

A joint normal-binary(probit) model

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Supplementary Materials

A Computation of the marginal joint longitudinal probit-normal model

We will sketch the derivation of (8). In order to do so, the random effects \mathbf{b}_i are integrated out of the joint density of \mathbf{y}_{1i} , \mathbf{y}_{2i} and \mathbf{b}_i .

Because the elements of \mathbf{y}_{1i} and \mathbf{y}_{2i} are independently normally distributed, conditional on the random effects, we have for the marginal probability:

$$\begin{aligned}
& f(\mathbf{y}_{1i}, \mathbf{y}_{2i} = 1) \\
&= \left(\int_{-\infty}^{+\infty} \right)^q \int_{t=-\infty}^{t=X_{2i}\beta+Z_{2i}\mathbf{b}_i} \frac{1}{(2\pi)^{(q+n_i+p_i)/2}|D|^{1/2}|\Sigma_i|^{1/2}} \exp \left\{ -\frac{1}{2} [\mathbf{b}'_i D^{-1} \mathbf{b}_i] \right\} \\
&\quad \exp \left\{ -\frac{1}{2} [(\mathbf{y}_{1i} - X_{1i}\beta - Z_{1i}\mathbf{b}_i)' \Sigma_i^{-1} (\mathbf{y}_{1i} - X_{1i}\beta - Z_{1i}\mathbf{b}_i) + \mathbf{t}' \mathbf{t}] \right\} d\mathbf{b}_i dt \\
&= \left(\int_{-\infty}^{+\infty} \right)^q \int_{s=-\infty}^{s=X_{2i}\beta} \frac{1}{(2\pi)^{(q+n_i+p_i)/2}|D|^{1/2}|\Sigma_i|^{1/2}} \exp \left\{ -\frac{1}{2} [\mathbf{b}'_i D^{-1} \mathbf{b}_i] \right\} \\
&\quad \exp \left\{ -\frac{1}{2} [(\gamma_i - Z_{1i}\mathbf{b}_i)' \Sigma_i^{-1} (\gamma_i - Z_{1i}\mathbf{b}_i) + (\mathbf{s} + Z_{2i}\mathbf{b}_i)' (\mathbf{s} + Z_{2i}\mathbf{b}_i)] \right\} d\mathbf{b}_i ds \\
&= \left(\int_{-\infty}^{+\infty} \right)^q \int_{s=-\infty}^{s=X_{2i}\beta} \frac{1}{(2\pi)^{(q+n_i+p_i)/2}|D|^{1/2}|\Sigma_i|^{1/2}} \exp \left\{ -\frac{1}{2} [\mathbf{b}_i' - u_i)' K_i^{-1} (\mathbf{b}_i - u_i) + V_i \right\} d\mathbf{b}_i ds.
\end{aligned}$$

Here, we have rewritten the variable $t = s + Z_{2i}\mathbf{b}_i$ and $\gamma_i = \mathbf{y}_{1i} - X_{1i}\beta$. Further,

$$\mathbf{b}'_i D^{-1} \mathbf{b}_i + (\gamma_i + Z_{1i}\mathbf{b}_i)' \Sigma_i^{-1} (\gamma_i + Z_{1i}\mathbf{b}_i) + (\mathbf{s} + Z_{2i}\mathbf{b}_i)' (\mathbf{s} + Z_{2i}\mathbf{b}_i) = (\mathbf{b}_i - u_i)' K_i^{-1} (\mathbf{b}_i - u_i) + \nu_i,$$

where

$$\begin{aligned} u_i &= -K_i l_i, \\ K_i^{-1} &= D^{-1} + Z_{1i}' \Sigma_i^{-1} Z_{1i} + Z_{2i}' Z_{2i}, \\ l'_i &= -\gamma_i' \Sigma_i^{-1} Z_{1i} + s' Z_{2i}, \\ \nu_i &= \gamma_i' \Sigma_i^{-1} \gamma_i - (Z_{2i}' s - Z_{1i}' \Sigma_i^{-1} \gamma_i)' K_i (Z_{2i}' s - Z_{1i}' \Sigma_i^{-1} \gamma_i) + s' s. \end{aligned}$$

Next, integration over the random effects produces

$$\begin{aligned} f(\mathbf{y}_{1i}, \mathbf{y}_{2i} = \mathbf{1}) &= \int_{s=-\infty}^{s=X_{2i}\beta} \frac{|K_i|^{1/2}}{|D|^{1/2} |\Sigma_i|^{1/2} (2\pi)^{(p_i+n_i)/2}} \exp\left\{-\frac{1}{2} \nu_i\right\} ds \\ &= \int_{s=-\infty}^{s=X_{2i}\beta} \frac{|K_i|^{1/2} |B_i|^{1/2}}{|D|^{1/2} (2\pi)^{(p_i+n_i)/2} |\Sigma_i|^{1/2} |B_i|^{1/2}} \exp\left\{-\frac{1}{2} [(s - \alpha_i)' B_i^{-1} (s - \alpha_i) + c_i]\right\} ds, \end{aligned}$$

where $\nu_i = (s - \alpha_i)' B_i^{-1} (s - \alpha_i) + c_i$, with

$$\begin{aligned} \alpha_i &= -B_i Z_{2i} K_i Z_{1i}' \Sigma_i^{-1} \gamma_i, \\ B_i^{-1} &= I - Z_{2i} K_i Z_{2i}', \\ c_i &= -\alpha_i' B_i^{-1} \alpha_i + \gamma_i' (\Sigma_i^{-1} - \Sigma_i^{-1} Z_{1i} K_i Z_{1i}' \Sigma_i^{-1}) \gamma_i. \end{aligned}$$

Further, write $s - \alpha_i = u$, which results in

$$f(\mathbf{y}_{1i}, \mathbf{y}_{2i} = \mathbf{1}) = \frac{|K_i|^{1/2} |B_i|^{1/2}}{|D|^{1/2} (2\pi)^{n_i/2} |\Sigma_i|^{1/2}} \exp\left(-\frac{c_i}{2}\right) \Phi(X_{2i}\beta - \alpha_i, B_i).$$

Now, consider: $\begin{bmatrix} K_i^{-1} & Z_{21i}' \\ Z_{21i} & I \end{bmatrix}$

and

Then

$$\begin{aligned} |K_i|^{-1} \cdot |I - Z_{2i} K_i Z_{2i}'| &= |I| \cdot |K_i^{-1} - Z_{2i}' Z_{2i}| \\ |K_i|^{-1} |B_i|^{-1} &= |D^{-1} + Z_{1i}' \Sigma_i^{-1} Z_{1i}| \\ |K_i|^{1/2} |B_i|^{1/2} &= |D^{-1} + Z_{1i}' \Sigma_i^{-1} Z_{1i}|^{-1/2}. \end{aligned}$$

Applying this result produces

$$f(\mathbf{y}_{1i}, \mathbf{y}_{2i} = \mathbf{1}) = \frac{|D^{-1} + Z_{1i}' \Sigma_i^{-1} Z_{1i}|^{-1/2}}{|D|^{1/2} (2\pi)^{n_i/2} |\Sigma_i|^{1/2}} \exp\left(-\frac{c_i}{2}\right) \Phi_{p_i}(X_{2i}\beta - \alpha_i, B_i).$$

Now write

$$\begin{aligned} c_i &= \gamma_i' (\Sigma_i^{-1} - \Sigma_i^{-1} Z_{1i} K_i (K_i^{-1} + Z_{2i}' B_i Z_{2i}) K_i Z_{1i}' \Sigma_i^{-1}) \gamma_i, \\ c_i &= \gamma_i' W_i^{-1} \gamma, \\ W_i^{-1} &= \Sigma_i^{-1} - \Sigma_i^{-1} Z_{1i} K_i [K_i^{-1} + Z_{2i}' B_i Z_{2i}] K_i Z_{1i}' \Sigma_i^{-1}. \end{aligned}$$

Further, consider $\begin{bmatrix} -D^{-1} & Z'_{1i} \\ Z_{1i} & I \end{bmatrix}$

Hence,

$$|D^{-1}| \cdot |\Sigma_i + Z_{1i}DZ'_{1i}| = |\Sigma_i| \cdot |D^{-1} + Z'_{1i}\Sigma_i^{-1}Z_{1i}|.$$

Hence,

$$\frac{1}{|D|^{1/2}|\Sigma_i|^{1/2}|D^{-1} + Z'_{1i}\Sigma_i^{-1}Z_{1i}|^{1/2}} = \frac{1}{|\Sigma_i + Z_{1i}DZ'_{1i}|^{1/2}}.$$

Applying this result results in

$$f(\mathbf{y}_{1i}, \mathbf{y}_{2i} = \mathbf{1}) = \frac{|W_i|^{1/2}}{|V_i|^{1/2}} \phi(X_{1i}\beta; W_i) \Phi(X_{2i}\beta - \alpha_i; B_i).$$

Now, consider

$$\begin{aligned} D^{-1} + Z'_{2i}Z_{2i} + Z'_{1i}\Sigma_i^{-1}Z_{1i} - Z'_{2i}Z_{2i} &= D^{-1} + Z'_{1i}\Sigma_i^{-1}Z_{1i} \\ D^{-1} + Z'_{2i}Z_{2i} + Z'_{1i}\Sigma_i^{-1}Z_{1i} - Z'_{2i}(I - Z_{2i}K_iZ'_{2i} + Z_{2i}K_iZ'_{2i})^{-1}Z_{2i} &= D^{-1} + Z'_{1i}\Sigma_i^{-1}Z_{1i}. \end{aligned}$$

Inserting $B_i^{-1} = I - Z_{2i}K_iZ'_{2i}$ and $K^{-1} = D^{-1} + Z'_{2i}Z_{2i} + Z'_{1i}\Sigma_i^{-1}Z_{1i}$ results in

$$K^{-1} - Z'_{2i}(B_i^{-1} + Z_{2i}K_iZ'_{2i})^{-1}Z_{2i} = D^{-1} + Z'_{1i}\Sigma_i^{-1}Z_{1i}.$$

Next, taking the inverse of both sides results in

$$\begin{aligned} K_i + K_iZ'_{2i}B_iZ_{2i}K_i &= (D^{-1} + Z'_{1i}\Sigma_i^{-1}Z_{1i})^{-1} \\ K_i[K_i^{-1} + Z'_{2i}B_iZ_{2i}]K_i &= (D^{-1} + Z'_{1i}\Sigma_i^{-1}Z_{1i})^{-1} \\ \Sigma_i^{-1} - \Sigma_i^{-1}Z_{1i}K_i[K_i^{-1} + Z'_{2i}B_iZ_{2i}]K_iZ'_{1i}\Sigma_i^{-1} &= \Sigma_i^{-1} - \Sigma_i^{-1}Z_{1i}(D^{-1} + Z'_{1i}\Sigma_i^{-1}Z_{1i})^{-1}Z'_{1i}\Sigma_i^{-1}. \end{aligned}$$

Since $V^{-1} = \Sigma_i^{-1} - \Sigma_i^{-1}Z_{1i}(D^{-1} + Z'_{1i}\Sigma_i^{-1}Z_{1i})^{-1}Z'_{1i}\Sigma_i^{-1}$, this results in

$$V^{-1} = W^{-1}.$$

Applying this result produces

$$f(\mathbf{y}_{1i}, \mathbf{y}_{2i} = \mathbf{1}) = \phi(X_{1i}\beta; V_i) \Phi(X_{2i}\beta - \alpha_i; B_i).$$

Note that (8) defines the joint probability of the continuous response with a vector of successes of the binary response $\mathbf{y}_{2i} = (1, 1, \dots, 1)'$. One can derive the joint probability with an arbitrary vector of successes and failures by applying appropriate contrasts. Precisely,

$$f(\mathbf{y}_{1i}, \mathbf{y}_{2i} = \mathbf{m}_i = (m_{i1}, \dots, m_{ip_i})') = \phi(X_{1i}\beta; V_i) \Phi(\mathbf{s} \diamond (X_{2i}\beta - \alpha_i); C \diamond B_i), \quad (16)$$

Where \diamond indicates pairwise multiplication. Further,

$$s_{ik} = \begin{cases} 1 & \text{if } m_{ik} \text{ is a success,} \\ -1 & \text{otherwise,} \end{cases}$$

$$C_{ikl} = \begin{cases} 1 & \text{if } m_{ik} \times m_{il} \text{ is even,} \\ -1 & \text{otherwise.} \end{cases}$$

B Conditional distribution of the the continuous response given the binary response

B.1 Expected value

Let us derive (10), the conditional expected value of the \tilde{n}_i -dimensional continuous subvector $\tilde{\mathbf{Y}}_{1i}$ given the \tilde{p}_i -dimensional binary subvector $\tilde{\mathbf{Y}}_{2i}$. This is equal to the integral over $\tilde{\mathbf{Y}}_{1i}$ multiplied by the conditional distribution, which is defined as the quotient of (8) and (4).

$$\begin{aligned}
& E[\tilde{\mathbf{Y}}_{1i} | \tilde{\mathbf{y}}_{2i} = 1] \\
&= \int_{\tilde{\mathbf{y}}_{1i}=-\infty}^{\tilde{\mathbf{y}}_{1i}=\infty} \tilde{\mathbf{y}}_{1i} \frac{\phi(X_{1i}\beta; V_i)\Phi(X_{2i}\beta - \alpha_i; B_i)}{\Phi(\tilde{X}_{2i}\beta, L^{-1})} d\tilde{\mathbf{y}}_{1i} \\
&= c \int_{\tilde{\mathbf{y}}_{1i}=-\infty}^{\tilde{\mathbf{y}}_{1i}=\infty} \int_{t=-\infty}^{t=\tilde{X}_{2i}\beta - H_i\tilde{\mathbf{y}}_{1i} + H_i\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{(\tilde{n}_i+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|V_i||B_i|}} \tilde{\mathbf{y}}_{1i} \\
&\quad \times \exp\left\{-\frac{1}{2}\left((\tilde{Y} - \tilde{X}_{1i}\beta)'V_i^{-1}(\tilde{Y} - \tilde{X}_{1i}\beta) + (t'B_i^{-1}t)\right)\right\} dt d\tilde{\mathbf{y}}_{1i} \\
&= c \int_{\tilde{\mathbf{y}}_{1i}=-\infty}^{\tilde{\mathbf{y}}_{1i}=\infty} \int_{s=-\infty}^{s=\tilde{X}_{2i}\beta + H_i\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{(\tilde{n}_i+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|V_i||B_i|}} \tilde{\mathbf{y}}_{1i} \\
&\quad \times \exp\left\{-\frac{1}{2}\left((\tilde{\mathbf{y}}_{1i} - \tilde{X}_{1i}\beta)'V_i^{-1}(\tilde{\mathbf{y}}_{1i} - \tilde{X}_{1i}\beta) + (s - H_i\tilde{\mathbf{y}}_{1i})'B_i^{-1}(s - H_i\mathbf{y}_{1i})\right)\right\} ds d\tilde{\mathbf{y}}_{1i} \\
&= c \int_{-\infty}^{\infty} \int_{s=-\infty}^{s=\tilde{X}_{2i}\beta + H_i\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{(\tilde{n}_i+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|V_i||B_i|}} \tilde{\mathbf{y}}_{1i} \exp\left\{-\frac{1}{2}\left((\tilde{\mathbf{y}}_{1i} - U_i)'E_i^{-1}(\tilde{\mathbf{y}}_{1i} - U_i) + O_i\right)\right\} ds d\tilde{\mathbf{y}}_{1i},
\end{aligned}$$

where

$$\begin{aligned}
H_i &= -B_i \tilde{Z}'_{2i} K_i \tilde{Z}_{1i} \Sigma_i^{-1} \\
c &= \frac{1}{\Phi(\tilde{X}_{2i}\beta, L^{-1})}.
\end{aligned}$$

