Co-Debruijn: Everybody's Got To Be Somewhere[2]

From Debruijn to co-Debruijn using Category Theory

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Outline

The Category of Scopes: Δ_+^X

Definition

Let Δ_+^X be the category of scopes.

- Objects: $\bar{x}, \bar{y}, \bar{s} \in |\Delta_+^X| = X^*$
- Morphisms: $f, g \in \Delta_+^X(\bar{x}, \bar{y})$ for $\bar{x}, \bar{y} \in X^*$ are inductively defined:

$$\frac{}{\varepsilon \sqsubseteq \varepsilon} \cdot \qquad\qquad \frac{\bar{x} \sqsubseteq \bar{y}}{\bar{x}x \sqsubseteq \bar{y}x} 1 \qquad\qquad \frac{\bar{x} \sqsubseteq \bar{y}}{\bar{x} \sqsubseteq \bar{y}y} 0$$

Corollary

The initial object of the Δ_+^X category is the empty scope ε with the $\bar{0}$ as the unique morphism.

Remark

Morphisms in Δ_+^X can be represented by *bit vectors* $\bar{b} \in \{0, 1\}^*$ with one bit per variable of the target scope telling whether it has been mapped to or skipped by the source scope.

Objects & Morphisms in $\Delta_+^\top$

Example

Let $X = \top$ (where $\top$ is the set with exactly one element $\langle\rangle$).
Thus, Objects $\bar{x} \in X^*$ represents numbers.

$$3 \xrightarrow{\;10101\;} 5$$
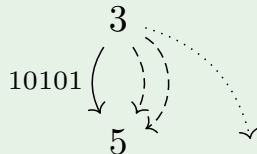
Identity and Composition in $\Delta_+^\top$

Example

$$5 \xrightarrow{\;id\;5\;} 5$$

Example

$$2 \xrightarrow{\;101\;} 3 \qquad 3 \xrightarrow{\;10101\;} 5 \qquad 2 \xrightarrow{\;10001\;} 5$$

Δ_+^X is in Fact a Category

Lemma

In Δ_+^X every object $\bar{x} \in X^*$ has an identity morphism, i.e. we can construct an identity morphism for $\bar{x}$ using the inference rules.

Proof.

$$
\begin{aligned}
id &: (\bar{x} : X^*) \to \bar{x} \sqsubseteq \bar{x} \\
id \; \varepsilon &= \cdot \\
id \; \bar{x}x &= (id \; \bar{x})1 \qquad\qquad \square
\end{aligned}
$$

Corollary

$$
\begin{aligned}
id-l &: id; f = f \\
id-r &: f; id = f
\end{aligned}
$$

Lemma

In Δ_+^X two morphisms $f : \bar{x} \sqsubseteq \bar{y}$ and $g : \bar{y} \sqsubseteq \bar{z}$ compose to a morphism $f; g : \bar{x} \sqsubseteq \bar{z}$, i.e. we can construct a morphism $f; g$ from f and g using the inference rules.

Proof.

$$
\begin{aligned}
\_;\_ \quad &: \bar{x} \sqsubseteq \bar{y} \to \bar{y} \sqsubseteq \bar{z} \to \bar{x} \sqsubseteq \bar{z} \\
\cdot \; ; \; \cdot &= \cdot \\
f1 \; ; \; g1 &= (f;g)1 \\
f0 \; ; \; g1 &= (f;g)0 \\
f \; ; \; g0 &= (f;g)0 \qquad\qquad \square
\end{aligned}
$$

Corollary

$$
\begin{aligned}
assoc \quad &: f;(g;h) = (f;g);h \\
antisym \quad &: (f : \bar{x} \sqsubseteq \bar{y}) \to (g : \bar{y} \sqsubseteq \bar{z}) \to \bar{x} = \bar{y} \wedge f = g = id \; \bar{x}
\end{aligned}
$$

Intrinsically Scoped Debruijn Syntax via $\Delta_+^\top$

Definition

Let $Tm : |\Delta_+^\top| \to Set$ be inductively defined:

$$\frac{\langle\rangle \sqsubseteq \bar{x}}{Tm\ \bar{x}}\ \#\qquad\qquad\frac{Tm\ \bar{x}\quad Tm\ \bar{x}}{Tm\ \bar{x}}\ \$\qquad\qquad\frac{Tm\ \bar{x}\langle\rangle}{Tm\ \bar{x}}\ \lambda$$

Example

Lifting Scope Indexed Terms using Composition in $\Delta_+^\top$

Lemma

Given an intrinsically scoped term $t \in Tm\ \bar{x}$ we can lift t to a $Tm\ \bar{y}$, if there exists a morphism $\bar{x} \sqsubseteq \bar{y} \in \Delta_+^\top(\bar{x}, \bar{y})$, i.e. $\bar{x}$ is a subscope of $\bar{y}$.

Proof.

$$\_\uparrow\_ \qquad : Tm\ \bar{x} \to \bar{x} \sqsubseteq \bar{y} \to Tm\ \bar{y}$$
$$(\#\ v)\quad \uparrow f\ =\ \#\ (v; f)$$
$$(t_1\ \$\ t_2) \uparrow f\ =\ (t_1 \uparrow f)\ \$\ (t_2 \uparrow f)$$
$$(\lambda\ t)\quad \uparrow f\ =\ \lambda\ (t \uparrow Sf) \qquad\qquad\qquad\qquad \square$$
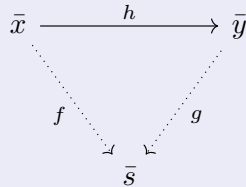
The Slice Category of Subscopes: $\Delta_+^X \setminus \bar{s}$

Definition

Let $\Delta_+^X \setminus \bar{s}$ be the category of subscopes for a given $\bar{s} \in X^*$.

