Marginal V1 Appendix

Abstract

Explicitly derives the math behind equations in the V1 whitepaper.

A Overview

A.1 Opening a position

Take the pool state before the position is opened to be (x, y) in reserves or (L, P) in liquidity and price, where the transformation between the two descriptions of the pool follows the Uniswap convention

$$x = L/\sqrt{P} \tag{1}$$
$$y = L\sqrt{P} \tag{2}$$

or

$$L = \sqrt{xy} \tag{3}$$
$$P = y/x \tag{4}$$

Assume the trader longs the X token relative to the Y token. The trader's position owes debts (d_x, d_y) to the pool at settlement, with collateral held in the position for future delivery of $(c_x + \delta x + \delta x' - i_x, 0)$. Insurance funds (i_x, i_y) are set aside by the pool to prevent liquidity shortfall in the event of a liquidation.

Debts are set such that if the trader were to immediately settle their position, the entire open $\rightarrow$ settle action would replicate a swap on the original L liquidity curve, with the trader sending d_y of Y token into the pool and receiving position size

$$s_x = \delta x + \delta x' - d_x - i_x \tag{5}$$

of X token out from the pool.

Aim to find expressions for $\delta x', d_x$ and d_y in terms of the liquidity fronted by the pool $(\delta x, \delta y)$ and the insurance balances (i_x, i_y) set in a later section.

From $A \rightarrow B$ in Figure 2, the pool internally replicates removing liquidity as the available liquidity drops from $L \rightarrow L - \delta L$, while maintaining the initial price P. As price remains the same through this process, the ratio of the reserves pulled must also be equal to this initial price (by assumption)

$$\delta y/\delta x = i_y/i_x = P \tag{6}$$

which implies $\delta y/\delta x = P = y/x \rightarrow \delta y/y = \delta x/x$.

Further, for $(\delta x, \delta y)$ of liquidity fronted by pool liquidity providers (LPs) to build the leveraged long position, the change to the liquidity curve δL will also follow the Uniswap transformations as price remains the same:

$$\begin{aligned} P &= (y - \delta y)/(x - \delta x) \\ &= (y/x)(1 - \delta y/y)/(1 - \delta x/x) \\ &= y/x \end{aligned} \tag{7}$$

and

$$\begin{aligned} L - \delta L &= \sqrt{(x - \delta x)(y - \delta y)} \\ &= \sqrt{xy}\sqrt{(1 - \delta x/x)(1 - \delta y/y)} \\ &= \sqrt{xy}\sqrt{(1 - \delta x/x)^2} \\ &= L \cdot (1 - \delta x/x) \end{aligned} \tag{8}$$

which implies

$$\delta L/L = \delta x/x = \delta y/y \tag{9}$$

or as $L = \sqrt{xy}$,

$$\begin{aligned} \delta L &= L \cdot \delta x/x \\ &= \sqrt{xy} \cdot (\delta x/x) \\ &= \sqrt{y/x} \cdot \delta x \\ &= \sqrt{\delta y/\delta x} \cdot \delta x \\ &= \sqrt{\delta x \cdot \delta y} \end{aligned} \tag{10}$$

From $B \to C$ in Figure 2, the pool internally replicates swapping $\delta y - i_y$ along the reduced liquidity curve $L - \delta L$ for more X token $\delta x'$. Available pool reserves at B are

$$B : (x - \delta x, y - \delta y) \tag{11}$$

and at C

$$\begin{aligned} C : (x - \delta x - \delta x', y - \delta y + (\delta y - i_y)) \\ = (x - \delta x - \delta x', y - i_y) \end{aligned} \tag{12}$$

as $y - i_y$ is sent in and $\delta x'$ is taken out. As the internal swap from $B \to C$ happens along the constant liquidity curve $L - \delta L$, the Uniswap invariant is obeyed

$$\begin{aligned} (L - \delta L)^2 &= (x - \delta x)(y - \delta y) \\ &= (x - \delta x - \delta x')(y - i_y) \end{aligned} \tag{13}$$

This determines $\delta x'$. Immediately note

$$\begin{aligned} x - \delta x - \delta x' &= (x - \delta x) \cdot \frac{y - \delta y}{y - i_y} \\ &= x \cdot (1 - \delta x / x) \cdot \frac{1 - \delta y / y}{1 - i_y / y} \\ &= x \cdot \frac{(1 - \delta y / y)^2}{1 - i_y / y} \end{aligned} \tag{14}$$

which will be helpful in deriving debts owed. Solving for the extra X tokens taken from reserves post internal swap

$$
\begin{aligned}
\delta x' &= x - \delta x - x \cdot \frac{(1-\delta y/y)^2}{1-i_y/y} \\
&= x \cdot (1-\delta x/x) - x \cdot \frac{(1-\delta y/y)^2}{1-i_y/y} \\
&= x \cdot (1-\delta x/x) - x \cdot \frac{(1-\delta x/x)(1-\delta y/y)}{1-i_y/y} \\
&= x \cdot (1-\delta x/x) \cdot \left[1 - \frac{1-\delta y/y}{1-i_y/y}\right] \\
&= x \cdot (1-\delta x/x) \cdot \left[\frac{1-i_y/y-(1-\delta y/y)}{1-i_y/y}\right] \\
&= x \cdot (1-\delta x/x) \cdot \left[\frac{\delta y/y - i_y/y}{1-i_y/y}\right] \\
&= x \cdot (1-\delta x/x) \cdot (\delta y/y) \cdot \left[\frac{1-i_y/\delta y}{1-i_y/y}\right] \\
&= (\delta y/P) \cdot (1-\delta x/x) \cdot \left[\frac{1-i_y/\delta y}{1-i_y/y}\right] \\
&= \delta x \cdot (1-\delta x/x) \cdot \left[\frac{1-i_y/\delta y}{1-i_y/y}\right] \qquad (15)
\end{aligned}
$$