In addition, we have substituted $s = t - H_i\tilde{\mathbf{y}}_{1i}$, and further

$$(\tilde{\mathbf{y}}_{1i} - \tilde{X}_{1i}\beta)'V_i^{-1}(\tilde{\mathbf{y}}_{1i} - \tilde{X}_{1i}\beta) + (s - H_i\tilde{\mathbf{y}}_{1i})'B_i^{-1}(s - H_i\mathbf{y}_{1i}) = (\tilde{\mathbf{y}}_{1i} - U_i)'E_i^{-1}(\tilde{\mathbf{y}}_{1i} - U_i) + O_i,$$

where

$$\begin{aligned}
E_i^{-1} &= H_i' B_i^{-1} H_i + V_i^{-1} \\
l'_i &= -s' B_i^{-1} H_i - \tilde{X}_{1i}\beta_1' V_i^{-1} \\
O_i &= s' B_i^{-1} s + \tilde{X}_{1i}\beta_1' V_i^{-1} \tilde{X}_{1i}\beta_1 - (-H_i' B_i^{-1} s - V_i^{-1} \tilde{X}_{1i}\beta_1)' E_i (-H_i' B_i^{-1} s - V_i^{-1} \tilde{X}_{1i}\beta_1) \\
U_i &= -E_i l_i.
\end{aligned}$$

Integration over $\tilde{\mathbf{y}}_{1i}$ produces

$$E[\tilde{Y}_{1i}|\tilde{y}_{2i}=1]$$

$$\begin{aligned}
&= c \int_{s=-\infty}^{s=\tilde{X}_{2i}\beta+H_i\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{\tilde{p}_i}{2}}} \frac{|E_i|^{1/2}}{\sqrt{|V_i||B_i|}} E_i(H_i' B_i^{-1} s + V_i^{-1} \tilde{X}_{1i}\beta) \exp\left\{-\frac{1}{2}O_i\right\} ds \\
&= c \int_{s=-\infty}^{s=\tilde{X}_{2i}\beta+H_i\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{\tilde{p}_i}{2}}} \frac{|E_i|^{1/2}}{\sqrt{|V_i||B_i|}} E_i(H_i' B_i^{-1} s + V_i^{-1} \tilde{X}_{1i}\beta) \exp\left\{-\frac{1}{2}\left((s-F_i)'T_i^{-1}(s-F_i) + G_i\right)\right\} ds \\
&= \int_{s=-\infty}^{s=\tilde{X}_{2i}\beta+H_i\tilde{X}_{1i}\beta} \frac{c}{(2\pi)^{\frac{\tilde{p}_i}{2}}} \frac{|E_i|^{1/2}}{\sqrt{|V_i||B_i|}} E_i H_i' B_i^{-1} s \exp\left\{-\frac{1}{2}\left((s-F_i)'T_i^{-1}(s-F_i) + G_i\right)\right\} ds \\
&\quad + c E_i V_i^{-1} \tilde{X}_{1i}\beta \int_{s=-\infty}^{s=\tilde{X}_{2i}\beta+H_i\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{\tilde{p}_i}{2}}} \exp\left\{-\frac{1}{2}\left((s-F_i)'T_i^{-1}(s-F_i) + G_i\right)\right\} ds,
\end{aligned}$$

where we have substituted $O_i = (s - F_i)'T_i^{-1}(s - F_i) + G_i$,

with

$$\begin{aligned}
T_i^{-1} &= B_i^{-1} - (H_i' B_i^{-1})' E_i (H_i' B_i^{-1}) \\
F_i &= T_i \cdot (H_i' B_i^{-1})' E_i (V_i^{-1} \tilde{X}_{1i}\beta) \\
G_i &= -F_i' T_i^{-1} F_i + (\tilde{X}_{1i}\beta)' V_i^{-1} (\tilde{X}_{1i}\beta) - (V_i^{-1} \tilde{X}_{1i}\beta)' E_i (V_i^{-1} \tilde{X}_{1i}\beta).
\end{aligned}$$

In order to solve the first integral, we use the formula of the expectation of the truncated normal distribution, described by Manjunath and Wilhelm (2012) in their Equation 17.

$$\begin{aligned}
&= c \exp\left\{-\frac{1}{2}G_i\right\} \sqrt{\frac{|E_i||T_i|}{|V_i||B_i|}} E_i H_i' B_i^{-1} T_i \Phi(o, F_i, T_i) [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)] \quad (17) \\
&\quad + c \exp\left\{-\frac{1}{2}G_i\right\} \sqrt{\frac{|E_i||T_i|}{|V_i||B_i|}} \Phi(\tilde{X}_{2i}\beta + H_i \tilde{X}_{1i}\beta, F_i, T_i) E_i H_i' B_i^{-1} F_i \\
&\quad + c \exp\left\{-\frac{1}{2}G_i\right\} \sqrt{\frac{|E_i||T_i|}{|V_i||B_i|}} \Phi(\tilde{X}_{2i}\beta + H_i \tilde{X}_{1i}\beta, F_i, T_i) E_i V_i^{-1} \tilde{X}_{1i}\beta,
\end{aligned}$$

where $o, F_i(x_i)$ and $\varphi(x_i)$ are defined in (18).

It is important to note that (10) is conditional on a vector of successes $(1, 1, \dots, 1)'$. In analogy with this derivation, we can derive the expected value given an arbitrary vector of successes and failures by using appropriate contrasts. The conditional expected value given an arbitrary vector is defined as

$$\begin{aligned}
E[\tilde{Y}_{1i}|\tilde{y}_{2i}] &= \mathbf{m}_i = (m_{i1}, \dots, m_{ip_i})' = \frac{\exp\left\{-\frac{1}{2}G_i\right\}}{\Phi(\tilde{X}_{2i}\beta^*, (L)^{-1})} \sqrt{\frac{|E_i||T_i|}{|V_i||B_i|}} \Phi(\tilde{X}_{2i}\beta^* + H_i \tilde{X}_{1i}\beta, F_i, T_i) \quad (18) \\
&\quad \times \left(E_i (V_i^{-1} \tilde{X}_{1i}\beta + H_i' B_i^{-1} F_i) - E_i H_i' B_i^{-1} T_i [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)] \right),
\end{aligned}$$

with

$$\begin{aligned}\tilde{X}_{2i}\beta^* &= s \diamond \tilde{X}_{2i}\beta \\ \tilde{Z}_{2i}^* &= J \diamond Z_{2i},\end{aligned}$$

where \diamond signals pairwise multiplication and s is defined in (16). Further

$$J_{ikl} = \begin{cases} 1 & \text{if } m_{ik} \text{ is a success,} \\ -1 & \text{otherwise.} \end{cases}$$

B.2 Prediction interval

The prediction interval of the expected values of a (sub)vector of the continuous response given a (sub)vector of the binary response is composed by the variability of the observations (second central moment) and the standard errors of the transformed parameters. We will first derive the second central moment and then derive the standard errors via the delta method.

The uncertainty of a new observation is defined as the second central moment

$$\begin{aligned}E\left[\left(\tilde{\mathbf{Y}}_{1i} - E[\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1}]\right)\left(\tilde{\mathbf{Y}}_{1i} - E[\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1}]\right)'\right] \\ = \int_{\tilde{\mathbf{y}}_{1i}=-\infty}^{\tilde{\mathbf{y}}_{1i}=+\infty} (\tilde{\mathbf{y}}_{1i} - E[\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1}])(\tilde{\mathbf{y}}_{1i} - E[\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1}])' \cdot f(\tilde{\mathbf{y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1}) d\tilde{\mathbf{y}}_{1i} \\ = \int_{\tilde{\mathbf{y}}_{1i}=-\infty}^{\tilde{\mathbf{y}}_{1i}=+\infty} \tilde{\mathbf{y}}_{1i}\tilde{\mathbf{y}}_{1i}'f(\tilde{\mathbf{y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1}) - E(\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1})E(\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1})'f(\tilde{\mathbf{y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1}) \\ - (\tilde{\mathbf{y}}_{1i}'E(\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1}) + E(\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1})\tilde{\mathbf{y}}_{1i}') \cdot f(\tilde{\mathbf{y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1}) d\tilde{\mathbf{y}}_{1i} \\ = \int_{\tilde{\mathbf{y}}_{1i}=-\infty}^{\tilde{\mathbf{y}}_{1i}=+\infty} \tilde{\mathbf{y}}_{1i}\tilde{\mathbf{y}}_{1i}'f(\tilde{\mathbf{y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1})d\tilde{\mathbf{y}}_{1i} - E(\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1})E(\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1})' \\ = c \int_{\tilde{\mathbf{y}}_{1i}=-\infty}^{\tilde{\mathbf{y}}_{1i}=+\infty} \int_{\mathbf{t}=-\infty}^{\mathbf{t}=\tilde{X}_{2i}\beta-H\tilde{\mathbf{y}}_{1i}+H\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{(\tilde{n}_i+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|V_i||B_i|}} \tilde{\mathbf{y}}_{1i}\tilde{\mathbf{y}}_{1i}' \\ \times \exp\left\{-\frac{1}{2}\left((\tilde{\mathbf{y}}_{1i} - \tilde{X}_{1i}\beta)'V_i^{-1}(\tilde{\mathbf{y}}_{1i} - \tilde{X}_{1i}\beta) + (\mathbf{t}'B_i^{-1}\mathbf{t})\right)\right\} dt d\tilde{\mathbf{y}}_{1i} - E(\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1})E(\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1})' \\ = c \int_{\tilde{\mathbf{y}}_{1i}=-\infty}^{\tilde{\mathbf{y}}_{1i}=+\infty} \int_{\mathbf{s}=-\infty}^{\mathbf{s}=\tilde{X}_{2i}\beta+H\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{(\tilde{n}_i+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|V_i||B_i|}} \tilde{\mathbf{y}}_{1i}\tilde{\mathbf{y}}_{1i}' \\ \times \exp\left\{-\frac{1}{2}\left((\tilde{\mathbf{y}}_{1i} - \tilde{X}_{1i}\beta)'V_i^{-1}(\tilde{\mathbf{y}}_{1i} - \tilde{X}_{1i}\beta) + ((\mathbf{s} - H\tilde{\mathbf{y}}_{1i})'B_i^{-1}(\mathbf{s} - H\tilde{\mathbf{y}}_{1i}))\right)\right\} ds d\tilde{\mathbf{y}}_{1i} \\ - E(\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1})E(\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1})' \\ = c \int_{\tilde{\mathbf{y}}_{1i}=-\infty}^{\tilde{\mathbf{y}}_{1i}=+\infty} \int_{\mathbf{s}=-\infty}^{\mathbf{s}=\tilde{X}_{2i}\beta+H\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{(\tilde{n}_i+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|V_i||B_i|}} \tilde{\mathbf{y}}_{1i}\tilde{\mathbf{y}}_{1i}' \\ \times \exp\left\{-\frac{1}{2}\left((\tilde{\mathbf{y}}_{1i} - u_i)'E_i^{-1}(\tilde{\mathbf{y}}_{1i} - u_i) + O_i\right)\right\} ds d\tilde{\mathbf{y}}_{1i} - E(\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1})E(\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1})'\end{aligned}$$

where we have substituted $s = t - H\tilde{\mathbf{y}}_{1i}$, and further

$$(\tilde{\mathbf{y}}_{1i} - \tilde{X}_{1i}\beta)'V_i^{-1}(\tilde{\mathbf{y}}_{1i} - \tilde{X}_{1i}\beta) + (s - H\tilde{\mathbf{y}}_{1i})'B_i^{-1}(s - H\mathbf{y}_{1i}) = (\tilde{\mathbf{y}}_{1i} - U_i)'E_i^{-1}(\tilde{\mathbf{y}}_{1i} - U_i) + O_i,$$

where

$$\begin{aligned} E_i^{-1} &= H'B_i^{-1}H + V_i^{-1} \\ l'_i &= -s'B_i^{-1}H - \tilde{X}_{1i}\beta'_1V_i^{-1} \\ O_i &= s'B_i^{-1}s + \tilde{X}_{1i}\beta'_1V_i^{-1}\tilde{X}_{1i}\beta_1 - (-H'B_i^{-1}s - V_i^{-1}\tilde{X}_{1i}\beta_1)'E(-H'B_i^{-1}s - V_i^{-1}\tilde{X}_{1i}\beta_1) \\ U_i &= -E_il_i. \end{aligned}$$

Now we can integrate over $\tilde{\mathbf{y}}_{1i}$, applying the formula of the second moment of a multivariate gaussian distribution:

$$\begin{aligned} &E\left[\left(\tilde{\mathbf{Y}}_{1i} - E[\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1}]\right)^2\right] \\ &= c \int_{\mathbf{s}=-\infty}^{\mathbf{s}=\tilde{X}_{2i}\beta+H\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{\tilde{p}_i}{2}}} \frac{|E_i|^{1/2}}{\sqrt{|V_i||B_i|}} u_i u'_i \exp\left\{-\frac{1}{2}O_i\right\} d\mathbf{s} \\ &\quad + E_i c \int_{\mathbf{s}=-\infty}^{\mathbf{s}=\tilde{X}_{2i}\beta+H\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{\tilde{p}_i}{2}}} \frac{|E_i|^{1/2}}{\sqrt{|V_i||B_i|}} \exp\left\{-\frac{1}{2}O_i\right\} d\mathbf{s} - E(\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1})E(\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1})' \\ &= c \int_{\mathbf{s}=-\infty}^{\mathbf{s}=\tilde{X}_{2i}\beta+H\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{\tilde{p}_i}{2}}} \frac{|E_i|^{1/2}}{\sqrt{|V_i||B_i|}} u_i u'_i \exp\left\{-\frac{1}{2}\left((\mathbf{s} - F_i)'T_i^{-1}(\mathbf{s} - F_i) + G_i\right)\right\} d\mathbf{s} \\ &\quad + E_i c \int_{\mathbf{s}=-\infty}^{\mathbf{s}=\tilde{X}_{2i}\beta+H\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{\tilde{p}_i}{2}}} \frac{|E_i|^{1/2}}{\sqrt{|V_i||B_i|}} \exp\left\{-\frac{1}{2}\left((\mathbf{s} - F_i)'T_i^{-1}(\mathbf{s} - F_i) + G_i\right)\right\} d\mathbf{s} \\ &\quad - E(\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1})E(\tilde{\mathbf{Y}}_{1i}|\tilde{\mathbf{y}}_{2i} = \mathbf{1})', \end{aligned}$$

where we have rewritten $O_i = (s - F_i)'T_i^{-1}(s - F_i) + G_i$, with

$$\begin{aligned} T_i^{-1} &= B_i^{-1} - (H'B_i^{-1})'E_i(H'B_i^{-1}) \\ F_i &= T_i \cdot (H'B_i^{-1})E_i(V_i^{-1}\tilde{X}_{1i}\beta) \\ G_i &= -F_i'T_i^{-1}F_i + (\tilde{X}_{1i}\beta)'V_i^{-1}(\tilde{X}_{1i}\beta) - (V_i^{-1}\tilde{X}_{1i}\beta)'E_i(V_i^{-1}\tilde{X}_{1i}\beta). \end{aligned}$$

