Appendix B

Rejection method

Rejection methods sample a variety of probability densities in a quite different way from the methods discussed in Chapter 2. The methods in Section 2.3 always returned a sample for each random number drawn. In a rejection method some draws do not return a sample: They are rejected. There are a number of rejection methods, some of which target specific densities. Of the general methods, von Neumann's is the most famous. It is quite general purposed, works in more than one dimension, and does not require the distribution of interest $f(x)$ to be normalized.

Suppose $f(x)$ equals $c\,p(x)$ where $p(x)$ is a probability density and c is the unknown normalization of $f(x)$, and $g(x)$ is another probability density. The procedure described in Algorithm 48 enables the sampling of x from $p(x)$. The loop generates an x until it is accepted.The other samples are rejected.

It follows directly from the **until** statement that because ζ is drawn from a uniform distribution over [0, 1], the probability of accepting an x is

$$P(\text{accept}|x) = P\left(\zeta < \frac{f(x)}{Cg(x)}\right) = \frac{cp(x)}{Cg(x)},$$

where C is a constant such that $Cg(x) \geq f(x)$.

Algorithm 48 Rejection method.

Input: Function $f(x)$ to be sampled, some density $g(x)$, and a constant C such that $Cg(x) \geq f(x)$.

repeat

Sample x from $g(x)$;

Compute the ratio $R(x) = \frac{f(x)}{Cg(x)}$ (≤ 1) ;

Draw a uniform random number $\zeta \in [0, 1]$;

until $\zeta < R(x)$.

return x.

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Next, we have

$$P(\text{accept}) = \int P(\text{accept}|x)g(x)dx$$
$$= \int \frac{cp(x)}{Cg(x)} g(x)dx = c/C.$$

We have $P(\text{accept}|x)$ but want $P(x|\text{accept})$. To find it we use Bayes's theorem:

$$P(x|\text{accept}) = \frac{P(x, \text{accept})}{P(\text{accept})}. \tag{B.1}$$

Because x is sampled independently of ζ, the two events in the joint probability are statistically independent, and we can write

$$P(x|\text{accept}) = \frac{P(\text{accept}|x)g(x)}{P(\text{accept})}$$
$$= \frac{[cp(x)/Cg(x)]g(x)}{c/C} = p(x),$$

which proves the validity of the algorithm.

The efficiency of the algorithm depends on the ratio c/C and the choice of $g(x)$. The first condition follows from (B.1): A small value of c/C implies low acceptance. The second is illustrated in Fig. B.1. Here, we assumed $f(x) = p(x)$ and $g(x)$ is uniform over the sampling interval (a, b). The algorithm is a variant of blindly throwing darts at a target and taking the ratio of the number of target hits relative to the number of times the bounding box is hit. For efficiency, $g(x)$ should "cover" $f(x)$ tightly and cost little to evaluate.

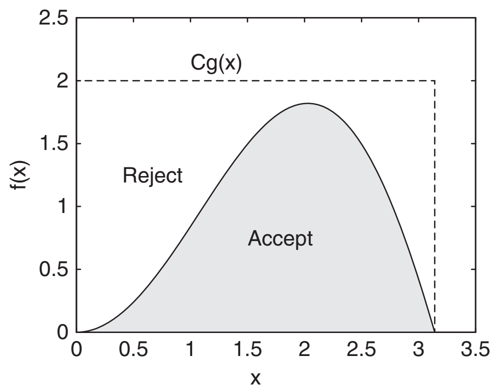

Figure B.1 The rejection algorithm. The function to be sampled is $f(x) = x\sin(x)$. The covering function $g(x)$ is uniform.

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