The pool moves to a new price P' at C after the internal swap. This new price is given by the same transformation equations as the ratio of Y to X token reserves as available liquidity at $L - \delta L$. Available pool reserves at C are $(x - \delta x - \delta x', y - i_y)$. The ratio gives

$$
\begin{aligned}
P' &= \frac{y - i_y}{x - \delta x - \delta x'} \\
&= \frac{y - i_y}{x \cdot \frac{(1-\delta y/y)^2}{1-i_y/y}} \\
&= \frac{y}{x} \cdot \frac{1-i_y/y}{\frac{(1-\delta y/y)^2}{1-i_y/y}} \\
&= P \cdot \left(\frac{1-i_y/y}{1-\delta y/y}\right)^2
\end{aligned}
\qquad (16)
$$

From $C \to D$ in Figure 2, the pool internally replicates adding liquidity as the available liquidity returns from $L - \delta L \to L$, while maintaining the new price P' post internal swap. As price remains the same through this process, the ratio of reserves added must also be equal to the new price. The pool internally has reserves d_x, i_x in X token and i_y in Y token set aside to provide the majority of necessary capital for the liquidity add. The trader delivers the remaining d_y to complete the full liquidity addition, so that from $C \to D$ amounts (d_x+i_x, d_y+i_y) are added to available pool reserves to return δL of liquidity at constant price P'. Available pool reserves at D are

$$\begin{aligned} D : (x - \delta x - \delta x' + (d_x + i_x), y - i_y + (d_y + i_y)) \\ = (x - \delta x - \delta x' + d_x + i_x, y + d_y) \\ = (x - s_x, y + d_y) \end{aligned} \tag{17}$$

Similar to the Uniswap transformations obeyed from $A \to B$, reserves added (d_x+i_x, d_y+i_y) from $C \to D$ at a constant price of P' for an increase in liquidity of δL to bring us back to the original curve L must follow

$$\delta L = \sqrt{(d_x + i_x)(d_y + i_y)} \tag{18}$$

$$P' = (d_y + i_y)/(d_x + i_x) \tag{19}$$

or

$$d_x + i_x = \delta L/\sqrt{P'} \tag{20}$$

$$d_y + i_y = \delta L\sqrt{P'} \tag{21}$$

as

$$\begin{aligned} (L - \delta L) + \delta L &= \sqrt{(x - \delta x - \delta x' + (d_x + i_x))(y - i_y + (d_y + i_y))} \\ &= \sqrt{(x - \delta x - \delta x')(y - i_y)}\sqrt{\left(1 + \frac{d_x + i_x}{x - \delta x - \delta x'}\right)\left(1 + \frac{d_y + i_y}{y - i_y}\right)} \\ &= (L - \delta L) \cdot \sqrt{\left(1 + \frac{d_x + i_x}{x - \delta x - \delta x'}\right)\left(1 + \frac{d_y + i_y}{y - i_y}\right)} \\ &= (L - \delta L) \cdot \sqrt{\left(1 + \frac{d_y + i_y}{y - i_y}\right)^2} \\ &= (L - \delta L) \cdot \left(1 + \frac{d_y + i_y}{y - i_y}\right) \end{aligned} \tag{22}$$

where we've used $(d_y + i_y)/(d_x + i_x) = P' = (y - i_y)/(x - \delta x - \delta x') \to (d_y + i_y)/(y - i_y) = (d_x + i_x)/(x - \delta x - \delta x')$.

Simplifying a bit more by dividing both sides by $L - \delta L$ and using

$$\frac{L - \delta L + \delta L}{L - \delta L} = 1 + \frac{\delta L}{L - \delta L} \tag{23}$$

we get after cancelling the 1's and multiplying through by $L - \delta L$ again

$$\delta L = (L - \delta L) \cdot \left(\frac{d_y + i_y}{y - i_y} \right) \tag{24}$$

But since the reserves at C obey the Uniswap invariant

$$L - \delta L = \sqrt{(x - \delta x - \delta x')(y - i_y)} \tag{25}$$

we find for $C \to D$

$$\begin{aligned}
\delta L &= (L - \delta L) \cdot \left(\frac{d_y + i_y}{y - i_y} \right) \\
&= \sqrt{(x - \delta x - \delta x')(y - i_y)} \cdot \left(\frac{d_y + i_y}{y - i_y} \right) \\
&= \sqrt{(x - \delta x - \delta x')/(y - i_y)} \cdot (d_y + i_y) \\
&= (d_y + i_y)/\sqrt{P'} \\
&= \frac{(d_y + i_y)}{\sqrt{\frac{d_y + i_y}{d_x + i_x}}} \\
&= \sqrt{(d_x + i_x)(d_y + i_y)}
\end{aligned} \tag{26}$$