Integration over \mathbf{s} produces

$$\begin{aligned}
& E \left[(\tilde{\mathbf{Y}}_{1i} - E[\tilde{\mathbf{Y}}_{1i} | \tilde{\mathbf{y}}_{2i} = \mathbf{1}]) (\tilde{\mathbf{Y}}_{1i} - E[\tilde{\mathbf{Y}}_{1i} | \tilde{\mathbf{y}}_{2i} = \mathbf{1}])' \right] \\
&= c \frac{1}{(2\pi)^{\frac{\tilde{p}_i}{2}}} \frac{|E_i|^{1/2}}{\sqrt{|V_i||B_i|}} \int_{\mathbf{s}=-\infty}^{\mathbf{s}=\tilde{X}_{2i}\beta + H\tilde{X}_{1i}\beta} \left(E_i H' B^{-1} \mathbf{s} \mathbf{s}' B^{-1} H E_i + E_i H' B^{-1} \mathbf{s} (\tilde{X}_{1i}\beta)' V_i^{-1} E_i \right. \\
&\quad \left. + E V_i^{-1} \tilde{X}_{1i}\beta \mathbf{s}' B^{-1} H E_i + E V_i^{-1} \tilde{X}_{1i}\beta (\tilde{X}_{1i}\beta)' V_i^{-1} E_i \right) \exp \left\{ -\frac{1}{2} \left((\mathbf{s} - F_i)' T_i^{-1} (\mathbf{s} - F_i) + G_i \right) \right\} d\mathbf{s} \\
&\quad + E_i \frac{\sqrt{|E_i||T_i|}}{\sqrt{|V_i||B_i|}} c \exp \left\{ -\frac{1}{2} G_i \right\} \Phi(\tilde{X}_{2i}\beta + H\tilde{X}_{1i}\beta, F_i, T_i) - E(\tilde{\mathbf{Y}}_{1i} | \tilde{\mathbf{y}}_{2i} = \mathbf{1}) E(\tilde{\mathbf{Y}}_{1i} | \tilde{\mathbf{y}}_{2i} = \mathbf{1})' \\
&= c \exp \left\{ -\frac{1}{2} G_i \right\} \frac{\sqrt{|E_i||T_i|}}{\sqrt{|V_i||B_i|}} \Phi(\tilde{X}_{2i}\beta + H\tilde{X}_{1i}\beta, F_i, T_i) \left(E_i + E_i V_i^{-1} \tilde{X}_{1i}\beta (\tilde{X}_{1i}\beta)' V_i^{-1} E_i \right. \\
&\quad \left. + E_i H' B^{-1} \left(N + J J' \right) B_i^{-1} H E_i + E V_i^{-1} \tilde{X}_{1i}\beta J' B^{-1} H E_i + E_i H' B^{-1} J (\tilde{X}_{1i}\beta)' V_i^{-1} E_i \right) \\
&\quad - E(\tilde{\mathbf{Y}}_{1i} | \tilde{\mathbf{y}}_{2i} = \mathbf{1}) E(\tilde{\mathbf{Y}}_{1i} | \tilde{\mathbf{y}}_{2i} = \mathbf{1})',
\end{aligned} \tag{19}$$

where J is the expected value of the truncated multivariate normal density, and N is the second moment of the latter density. More specifically,

$$\begin{aligned}
a &= \tilde{\mathbf{X}}_{2i}\beta + \mathbf{H}_i \tilde{\mathbf{X}}_{1i}\beta \\
J &= \mathbf{T}_i [-F_1(a_1) \quad -F_2(a_2) \quad \dots \quad -F_{\tilde{p}_i}(a_{\tilde{p}_i})] + \mathbf{F}_i, \\
N_{i,j} &= T_{i,j} \sum_{k=1}^{\tilde{p}_i} T_{i,k} \frac{-T_{i,j,k} a_k F_k(a_k)}{T_{i,k,k}} + \sum_{k=1}^{\tilde{p}_i} T_{i,k} \sum_{q \neq k} \left(T_{i,j,q} - \frac{T_{i,k,q} T_{i,j,k}}{T_{i,k,k}} \right) \cdot -F_{k,q}(a_k, a_q) - J_i J_k, \\
F_{k,q}(x, y) &= \int_{-\infty}^{a_1} \dots \int_{-\infty}^{a_{k-1}} \int_{-\infty}^{a_{k+1}} \dots \int_{-\infty}^{a_{q-1}} \int_{-\infty}^{a_{q+1}} \dots \int_{-\infty}^{a_{\tilde{n}_i}} \phi(x, y, x_{-k-, -q}, \mathbf{F}_i, \mathbf{T}_i) dx_{-k-, -q}, \\
F_i(x_i) &= \int_{-\infty}^{a_1} \dots \int_{-\infty}^{a_{i-1}} \int_{-\infty}^{a_{i+1}} \dots \int_{-\infty}^{a_{\tilde{p}_i}} \varphi(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_{\tilde{p}_i}) dx_{\tilde{n}_i}, \dots, dx_{i+1} dx_{i-1} \dots dx_1 \\
\varphi(x) &= \begin{cases} \frac{\phi(x, \mathbf{F}_i, \mathbf{T}_i)}{\Phi(\tilde{\mathbf{X}}_{2i}\beta + \mathbf{H}_i \tilde{\mathbf{X}}_{1i}\beta, \mathbf{F}_i, \mathbf{T}_i)}, & \text{for } \mathbf{x} \leq \tilde{\mathbf{X}}_{2i}\beta + \mathbf{H}_i \tilde{\mathbf{X}}_{1i}\beta, \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

The above equations are deduced from Manjutah and Wilhelm (2012).

Agresti (2002) derived the distribution of transformed maximum likelihood parameters via the delta method

$$G(\hat{\beta}) \rightarrow N \left(\beta, \left(\frac{\partial G(\beta)}{\partial \beta} \right)' V \text{ar}(\hat{\beta}) \frac{\partial G(\beta)}{\partial \beta} \right).$$

We will first sketch the derivative of the expected value with respect to a coefficient β_{12} of a predictor X_{12} of the continuous response and next sketch the derivation of a coefficient β_{22} of a predictor X_{22}

of the binary response.

$$\begin{aligned} \frac{\mathbb{E}(\tilde{Y}_{1i}|\tilde{y}_{2i}=1)}{\partial\beta_{12}} &= \frac{1}{\Phi(\tilde{X}_{2i}\beta, L^{-1})} \sqrt{\frac{|E_i||T_i|}{|V_i||B_i|}} \left\{ \Lambda\Phi(\tilde{X}_{2i}\beta + H\tilde{X}_{1i}\beta, F_i, T_i)E_i(V_i^{-1}\tilde{X}_{1i}\beta + H'B_i^{-1}F_i) \right. \\ &\quad + e^{-0.5G}\lambda E_i(V_i^{-1}\tilde{X}_{1i}\beta + H'B_i^{-1}F_i) \\ &\quad + e^{-0.5G}\Phi(\tilde{X}_{2i}\beta + H\tilde{X}_{1i}\beta, F_i, T_i) \left(EV_i^{-1}\tilde{X}'_{12i} + E_iH'B_i^{-1}T_iB_i^{-1}HE_iV_i^{-1}\tilde{X}'_{12i} \right) \\ &\quad \left. + E_iH'B_i^{-1}\cdot\Lambda\cdot T_i\Phi(o, T_i) \begin{bmatrix} -F_1(o_1) & -F_2(o_2) & \dots & -F_p(o_p) \end{bmatrix} + E_iH'B_i^{-1}e^{-0.5G}\nu \right\}, \end{aligned}$$

with

$$\begin{aligned} \Lambda &= \exp\left\{-\frac{1}{2}G_i\right\} \left(-(\tilde{X}'_{1ij}\beta)'V_i^{-1}\tilde{X}'_{12i} + (\tilde{X}'_{1ij}\beta)'V_i^{-1}E_iV_i^{-1}\tilde{X}'_{12i} \right. \\ &\quad \left. + (\tilde{X}'_{1ij}\beta)'V_i^{-1}E_iH'B_i^{-1}T_iB_i^{-1}HE_iV_i^{-1}\tilde{X}'_{12i} \right), \\ \lambda &= \sum_{k=1}^{\tilde{p}_i} (H\tilde{X}'_{12i} - T_i \cdot B_i^{-1}HE_iV_i^{-1}\tilde{X}_{12ij})_k \phi((\tilde{X}'_{22ik}\beta + H\tilde{X}_{1i}\beta)_k, F_{i,k}, T_{i,kk}) \\ &\quad \times \Phi((\tilde{X}'_{22i}\beta + H\tilde{X}_{1i}\beta)_{-k}, F_{i,-k}, T_{i,-k-k}), \\ \nu &= \sum_{k=1}^{\tilde{p}_i} (H\tilde{X}'_{12i} - T_i \cdot B_i^{-1}HE_iV_i^{-1}\tilde{X}_{12ij})_k g_k(o_k) \\ g_k(x_k) &= \int_{-\infty}^{o_1} \dots \int_{-\infty}^{o_{i-1}} \int_{-\infty}^{o_{i+1}} \dots \int_{-\infty}^{o_{\tilde{p}_i}} [x_1 \dots x_{k-1} o_k x_{k+1} \dots x_{\tilde{p}_i}]' \phi([x_1 \dots x_{k-1} o_k x_{k+1} \dots x_{\tilde{p}_i}]', T_i) dx_{-k}. \end{aligned}$$

Next, for a coefficient β_{22} of a predictor X_{22} of the binary response the derivative is the following

$$\begin{aligned} \frac{\mathbb{E}(\tilde{Y}_{1i}|\tilde{y}_{2i}=1)}{\partial\beta_{22}} &= \frac{1}{\Phi^2(\tilde{X}_{2i}\beta, L^{-1})} \sqrt{\frac{|E_i||T_i|}{|V_i||B_i|}} \exp\left\{-\frac{1}{2}G_i\right\} \\ &\quad \times \left[E_i(V_i^{-1}\tilde{X}_{1i}\beta + H'B_i^{-1}F_i) \left(\Omega\Phi(\tilde{X}_{2i}\beta, L^{-1}) - \omega\Phi(\tilde{X}_{2i}\beta + H\tilde{X}_{1i}\beta, F_i, T_i) \right) \right. \\ &\quad \left. + E_iH'B_i^{-1} \left(\xi\Phi(\tilde{X}_{2i}\beta, L^{-1}) - \omega \cdot T_i\Phi(o, T_i) \begin{bmatrix} -F_1(o_1) & -F_2(o_2) & \dots & -F_p(o_p) \end{bmatrix} \right) \right], \end{aligned} \tag{20}$$

where

$$\begin{aligned} \omega &= \sum_{k=1}^{\tilde{p}} \tilde{X}'_{22ik} \phi((\tilde{X}_{2i}\beta)_k, L_{kk}^{-1}) \Phi((\tilde{X}_{2i}\beta)_{-k}, L_{-k,-k}^{-1}) \\ \Omega &= \sum_{k=1}^{\tilde{p}} \tilde{X}'_{22ik} \phi((\tilde{X}_{2i}\beta + H\tilde{X}_{1i}\beta - F_i)_k, T_{kk}) \Phi((\tilde{X}_{2i}\beta + H\tilde{X}_{1i}\beta - F_i)_{-k}, T_{i,-k,-k}) \\ \xi &= \sum_{k=1}^{\tilde{p}_i} \tilde{X}'_{22ik} g_k(o_k) \end{aligned}$$

Next, the 95% prediction interval can be computed with the following general formula

$$\begin{aligned} & \left[E[\tilde{\mathbf{Y}}_{1i} | \tilde{\mathbf{y}}_{2i} = 1] - 1.96 \sqrt{E[(\tilde{\mathbf{Y}}_{1i} - E[\tilde{\mathbf{Y}}_{1i} | \tilde{\mathbf{y}}_{2i} = 1])^2] + \frac{\partial G(\beta)}{\partial \beta'} Var(\hat{\beta}) \frac{\partial G(\beta)}{\partial \beta}}, \right. \\ & \left. E[\tilde{\mathbf{Y}}_{1i} | \tilde{\mathbf{y}}_{2i} = 1] + 1.96 \sqrt{E[(\tilde{\mathbf{Y}}_{1i} - E[\tilde{\mathbf{Y}}_{1i} | \tilde{\mathbf{y}}_{2i} = 1])^2] + \frac{\partial G(\beta)}{\partial \beta'} Var(\hat{\beta}) \frac{\partial G(\beta)}{\partial \beta}} \right]. \end{aligned}$$

C Conditional distribution of a subvector of the binary response given a subvector of the continuous response

C.1 Expected value

Let us derive (10), the conditional probability for the \tilde{p}_i -dimensional subvector of successes given a subvector of the continuous response $\tilde{\mathbf{Y}}_{1i}$. Since the probability is equal to the conditional density, this is defined as

$$f(\tilde{\mathbf{Y}}_{2i} = \mathbf{1} | \tilde{\mathbf{y}}_{1i}) = \frac{\phi(\tilde{X}_{1i}\beta; V_i)\Phi(\tilde{X}_{2i}\beta - \alpha_i; B_i)}{\phi(\tilde{X}_{1i}\beta; V_i)} = \Phi(\tilde{X}_{2i}\beta - \alpha_i; B_i).$$

C.2 Confidence interval

In analogy with Appendix B.2, the gradients of the transformed parameters can be computed by the delta method in order to construct the confidence interval. First the standard error of a coefficient β_{12} of a predictor of the continuous response X_{12} is derived:

$$\frac{f(\tilde{\mathbf{Y}}_{2i} = \mathbf{1} | \tilde{\mathbf{y}}_{1i})}{\partial \beta_{12}} = \sum_{k=1}^{\tilde{p}_i} \gamma_{kk}^{-1/2} H_{ik} X_{12i} \phi(d_k) \Phi^{\tilde{R}_{B_i}^{(k)}}(\tilde{d}^{(k)}) \quad (21)$$

with

$$\begin{aligned} \gamma &= \text{Diagonal matrix composed of the variances of } B_i \\ d &= \gamma^{-1/2}(X_{2i}\beta - H_i \tilde{\mathbf{y}}_{1i} + H_i X_{1i}\beta) \\ R_{B_i} &= \gamma^{-1/2} B_i \gamma^{-1/2} = (r_{ij})_{i,j=1}^{\tilde{p}_i} \\ \tilde{d}^{(i)} &= \left(\frac{d_1 - r_{1,i}d_i}{\sqrt{1 - r_{1,i}^2}}, \dots, \frac{d_{i-1} - r_{i-1,i}d_i}{\sqrt{1 - r_{i-1,i}^2}}, \frac{d_{i+1} - r_{i+1,i}d_i}{\sqrt{1 - r_{i+1,i}^2}}, \dots, \frac{d_{\tilde{p}_i} - r_{\tilde{p}_i,i}d_i}{\sqrt{1 - r_{\tilde{p}_i,i}^2}} \right). \end{aligned}$$

The $(\tilde{p}_i - 1) \times (\tilde{p}_i - 1)$ -dimensional correlation matrix $\tilde{R}_{B_i}^{(i)}$ has entries

$$\tilde{r}_{j,k}^{(i)} = \frac{r_{j',k'} - r_{j',i}r_{k',i}}{\sqrt{1 - r_{j',i}^2}\sqrt{1 - r_{k',i}^2}} \quad i = 1 \dots \tilde{p}_i; \quad j, k = 1 \dots \tilde{p}_i - 1$$

with

$$j' = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases} \quad k' = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i. \end{cases}$$

Next, the gradient of a coefficients β_{22} of one of the predictors of the binary response X_{22} is defined as

$$\frac{f(\tilde{\mathbf{Y}}_{2i} = \mathbf{1} | \tilde{\mathbf{y}}_{1i})}{\partial \beta_{22}} = \sum_{k=1}^{\tilde{p}_i} \gamma_{kk}^{-1/2} X_{22i} \phi(d_k) \Phi^{\tilde{R}_{B_i}^{(k)}}(\tilde{d}^{(k)}), \quad (22)$$

where (21) and (22) have been derived from Prékopa (1995) p.204.

