\boldsymbol{b}_i' D^{-1}\boldsymbol{b}_i\big]\Big\} \\
&\quad \times \mathsf{exp}\Big\{-\frac{1}{2}\big[(\boldsymbol{y}_{1i} - X_{1i}\beta - Z_{1i}\boldsymbol{b}_i)'\Sigma_i^{-1} \\
&\quad (\boldsymbol{y}_{1i} - X_{1i}\beta - Z_{1i}\boldsymbol{b}_i) + \boldsymbol{t}'\boldsymbol{t}\big]\Big\} d\boldsymbol{b}_i \mathsf{d}\boldsymbol{t} \\
&= \phi(y_{1i}; X_{1i}\boldsymbol{\beta}; V_i)\Phi(X_{2i}\boldsymbol{\beta} - \alpha_i; B_i)
\end{aligned}
$$

4 Correlation function

$$\rho_{Y_{1ij},Y_{2ik}} = \frac{\left(\frac{1}{|D|^{1/2}}\frac{1}{|M|^{1/2}}\frac{1}{L^{1/2}} - 1\right)\boldsymbol{x}'_{1ij}\boldsymbol{\beta}\Phi(L^{1/2}\boldsymbol{x}'_{2ik}\boldsymbol{\beta})}{\sqrt{(Z'_{1ij}DZ_{1ij} + \Sigma_{1ij})\Phi(L^{1/2}\boldsymbol{x}'_{2ik}\boldsymbol{\beta})(1 - \Phi(L^{1/2}\boldsymbol{x}'_{2ik}\boldsymbol{\beta}))}} + \frac{\frac{1}{|D|^{1/2}}\frac{1}{|M|^{1/2}}\frac{1}{L}Z'_{1ij}M^{-1}Z_{2ik}\phi(L^{1/2}\boldsymbol{x}'_{2ik}\boldsymbol{\beta})}{\sqrt{(Z'_{1ij}DZ_{1ij} + \Sigma_{1ij})\Phi(L^{1/2}\boldsymbol{x}'_{2ik}\boldsymbol{\beta})(1 - \Phi(L^{1/2}\boldsymbol{x}'_{2ik}\boldsymbol{\beta}))}}$$

4 Conditional distributions of the joint model

$$
\begin{aligned}
f(\widetilde{\boldsymbol{y}}_{\boldsymbol{2i}} = \mathbf{1} | \widetilde{\boldsymbol{y}}_{\boldsymbol{1i}}) &= \frac{\phi(\widetilde{X}_{1i}\boldsymbol{\beta}; V_i)\Phi(\widetilde{X}_{2i}\boldsymbol{\beta} - \alpha_i; B_i)}{\phi(\widetilde{X}_{1i}\boldsymbol{\beta}; V_i)} \\
&= \Phi(\widetilde{X}_{2i}\boldsymbol{\beta} - \alpha_i; B_i)
\end{aligned}
$$

4 Conditional distributions of the joint model

$$
\begin{aligned}
E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{1i}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{2i}} = \boldsymbol{1}] &= \int_{\widetilde{\boldsymbol{y}}_{\boldsymbol{1i}}=-\infty}^{\widetilde{\boldsymbol{y}}_{\boldsymbol{1i}}=\infty} \widetilde{\boldsymbol{y}}_{\boldsymbol{1i}} \frac{\phi(X_{1i}\beta; W_i)\Phi(X_{2i}\beta - \alpha_i; B_i)}{\Phi(\widetilde{X}'_{2i}\beta, L^{-1})} d\widetilde{\boldsymbol{y}}_{\boldsymbol{1i}} \\
&= \frac{e^{-\frac{1}{2}G_i}}{\Phi(\widetilde{X}_{2i}\boldsymbol{\beta}; L^{-1})} \sqrt{\frac{|E_i||T_i|}{|V_i||B_i|}} \Phi(\widetilde{X}'_{2i}\boldsymbol{\beta} + H_i\widetilde{X}'_{1i}\boldsymbol{\beta}, F_i, T_i) \\
& \Bigg(E_i(V_i^{-1}\widetilde{X}'_{1i}\boldsymbol{\beta} + H'_i B_i^{-1} F_i) + \\
& E_i H'_i B_i^{-1} T_i \big[-F_1(o_1) \quad -F_2(o_2) \quad \ldots \quad -F_p(o_p) \big] \Bigg)
\end{aligned}
$$