The Uniswap transformations applied to reserves added from $C \to D$ make it relatively simple to determine expressions for the required debts (d_x, d_y)

$$\begin{aligned}
d_x + i_x &= \delta L / \sqrt{P'} \\
&= (\delta x \sqrt{P})/\sqrt{P'} \\
&= \delta x \cdot \sqrt{\left(\frac{1 - \delta y / y}{1 - i_y / y} \right)^2} \\
&= \delta x \cdot \left(\frac{1 - \delta y / y}{1 - i_y / y} \right)
\end{aligned} \tag{27}$$

and

$$
\begin{aligned}
d_y &= \delta L\sqrt{P'} - i_y \\
&= (\delta y/\sqrt{P})\sqrt{P'} - i_y \\
&= \delta y \cdot \left(\frac{1 - i_y/y}{1 - \delta y/y}\right) - i_y \\
&= \delta y \cdot \left(\frac{1 - i_y/y}{1 - \delta y/y} - \frac{i_y}{\delta y}\right) \\
&= \delta y \cdot \left(\frac{1 - i_y/y}{1 - \delta y/y} - \frac{\frac{i_y}{\delta y}(1 - \delta y/y)}{1 - \delta y/y}\right) \\
&= \delta y \cdot \left(\frac{1 - i_y/y - i_y/\delta y + i_y/y}{1 - \delta y/y}\right) \\
&= \delta y \cdot \left(\frac{1 - i_y/\delta y}{1 - \delta y/y}\right) \qquad (28)
\end{aligned}
$$

Finally, size reduces to

$$
\begin{aligned}
s_x &= \delta x + \delta x' - (d_x + i_x) \\
&= \delta x + \delta x \cdot (1 - \delta x/x) \cdot \left(\frac{1 - i_y/\delta y}{1 - i_y/y}\right) - \delta x \cdot \left(\frac{1 - \delta y/y}{1 - i_y/y}\right) \\
&= \delta x \cdot \left[1 + (1 - \delta x/x) \cdot \left(\frac{1 - i_y/\delta y}{1 - i_y/y}\right) - \left(\frac{1 - \delta y/y}{1 - i_y/y}\right)\right] \\
&= \frac{\delta x}{1 - i_y/y} \cdot \left[1 - i_y/y + (1 - \delta x/x) \cdot (1 - i_y/\delta y) - (1 - \delta y/y)\right] \\
&= \frac{\delta x}{1 - i_y/y} \cdot \left[1 - i_y/y + (1 - \delta y/y) \cdot (1 - i_y/\delta y) - (1 - \delta y/y)\right] \\
&= \frac{\delta x}{1 - i_y/y} \cdot \left[1 - i_y/y + 1 - i_y/\delta y - \delta y/y + i_y/y - 1 + \delta y/y\right] \\
&= \delta x \cdot \left(\frac{1 - i_y/\delta y}{1 - i_y/y}\right) \qquad (29)
\end{aligned}
$$

A.2 Settling a position

For the AMM to be viable for LPs, the liquidity returned to the pool when a position is settled at any time in the future must be at least as large as the original liquidity fronted by LPs δL. Otherwise, there is a liquidity shortfall for LPs in the pool (i.e. bad debt). We take advantage of the liability mismatch between the δL lent out by LPs to form the position and the physical amounts to be sent by the position to the pool at settlement. Passively holding those

X and Y tokens set aside to collateralize the position will outperform having the same tokens remaining in the swappable pool reserves when price deviates significantly due to impermanent loss.

Meaning, we should expect amounts $(d_x + i_x, d_y + i_x)$ with ratio P' to return at least δL of liquidity back to the pool even when the price of the pool at the time of settlement is *not* P'. When the price of the pool immediately before settling is P', we should get exactly δL returned to the pool due to how we set debt values at the end of the last section.

Imagine the Marginal pool is in a general state prior to position settlement of $(\tilde{L}, \tilde{P})$. When the trader chooses to settle with the pool, they deliver d_y of debt owed to the pool and receive s_x of size back plus their original margin c_x. $d_x + i_x$ of X token and $d_y + i_y$ of Y token are then internally added to pool liquidity: $E \to F$ in Figure 3.

Available reserves in the pre-settlement state at E are assumed to be

$$E : (\tilde{x}, \tilde{y}) \tag{30}$$

where the Uniswap transformations are obeyed

$$\begin{aligned} \tilde{L} &= \sqrt{\tilde{x}\tilde{y}} && (31) \\ \tilde{P} &= \tilde{y}/\tilde{x} && (32) \end{aligned}$$

and

$$\begin{aligned} \tilde{x} &= \tilde{L}/\sqrt{\tilde{P}} && (33) \\ \tilde{y} &= \tilde{L}\sqrt{\tilde{P}} && (34) \end{aligned}$$

Available reserves post-settlement at F are

$$F : (\tilde{x} + d_x + i_x, \tilde{y} + d_y + i_y) \tag{35}$$

From $E \to F$ in Figure 3, pool liquidity transitions from $\tilde{L} \to \tilde{L} + \delta\tilde{L}$, where $\delta\tilde{L}$ is the *actual* amount of liquidity added to the pool via returning amounts $(d_x + i_x, d_y + i_y)$ at E. We'd like for $\delta\tilde{L} \geq \delta L$ for all possible values of $(\tilde{L}, \tilde{P})$. Checking the Uniswap invariant at F,

$$
\begin{aligned}
(\tilde{L}+\delta\tilde{L})^2 &= (\tilde{x}+d_x+i_x)(\tilde{y}+d_y+i_y) \\
&= \tilde{x}\tilde{y}\cdot[1+(d_x+i_x)/\tilde{x}][1+(d_y+i_y)/\tilde{y}] \\
&= \tilde{L}^2\cdot[1+(d_x+i_x)/\tilde{x}][1+(d_y+i_y)/\tilde{y}] \\
&= \tilde{L}^2\cdot[1+(d_x+i_x)/\tilde{x}+(d_y+i_y)/\tilde{y}+(d_x+i_x)(d_y+i_y)/(\tilde{x}\tilde{y})] \\
&= \tilde{L}^2\cdot[1+(d_x+i_x)/\tilde{x}+(d_y+i_y)/\tilde{y}+(\delta L/\tilde{L})^2] \\
&= \tilde{L}^2\cdot[1+(\delta L/\sqrt{P'})/(\tilde{L}/\sqrt{\tilde{P}})+(\delta L\sqrt{P'})/(\tilde{L}\sqrt{\tilde{P}})+(\delta L/\tilde{L})^2] \\
&= \tilde{L}^2\cdot\left[1+(\delta L/\tilde{L})\left(\sqrt{\frac{\tilde{P}}{P'}}+\sqrt{\frac{P'}{\tilde{P}}}\right)+(\delta L/\tilde{L})^2\right] \qquad (36)
\end{aligned}
$$