D Conditional distribution of a subvector of the continuous response given the binary response and a subvector of the continuous response

D.1 Expected value

Let us derive (11), the conditional expected value of the continuous subvector $\tilde{\mathbf{Y}}_{1i}^a = (Y_{1i}^{a_1} Y_{1i}^{a_2} \dots Y_{1i}^{a_{n_a}})$ given a distinct subvector of continuous responses $\tilde{\mathbf{Y}}_{1i}^b = (Y_{1i}^{b_1} Y_{1i}^{b_2} \dots Y_{1i}^{b_{n_b}})$ and the \tilde{p}_i -dimensional binary subvector $\tilde{\mathbf{Y}}_{2i}$. In the following calculations, superscripts will indicate subvectors and submatrices. The superscript a denotes the rows a_1 until a_{n_a} and the superscript b denotes the rows b_1 until b_{n_b} . Analogously, the superscript bb denotes the submatrix with rows b_1 until b_{n_b} and columns b_1 until b_{n_b} . The superscript ab denotes the submatrix with rows a_1 until a_{n_a} and columns b_1 until b_{n_b} . Lastly, the superscript $.b$ denotes the columns b_1 until b_{n_b} from a matrix. The conditional expected value is equal to the integral over $\tilde{\mathbf{Y}}_{1i}^a$ multiplied by the conditional distribution, which is defined as the joint distribution of $\tilde{\mathbf{Y}}_{1i} = [\tilde{\mathbf{Y}}_{1i}^a, \tilde{\mathbf{Y}}_{1i}^b]$ and $\tilde{\mathbf{Y}}_{2i}$ divided by the joint distribution of $\tilde{\mathbf{Y}}_{1i}^b$ and $\tilde{\mathbf{Y}}_{2i}$:

$$\begin{aligned}
& E[\tilde{\mathbf{Y}}_{1i}^a | \tilde{\mathbf{Y}}_{1i}^b = \tilde{\mathbf{y}}_{1i}^b, \tilde{\mathbf{y}}_{2i} = 1] \\
&= \int_{\tilde{\mathbf{y}}_{1i}^a = -\infty}^{\tilde{\mathbf{y}}_{1i}^a = \infty} \tilde{\mathbf{y}}_{1i}^a \frac{\phi(\tilde{\mathbf{y}}_{1i}; X_{1i}\beta; V_i) \Phi(X_{2i}\beta - H\tilde{\mathbf{y}}_{1i} + HX_{1i}\beta; B_i)}{f(\tilde{\mathbf{y}}_{1i}^b, \tilde{\mathbf{y}}_{2i})} d\tilde{\mathbf{y}}_{1i}^a \\
&= \frac{1}{c} \int_{\tilde{\mathbf{y}}_{1i}^a = -\infty}^{\tilde{\mathbf{y}}_{1i}^a = \infty} \int_{\mathbf{t} = -\infty}^{\mathbf{t} = \tilde{X}_{2i}\beta - H \begin{bmatrix} \tilde{\mathbf{y}}_{1i}^a \\ \tilde{\mathbf{y}}_{1i}^b \end{bmatrix} + H\tilde{\mathbf{y}}_{1i}\beta} \frac{1}{(2\pi)^{\frac{(n_a+n_b+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|V_i||B_i|}} \tilde{\mathbf{y}}_{1i}^a \\
&\quad \exp \left\{ -\frac{1}{2} \left((\begin{bmatrix} \tilde{\mathbf{y}}_{1i}^a \\ \tilde{\mathbf{y}}_{1i}^b \end{bmatrix} - \tilde{X}_{1i}\beta)' V_i^{-1} (\begin{bmatrix} \tilde{\mathbf{y}}_{1i}^a \\ \tilde{\mathbf{y}}_{1i}^b \end{bmatrix} - \tilde{X}_{1i}\beta) + \mathbf{t}' B_i^{-1} \mathbf{t} \right) \right\} d\tilde{\mathbf{y}}_{1i}^a d\mathbf{t} \\
&= \frac{1}{c} \int_{\tilde{\mathbf{y}}_{1i}^a = -\infty}^{\tilde{\mathbf{y}}_{1i}^a = \infty} \int_{\mathbf{s} = -\infty}^{\mathbf{s} = \tilde{X}_{2i}\beta + H\tilde{\mathbf{y}}_{1i}\beta} \frac{1}{(2\pi)^{\frac{(n_a+n_b+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|V_i||B_i|}} \tilde{\mathbf{y}}_{1i}^a \\
&\quad \exp \left\{ -\frac{1}{2} \left((\begin{bmatrix} \tilde{\mathbf{y}}_{1i}^a \\ \tilde{\mathbf{y}}_{1i}^b \end{bmatrix} - \tilde{X}_{1i}\beta)' V_i^{-1} (\begin{bmatrix} \tilde{\mathbf{y}}_{1i}^a \\ \tilde{\mathbf{y}}_{1i}^b \end{bmatrix} - \tilde{X}_{1i}\beta) + (\mathbf{s} - H \begin{bmatrix} \tilde{\mathbf{y}}_{1i}^a \\ \tilde{\mathbf{y}}_{1i}^b \end{bmatrix})' B_i^{-1} (\mathbf{s} - H \begin{bmatrix} \tilde{\mathbf{y}}_{1i}^a \\ \tilde{\mathbf{y}}_{1i}^b \end{bmatrix}) \right) \right\} d\tilde{\mathbf{y}}_{1i}^a d\mathbf{s} \\
&= \frac{1}{c} \int_{\tilde{\mathbf{y}}_{1i}^a = -\infty}^{\tilde{\mathbf{y}}_{1i}^a = \infty} \int_{\mathbf{s} = -\infty}^{\mathbf{s} = \tilde{X}_{2i}\beta + H\tilde{\mathbf{y}}_{1i}\beta} \frac{1}{(2\pi)^{\frac{(n_a+n_b+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|V_i||B_i|}} \tilde{\mathbf{y}}_{1i}^a \\
&\quad \exp \left\{ -\frac{1}{2} \left((\begin{bmatrix} \tilde{\mathbf{y}}_{1i}^a \\ \tilde{\mathbf{y}}_{1i}^b \end{bmatrix} - u_i)' \mathbf{E}_i^{-1} (\begin{bmatrix} \tilde{\mathbf{y}}_{1i}^a \\ \tilde{\mathbf{y}}_{1i}^b \end{bmatrix} - u_i) + O_i \right) \right\} d\tilde{\mathbf{y}}_{1i}^a d\mathbf{s},
\end{aligned}$$

where

$$\begin{aligned}
c &= f(\tilde{\mathbf{y}}_{1i}^b, \tilde{\mathbf{y}}_{2i}), \text{ the marginal joint distribution} \\
\mathbf{s} &= \mathbf{t} - H\tilde{\mathbf{y}}_{1i}.
\end{aligned}$$

Further,

$$(\begin{bmatrix} \tilde{\mathbf{y}}_{1i}^a \\ \tilde{\mathbf{y}}_{1i}^b \end{bmatrix} - \tilde{X}_{1i}\beta)' V_i^{-1} (\begin{bmatrix} \tilde{\mathbf{y}}_{1i}^a \\ \tilde{\mathbf{y}}_{1i}^b \end{bmatrix} - \tilde{X}_{1i}\beta) + (\mathbf{s} - H \begin{bmatrix} \tilde{\mathbf{y}}_{1i}^a \\ \tilde{\mathbf{y}}_{1i}^b \end{bmatrix})' B_i^{-1} (\mathbf{s} - H \begin{bmatrix} \tilde{\mathbf{y}}_{1i}^a \\ \tilde{\mathbf{y}}_{1i}^b \end{bmatrix}) = (\begin{bmatrix} \tilde{\mathbf{y}}_{1i}^a \\ \tilde{\mathbf{y}}_{1i}^b \end{bmatrix} - u_i)' \mathbf{E}_i^{-1} (\begin{bmatrix} \tilde{\mathbf{y}}_{1i}^a \\ \tilde{\mathbf{y}}_{1i}^b \end{bmatrix} - u_i) + O_i,$$

with

$$\begin{aligned}
\mathbf{E}_i^{-1} &= H'B_i^{-1}H + V_i^{-1} \\
l'_i &= -s'B_i^{-1}H - \tilde{X}_{1i}\beta'_1V_i^{-1} \\
O_i &= s'B_i^{-1}s + \tilde{X}_{1i}\beta'_1V_i^{-1}\tilde{X}_{1i}\beta_1 - (-H'B_i^{-1}s - V_i^{-1}\tilde{X}_{1i}\beta_1)' \mathbf{E}_i(-H'B_i^{-1}s - V_i^{-1}\tilde{X}_{1i}\beta_1) \\
u_i &= -\mathbf{E}_i l_i.
\end{aligned}$$

Integrating over $\tilde{\mathbf{y}}_{1i}^a$ results in

$$\begin{aligned}
&E[\tilde{\mathbf{Y}}_{1i}^a | \tilde{\mathbf{Y}}_{1i}^b = \tilde{\mathbf{y}}_{1i}^b, \tilde{\mathbf{y}}_{2i} = 1] \\
&= \frac{1}{c} \int_{s=-\infty}^{s=\tilde{X}_{2i}\beta+H\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{\tilde{p}_i}{2}}} \frac{\sqrt{|\mathbf{E}_i|}}{\sqrt{|V_i||B_i|}} \left((\mathbf{u}_i^a + \mathbf{E}_i^{ab} \mathbf{E}_i^{bb-1}(\tilde{\mathbf{y}}_{1i}^b - \mathbf{u}_i^b)) \phi(\tilde{\mathbf{y}}_{1i}^b, \mathbf{u}_i^b, \mathbf{E}_i^{bb}) \right) \exp\left\{-\frac{1}{2}O_i\right\} ds \\
&= \frac{1}{c} \int_{s=-\infty}^{s=\tilde{X}_{2i}\beta+H\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{(\tilde{p}_i+n_b)}{2}}} \frac{\sqrt{|\mathbf{E}_i|}}{\sqrt{|V_i||B_i||\mathbf{E}_i^{bb}|}} (\mathbf{u}_i^a + \mathbf{E}_i^{ab} \mathbf{E}_i^{bb-1}(\tilde{\mathbf{y}}_{1i}^b - \mathbf{u}_i^b)) \\
&\quad \exp\left\{-\frac{1}{2}\left((s - F_i)' \mathbf{T}_i^{-1}(s - F_i) + G_i\right)\right\} ds
\end{aligned}$$

where we have substituted $O_i + (\tilde{\mathbf{y}}_{1i}^b - \mathbf{u}_i^b)'(\mathbf{E}_i^{bb})^{-1}(\tilde{\mathbf{y}}_{1i}^b - \mathbf{u}_i^b) = (s - F_i)' \mathbf{T}_i^{-1}(s - F_i) + G_i$,

with

$$\begin{aligned}
T_i^{-1} &= (\mathbf{E}_i H_i' B_i^{-1})^{b'} (\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i H_i' B_i^{-1})^b + B_i^{-1} - (H_i' B_i^{-1})' \mathbf{E}_i (H_i' B_i^{-1}) \\
F_i &= T_i \cdot \left((\mathbf{E}_i H_i' B_i^{-1})^{b'} (\mathbf{E}_i^{bb})^{-1} (\tilde{\mathbf{y}}_{1i}^b - (\mathbf{E}_i V^{-1} \tilde{X}_{1i} \beta_1)^b) + (H_i' B_i^{-1})' \mathbf{E}_i (V_i^{-1} \tilde{X}_{1i} \beta) \right) \\
G_i &= \left(\tilde{\mathbf{y}}_{1i}^b - (\mathbf{E}_i V_i^{-1} \tilde{X}_{1i} \beta_1)^b \right)' (\mathbf{E}_i^{bb})^{-1} \left(\tilde{\mathbf{y}}_{1i}^b - (\mathbf{E}_i V^{-1} \tilde{X}_{1i} \beta_1)^b \right) - F_i' T_i^{-1} F_i + \\
&\quad (\tilde{X}_{1i} \beta)' V_i^{-1} (\tilde{X}_{1i} \beta) - (V_i^{-1} \tilde{X}_{1i} \beta)' \mathbf{E}_i (V_i^{-1} \tilde{X}_{1i} \beta).
\end{aligned}$$

Integrating over s results in

$$\begin{aligned}
E[\tilde{\mathbf{Y}}_{1i}^a | \tilde{\mathbf{y}}_{1i}^b = \tilde{\mathbf{y}}_{1i}^b, \tilde{\mathbf{y}}_{2i} = 1] &= \frac{e^{-0.5G_i}}{c(2\pi)^{\frac{n_b}{2}}} \frac{\sqrt{|\mathbf{E}_i||T_i|}}{\sqrt{|V_i||B_i||\mathbf{E}_i^{bb}|}} \Phi(\tilde{X}_{2i}\beta + H\tilde{X}_{1i}\beta, F_i, T_i) \\
&\quad \left\{ \left((\mathbf{E}_i V^{-1} \tilde{X}_{1i} \beta_1)^a + E^{ab} (E^{bb})^{-1} (\tilde{\mathbf{y}}_{1i}^b - (\mathbf{E}_i V^{-1} \tilde{X}_{1i} \beta_1)^b) \right) \right. \\
&\quad + \left. \left((EH' B^{-1})^a - E^{ab} (E^{bb})^{-1} (\mathbf{E}_i H B_i^{-1})^b \right) \right. \\
&\quad \times \left. \left(T_i [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)] + F_i \right) \right\},
\end{aligned}$$

where $o_i F_i(x_i)$ and $\varphi(x_i)$ are defined in (18).

