Appendix: Supplementary Materials for `DieRouter+`

I. Theoretical Results on Scheduler-Driven DP

For convenience, we summarize the key notations used in the main paper.

- N_j^+: The number of nets traversing edge j in the positive direction.
- $\{0, \ldots, N_j^+ - 1\}$: The index set of nets traversing edge j in the positive direction, ordered such that their continuous TDM ratios satisfy
$$r_{0,j} \leq r_{1,j} \leq \cdots \leq r_{N_j^+-1,j}.$$
- (α, β): A state representing the allocation of nets $\{0, \ldots, \alpha\}$ to β wires.
- $J^+(\alpha, \beta)$: The optimal state cost of (α, β), defined as the minimum possible value of the maximal displacement among all feasible allocation schemes.
- $J^-(\alpha, \beta)$: The optimal state cost in the negative direction, defined analogously.
- $V^+(\beta)$: The optimal cost of assigning all positive-direction nets to β wires, given by
$$V^+(\beta) = J^+(N_j^+ - 1, \beta).$$
- $V^-(\beta)$: The optimal cost of assigning all negative-direction nets to β wires, given by
$$V^-(\beta) = J^-(N_j^- - 1, \beta).$$
- $g(\beta)$: The function
$$g(\beta) = \max\{V^+(\beta), V^-(n_j - \beta)\},$$
which evaluates the worst-case cost for a given partitioning of wires.
- β^*: The optimal β, determined by
$$\beta^* = \arg \min_{\beta \in \{1, \ldots, n_j - 1\}} g(\beta). \tag{1}$$

Lemma 1. *Let β^+ and β^- be integers satisfying $\beta^+ + \beta^- < n_j$. Then the following statements hold: (a) If $V^+(\beta^+) \geq V^-(\beta^-)$, then $g(\beta^+ + 1) \leq g(\beta^+)$; (b) If $V^+(\beta^+) \leq V^-(\beta^-)$, then $g(n_j - \beta^- - 1) \leq g(n_j - \beta^-)$.*

Proof. We provide a proof for part (a) of Lemma 1. The proof of part (b) follows a similar argument.

Given $\beta^+ + \beta^- < n_j$, it follows that $\beta^- \leq n_j - 1 - \beta^+$. Consequently, we have:
$$V^-(n_j - \beta^+ - 1) \leq V^-(\beta^-) \leq V^+(\beta^+), \tag{2}$$
where the first inequality holds since V^- is a non-increasing function.

Furthermore, since $V^+(\beta^+ + 1) \leq V^+(\beta^+)$, we can establish the following:
$$\begin{aligned} g(\beta^+ + 1) &= \max\{V^+(\beta^+ + 1), V^-(n_j - \beta^+ - 1)\} && (3)\\ &\leq V^+(\beta^+) && (4)\\ &\leq \max\{V^+(\beta^+), V^-(n_j - \beta^+)\} && (5)\\ &= g(\beta^+). && (6) \end{aligned}$$

This completes the proof of part (a). □

Theorem 1. *Once the scheduler has allocated all wires, that is, when $\beta^+ + \beta^- = n_j$, β^+ is an optimal solution to* (1).

Proof. We first consider the case $n_j = 2$. In this scenario, the scheduler terminates with $\beta^+ = \beta^- = 1$. Since each direction requires at least one resource, it is clear that $\beta^+ = 1$ is an optimal solution.

Next, we examine the case $n_j > 2$. The crux of the argument relies on the fact that there always exists an optimal solution β^* of (1) in the set $\{\beta^+, \ldots, n_j - \beta^-\}$, and this statement serves as a loop invariant.

This property holds trivially when $\beta^+ = \beta^- = 1$. Suppose it holds at some β^+, β^-. We show that it remains true after a round of resource allocation. Note that a new resource can only be allocated if $\beta^+ + \beta^- < n_j$. We consider two cases:

Case 1. $V^+(\beta^+) \geq V^-(\beta^-)$. By Lemma 1(a), we have $g(\beta^+ + 1) \leq g(\beta^+)$. Hence, there exists an optimal solution within $\{\beta^+ + 1, \ldots, n_j - \beta^-\}$.