where we've used $d_x + i_x = \delta L/\sqrt{P'}$ and $d_y + i_y = \delta L\sqrt{P'}$ from the end of the last section. For $\delta\tilde{L} \geq \delta L$,

$$
(\tilde{L}+\delta\tilde{L})^2 \geq (\tilde{L}+\delta L)^2 = \tilde{L}^2[1+2\delta L/\tilde{L}+(\delta L/\tilde{L})^2] \qquad (37)
$$

Or when comparing with the expression for $(\tilde{L}+\delta\tilde{L})^2$

$$
\sqrt{\frac{\tilde{P}}{P'}}+\sqrt{\frac{P'}{\tilde{P}}} \geq 2 \qquad (38)
$$

which is independent of pre-settlement liquidity $\tilde{L}$ (i.e. pool liquidity prior to settlement is irrelevant for the comparison).

Let

$$
p = \ln(\tilde{P}/P') \qquad (39)
$$

such that

$$
\begin{aligned}
\sqrt{\tilde{P}/P'} &= e^{p/2} \qquad (40) \\
\sqrt{P'/\tilde{P}} &= e^{-p/2} \qquad (41)
\end{aligned}
$$

The left-hand side of the prior constraint becomes

$$
\begin{aligned}
\sqrt{\frac{\tilde{P}}{P'}}+\sqrt{\frac{P'}{\tilde{P}}} &= e^{p/2}+e^{-p/2} \\
&= 2\cosh(p/2) \qquad (42)
\end{aligned}
$$

which means that for $\delta\tilde{L} \geq \delta L$, we'll need

$$\cosh(p/2) \geq 1 \tag{43}$$

But this is true for all possible p values (and thus $\tilde{P}$ values) given the range of the hyperbolic cosine function. So post-settlement, liquidity added to the pool will be at least the original liquidity fronted by LPs

$$\delta\tilde{L} \geq \delta L \tag{44}$$

for any pre-settlement state at E of $(\tilde{L}, \tilde{P})$.

However, if the pre-settlement price is not the same as the price after opening the position $\tilde{P} \neq P'$, the act of settling will alter the pre-settlement price in the direction of P'. This presents an arbitrage opportunity against the pool, that should quickly be taken advantage of to move the price back toward $\tilde{P}$ and in line with spot. From the perspective of LPs in the pool, this won't really matter as they still get at least δL back post-settlement.

B Maintenance requirements

B.1 Safety of a position

From the perspective of the trader, the value of the leverage position is the value of the collateral backing the position less the debt needed to settle. The collateral backing the position less the X token debt returned internally to the pool at settlement is

$$\begin{aligned}(c_x + \delta x + \delta x' - i_x) - d_x &= c_x + s_x + d_x - d_x \\ &= c_x + s_x\end{aligned} \tag{45}$$

The position still owes d_y in Y token debt to be delivered by the trader at settlement in return for the margin c_x and size s_x behind the position.

In Y token terms, the value of the position is then

$$V_y(P_t) = (c_x + s_x) \cdot P_t - d_y \tag{46}$$

where P_t is the pool price at some time t.

We set the liquidation condition (i.e. whether the position is safe) to require some minimum maintenance buffer

$$M \cdot d_y \tag{47}$$

before the position value turns negative, in case liquidators are slow to liquidate. M is the maintenance margin factor specific to the pool. We also use the spot oracle time-weighted average price $\bar{P}_{t-\tau,t}$ to mitigate manipulation of the safety check. Accounting for the maintenance margin, the position is considered safe if the value of the position exceeds the maintenance buffer

$$V_y(\bar{P}_{t-\tau,t}) \geq M \cdot d_y \tag{48}$$

Or, after adding $+d_y$ to both sides,

$$(c_x + s_x) \cdot \bar{P}_{t-\tau,t} \geq (1 + M) \cdot d_y \tag{49}$$

Averaging over a longer time τ for the spot oracle benefits the trader due to higher cost of attack to manipulate in addition to more time for the trader to return the position to safety prior to being liquidated, given the time lag relative to the instantaneous pool price. The downside is a higher likelihood of a negative equity value at liquidation in instantaneous price terms.

The larger the fraction of $\delta L/L$ borrowed originally from the pool to front the leverage at open, the higher the "slippage" experienced by the trader in the form of increased debt. As slippage trends toward zero, the debt the trader owes the pool in the Y token trends toward position size in the X token times the original price P.

