$$Y = \left(\left(\prod_{i=1}^{n} S_{t_i}\right)^{1/n} - K\right)^+ \qquad T=1 \quad K=1$$

On a $W = (W_{t_n})_{n\in\mathbb{N}}$ tel que $W_0 = 0$ et $W_{t_{n+1}} = W_{t_n} + \sqrt{t_{n+1}-t_n}\, Z_{n+1}$

où $(Z_n)_{n\in\mathbb{N}} \overset{iid}{\sim} \mathcal{N}(0,1)$

Donc, $\forall i \in \mathbb{N}$, $W_{t_i} = W_{t_i} - W_{t_0} = \sum_{k=0}^{i-1} W_{t_{k+1}} - W_{t_k} = \sum_{k=0}^{i-1} \sqrt{t_{k+1}-t_k}\, Z_{k+1}$

$\boxed{W_{t_i} = \frac{1}{\sqrt{n}} \sum_{k=1}^{i} Z_k}$ car $t_k = \frac{T}{n} k$ donc $t_{k+1} - t_k = \frac{T}{n} = \frac{1}{n}$

$S = (S_{t_n})_{n\in\mathbb{N}}$ est tel que $S_{t_i} = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t_i + \sigma W_{t_i}\right)$, $\forall i \in \mathbb{N}$.

Or $S_0 = 1$ et $r = 0$, donc $S_{t_i} = \exp\left(-\frac{\sigma^2}{2}\frac{i}{n} + \sigma \frac{1}{\sqrt{n}} \sum_{k=1}^{i} Z_k\right)$

Donc $\left(\prod_{i=1}^{n} S_{t_i}\right)^{1/n} = \exp\left(\frac{1}{n}\sum_{i=1}^{n}\left(-\frac{\sigma^2}{2}\frac{i}{n} + \frac{\sigma}{\sqrt{n}} \sum_{k=1}^{i} Z_k\right)\right)$

$$= \exp\left(\frac{-\sigma^2}{2n^2}\sum_{i=1}^{n} i + \frac{\sigma}{n\sqrt{n}} \sum_{i=1}^{n}\sum_{k=1}^{i} Z_k\right)$$

Or $\sum_{i=1}^{n}\sum_{k=1}^{i} Z_k = \sum_{i=1}^{n} i Z_i$ et on a $(Z_i) \overset{iid}{\sim} \mathcal{N}(0,1)$

donc $\sum_{i=1}^{n} i Z_i \sim \mathcal{N}\left(0, \sum_{i=1}^{n} i^2\right) \equiv \mathcal{N}\left(0, \frac{n(n+1)(2n+1)}{6}\right)$

donc $\sum_{i=1}^{n}\sum_{k=1}^{i} Z_k = \sqrt{\frac{n(n+1)(2n+1)}{6}}\, N$ où $N \sim \mathcal{N}(0,1)$

Donc, $\left(\prod_{i=1}^{n} S_{t_i}\right)^{1/n} = e^{-\frac{\sigma^2(n+1)}{4n}} \exp\left(\frac{\sigma}{n}\sqrt{\frac{(n+1)(2n+1)}{6}}\, N\right)$

$=: \phi(N)$ où $\phi : x \longmapsto e^{-\frac{\sigma^2(n+1)}{4n}} \exp\left(\frac{\sigma}{n}\sqrt{\frac{(n+1)(2n+1)}{6}}\, x\right)$

Donc $Y = \max(0, \phi(N) - 1)$

Donc

$$\mathbb{E}[Y] = \mathbb{E}\left[(\phi(N)-1)\,\mathbb{1}_{\phi(N)\geq 1}\right]$$

$$= \mathbb{E}[\hat{\phi}(N)] \quad \text{où} \quad \hat{\phi} : x \longmapsto (\phi(x)-1)\,\mathbb{1}_{\phi(x)\geq 1}$$

mesurable

Par la formule de transfert,

$$\mathbb{E}[Y] = \int_{-\infty}^{+\infty} (\phi(x)-1)\,\mathbb{1}_{\phi(x)\geq 1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$= e^{-\frac{\sigma^2(n+1)}{4n}} \int_{\ell}^{+\infty} \exp\left(\frac{\sigma}{n}\sqrt{\frac{(n+1)(2n+1)}{6}}\,x\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$- \int_{\ell}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

or
$$\frac{\sigma}{n}\sqrt{\frac{(n+1)(2n+1)}{6}}\,x - \frac{x^2}{2} = \frac{-1}{2}\left(x^2 - \frac{2\sigma}{n}\sqrt{\frac{(n+1)(2n+1)}{6}}\,x\right)$$

$$= \frac{-1}{2}\left(x^2 - 2mx\right) \quad \text{avec} \quad \boxed{m := \frac{\sigma}{n}\sqrt{\frac{(n+1)(2n+1)}{6}}}$$

$$= \frac{-1}{2}(x-m)^2 + \frac{m^2}{2}$$

$$\phi(x) \geq 1 \iff \exp\left(\frac{\sigma}{n}\sqrt{\frac{(n+1)(2n+1)}{6}}\,x\right) \geq e^{\frac{\sigma^2(n+1)}{4n}}$$

$$\iff x \geq \sqrt{\frac{6}{(n+1)(2n+1)}}\,\frac{\sigma(n+1)}{4n} = \boxed{\frac{\sqrt{6(n+1)}\,\sigma}{4n\sqrt{2n+1}} =: \ell}$$

Donc
$$\mathbb{E}[Y] = e^{-\frac{\sigma^2(n+1)}{4n} + \frac{m^2}{2}} \int_{\ell}^{+\infty} \exp\left(\frac{-1}{2}(x-m)^2\right) \frac{1}{\sqrt{2\pi}}\,dx - \left(1 - F_{\mathcal{N}(0,1)}(\ell)\right)$$

$$= e^{-\frac{\sigma^2(n+1)}{4n} + \frac{m^2}{2}} \left(1 - F_{\mathcal{N}(m,1)}(\ell)\right) - \left(1 - F_{\mathcal{N}(0,1)}(\ell)\right)$$