(11) is conditional on a sequence of successes $(1,1,\dots,1)'$. The expected value conditional on a arbitrary vector of failures and successes can be derived with appropriate contrasts. Given an arbitrary vector of failures and successes, the conditional expected value equals

$$\begin{aligned} E[\tilde{\mathbf{Y}}_{1i}^a | \tilde{\mathbf{Y}}_{1i}^b = \tilde{\mathbf{y}}_{1i}^b, \tilde{\mathbf{y}}_{2i}] &= \frac{e^{-0.5G_i}}{c(2\pi)^{\frac{n_b}{2}}} \frac{\sqrt{|\mathbf{E}_i||T_i|}}{\sqrt{|V_i||B_i||E_i^{bb}|}} \Phi(\tilde{X}_{2i}\beta^* + H\tilde{X}_{1i}\beta, F_i, T_i) \\ &\quad \left\{ \left((\mathbf{E}_i V^{-1} \tilde{X}_{1i} \beta_1)^a + E^{ab} (E^{bb})^{-1} (\tilde{\mathbf{y}}_{1i}^b - (\mathbf{E}_i V^{-1} \tilde{X}_{1i} \beta_1)^b) \right) \right. \\ &\quad + \left. \left((EH'B^{-1})^a - E^{ab} (E^{bb})^{-1} (\mathbf{E}_i H' B_i^{-1})^b \right) \right. \\ &\quad \times \left. \left(T_i [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)] + F_i \right) \right\}, \end{aligned}$$

with

$$\begin{aligned} \tilde{X}_{2i}\beta^* &= s \diamond \tilde{X}_{2i}\beta \\ \tilde{Z}_{2i}^* &= J \diamond Z_{2i}, \end{aligned}$$

where \diamond indicates pairwise multiplication and s and J are defined as in (16) and (17) respectively.

D.2 Prediction interval

The prediction interval of a subvector of continuous responses conditional on another subvector of continuous responses and a subvector of binary responses is defined analogously to Appendix B.2. We will first derive the second central moment of the conditional distribution and next derive the standard errors of the transformed parameters.

The second central moment of (11) equals

$$\begin{aligned}
& E \left[(\tilde{Y}_{1i}^a - E[\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1]) (\tilde{Y}_{1i}^a - E[\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1])' \right] \\
&= \int_{\tilde{y}_{1i}^a = -\infty}^{\tilde{y}_{1i}^a = \infty} \left[\tilde{y}_{1i}^a - E[\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1] \right] \left[\tilde{y}_{1i}^a - E[\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1] \right]' \cdot f(\tilde{y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1) d\tilde{y}_{1i}^a \\
&= \int_{\tilde{y}_{1i}^a = -\infty}^{\tilde{y}_{1i}^a = +\infty} \tilde{y}_{1i} \tilde{y}_{1i}' f(\tilde{y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1) \\
&\quad - E[\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1] E[\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1]' f(\tilde{y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1) \\
&\quad - \left(\tilde{y}_{1i}^a'E[\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1] + E[\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1] \tilde{y}_{1i}^a' \right) \cdot f(\tilde{y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1) d\tilde{y}_{1i}^a \\
&= \int_{\tilde{y}_{1i}^a = -\infty}^{\tilde{y}_{1i}^a = +\infty} \tilde{y}_{1i}^a \tilde{y}_{1i}' f(\tilde{y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1) d\tilde{y}_{1i}^a - E[\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1] E[\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1]' \\
&= \frac{1}{c} \int_{\tilde{y}_{1i}^a = -\infty}^{\tilde{y}_{1i}^a = \infty} \int_{t=-\infty}^{t=\tilde{X}_{2i}\beta-H} \frac{\begin{bmatrix} \tilde{y}_{1i}^a \\ \tilde{y}_{1i}^b \end{bmatrix} + H\tilde{X}_{1i}\beta}{(2\pi)^{\frac{(n_a+n_b+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|V_i||B_i|}} \tilde{y}_{1i}^a \tilde{y}_{1i}' \\
&\quad \times \exp \left\{ -\frac{1}{2} \left(\left(\begin{bmatrix} \tilde{y}_{1i}^a \\ \tilde{y}_{1i}^b \end{bmatrix} - \tilde{X}_{1i}\beta \right)' V_i^{-1} \left(\begin{bmatrix} \tilde{y}_{1i}^a \\ \tilde{y}_{1i}^b \end{bmatrix} - \tilde{X}_{1i}\beta \right) + t'B_i^{-1}t \right) \right\} d\tilde{y}_{1i}^a dt \\
&\quad - E[\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1] E[\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1]' \\
&= \frac{1}{c} \int_{\tilde{y}_{1i}^a = -\infty}^{\tilde{y}_{1i}^a = \infty} \int_{s=-\infty}^{s=\tilde{X}_{2i}\beta+H\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{(n_a+n_b+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|V_i||B_i|}} \tilde{y}_{1i}^a \tilde{y}_{1i}' \\
&\quad \times \exp \left\{ -\frac{1}{2} \left(\left(\begin{bmatrix} \tilde{y}_{1i}^a \\ \tilde{y}_{1i}^b \end{bmatrix} - \tilde{X}_{1i}\beta \right)' V_i^{-1} \left(\begin{bmatrix} \tilde{y}_{1i}^a \\ \tilde{y}_{1i}^b \end{bmatrix} - \tilde{X}_{1i}\beta \right) + (s - H \begin{bmatrix} \tilde{y}_{1i}^a \\ \tilde{y}_{1i}^b \end{bmatrix})' B_i^{-1} (s - H \begin{bmatrix} \tilde{y}_{1i}^a \\ \tilde{y}_{1i}^b \end{bmatrix}) \right) \right\} d\tilde{y}_{1i}^a ds \\
&\quad - E[\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1] E[\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1]' \\
&= \frac{1}{c} \int_{\tilde{y}_{1i}^a = -\infty}^{\tilde{y}_{1i}^a = \infty} \int_{s=-\infty}^{s=\tilde{X}_{2i}\beta+H\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{(n_a+n_b+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|V_i||B_i|}} \tilde{y}_{1i}^a \tilde{y}_{1i}' \\
&\quad \exp \left\{ -\frac{1}{2} \left(\left(\begin{bmatrix} \tilde{y}_{1i}^a \\ \tilde{y}_{1i}^b \end{bmatrix} - u_i \right)' E_i^{-1} \left(\begin{bmatrix} \tilde{y}_{1i}^a \\ \tilde{y}_{1i}^b \end{bmatrix} - u_i \right) + O_i \right) \right\} d\tilde{y}_{1i}^a ds - E[\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1] E[\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1]',
\end{aligned}$$

where

$$\begin{aligned}
c &= f(\tilde{y}_{1i}^b, \tilde{y}_{2i}), \text{ the marginal joint distribution} \\
s &= t - H\tilde{y}_{1i}.
\end{aligned}$$

Further,

$$\left(\begin{bmatrix} \tilde{y}_{1i}^a \\ \tilde{y}_{1i}^b \end{bmatrix} - \tilde{X}_{1i}\beta \right)' V_i^{-1} \left(\begin{bmatrix} \tilde{y}_{1i}^a \\ \tilde{y}_{1i}^b \end{bmatrix} - \tilde{X}_{1i}\beta \right) + (s - H \begin{bmatrix} \tilde{y}_{1i}^a \\ \tilde{y}_{1i}^b \end{bmatrix})' B_i^{-1} (s - H \begin{bmatrix} \tilde{y}_{1i}^a \\ \tilde{y}_{1i}^b \end{bmatrix}) = \left(\begin{bmatrix} \tilde{y}_{1i}^a \\ \tilde{y}_{1i}^b \end{bmatrix} - u_i \right)' E_i^{-1} \left(\begin{bmatrix} \tilde{y}_{1i}^a \\ \tilde{y}_{1i}^b \end{bmatrix} - u_i \right) + O_i,$$

with

$$\begin{aligned}
E_i^{-1} &= H'B_i^{-1}H + V_i^{-1} \\
l'_i &= -s'B_i^{-1}H - \tilde{X}_{1i}\beta'_1V_i^{-1} \\
O_i &= s'B_i^{-1}s + \tilde{X}_{1i}\beta'_1V_i^{-1}\tilde{X}_{1i}\beta_1 - (-H'B_i^{-1}s - V_i^{-1}\tilde{X}_{1i}\beta_1)'E(-H'B_i^{-1}s - V_i^{-1}\tilde{X}_{1i}\beta_1) \\
u_i &= -E_il_i.
\end{aligned}$$

Applying the formulas of the conditional mean and variance in the multivariate normal distribution, integration over $\tilde{\mathbf{y}}_{1i}^a$ results in

$$\begin{aligned}
&E\left[\left(\tilde{\mathbf{Y}}_{1i}^a - E[\tilde{\mathbf{Y}}_{1i}^a|\tilde{\mathbf{y}}_{1i}^b, \tilde{\mathbf{y}}_{2i} = 1]\right)\left(\tilde{\mathbf{Y}}_{1i}^a - E[\tilde{\mathbf{Y}}_{1i}^a|\tilde{\mathbf{y}}_{1i}^b, \tilde{\mathbf{y}}_{2i} = 1]\right)'\right] \\
&= \frac{1}{c} \int_{\mathbf{s}=-\infty}^{\mathbf{s}=\tilde{X}_{2i}\beta+H\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{\tilde{p}}{2}}} \frac{\sqrt{|E_i|}}{\sqrt{|V_i||B_i|}} \left\{ E_i^{aa} - E_i^{ab}(E_i^{bb})^{-1}E_i^{ba} + \left(u_i^a + E_i^{ab}(E_i^{bb})^{-1}(\tilde{\mathbf{y}}_{1i}^b - u_i^b)\right)' \right. \\
&\quad \times \left. \left(u_i^a + E_i^{ab}(E_i^{bb})^{-1}(\tilde{\mathbf{y}}_{1i}^b - u_i^b)\right) \right\} \phi(\tilde{\mathbf{y}}_{1i}^b, u_i^b, E_i^{bb}) \exp\left\{-\frac{1}{2}O_i\right\} d\mathbf{s} \\
&\quad - E[\tilde{\mathbf{Y}}_{1i}^a|\tilde{\mathbf{y}}_{1i}^b, \tilde{\mathbf{y}}_{2i} = 1] E[\tilde{\mathbf{Y}}_{1i}^a|\tilde{\mathbf{y}}_{1i}^b, \tilde{\mathbf{y}}_{2i} = 1]' \\
&= \frac{1}{c} \int_{\mathbf{s}=-\infty}^{\mathbf{s}=\tilde{X}_{2i}\beta+H\tilde{X}_{1i}\beta} \frac{1}{(2\pi)^{\frac{n_b+\tilde{p}}{2}}} \frac{\sqrt{|E_i|}}{\sqrt{|V_i||B_i|}} \left\{ E_i^{aa} - E_i^{ab}(E_i^{bb})^{-1}E_i^{ba} + \left(u_i^a + E_i^{ab}(E_i^{bb})^{-1}(\tilde{\mathbf{y}}_{1i}^b - u_i^b)\right)' \right. \\
&\quad \times \left. \left(u_i^a + E_i^{ab}(E_i^{bb})^{-1}(\tilde{\mathbf{y}}_{1i}^b - u_i^b)\right) \right\} \exp\left\{-\frac{1}{2}\left((\mathbf{s} - F_i)'T_i^{-1}(\mathbf{s} - F_i) + G_i\right)\right\} d\mathbf{s} \\
&\quad - E[\tilde{\mathbf{Y}}_{1i}^a|\tilde{\mathbf{y}}_{1i}^b, \tilde{\mathbf{y}}_{2i} = 1] E[\tilde{\mathbf{Y}}_{1i}^a|\tilde{\mathbf{y}}_{1i}^b, \tilde{\mathbf{y}}_{2i} = 1].
\end{aligned}$$

Here we have rewritten $\phi(\tilde{\mathbf{y}}_{1i}^b, u_i^b, E_i^{bb}) \exp\left\{-\frac{1}{2}O_i\right\} = \exp\left\{-\frac{1}{2}\left((\mathbf{s} - F_i)'T_i^{-1}(\mathbf{s} - F_i) + G_i\right)\right\}$, with

$$\begin{aligned}
T_i^{-1} &= (E_iH'B_i^{-1})'^b(E_i^{bb})^{-1}(E_iH'B_i^{-1})^b + B_i^{-1} - (H'B_i^{-1})'E_i(H'B_i^{-1}) \\
F_i &= T_i \cdot \left((E_iH'B_i^{-1})'^b(E_i^{bb})^{-1}(\tilde{\mathbf{y}}_{1i}^b - (E_iV_i^{-1}\tilde{X}_{1i}\beta_1)^b) + (H'B_i^{-1})'E_i(V_i^{-1}\tilde{X}_{1i}\beta) \right) \\
G_i &= \left(\tilde{\mathbf{y}}_{1i}^b - (E_iV_i^{-1}\tilde{X}_{1i}\beta_1)^b \right)'(E_i^{bb})^{-1}\left(\tilde{\mathbf{y}}_{1i}^b - (E_iV_i^{-1}\tilde{X}_{1i}\beta_1)^b \right) - F_i'T_i^{-1}F_i + \\
&\quad (\tilde{X}_{1i}\beta)'V_i^{-1}(\tilde{X}_{1i}\beta) - (V_i^{-1}\tilde{X}_{1i}\beta)'E_i(V_i^{-1}\tilde{X}_{1i}\beta).
\end{aligned}$$