Expressing X size in terms of the Y debt owed at settlement, recall

$$\begin{aligned}
s_x &= \delta x \cdot \left(\frac{1 - i_y/\delta y}{1 - i_y/y} \right) \\
d_y &= \delta y \cdot \left(\frac{1 - i_y/\delta y}{1 - \delta y/y} \right) \\
\sqrt{\frac{P'}{P}} &= \frac{1 - i_y/y}{1 - \delta y/y}
\end{aligned}$$

Plugging the second and third into the first equation, we get

$$
\begin{aligned}
s_x &= \delta x \cdot \left(\frac{1 - i_y/\delta y}{1 - i_y/y}\right) \\
&= \frac{\delta y}{P} \cdot \left(\frac{1 - i_y/\delta y}{1 - \delta y/y}\right)\left(\frac{1 - \delta y/y}{1 - i_y/y}\right) \\
&= \frac{\delta y}{P} \cdot \left(\frac{1 - i_y/\delta y}{1 - \delta y/y}\right)\left(\frac{1 - \delta y/y}{1 - i_y/y}\right) \\
&= \frac{d_y}{P} \cdot \left(\frac{1 - \delta y/y}{1 - i_y/y}\right) \\
&= \frac{d_y}{P} \cdot \sqrt{\frac{P}{P'}} \\
&= \frac{d_y}{\sqrt{P \cdot P'}} \qquad (50)
\end{aligned}
$$

Take the limit as $\delta y/y \to 0$, assuming we will set insurance values such that this also implies $i_y/y \to 0$ such that smaller liquidity amounts fronted by the pool lead to less slippage for the pool price

$$
\lim_{\delta y \to 0} P' = P \qquad (51)
$$

Thus, the X value of the debt will equal the size taken out on the pool for small sizes borrowed

$$
\lim_{\delta y \to 0} s_x = \frac{d_y}{\sqrt{P \cdot P'}} = \frac{d_y}{P} \qquad (52)
$$

B.2 Liquidating a position

Liquidation causes funds collateralizing the position to be added back to available pool liquidity. Collateral amounts directly backing the position are $(c_x + s_x + d_x, 0)$. Insurance funds set aside by the pool in case of liquidation are (i_x, i_y) in *both* X and Y tokens.

For a position liquidated almost immediately after the trader opens, liquidation transitions the pool reserves from $C \to G$ in Figure 5. Again, after position open but before liquidation, available pool reserves at C are

$$
\begin{aligned}
&C : (x - \delta x - \delta x', y - i_y) \\
&= (x - s_x - d_x - i_x, y - i_y) \qquad (53)
\end{aligned}
$$

Available pool reserves at G after position liquidation are

$$
\begin{aligned}
G : (x - s_x - d_x - i_x + (c_x + s_x + d_x + i_x), y - i_y + i_y) \\
= (x + c_x, y)
\end{aligned} \tag{54}
$$

The remaining position margin c_x is taken as a liquidation fee by the pool, increasing pool liquidity past the original state at A. Price decreases back down from P' to slightly less than the original pool price P before the position was ever opened.

C Insurance funds

For the AMM to be viable for LPs, the liquidity returned to the pool when a position is liquidated at any time in the future must be at least as large as the original liquidity fronted by LPs δL, to avoid a liquidity shortfall and bad debt for the pool. When setting insurance balances to backstop the pool, we again take advantage of the liability mismatch: passively holding X and Y tokens will outperform having the same tokens remaining in the swappable pool reserves as price deviates significantly due to impermanent loss.

Meaning, to completely overcome the possibility of negative position value at liquidation, insurance balances (i_x, i_y) set aside at open and margin requirements on c_x are determined such that at least δL of liquidity is returned to the pool regardless of the pool state $(\tilde{L}, \tilde{P})$ prior to liquidation. The impermanent "gain" mechanism kicks in as price drops significantly on the long position, increasing the contribution to additional liquidity from the insurance amounts passively held internally in the pool.

If we can set insurance balances such that the pool is guaranteed to receive back at least δL of liquidity post-liquidation irrespective of the pre-liquidation pool state, we technically no longer need a fresh oracle price to prevent negative position value scenarios as liquidations become deterministically safe at any price *from the perspective of the pool*, since the original liability δL will always be covered. Therefore, we can significantly increase the time averaged over with the spot oracle in the safety condition to decrease the possibility of manipulation even further. This guarantee also allows us to safely accommodate pools with longer tailed assets, as LPs won't experience a liquidity shortfall (ignoring potential funding payments from the pool).

From $H \to I$ in Figure 6, liquidation causes pool liquidity to change from $\tilde{L} \to \tilde{L} + \delta\tilde{L}$. Available reserves at H are

$$
H : (\tilde{x}, \tilde{y}) \tag{55}
$$

obeying the same Uniswap transformations as in position settlement analysis. Available reserves at I, however, are

$$I : (\tilde{x} + c_x + s_x + d_x + i_x, \tilde{y} + i_y) \tag{56}$$

Goal is to find insurance and margin requirements such that $\delta\tilde{L} \geq \delta L$. Checking the Uniswap invariant at I,