Case 2. $V^+(\beta^+) < V^-(\beta^-)$. By Lemma 1(b), we obtain $g(n_j - \beta^- - 1) \leq g(n_j - \beta^-)$, implying that there is an optimal solution in $\{\beta^+, \ldots, n_j - \beta^- - 1\}$.

These two cases justify the scheduler's decision to allocate a resource to the direction with a higher cost while preserving the loop invariant.

Finally, when $\beta^+ + \beta^- = n_j$, the search space containing an optimal solution reduces to a single element. That element must be optimal. □

II. Ablation Study

We present the ablation study in Table I, evaluating `DieRouter+` with different initial routing algorithms and reporting maximum net delay at each stage. For clarity, the table uses the following abbreviations: *Init* for Initial Routing, *R&R* for violation-driven and performance-driven rip-up-and-rerouting, *Conti* for solving continuous TDM ratios via SOCP, and *Leg* for legalization via scheduler-driven DP.

We compare our SPT-based initial routing with the hybrid approach in `DieRouter`, where nets are classified as critical

TABLE I
ABLATION STUDY RESULTS.

Test Case	DieRouter+ w/ MST-SPT				DieRouter+ w/ SMT-SPT				DieRouter+ w/ SPT			
	Init	R& R	Conti	Leg	Init	R& R	Conti	Leg	Init	R& R	Conti	Leg
1	2.51	2.51	6.5	**6.5**	2.51	2.51	6.5	**6.5**	2.51	2.51	6.5	**6.5**
2	6.93	5.65	13	13	6.64	6.64	14	14	3.84	3.83	7.5	**7.5**
3	12.1	12.1	10.5	13.5	17.2	16.6	13	13.5	10.7	10.7	9.5	**11.5**
4	29.7	18.1	14.4	**18.5**	30	18.1	14.32	**18.5**	16.7	16.5	15.49	**18.5**
5	139.71	138.48	125.77	**133**	150.1	137.57	125.8	**133**	137.69	137.51	125.77	**133**
$\Delta\mathsf{Delay}_{avg}$	-20.28%	-10.66%	-8.85%	**-11.42%**	-26.51%	-17.35%	-13.04%	**-12.25%**	0%	0%	0%	0%
6	279.75	621.78	255.85	264	279.87	548.19	189.27	199.5	92.27	460.99	181.05	**187.5**
7	233.39	223.99	75.43	81.50	237.64	188.39	66.82	**73.5**	106.69	98.21	66.58	**73.5**
8	360.03	428.81	159.92	169.50	361.81	310.49	97.57	109	158.03	142.72	99.47	**105.5**
9	216.01	410.19	139.23	148.50	217.48	458.92	134.31	142.50	89.96	238.94	132.92	**139.5**
10	1699.65	7274.17	4402.39	**4420**	1699.76	7323.2	4432.11	4449	1404.81	7354.09	4465.99	4485
$\Delta\mathsf{Delay}_{avg}$	-50.62%	-37.88%	-16.37%	**-16.23%**	-50.89%	-33.06%	-0.61%	**-2.10%**	0%	0%	0%	0%

or non-critical, with critical nets routed by MSTs and non-critical nets by SPTs. We also test an approximated SMT-based approach [1] for routing critical nets, denoted as "SMT-SPT". This naming convention is applied consistently across all cases. For net classification, we use the same hyperparameters as `DieRouter`.

From Table I, we conclude that SPT-based initial routing reduces average maximum net delay by **11.42%/16.23%** over MST-SPT and **12.25%/2.1%** over SMT-SPT on easier/ challenging test cases, highlighting the impact of the initial routing topology.

Generally, a lower initial delay correlates with a lower final delay. However, when initial delays are similar, final performance becomes less predictable. In test case 10, for instance, despite a slightly lower initial delay with SPT-based routing, its final maximum net delay exceeds that of the other two methods. Nonetheless, maximum net delay after the rip-up and rerouting phase remains a useful indicator of final performance.

REFERENCES

[1] K. Mehlhorn, "A faster approximation algorithm for the steiner problem in graphs," *Information Processing Letters*, vol. 27, no. 3, pp. 125–128, 1988.