Integrating over s results in

$$\begin{aligned}
& E \left[(\tilde{\mathbf{Y}}_{1i}^a - E[\tilde{\mathbf{Y}}_{1i}^a | \tilde{\mathbf{y}}_{1i}^b, \tilde{\mathbf{y}}_{2i} = 1])^2 \right] \\
&= \nu (E_i^{aa} - E_i^{ab} (E_i^{bb})^{-1} E_i^{ba}) + \nu \left\{ (E_i H' B_i^{-1})^a (N + J J') (E_i H' B_i^{-1})^{a'} \right. \\
&\quad + (E_i H' B_i^{-1})^a J ((\tilde{X}_{1i} \beta_1)' V_i^{-1} E_i)^a + (E_i V_i^{-1} \tilde{X}_{1i} \beta_1)^a J' (E_i H' B_i^{-1})^{a'} + (E_i V_i^{-1} \tilde{X}_{1i} \beta_1)^a (E_i V_i^{-1} \tilde{X}_{1i} \beta_1)^{a'} \Big\} \\
&\quad + \nu \left\{ (E_i H' B_i^{-1})^a J \left(\tilde{\mathbf{y}}_{1i}^b - ((\tilde{X}_{1i} \beta_1)' V_i^{-1} E_i)^b \right) - (E_i H' B_i^{-1})^a (N + J J') (E_i H' B_i^{-1})^{b'} \right. \\
&\quad + (E_i V_i^{-1} \tilde{X}_{1i} \beta_1)^a \left(\tilde{\mathbf{y}}_{1i}^b - ((\tilde{X}_{1i} \beta_1)' V_i^{-1} E_i)^b \right)' - (E_i V_i^{-1} \tilde{X}_{1i} \beta_1)^a J' (E_i H' B_i^{-1})^{b'} \Big\} (E_i^{bb})^{-1} E_i^{ba} \\
&\quad + \nu E_i^{ab} (E_i^{bb})^{-1} \left\{ \left(\tilde{\mathbf{y}}_{1i}^b - E_i V_i^{-1} \tilde{X}_{1i} \beta_1 \right)^b ((\tilde{X}_{1i} \beta_1)' V_i E_i)^a - (E_i H' B_i^{-1})^b J ((\tilde{X}_{1i} \beta_1)' V_i E_i)^a \right. \\
&\quad + \left(\tilde{\mathbf{y}}_{1i}^b - (E_i V_i^{-1} \tilde{X}_{1i} \beta_1)^b \right) J' (E_i H' B_i^{-1})^{a'} - (E_i H' B_i^{-1})^b (N + J J') (E_i H' B_i^{-1})^{a'} \Big\} \\
&\quad + \nu E_i^{ab} (E_i^{bb})^{-1} \left\{ (E_i H' B_i^{-1})^b (N + J J') (E_i H' B_i^{-1})^{b'} - (E_i H' B_i^{-1})^b J \left(\tilde{\mathbf{y}}_{1i}^b - (E_i V_i^{-1} \tilde{X}_{1i} \beta_1)^b \right)' \right. \\
&\quad - \left(\tilde{\mathbf{y}}_{1i}^b - (E_i V_i^{-1} \tilde{X}_{1i} \beta_1)^b \right) J' (E_i H' B_i^{-1})^{b'} + \left(\tilde{\mathbf{y}}_{1i}^b - (E_i V_i^{-1} \tilde{X}_{1i} \beta_1)^b \right) \left(\tilde{\mathbf{y}}_{1i}^b - (E_i V_i^{-1} \tilde{X}_{1i} \beta_1)^b \right)' \Big\} \\
&\quad \times (E_i^{bb})^{-1} E_i^{ba} - E[\tilde{\mathbf{Y}}_{1i}^a | \tilde{\mathbf{y}}_{1i}^b, \tilde{\mathbf{y}}_{2i} = 1] E[\tilde{\mathbf{Y}}_{1i}^a | \tilde{\mathbf{y}}_{1i}^b, \tilde{\mathbf{y}}_{2i} = 1]' ,
\end{aligned}$$

with J as the expected value of the truncated multivariate normal density, and N is the second central moment of the latter density. These are defined in (19).

The standard errors of the transformed parameters are derived analogously to Appendix B.2 with the delta method. The derivative of the expected value with respect to β_{12} , and arbitrary coefficient of

a predictor of the continuous response X_{12} is the following

$$\begin{aligned}
\frac{\mathbb{E}(\tilde{Y}_{1i}^a | \tilde{y}_{1i}^b, \tilde{y}_{2i} = 1)}{\partial \beta_{12}} &= \frac{1}{c^2(2\pi)^{\frac{n_b}{2}}} \frac{\sqrt{|E_i||T_i|}}{\sqrt{|V_i||B_i||E_i^{bb}|}} \left[c\Lambda\Phi(\tilde{X}_{2i}\beta + H\tilde{X}_{1i}\beta, F_i, T_i) \right. \\
&\times \left\{ \left((EV^{-1}\tilde{X}_{1i}\beta_1)^a + E_i^{ab}(E_i^{bb})^{-1}(\tilde{y}_{1i}^b - (EV^{-1}\tilde{X}_{1i}\beta_1)^b) \right) \right. \\
&+ \left((E_iH'B_i^{-1})^a - E^{ab}(E^{bb})^{-1}(E_iH'B_i^{-1})^b \right) \\
&\left. \left(T_i[-F_1(a_1) - F_2(a_2) \dots - F_p(a_p)] + F_i \right) \right\} \\
&+ ce^{-0.5G}\Theta \left\{ \left((EV^{-1}\tilde{X}_{1i}\beta_1)^a + E_i^{ab}(E_i^{bb})^{-1}(\tilde{y}_{1i}^b - (EV^{-1}\tilde{X}_{1i}\beta_1)^b) \right) \right. \\
&+ \left((E_iH'B_i^{-1})^a - E^{ab}(E^{bb})^{-1}(E_iH'B_i^{-1})^b \right) F_i \left. \right\} \\
&+ ce^{-0.5G}\Phi(\tilde{X}_{2i}\beta + H\tilde{X}_{1i}\beta, F_i, T_i) \left\{ (E_iV_i^{-1})^a \tilde{X}_{12i} - E_i^{ab}(E_i^{bb})^{-1}(E_iV_i^{-1})^b \tilde{X}_{12i} \right\} \\
&+ ce^{-0.5G} \left\{ \left((E_iH'B_i^{-1})^a - E^{ab}(E^{bb})^{-1}(E_iH'B_i^{-1})^b \right) \nu \right\} \\
&- \lambda e^{-0.5G}\Phi(\tilde{X}_{2i}\beta + H\tilde{X}_{1i}\beta, F_i, T_i) \\
&\times \left\{ \left((EV^{-1}\tilde{X}_{1i}\beta_1)^a + E_i^{ab}(E_i^{bb})^{-1}(\tilde{y}_{1i}^b - (EV^{-1}\tilde{X}_{1i}\beta_1)^b) \right) \right. \\
&+ \left((E_iH'B_i^{-1})^a - E^{ab}(E^{bb})^{-1}(E_iH'B_i^{-1})^b \right) \\
&\left. \left(T_i[-F_1(a_1) - F_2(a_2) \dots - F_p(a_p)] + F_i \right) \right\},
\end{aligned} \tag{23}$$

with

$$\begin{aligned}
\Lambda &= -e^{-\frac{1}{2}G} \left\{ (\tilde{y}_{1i}^b - (EV^{-1}\tilde{X}_{1i}\beta_1)^b)'(E^{bb})^{-1}(EV^{-1}\tilde{X}_{12i})^b \right. \\
&+ \left. \tilde{X}_{1i}\beta_1'V^{-1}\tilde{X}_{12i} - (V^{-1}\tilde{X}_{1i}\beta_1)'E_iV^{-1}\tilde{X}_{12i} - F_iT_i^{-1}\delta_i' \right\} \\
\Theta &= \sum_{k=1}^{\tilde{p}} \Gamma_{kk}^{-1/2} (H_k X_{12i} - F_{ik}') \phi[(X_{2i}\beta + H X_{1i}\beta - F_i)_k, T_{kk}] \Phi^{\tilde{R}_{T_i}^{(k)}}(\tilde{z}^{(k)}),
\end{aligned}$$

with

$$\begin{aligned}
\delta'_i &= T_i \left(- (E_i H' B^{-1})^{bb} (E_i^{bb})^{-1} (E_i V^{-1})^b \tilde{X}_{12i} + (H' B_i^{-1})' E_i (V_i^{-1} \tilde{X}_{12i}) \right) \\
\Gamma &= \text{Diagonal matrix composed of the variances of } T_i \\
z &= \Gamma^{-1/2} (X_{2i}\beta + H X_{1i}\beta - F_i) \\
R_{T_i} &= \Gamma^{-1/2} T_i \Gamma^{-1/2} = (r_{ij})_{i,j=1}^{\tilde{p}_i} \\
\tilde{z}^{(i)} &= \left(\frac{z_1 - r_{1,i} z_i}{\sqrt{1 - r_{1,i}^2}}, \dots, \frac{z_{i-1} - r_{i-1,i} z_i}{\sqrt{1 - r_{i-1,i}^2}}, \frac{z_{i+1} - r_{i+1,i} z_i}{\sqrt{1 - r_{i+1,i}^2}}, \dots, \frac{z_{\tilde{p}_i} - r_{\tilde{p}_i,i} z_i}{\sqrt{1 - r_{\tilde{p}_i,i}^2}} \right).
\end{aligned}$$

The $(\tilde{p}_i - 1) \times (\tilde{p}_i - 1)$ -dimensional correlation matrix $\tilde{R}_{T_i}^{(i)}$ has entries

$$\tilde{r}_{j,k}^{(i)} = \frac{r_{j',k'} - r_{j',i} r_{k',i}}{\sqrt{1 - r_{j',i}^2} \sqrt{1 - r_{k',i}^2}} \quad i = 1 \dots \tilde{p}_i; \quad j, k = 1 \dots \tilde{p}_i - 1$$

with

$$j' = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases} \quad k' = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i. \end{cases}$$

Further,

$$\begin{aligned}
\nu &= \sum_{k=1}^{\tilde{p}_i} (\mathbf{H}_i \tilde{\mathbf{X}}_{12i} + \boldsymbol{\delta}_i)_k g_k(a_k) + \Phi(\tilde{\mathbf{X}}_{2i}\beta + \mathbf{H}_i \tilde{\mathbf{X}}_{1i}\beta, \mathbf{F}_i, \mathbf{T}_i) \boldsymbol{\delta}_{ik}, \\
a &= \tilde{X}_{2i}\beta + H \tilde{X}_{1i}\beta, \\
g_k(x_k) &= \int_{-\infty}^{a_1} \dots \int_{-\infty}^{a_{i-1}} \int_{-\infty}^{a_{i+1}} \dots \int_{-\infty}^{a_{\tilde{p}_i}} [x_1 \dots x_{k-1} a_k x_{k+1} \dots x_{\tilde{p}_i}]' \phi([x_1 \dots x_{k-1} a_k x_{k+1} \dots x_{\tilde{p}_i}]', T_i) dx_{-k}.
\end{aligned}$$

Next,

$$\begin{aligned}
\lambda &= \sum_{k=1}^{\tilde{p}} \gamma_{kk}^{-1/2} H_k^* X_{12i}^b \phi[(X_{2i}\beta - H_i^* \tilde{\mathbf{y}}_{1i}^b + H_i^* X_{1i}^b \beta)_k, B_{kk}] \Phi^{\tilde{R}_{B_i^*}^{(k)}}(\tilde{d}^{(k)}) \phi(\tilde{\mathbf{y}}_{1i}^b, X_{1i}^b \beta, V_i^{bb}) \\
&\quad + \Phi(X_{2i}\beta - H^* \tilde{\mathbf{y}}_{1i}^b + H^* X_{1i}^b \beta, B_i^*) (\tilde{\mathbf{y}}_{1i}^b - X_{1i}^b \beta)' (V_i^{bb})^{-1} X_{12i}^b \phi(\tilde{\mathbf{y}}_{1i}^b, X_{1i}^b \beta, V_i^{bb})
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
B_i^{*-1} &= I - Z_{2i} K_i^* Z_{2i}', \\
K_i^{*-1} &= D^{-1} + Z_{1i}' (\Sigma_i^{bb})^{-1} Z_{1i}^b + Z_{2i}' Z_{2i}, \\
H_i^* &= -B_i^* Z_{2i} K_i^* Z_{1i}' (\Sigma_i^{bb})^{-1}. \\
\gamma &= \text{Diagonal matrix composed of the variances of } B_i^* \\
d &= \gamma^{-1/2} (X_{2i} \beta - H_i^* \tilde{\mathbf{y}}_{1i}^b + H_i^* X_{1i}^b \beta) \\
R_{B_i^*} &= \gamma^{-1/2} B_i^* \gamma^{-1/2} = (r_{ij})_{i,j=1}^{\tilde{p}_i} \\
\tilde{d}^{(i)} &= \left(\frac{d_1 - r_{1,i} d_i}{\sqrt{1 - r_{1,i}^2}}, \dots, \frac{d_{i-1} - r_{i-1,i} d_i}{\sqrt{1 - r_{i-1,i}^2}}, \frac{d_{i+1} - r_{i+1,i} d_i}{\sqrt{1 - r_{i+1,i}^2}}, \dots, \frac{d_{\tilde{p}_i} - r_{\tilde{p}_i,i} d_i}{\sqrt{1 - r_{\tilde{p}_i,i}^2}} \right).
\end{aligned}$$

The $(\tilde{p}_i - 1) \times (\tilde{p}_i - 1)$ -dimensional correlation matrix $\tilde{R}_{B_i^*}^{(i)}$ has entries

$$\tilde{r}_{j,k}^{(i)} = \frac{r_{j',k'} - r_{j',i} r_{k',i}}{\sqrt{1 - r_{j',i}^2} \sqrt{1 - r_{k',i}^2}} \quad i = 1 \dots \tilde{p}_i; \quad j, k = 1 \dots \tilde{p}_i - 1$$

with

$$j' = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases} \quad k' = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i. \end{cases}$$

The derivative of the expected value with respect to a coefficient β_{22} of a predictor X_{22} of the binary response

$$\begin{aligned}
\frac{\mathbb{E}(\tilde{\mathbf{Y}}_{1i}^a | \tilde{\mathbf{y}}_{1i}^b, \tilde{\mathbf{y}}_{2i} = \mathbf{1})}{\partial \beta_{22}} &= \frac{e^{-0.5G_i}}{c^2 (2\pi)^{\frac{n_b}{2}}} \frac{\sqrt{|E_i||T_i|}}{\sqrt{|V_i||B_i||E_i^{bb}|}} \left[c\omega \left\{ \left((E_i V^{-1} \tilde{X}_{1i} \beta_1)^a + \right. \right. \right. \\
&\quad \left. \left. \left. E_i^{ab} (E_i^{bb})^{-1} (\tilde{\mathbf{y}}_{1i}^b - (E_i V^{-1} \tilde{X}_{1i} \beta_1)^b) \right) + \left((E_i H' B_i^{-1})^a - E_i^{ab} (E_i^{bb})^{-1} (E_i H' B_i^{-1})^b \right) \right. \\
&\quad \times \left. \left(T_i [-F_1(a_1) \quad -F_2(a_2) \quad \dots \quad -F_p(a_p)] + F_i \right) \right\} \\
&+ c \left\{ \left((E_i H' B_i^{-1})^a - E_i^{ab} (E_i^{bb})^{-1} (E_i H' B_i^{-1})^b \right) \zeta \right\} \\
&- \Omega \Phi(\tilde{X}_{2i} \beta + H \tilde{X}_{1i} \beta, F_i, T_i) \left\{ \left((E_i V^{-1} \tilde{X}_{1i} \beta_1)^a + E_i^{ab} (E_i^{bb})^{-1} (\tilde{\mathbf{y}}_{1i}^b - (E_i V^{-1} \tilde{X}_{1i} \beta_1)^b) \right) \right. \\
&+ \left. \left((E_i H' B_i^{-1})^a - E_i^{ab} (E_i^{bb})^{-1} (E_i H' B_i^{-1})^b \right) \right. \\
&\quad \left. \left(T_i [-F_1(a_1) \quad -F_2(a_2) \quad \dots \quad -F_p(a_p)] + F_i \right) \right\} \right], \tag{25}
\end{aligned}$$

with

$$\begin{aligned}
\zeta &= \sum_{k=1}^{\tilde{p}_i} \tilde{X}'_{22ik} g_k(o_k) \\
\Omega &= \phi(\tilde{\mathbf{y}}_{1i}^b, X_{1i}^b \beta; V_i^{bb}) \sum_{k=1}^{\tilde{p}_i} \gamma_{kk}^{-1/2} X_{22ik} \phi[(X_{2i} \beta - H_i^* \tilde{\mathbf{y}}_{1i}^b + H_i^* X_{1i}^b \beta)_k, B_{kk}] \Phi^{\tilde{R}_{B_i}^{(k)}}(\tilde{d}^{(k)}), \\
\omega &= \sum_{k=1}^{\tilde{p}} \Gamma_{kk}^{-1/2} X_{22ik} \phi(z_{1k}) \Phi^{\tilde{R}_{T_i}^{(k)}}(\tilde{z}_1^{(k)}),
\end{aligned}$$

(24), (24), (26) and (26) have been derived from Prékopa (1995) p.204.