$$\begin{aligned}
(\tilde{L} + \delta\tilde{L})^2 &= (\tilde{x} + c_x + s_x + d_x + i_x)(\tilde{y} + i_y) \\
&= \tilde{x}\tilde{y} \cdot [1 + (c_x + s_x + d_x + i_x)/\tilde{x}][1 + i_y/\tilde{y}] \\
&= \tilde{L}^2 \cdot [1 + (c_x + s_x + d_x + i_x)/\tilde{x}][1 + i_y/\tilde{y}] \\
&= \tilde{L}^2 \cdot [1 + (c_x + s_x + d_x + i_x)/\tilde{x} + i_y/\tilde{y} + i_y \cdot (c_x + s_x + d_x + i_x)/(\tilde{x}\tilde{y})] \\
&= \tilde{L}^2 \cdot [1 + (d_x + i_x)/\tilde{x} + (c_x + s_x)/\tilde{x} + (d_y + i_y)/\tilde{y} \\
&\quad -d_y/\tilde{y} + i_y \cdot (c_x + s_x)/(\tilde{x}\tilde{y}) + i_y \cdot (d_x + i_x)/(\tilde{x}\tilde{y})] \\
&= \tilde{L}^2 \cdot [1 + (d_x + i_x)/\tilde{x} + (d_y + i_y)/\tilde{y} + (c_x + s_x)/\tilde{x} \\
&\quad -d_y/\tilde{y} + i_y \cdot (c_x + s_x)/(\tilde{x}\tilde{y}) + (d_y + i_y)(d_x + i_x)/(\tilde{x}\tilde{y}) \\
&\quad -d_y \cdot (d_x + i_x)/(\tilde{x}\tilde{y})] \\
&= \tilde{L}^2 \cdot [1 + (d_x + i_x)/\tilde{x} + (d_y + i_y)/\tilde{y} + (d_y + i_y)(d_x + i_x)/(\tilde{x}\tilde{y}) \\
&\quad +\tilde{y} \cdot (c_x + s_x)/(\tilde{x}\tilde{y}) - \tilde{x} \cdot d_y/(\tilde{x}\tilde{y}) + i_y \cdot (c_x + s_x)/(\tilde{x}\tilde{y}) \\
&\quad -d_y \cdot (d_x + i_x)/(\tilde{x}\tilde{y})] \\
&= \tilde{L}^2 \cdot [1 + (d_x + i_x)/\tilde{x} + (d_y + i_y)/\tilde{y} + (d_y + i_y)(d_x + i_x)/(\tilde{x}\tilde{y})] \\
&\quad +(\tilde{L}^2/(\tilde{x}\tilde{y}))[\tilde{y} \cdot (c_x + s_x) - \tilde{x} \cdot d_y + i_y \cdot (c_x + s_x) - d_y \cdot (d_x + i_x)]
\end{aligned} \tag{57}$$

But from the position settle analysis in the prior section,

$$\begin{aligned}
&\tilde{L}^2 \cdot [1 + (d_x + i_x)/\tilde{x} + (d_y + i_y)/\tilde{y} + (d_x + i_x)(d_y + i_y)/(\tilde{x}\tilde{y})] \\
&\qquad = \tilde{L}^2 \cdot \left[1 + (\delta L/\tilde{L})\left(\sqrt{\frac{\tilde{P}}{P'}} + \sqrt{\frac{P'}{\tilde{P}}}\right) + (\delta L/\tilde{L})^2\right] \\
&\qquad = \tilde{L}^2 \cdot [1 + (2\delta L/\tilde{L})\cosh(p/2) + (\delta L/\tilde{L})^2]
\end{aligned} \tag{58}$$

where $p = \ln(\tilde{P}/P')$. Considering $\tilde{x}\tilde{y} = \tilde{L}^2$ as well, these expressions further reduce the invariant at I to

$$\begin{aligned}
(\tilde{L} + \delta\tilde{L})^2 &= \tilde{L}^2 \cdot [1 + (2\delta L/\tilde{L})\cosh(p/2) + (\delta L/\tilde{L})^2] \\
&\quad + \tilde{y} \cdot (c_x + s_x) - \tilde{x} \cdot d_y + i_y \cdot (c_x + s_x) - d_y \cdot (d_x + i_x) \\
&= \tilde{L}^2 \cdot [1 + (2\delta L/\tilde{L})\cosh(p/2) + (\delta L/\tilde{L})^2] \\
&\quad + (c_x + s_x)[\tilde{y} + i_y] - d_y \cdot [\tilde{x} + d_x + i_x] \\
&= \tilde{L}^2 \cdot [1 + 2\delta L/\tilde{L} + (\delta L/\tilde{L})^2] + (2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] \\
&\quad + (c_x + s_x)[\tilde{y} + i_y] - d_y \cdot [\tilde{x} + d_x + i_x] \\
&= (\tilde{L} + \delta L)^2 + (2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] \\
&\quad + (c_x + s_x)[\tilde{y} + i_y] - d_y \cdot [\tilde{x} + d_x + i_x] \qquad (59)
\end{aligned}$$

Thus for at least the original liquidity liability to be returned to the pool post-liquidation $(\tilde{L} + \delta\tilde{L})^2 - (\tilde{L} + \delta L)^2 \geq 0$, we need

$$(2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] + (c_x + s_x)[\tilde{y} + i_y] - d_y \cdot [\tilde{x} + d_x + i_x] \geq 0 \qquad (60)$$

where, given $\cosh(p/2) \geq 1$ for all p, the first term is always greater than zero for any pre-liquidation state

$$(2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] \geq 0 \qquad (61)$$

We set insurance balances (i_x, i_y) to ensure the other terms are greater than zero when pre-liquidation price is greater than or equal to the position bankruptcy price $\tilde{P} \geq P_b$, where the bankruptcy price

$$P_b = \frac{d_y}{c_x + s_x} \qquad (62)$$

is the pool price at which position value equals zero $V_y(P_b) = 0$. If we don't allow the trader to take out more leverage than the initial max leverage for the position even if the position is in profit, this will require that margin c_x satisfies the minimum margin condition

$$(c_x + s_x) \geq (1 + M) \cdot \frac{d_y}{P} \qquad (63)$$

at all times. The pool contract enforces this requirement by storing the pool tick before position open. One then has that the largest possible bankruptcy pool price is

$$\frac{d_y}{c_x + s_x} \leq \frac{P}{1 + M} = P_{b_{max}} \qquad (64)$$

Returning to (60), we'll set insurance balances such that the last two terms at bankruptcy are also greater than zero

$$\begin{aligned}
&\lim_{\tilde{P}\to P_b}\left[(c_x+s_x)[\tilde{y}+i_y]-d_y\cdot[\tilde{x}+d_x+i_x]\right]\\
&=(c_x+s_x)[\tilde{y}+i_y]-\tilde{P}(c_x+s_x)\cdot[\tilde{x}+d_x+i_x]\\
&=(c_x+s_x)\left[\tilde{y}+i_y-\tilde{P}\tilde{x}-\tilde{P}(d_x+i_x)\right]\\
&=(c_x+s_x)\left[i_y-\tilde{P}(d_x+i_x)\right]\geq 0
\end{aligned}\tag{65}$$