4 Conditional distributions of the joint model

$$f(\widetilde{\boldsymbol{y}}_{\boldsymbol{2i}}^{\boldsymbol{a}} = \mathbf{1} | \widetilde{\boldsymbol{y}}_{\mathbf{1}i}, \widetilde{\boldsymbol{y}}_{\boldsymbol{2i}}^{\boldsymbol{b}} = \mathbf{1}) = \frac{\Phi(\widetilde{X}_{2i}\boldsymbol{\beta} - H_i(\widetilde{\mathbf{Y}_{\mathbf{1}\boldsymbol{i}}} - \widetilde{X}_{1i}\boldsymbol{\beta}); B_i)}{\Phi(\widetilde{X}_{2i}^b\boldsymbol{\beta} - H_i^b(\widetilde{\mathbf{Y}_{\mathbf{1}\boldsymbol{i}}} - \widetilde{X}_{1i}\boldsymbol{\beta}); B_i^{bb})}$$

4 Conditional distributions of the joint model

$$
\begin{aligned}
&E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{1i}}|\widetilde{\boldsymbol{Y}}^{\boldsymbol{b}}_{\boldsymbol{1i}} = \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{1i}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{2i}} = \mathbf{1}] = \\
&\frac{e^{-0.5G_i}}{(2\pi)^{\frac{n_b}{2}} f(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{1i}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{2i}} = \mathbf{1})} \frac{\sqrt{|E_i||T_i|}}{\sqrt{|V_i||B_i||E_i^{bb}|}} \Phi(\widetilde{X}'_{2i}\boldsymbol{\beta} + H_i\widetilde{X}'_{1i}\boldsymbol{\beta}, F_i, T_i) \\
&\Bigg\{\Bigg((E_iV_i^{-1}\widetilde{X}_{1i}\boldsymbol{\beta}_1)^a + E_i^{ab}(E_i^{bb})^{-1}(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{1i}} - (E_iV_i^{-1}\widetilde{X}_{1i}\boldsymbol{\beta}_1)^b)\Bigg) \\
&+ \Bigg((E_iH_i'B_i^{-1})^a - E_i^{ab}(E_i^{bb})^{-1}(E_iH_iB_i^{-1})^b\Bigg) \\
&\times \Bigg(T_i\big[F_1(o_1) \quad F_2(o_2) \quad \dots \quad F_p(o_p)\big] + F_i\Bigg)\Bigg\}
\end{aligned}
$$

5 Results

	year(ABPA)								
year(FEV)	0	1	2	3	4	5	6	7	8
0	0.146	0.148	0.150	0.151	0.151	0.151	0.150	0.149	0.147
1	0.155	0.157	0.159	0.159	0.159	0.159	0.158	0.157	0.155
2	0.163	0.165	0.166	0.167	0.167	0.166	0.165	0.163	0.161
3	0.169	0.171	0.172	0.173	0.172	0.171	0.170	0.168	0.166
4	0.174	0.177	0.177	0.177	0.177	0.176	0.174	0.172	0.170
5	0.179	0.180	0.181	0.181	0.180	0.179	0.177	0.175	0.173
6	0.182	0.183	0.184	0.184	0.183	0.181	0.180	0.177	0.175
7	0.184	0.185	0.186	0.185	0.184	0.183	0.181	0.178	0.176
8	0.185	0.186	0.187	0.186	0.185	0.184	0.181	0.179	0.176

5 Results

				No ABPA		ABPA	
j	$Y_{1i(j-3)}$	$Y_{1i(j-2)}$	$Y_{1i(j-1)}$	$E[Y_{1ij}]$	PI Y_{1ij}	$E[Y_{1ij}]$	PI Y_{1ij}
3	64.9	64.9	64.9	62.70	[49.9;78.7]	62.43	[49.2;79.3]
4	64.9	64.9	64.9	62.41	[49.7;78.4]	62.18	[48.9;79.1]
5	64.9	64.9	64.9	62.17	[49.6;77.9]	61.98	[48.7;78.9]
6	64.9	64.9	64.9	61.97	[49.3;77.9]	61.82	[48.6;78.7]
7	64.9	64.9	64.9	61.83	[49.1;77.9]	61.72	[48.5;78.6]
8	64.9	64.9	64.9	61.74	[49.1;77.6]	61.66	[48.4;78.5]
3	84	84	84	81.66	[68.6;97.2]	81.48	[67.5;98.4]
4	84	84	84	81.62	[68.5;97.2]	81.47	[67.4;98.4]
5	84	84	84	81.61	[68.3;97.5]	81.48	[67.4;98.5]
6	84	84	84	81.61	[68.7;96.9]	81.51	[67.4;98.5]
7	84	84	84	81.65	[68.5;97.3]	81.57	[67.5;98.6]
8	84	84	84	81.71	[68.4;97.6]	81.65	[67.6;98.6]

6 Conclusion and discussion

- Latent versus manifest correlations
- Time dependent covariates
  - Missing data
  - Characterization of the lag relationship
  - Endogenous or exogenous
  - Intermediate variable
- Random effects structure