E Conditional distribution of a subvector of the binary response given a subvector of the continuous response and a subvector of the binary response

E.1 Expected value

Let us derive (12), the conditional probability for the p_a -dimensional subvector of successes $\tilde{\mathbf{Y}}_{2i}^a$ given a subvector of the \tilde{n}_i -dimensional subvector of continuous responses $\tilde{\mathbf{Y}}_{1i}$ and a p_b -dimensional subvector of successes $\tilde{\mathbf{Y}}_{2i}^b$. In the following calculations, superscripts will indicate subvectors and submatrices. The superscript a denotes the rows a_1 until a_{p_a} and the superscript b denotes the rows b_1 until p_b . Analogously, the superscript bb denotes the submatrix with rows b_1 until p_b and columns b_1 until p_b . The superscript ab denotes the submatrix with rows a_1 until a_{p_a} and columns b_1 until p_b .

Since conditional probability is equal to the conditional density, the conditional probability equals the joint distribution of $\tilde{\mathbf{Y}}_{2i} = [\tilde{\mathbf{Y}}_{2i}^a, \tilde{\mathbf{Y}}_{2i}^b]$ and $\tilde{\mathbf{Y}}_{1i}$ divided by the joint distribution of $\tilde{\mathbf{Y}}_{1i}$ and $\tilde{\mathbf{Y}}_{2i}^b$.

$$\begin{aligned} f(\tilde{\mathbf{Y}}_{2i}^a = \mathbf{1} | \tilde{\mathbf{Y}}_{1i}, \tilde{\mathbf{y}}_{2i}^b = \mathbf{1}) &= \frac{\phi(\tilde{\mathbf{X}}_{1i}\beta; V_i)\Phi(\tilde{\mathbf{X}}_{2i}\beta - H_i(\tilde{\mathbf{Y}}_{1i} - \tilde{\mathbf{X}}_{1i}\beta); B_i)}{\phi(\tilde{\mathbf{X}}_{1i}\beta; V_i)\Phi(\tilde{\mathbf{X}}_{2i}^b\beta - H_i^b(\tilde{\mathbf{Y}}_{1i} - \tilde{\mathbf{X}}_{1i}\beta); B_i^{bb})} \\ &= \frac{\Phi(\tilde{\mathbf{X}}_{2i}\beta - H_i(\tilde{\mathbf{Y}}_{1i} - \tilde{\mathbf{X}}_{1i}\beta); B_i)}{\Phi(\tilde{\mathbf{X}}_{2i}^b\beta - H_i^b(\tilde{\mathbf{Y}}_{1i} - \tilde{\mathbf{X}}_{1i}\beta); B_i^{bb})}, \end{aligned}$$

where $\tilde{\mathbf{X}}_{2i}\beta = [\tilde{\mathbf{X}}_{2i}^a\beta, \tilde{\mathbf{X}}_{2i}^b\beta]$ and $\tilde{p}_i = p_a + p_b$.

E.2 Confidence interval

In order to construct the confidence interval, the standard errors of the transformed parameters can be computed by the delta method. First the gradient of a coefficient β_{12} of a predictor of the continuous response X_{12} is derived:

$$\begin{aligned} \frac{f(\tilde{\mathbf{Y}}_{2i}^a = \mathbf{1} | \tilde{\mathbf{y}}_{1i}, \tilde{\mathbf{y}}_{2i}^b = \mathbf{1})}{\partial \beta_{12}} &= \left(\sum_{k=1}^{\tilde{p}_i} \gamma_{kk}^{-1/2} H_k X_{12i} \phi(d_k) \Phi^{\tilde{R}_{B_i}^{(k)}}(\tilde{d}^{(k)}) \Phi(\tilde{\mathbf{X}}_{2i}^b\beta - H^b(\tilde{\mathbf{Y}}_{1i} - \tilde{\mathbf{X}}_{1i}\beta); B_i^{bb}) \right. \\ &\quad \left. - \sum_{k=1}^{\tilde{p}_i^b} \gamma_{kk}^{-1/2} H_k^b X_{12i} \phi(d_k) \Phi^{\tilde{R}_{B_i^{bb}}^{(k)}}(\tilde{d}^{(k)}) \Phi(\tilde{\mathbf{X}}_{2i}\beta - H^b(\tilde{\mathbf{Y}}_{1i} - \tilde{\mathbf{X}}_{1i}\beta); B_i) \right) \\ &\quad \times \left(\Phi(\tilde{\mathbf{X}}_{2i}^b\beta - H^b(\tilde{\mathbf{Y}}_{1i} - \tilde{\mathbf{X}}_{1i}\beta); B_i^{bb}) \right)^{-2}, \end{aligned}$$

where γ , d , R_{B_i} , $\tilde{d}^{(i)}$ and $\tilde{R}_{B_i}^{(i)}$ are defined in (21).

Next, the gradient of a coefficient β_{22} of one of the predictors of the binary response X_{22} is defined

as

$$\begin{aligned}
\frac{f(\tilde{\mathbf{Y}}_{2i}^a = \mathbf{1} | \tilde{\mathbf{y}}_{1i}, \tilde{\mathbf{y}}_{2i}^b = \mathbf{1})}{\partial \beta_{22}} &= \left(\sum_{k=1}^{\tilde{p}_i} \gamma_{kk}^{-1/2} X_{22ik} \phi(d_k) \Phi^{\tilde{R}_{B_i}^{(k)}}(\tilde{d}^{(k)}) \Phi(\tilde{X}_{2i}^b \boldsymbol{\beta} - H^b(\tilde{\mathbf{Y}}_{1i} - \tilde{X}_{1i} \boldsymbol{\beta}); R_{\tau}^{bb}) \right. \\
&- \left. \sum_{k=1}^{\tilde{p}_i^b} \gamma_{kk}^{-1/2} X_{22ik}^b \phi(d_k) \Phi^{\tilde{R}_{B_i}^{(k)}}(\tilde{d}^{(k)}) \Phi(\tilde{X}_{2i}^b \boldsymbol{\beta} - H(\tilde{\mathbf{Y}}_{1i} - \tilde{X}_{1i} \boldsymbol{\beta}); B_i) \right) \\
&\times \left(\Phi(\tilde{X}_{2i}^b \boldsymbol{\beta} - H^b(\tilde{\mathbf{Y}}_{1i} - \tilde{X}_{1i} \boldsymbol{\beta}); B_i^{bb}) \right)^{-2},
\end{aligned}$$

where (26) and (27) have been derived from Prékopa (1995) p.204.

F Computation of the correlation of single responses

In this section (13) will be derived. Because the elements of \mathbf{Y}_{1i} and \mathbf{Y}_{2i} are independent conditional on the random effects, we have for the expected value of the product:

$$\begin{aligned}
& E[Y_{1ij}Y_{2ik}] \\
&= E_{\mathbf{b}_i}[E(Y_{1ij}, Y_{2ik}|\mathbf{b}_i)] \\
&= E_{\mathbf{b}_i}[E(Y_{1ij}|\mathbf{b}_i)E(Y_{2ik}|\mathbf{b}_i)] \\
&= E_{\mathbf{b}_i}[(X'_{1ij}\beta + Z'_{1ij}\mathbf{b}_i) \cdot \Phi(X'_{2ij}\beta + Z_{2ij}\mathbf{b}_i)] \\
&= \left(\int_{-\infty}^{+\infty} \right)^q \int_{t=-\infty}^{t=X'_{2ik}\beta + Z'_{2ik}\mathbf{b}_i} \frac{1}{(2\pi)^{q/2}|D|^{1/2}\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}(\mathbf{b}'_i D^{-1} \mathbf{b}_i + t^2) \right\} (X'_{1ij}\beta + Z'_{1ij}\mathbf{b}_i) d\mathbf{b}_i dt \\
&= \left(\int_{-\infty}^{+\infty} \right)^q \int_{s=-\infty}^{s=X'_{2ik}\beta} \frac{1}{(2\pi)^{q/2}|D|^{1/2}\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}(\mathbf{b}'_i D^{-1} \mathbf{b}_i + (s + z'_{2ik}\mathbf{b}_i)^2) \right\} (X'_{1ij}\beta + Z'_{1ij}\mathbf{b}_i) d\mathbf{b}_i ds \\
&= \left(\int_{-\infty}^{+\infty} \right)^q \int_{s=-\infty}^{s=X'_{2ik}\beta} \frac{1}{(2\pi)^{q/2}|D|^{1/2}\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left(\frac{s}{(1 - (Z'_{2ik}(D^{-1} + Z_{2ik}Z'_{2ik})^{-1}Z_{2ik}))^{-1/2}} \right)^2 \right\} \\
&\quad \times \exp\left\{ -\frac{1}{2}((\mathbf{b}_i - k)'[(D^{-1} + Z_{2ik}Z'_{2ik})^{-1}]^{-1}(\mathbf{b}_i - k) \right\} (X'_{1ij}\beta + Z'_{1ij}\mathbf{b}_i) d\mathbf{b}_i ds. \tag{28}
\end{aligned}$$

Here, we have rewritten the variable $t = s + z'_{2ik}\mathbf{b}_i$ and $\mathbf{b}'_i D^{-1} \mathbf{b}_i + t^2 = (\mathbf{b}_i - k)'[(D^{-1} + Z_{2ik}Z'_{2ik})^{-1}]^{-1}(\mathbf{b}_i - k) + \frac{s}{(1 - (Z'_{2ik}(D^{-1} + Z_{2ik}Z'_{2ik})^{-1}Z_{2ik}))^{-1/2}})^2$, where

$$k = -(D^{-1} + Z'_{2ik}Z_{2ik})^{-1}Z'_{2ik}s$$

Next, we integrate over the random effects, which results in

$$\begin{aligned}
& E[Y_{1ij}Y_{2ik}] \\
&= \int_{s=-\infty}^{s=X'_{2ik}\beta} \exp\left\{ -\frac{1}{2} \left(\frac{s}{(1 - (Z'_{2ik}(D^{-1} + Z_{2ik}Z'_{2ik})^{-1}Z_{2ik}))^{-1/2}} \right)^2 \right\} \\
&\quad \times \left(\frac{1}{|D|^{1/2}\sqrt{2\pi}} X'_{1ij}\beta \frac{1}{|D^{-1} + Z_{2ik}Z'_{2ik}|^{1/2}} + Z'_{1ij} \frac{1}{|D|^{1/2}\sqrt{2\pi}} \frac{1}{|D^{-1} + Z_{2ik}Z'_{2ik}|^{1/2}} k \right) ds \\
&= \int_{u=-\infty}^{u=\frac{X'_{2ik}\beta}{L^{-1/2}}} L^{-1/2} \exp\left\{ -\frac{1}{2}u^2 \right\} \left(\frac{1}{|D|^{1/2}\sqrt{2\pi}} X'_{1ij}\beta \frac{1}{|M|^{1/2}} \right) du \\
&\quad + \int_{u=-\infty}^{u=\frac{X'_{2ik}\beta}{L^{-1/2}}} L^{-1} u \exp\left\{ -\frac{1}{2}u^2 \right\} \left(Z'_{1ij} \frac{1}{|D|^{1/2}\sqrt{2\pi}} \frac{1}{|M|^{1/2}} (-M^{-1}Z_{2ik}) \right) du
\end{aligned}$$

Here, we have rewritten the variables

$$\begin{aligned} u &= \frac{s}{L^{-1/2}}, \\ L &= (1 - (Z'_{2ik}(D^{-1} + Z_{2ik}Z'_{2ik})^{-1}Z_{2ik})), \\ M &= D^{-1} + \tilde{Z}_{2ik}\tilde{Z}'_{2ik}. \end{aligned}$$

Next, integration over u produces

$$E[Y_{1ij}Y_{2ik}] = \frac{1}{|D|^{1/2}} \frac{1}{|M|^{1/2}} \frac{1}{L^{1/2}} X'_{1ij} \beta \Phi(L^{1/2} X'_{2ik} \beta) + \frac{1}{|D|^{1/2}} \frac{1}{|M|^{1/2}} \frac{1}{L} Z'_{1ij} M^{-1} Z_{2ik} \phi(L^{1/2} X'_{2ik} \beta).$$

Hence, the covariance equals

$$\text{Cov}[Y_{1ij}Y_{2ik}] = \left(\frac{1}{|D|^{1/2}} \frac{1}{|M|^{1/2}} \frac{1}{L^{1/2}} - 1 \right) X'_{1ij} \beta \Phi(L^{1/2} X'_{2ik} \beta) + \frac{1}{|D|^{1/2}} \frac{1}{|M|^{1/2}} \frac{1}{L} Z'_{1ij} M^{-1} Z_{2ik} \phi(L^{1/2} X'_{2ik} \beta).$$

As a result, the correlation is equal to

$$\rho_{Y_{1ij}, Y_{2ik}} = \frac{\left(\frac{1}{|D|^{1/2}} \frac{1}{|M|^{1/2}} \frac{1}{L^{1/2}} - 1 \right) X'_{1ij} \beta \Phi(L^{1/2} X'_{2ik} \beta) + \frac{1}{|D|^{1/2}} \frac{1}{|M|^{1/2}} \frac{1}{L} Z'_{1ij} M^{-1} Z_{2ik} \phi(L^{1/2} X'_{2ik} \beta)}{\sqrt{(Z'_{1ij} D Z_{1ij} + \Sigma_{1ij}) \Phi(L^{1/2} X'_{2ik} \beta) (1 - \Phi(L^{1/2} X'_{2ik} \beta))}}$$