which occurs when

$$i_y \geq P_b\cdot(d_x+i_x)\tag{66}$$

But from a prior section,

$$d_x+i_x=\delta x\cdot\left(\frac{1-\delta y/y}{1-i_y/y}\right)$$

So

$$\begin{aligned}
i_y &\geq \frac{P_b}{P}\cdot P\cdot(d_x+i_x)\\
&= \frac{P_b}{P}\cdot P\cdot\delta x\cdot\left(\frac{1-\delta y/y}{1-i_y/y}\right)\\
&= \frac{P_b}{P}\cdot\delta y\cdot\left(\frac{1-\delta y/y}{1-i_y/y}\right)
\end{aligned}\tag{67}$$

If we set insurance balances to be their largest possible amounts to satisfy this condition on i_y, we'd need to take $P_b\to P_{b_{max}}$. Multiplying both sides by $1-i_y/y$, we have

$$\begin{aligned}
i_y\cdot(1-i_y/y) &\geq \lim_{P_b\to P_{b_{max}}}\frac{P_b}{P}\cdot\delta y\cdot(1-\delta y/y)\\
&= \frac{\delta y}{1+M}\cdot(1-\delta y/y)
\end{aligned}\tag{68}$$

We choose the Y token insurance value to equal the bound. Carrying over terms to the right hand side and expanding

$$0 = (i_y/y)^2 - (i_y/y) + \frac{\delta y/y}{1+M} \cdot (1 - \delta y/y) \tag{69}$$

Solutions to the quadratic are

$$\frac{1}{2}\left[1 \pm \sqrt{1 - \frac{4}{1+M}\frac{\delta y}{y}\left(1 - \frac{\delta y}{y}\right)}\right]$$

which requires we take the $-$ negative root to ensure $i_y \leq \delta y \leq y$. We then set

$$i_y = \frac{y}{2}\left[1 - \sqrt{1 - \frac{4}{1+M}\frac{\delta y}{y}\left(1 - \frac{\delta y}{y}\right)}\right] \tag{70}$$

to ensure at least the original liquidity is returned to the pool at pre-liquidation pool prices greater than or equal to the highest bankruptcy price for the position given maintenance requirements. We set $i_x = i_y/P$ to maintain token ratios removed from pool at open.

Accounting for all terms together in (60) with the insurance function (70) shows that liquidity returned to the pool will actually be at least the original liquidity fronted at open for *any* pre-liquidation pool state, as

$$i_y \cdot (1 - i_y/y) = \frac{\delta y}{1+M} \cdot (1 - \delta y/y)$$

implies

$$i_y = P_{b_{max}}(d_x + i_x)$$

Given the minimum maintenance requirement enforced by the pool contract of

$$c_x \geq (1+M) \cdot \frac{d_y}{P} - s_x = \frac{d_y}{P_{b_{max}}} - s_x \tag{71}$$

we have

$$
\begin{aligned}
&(2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] + (c_x + s_x)[\tilde{y} + i_y] - d_y \cdot [\tilde{x} + d_x + i_x] \\
&= (2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] + (d_y / P_{b_{max}})[\tilde{y} + i_y] - d_y \cdot [\tilde{x} + d_x + i_x] \\
&= (2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] + d_y \cdot \left[\frac{\tilde{y}}{P_{b_{max}}} + \frac{i_y}{P_{b_{max}}} - \tilde{x} - (d_x + i_x)\right] \\
&= (2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] + d_y \cdot \left[(\tilde{y}/P_{b_{max}}) + (d_x + i_x) - \tilde{x} - (d_x + i_x)\right] \\
&= (2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] + d_y \cdot [(\tilde{y}/P_{b_{max}}) - \tilde{x}] \\
&= (2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] + \tilde{x} \cdot d_y \cdot [(\tilde{P}/P_{b_{max}}) - 1] \\
&= (2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] + \tilde{x} \cdot (d_y + i_y) \cdot [(\tilde{P}/P_{b_{max}}) - 1] \\
&\quad - \tilde{x} \cdot i_y \cdot [(\tilde{P}/P_{b_{max}}) - 1] \\
&= (2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] + \tilde{x} \cdot \delta L \sqrt{P'} \cdot [(\tilde{P}/P_{b_{max}}) - 1] \\
&\quad - \tilde{x} \cdot i_y \cdot [(\tilde{P}/P_{b_{max}}) - 1] \qquad (72)
\end{aligned}
$$

where we used in the last line

$$d_y + i_y = \delta L \sqrt{P'}$$

Further, since

$$
\begin{aligned}
d_x + i_x &= \delta L / \sqrt{P'} \\
i_y &= P_{b_{max}}(d_x + i_x)
\end{aligned}
$$

we can express i_y in terms of δL liquidity fronted at open and the resulting price P'

$$
\begin{aligned}
i_y &= P_{b_{max}}(d_x + i_x) \\
&= P_{b_{max}}(\delta L / \sqrt{P'}) \qquad (73)
\end{aligned}
$$