G Correlation of vectors of responses

Let us derive the manifest variance-covariance matrix between subvectors of the two responses \tilde{Y}_{1i} of length \tilde{n}_i and \tilde{Y}_{2i} of length \tilde{p}_i . Since the subvectors are independent conditional on the random effects, the expected value of the product equals

$$E[Y_{1i}Y_{2i}]$$

$$\begin{aligned}
&= \left(\int_{-\infty}^{+\infty} \right)^q \left(\int_{t=-\infty}^{t=X'_{2ik}\beta + \tilde{Z}_{2i}\mathbf{b}_i} \right) \tilde{p}_i \frac{(X'_{1ij}\beta + \tilde{Z}_{1i}\mathbf{b}_i)}{(2\pi)^{(q+1)/2}|D|^{1/2}} \exp \left\{ -\frac{1}{2}(\mathbf{b}'_i D^{-1} \mathbf{b}_i + \mathbf{t}' \mathbf{t}) \right\} d\mathbf{b}_i dt \\
&= \left(\int_{-\infty}^{+\infty} \right)^q \left(\int_{t=-\infty}^{t=X'_{2ik}\beta + \tilde{Z}_{2i}\mathbf{b}_i} \right) \tilde{p}_i \frac{\tilde{Z}_{1i}\mathbf{b}_i}{(2\pi)^{(q+1)/2}|D|^{1/2}} \exp \left\{ -\frac{1}{2}(\mathbf{b}'_i D^{-1} \mathbf{b}_i + \mathbf{t}' \mathbf{t}) \right\} d\mathbf{b}_i dt \\
&\quad + \tilde{X}'_{1i}\beta \begin{bmatrix} \Phi(\tilde{X}_{2i}^1\beta, \frac{1}{L^1}) & \Phi(\tilde{X}_{2i}^2\beta, \frac{1}{L^2}) & \dots & \Phi(\tilde{X}_{2i}^{\tilde{p}_i}\beta, \frac{1}{L^{\tilde{p}_i}}) \end{bmatrix}' \\
&= \left(\int_{-\infty}^{+\infty} \right)^q \left(\int_{s=-\infty}^{s=X'_{2ik}\beta} \right) \tilde{p}_i \frac{\tilde{Z}_{1i}\mathbf{b}_i}{(2\pi)^{(q+1)/2}|D|^{1/2}} \exp \left\{ -\frac{1}{2} \left(\mathbf{b}'_i D^{-1} \mathbf{b}_i + (\mathbf{s} + \tilde{Z}_{2i}\mathbf{b}_i)'(\mathbf{s} + \tilde{Z}_{2i}\mathbf{b}_i) \right) \right\} d\mathbf{b}_i ds \\
&\quad + \tilde{X}'_{1i}\beta \begin{bmatrix} \Phi(\tilde{X}_{2i}^1\beta, \frac{1}{L^1}) & \Phi(\tilde{X}_{2i}^2\beta, \frac{1}{L^2}) & \dots & \Phi(\tilde{X}_{2i}^{\tilde{p}_i}\beta, \frac{1}{L^{\tilde{p}_i}}) \end{bmatrix}' \\
&= \left(\int_{-\infty}^{+\infty} \right)^q \frac{1}{(2\pi)^{(q+1)/2}|D|^{1/2}} \tilde{Z}_{1i} \begin{bmatrix} \int_{s=-\infty}^{s=\tilde{X}_{2i}^1\beta} \mathbf{b}_i \exp \left\{ -\frac{1}{2} \left(\mathbf{b}'_i D^{-1} \mathbf{b}_i + (s + \tilde{Z}_{2i}^1 b_i)^2 \right) \right\} ds d\mathbf{b}_i \\ \int_{s=-\infty}^{s=\tilde{X}_{2i}^2\beta} \mathbf{b}_i \exp \left\{ -\frac{1}{2} \left(\mathbf{b}'_i D^{-1} \mathbf{b}_i + (s + \tilde{Z}_{2i}^2 b_i)^2 \right) \right\} ds d\mathbf{b}_i \\ \dots \\ \int_{s=-\infty}^{s=\tilde{X}_{2i}^{\tilde{p}_i}\beta} \mathbf{b}_i \exp \left\{ -\frac{1}{2} \left(\mathbf{b}'_i D^{-1} \mathbf{b}_i + (s + \tilde{Z}_{2i}^{\tilde{p}_i} b_i)^2 \right) \right\} ds d\mathbf{b}_i \end{bmatrix}' \\
&\quad + \tilde{X}'_{1i}\beta \begin{bmatrix} \Phi(\tilde{X}_{2i}^1\beta, (L^{-1})_1) & \Phi(\tilde{X}_{2i}^2\beta, (L^{-1})_2) & \dots & \Phi(\tilde{X}_{2i}^{\tilde{p}_i}\beta, (L^{-1})_{\tilde{p}_i}) \end{bmatrix} \\
&= \left(\int_{-\infty}^{+\infty} \right)^q \frac{1}{(2\pi)^{(q+1)/2}|D|^{1/2}} \tilde{Z}_{1i} \begin{bmatrix} \int_{s=-\infty}^{s=\tilde{X}_{2i}^1\beta} \mathbf{b}_i \exp \left\{ -\frac{1}{2} \left((\mathbf{b}_i - u_1)' E_1^{-1} (\mathbf{b}_i - u_1) + O_1 \right) \right\} ds d\mathbf{b}_i \\ \int_{s=-\infty}^{s=\tilde{X}_{2i}^2\beta} \mathbf{b}_i \exp \left\{ -\frac{1}{2} \left((\mathbf{b}_i - u_2)' E_2^{-1} (\mathbf{b}_i - u_2) + O_2 \right) \right\} ds d\mathbf{b}_i \\ \dots \\ \int_{s=-\infty}^{s=\tilde{X}_{2i}^{\tilde{p}_i}\beta} \mathbf{b}_i \exp \left\{ -\frac{1}{2} \left((\mathbf{b}_i - u_{\tilde{p}_i})' E_{\tilde{p}_i}^{-1} (\mathbf{b}_i - u_{\tilde{p}_i}) + O_{\tilde{p}_i} \right) \right\} ds d\mathbf{b}_i \end{bmatrix}' \\
&\quad + \tilde{X}'_{1i}\beta \begin{bmatrix} \Phi(\tilde{X}_{2i}^1\beta, (L^{-1})_1) & \Phi(\tilde{X}_{2i}^2\beta, (L^{-1})_2) & \dots & \Phi(\tilde{X}_{2i}^{\tilde{p}_i}\beta, (L^{-1})_{\tilde{p}_i}) \end{bmatrix}
\end{aligned}$$

where the superscript denotes the row of the submatrix. Here we have rewritten the variables $\mathbf{t} = \mathbf{s} + \tilde{Z}_{2i}\mathbf{b}_i$ and $\mathbf{b}'_i D^{-1} \mathbf{b}_i + (s + \tilde{Z}_{2i}^2 b_i)^2 = (\mathbf{b}_i - u_{\tilde{p}_i})' E_{\tilde{p}_i}^{-1} (\mathbf{b}_i - u_{\tilde{p}_i}) + O_{\tilde{p}_i}$, with

$$\begin{aligned}
E_{\tilde{p}_i}^{-1} &= \tilde{Z}^{\tilde{p}_i'}_{2i} \tilde{Z}_{2i}^{\tilde{p}_i} + D^{-1}, \\
O_{\tilde{p}_i} &= s^2 - (\tilde{Z}^{\tilde{p}_i'}_{2i} s) E_{\tilde{p}_i} (\tilde{Z}_{2i}^{\tilde{p}_i} s)', \\
u_{\tilde{p}_i} &= -E_{\tilde{p}_i} \tilde{Z}_{2i}^{\tilde{p}_i} s.
\end{aligned}$$

Next, integrating over b_i produces

$$\begin{aligned}
& E[Y_{1ij} Y_{2ik}] \\
&= \frac{1}{(2\pi)^{1/2} |D|^{1/2}} \tilde{Z}_{1i} \left[\begin{array}{l} \int_{s=-\infty}^{s=\tilde{X}_{2i}^1 \beta} |E_1|^{1/2} u_1 \exp \left\{ -\frac{1}{2} O_1 \right\} ds \\ \int_{s=-\infty}^{s=\tilde{X}_{2i}^2 \beta} |E_2|^{1/2} u_2 \exp \left\{ -\frac{1}{2} O_2 \right\} ds \\ \dots \\ \int_{s=-\infty}^{s=\tilde{X}_{2i}^{\tilde{p}_i} \beta} |E_{\tilde{p}_i}|^{1/2} u_{\tilde{p}_i} \exp \left\{ -\frac{1}{2} O_{\tilde{p}_i} \right\} ds \end{array} \right]' \\
&+ \tilde{X}'_{1i} \beta \begin{bmatrix} \Phi(\tilde{X}_{2i}^1 \beta, \frac{1}{L^1}) & \Phi(\tilde{X}_{2i}^2 \beta, \frac{1}{L^2}) & \dots & \Phi(\tilde{X}_{2i}^{\tilde{p}_i} \beta, \frac{1}{L^{\tilde{p}_i}}) \end{bmatrix} \\
&= \frac{-1}{(2\pi)^{1/2} |D|^{1/2}} \tilde{Z}_{1i} \left[\begin{array}{l} \int_{s=-\infty}^{s=\tilde{X}_{2i}^1 \beta} |E_1|^{1/2} E_1 \tilde{Z}_{2i}^{1'} s \exp \left\{ -\frac{1}{2} \left(s(1 - \tilde{Z}_{2i}^1 E_1 \tilde{Z}_{2i}^{1'}) s \right) \right\} ds \\ \int_{s=-\infty}^{s=\tilde{X}_{2i}^2 \beta} |E_2|^{1/2} E_2 \tilde{Z}_{2i}^{2'} s \exp \left\{ -\frac{1}{2} \left(s(1 - \tilde{Z}_{2i}^2 E_2 \tilde{Z}_{2i}^{2'}) s \right) \right\} ds \\ \dots \\ \int_{s=-\infty}^{s=\tilde{X}_{2i}^{\tilde{p}_i} \beta} |E_{\tilde{p}_i}|^{1/2} E_{\tilde{p}_i} \tilde{Z}_{2i}^{\tilde{p}_i'} s \exp \left\{ -\frac{1}{2} \left(s(1 - \tilde{Z}_{2i}^{\tilde{p}_i} E_{\tilde{p}_i} \tilde{Z}_{2i}^{\tilde{p}_i'}) s \right) \right\} ds \end{array} \right]' \\
&+ \tilde{X}'_{1i} \beta \begin{bmatrix} \Phi(\tilde{X}_{2i}^1 \beta, \frac{1}{L^1}) & \Phi(\tilde{X}_{2i}^2 \beta, \frac{1}{L^2}) & \dots & \Phi(\tilde{X}_{2i}^{\tilde{p}_i} \beta, \frac{1}{L^{\tilde{p}_i}}) \end{bmatrix} \\
&= \frac{\tilde{Z}_{1i}}{\sqrt{|D|}} \left[\begin{array}{l} \sqrt{|E_1|} E_1 \tilde{Z}_{2i}^{1'} \phi \left(\tilde{X}_{2i}^1 \beta \cdot \sqrt{1 - \tilde{Z}_{2i}^1 E_1 \tilde{Z}_{2i}^{1'}} \right) \left(1 - \tilde{Z}_{2i}^1 E_1 \tilde{Z}_{2i}^{1'} \right)^{-1} \\ \sqrt{|E_2|} E_2 \tilde{Z}_{2i}^{2'} \phi \left(\tilde{X}_{2i}^2 \beta \cdot \sqrt{1 - \tilde{Z}_{2i}^2 E_2 \tilde{Z}_{2i}^{2'}} \right) \left(1 - \tilde{Z}_{2i}^2 E_2 \tilde{Z}_{2i}^{2'} \right)^{-1} \\ \dots \\ \sqrt{|E_{\tilde{p}_i}|} E_{\tilde{p}_i} \tilde{Z}_{2i}^{\tilde{p}_i'} \phi \left(\tilde{X}_{2i}^{\tilde{p}_i} \beta \cdot \sqrt{1 - \tilde{Z}_{2i}^{\tilde{p}_i} E_{\tilde{p}_i} \tilde{Z}_{2i}^{\tilde{p}_i'}} \right) \left(1 - \tilde{Z}_{2i}^{\tilde{p}_i} E_{\tilde{p}_i} \tilde{Z}_{2i}^{\tilde{p}_i'} \right)^{-1} \end{array} \right]' \\
&+ \tilde{X}'_{1i} \beta \begin{bmatrix} \Phi(\tilde{X}_{2i}^1 \beta, (L^{-1})_1) & \Phi(\tilde{X}_{2i}^2 \beta, (L^{-1})_2) & \dots & \Phi(\tilde{X}_{2i}^{\tilde{p}_i} \beta, (L^{-1})_{\tilde{p}_i}) \end{bmatrix}'.
\end{aligned}$$

Hence, the covariance equals

$$\frac{\tilde{Z}_{1i}}{\sqrt{|D|}} \begin{bmatrix} \sqrt{|E_1|} E_1 \tilde{Z}_{2i}^{1'} \phi \left(\tilde{X}_{2i}^1 \beta \cdot \sqrt{1 - \tilde{Z}_{2i}^1 E_1 \tilde{Z}_{2i}^{1'}} \right) \left(1 - \tilde{Z}_{2i}^1 E_1 \tilde{Z}_{2i}^{1'} \right)^{-1} \\ \sqrt{|E_2|} E_2 \tilde{Z}_{2i}^{2'} \phi \left(\tilde{X}_{2i}^2 \beta \cdot \sqrt{1 - \tilde{Z}_{2i}^2 E_2 \tilde{Z}_{2i}^{2'}} \right) \left(1 - \tilde{Z}_{2i}^2 E_2 \tilde{Z}_{2i}^{2'} \right)^{-1} \\ .. \\ \sqrt{|E_{\tilde{p}_i}|} E_{\tilde{p}_i} \tilde{Z}_{2i}^{\tilde{p}_i} \phi \left(\tilde{X}_{2i}^{\tilde{p}_i} \beta \cdot \sqrt{1 - \tilde{Z}_{2i}^{\tilde{p}_i} E_{\tilde{p}_i} \tilde{Z}_{2i}^{\tilde{p}_i}} \right) \left(1 - \tilde{Z}_{2i}^{\tilde{p}_i} E_{\tilde{p}_i} \tilde{Z}_{2i}^{\tilde{p}_i} \right)^{-1} \end{bmatrix}'$$