Plugging back into (72),

$$
\begin{aligned}
&(2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] + (c_x + s_x)[\tilde{y} + i_y] - d_y \cdot [\tilde{x} + d_x + i_x] \\
&\quad = (2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] + \tilde{x} \cdot \delta L\sqrt{P'} \cdot [(\tilde{P}/P_{b_{max}}) - 1] \\
&\qquad\qquad - \tilde{x} \cdot i_y \cdot [(\tilde{P}/P_{b_{max}}) - 1] \\
&\quad = (2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] + \tilde{x} \cdot \delta L\sqrt{P'} \cdot [(\tilde{P}/P_{b_{max}}) - 1] \\
&\qquad\qquad - \tilde{x} \cdot P_{b_{max}}(\delta L/\sqrt{P'}) \cdot [(\tilde{P}/P_{b_{max}}) - 1] \\
&\quad = (2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] + \tilde{x} \cdot \delta L\sqrt{P'} \cdot \Big[(\tilde{P}/P_{b_{max}}) - 1 \\
&\qquad\qquad - (P_{b_{max}}/P') \cdot [(\tilde{P}/P_{b_{max}}) - 1]\Big] \\
&\quad = (2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] + \tilde{x} \cdot \delta L\sqrt{P'} \cdot \Big[(\tilde{P}/P_{b_{max}}) - 1 \\
&\qquad\qquad - (\tilde{P}/P') + (P_{b_{max}}/P')\Big] \qquad (74)
\end{aligned}
$$

Expanding the first term using $p = \ln(\tilde{P}/P')$ and

$$
2\cosh(p/2) = \sqrt{(\tilde{P}/P')} + \sqrt{(P'/\tilde{P})} \qquad (75)
$$

we have

$$
\begin{aligned}
(2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] &= (\delta L \cdot \tilde{L})\left[\sqrt{(\tilde{P}/P')} + \sqrt{(P'/\tilde{P})} - 2\right] \\
&= \delta L \cdot \tilde{L} \cdot \left[\sqrt{(\tilde{P}/P')} + \sqrt{(P'/\tilde{P})} - 2\right] \\
&= \delta L \cdot \tilde{x} \cdot \sqrt{\tilde{P}} \cdot \left[\sqrt{(\tilde{P}/P')} + \sqrt{(P'/\tilde{P})} - 2\right] \\
&= \tilde{x} \cdot \delta L\sqrt{P'} \cdot \sqrt{(\tilde{P}/P')} \cdot \left[\sqrt{(\tilde{P}/P')} + \sqrt{(P'/\tilde{P})} - 2\right] \\
&= \tilde{x} \cdot \delta L\sqrt{P'} \cdot \left[(\tilde{P}/P') + 1 - 2\sqrt{(\tilde{P}/P')}\right] \qquad (76)
\end{aligned}
$$

Returning to (74),

$$
\begin{aligned}
&(2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] + (c_x + s_x)[\tilde{y} + i_y] - d_y \cdot [\tilde{x} + d_x + i_x] \\
&= (2\delta L \cdot \tilde{L})[\cosh(p/2) - 1] + \tilde{x} \cdot \delta L\sqrt{P'} \cdot \Big[(\tilde{P}/P_{b_{max}}) - 1 \\
&\qquad - (\tilde{P}/P') + (P_{b_{max}}/P')\Big] \\
&= \tilde{x} \cdot \delta L\sqrt{P'} \cdot \left[(\tilde{P}/P') + 1 - 2\sqrt{(\tilde{P}/P')}\right] + \tilde{x} \cdot \delta L\sqrt{P'} \cdot \Big[(\tilde{P}/P_{b_{max}}) - 1 \\
&\qquad - (\tilde{P}/P') + (P_{b_{max}}/P')\Big] \\
&= \tilde{x} \cdot \delta L\sqrt{P'} \cdot \Big[(\tilde{P}/P') + 1 - 2\sqrt{(\tilde{P}/P')} + (\tilde{P}/P_{b_{max}}) - 1 \\
&\qquad - (\tilde{P}/P') + (P_{b_{max}}/P')\Big] \\
&= \tilde{x} \cdot \delta L\sqrt{P'} \cdot \left[(\tilde{P}/P_{b_{max}}) - 2\sqrt{(\tilde{P}/P')} + (P_{b_{max}}/P')\right] \\
&= \tilde{x} \cdot \frac{\delta L\sqrt{P'}}{P_{b_{max}}} \cdot \left[\tilde{P} - 2\sqrt{\tilde{P}} \cdot (P_{b_{max}}/\sqrt{P'}) + (P_{b_{max}}/\sqrt{P'})^2\right] \\
&= \tilde{x} \cdot \frac{\delta L\sqrt{P'}}{P_{b_{max}}} \cdot \left(\sqrt{\tilde{P}} - \frac{P_{b_{max}}}{\sqrt{P'}}\right)^2 \\
&= \tilde{L} \cdot \frac{\delta L}{P_{b_{max}}} \cdot \sqrt{\frac{P'}{\tilde{P}}} \cdot \left(\sqrt{\tilde{P}} - \frac{P_{b_{max}}}{\sqrt{P'}}\right)^2 \geq 0
\end{aligned} \tag{77}
$$

which is greater than or equal to zero for all possible values of pre-liquidation pool state $(\tilde{L}, \tilde{P})$. Thus,

$$
(\tilde{L} + \delta\tilde{L})^2 \geq (\tilde{L} + \delta L)^2 \tag{78}
$$

for any pre-liquidation pool state given the manner in which we set insurance balances (i_x, i_y) as well as the minimum margin constraint on c_x. LPs are guaranteed to get at least the original liquidity fronted back upon position liquidation.

Both the prior liquidation condition determining position safety as a function of oracle TWAP and the minimum margin requirement c_x taken together mean the trader must maintain a collateral amount c_x that satisfies the more conservative bound between the oracle TWAP price and the pool price pre-open

$$
c_{x_{safe}} \geq (1 + M) \cdot \frac{d_y}{\min(P, \bar{P}_{t-\tau,t})} - s_x \tag{79}
$$

for the position to remain safe. This conservatively constrains maximum position leverage traders can take for all t.