- Objects: $\bar{b}, (\bar{x}, f) \in |\Delta_+^X \setminus \bar{s}| = (\bar{x} : X^* \times \Delta_+^X(\bar{x}, \bar{s}))$
- Morphisms: $h \in [\Delta_+^X \setminus \bar{s}]((\bar{x}, f), (\bar{y}, g))$ such that $f = h; g$

$$\bar{x} \xrightarrow{\quad h \quad} \bar{y}$$

$$f \searrow \qquad \swarrow g$$
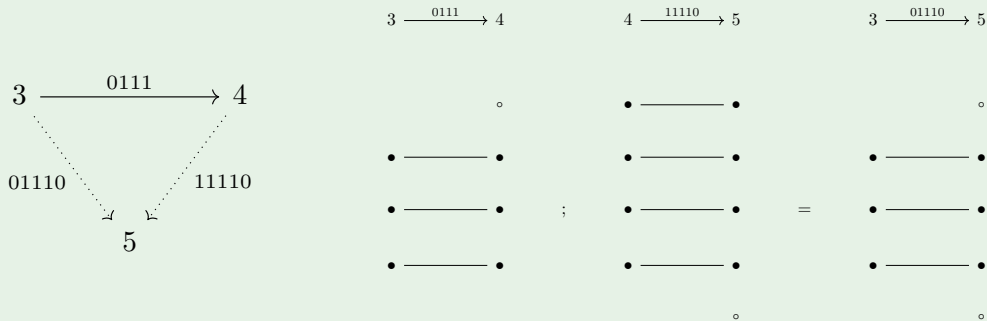
$$\bar{s}$$

Corollary

The initial object of the $\Delta_+^X \setminus \bar{s}$ category is the empty subscope $(\varepsilon, \bar{0})$.

Remark

Objects in $\Delta_+^X \setminus \bar{s}$ can be represented by *bit vectors* $\bar{b} \in \{0, 1\}^*$ with one bit per variable of scope $\bar{s}$, telling whether it has been selected.

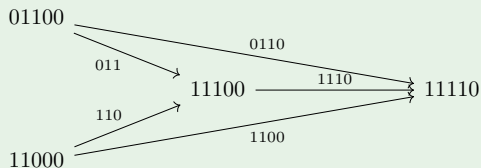# Objects & Morphisms in $\Delta_+^T \setminus 5$

Example

$$3 \xrightarrow{\;0111\;} 4 \qquad\qquad 4 \xrightarrow{\;11110\;} 5 \qquad\qquad 3 \xrightarrow{\;01110\;} 5$$

$$3 \xrightarrow{\;0111\;} 4$$

$$01110 \qquad\qquad 11110$$

$$5$$

Alternatively:

- $(3, 01110) \xrightarrow{\;0111\;} (4, 11110)$

- $01110 \xrightarrow{\;0111\;} 11110$

Coproducts in $\Delta_+^X \setminus \bar{s}$

Theorem

Objects in the slice category $\bar{b}_1, \bar{b}_2 \in |\Delta_+^X \setminus \bar{s}|$ have a coproduct object $\bar{b}_1 + \bar{b}_2$ if there exist morphisms $l \in [\Delta_+^X \setminus \bar{s}](\bar{b}_1, \bar{b}_1 + \bar{b}_2)$ and $r \in [\Delta_+^X \setminus \bar{s}](\bar{b}_2, \bar{b}_1 + \bar{b}_2)$. Then for every $f \in [\Delta_+^X \setminus \bar{s}](\bar{b}_1, \bar{b}_3)$ and $g \in [\Delta_+^X \setminus \bar{s}](\bar{b}_2, \bar{b}_3)$, there exists a unqiue $h \in [\Delta_+^X \setminus \bar{s}](\bar{b}_1 + \bar{b}_2, \bar{b}_3)$ such that $f = l; h$ and $g = r; h$.

Example

Remark

The coproduct $\bar{b}_1 + \bar{b}_2$ of two subscopes $\bar{b}_1, \bar{b}_2$ corresponds to the minimal subscope covering both $\bar{b}_1$ and $\bar{b}_2$. The coproduct $\bar{b}_1 + \bar{b}_2$ can be computed by pointswise disjunction of $\bar{b}_1$ and $\bar{b}_2$.

Category of Sets Indexed by Scopes

Definition

Let Set_X be the category of sets indexed by scopes $\bar{x} \in X^*$.

- Objects: $T, S \in |Set_X| = X^* \to Set = \bar{X}$
- Morphisms: $f_. \in Set_X(T, S) = (\bar{x} \in X^*) \to T(\bar{x}) \to S(\bar{x}) = T \overset{.}{\to} S$

Definition

Let $\_ \Uparrow \_ : \bar{X} \to \bar{X} = (T, \bar{x}) \mapsto (T(\bar{s}) \times \bar{s} \sqsubseteq \bar{x})$.
We define $Ref : Set_X \overset{.}{\to} Set_X$ to be the endofunctor induced by the mapping

- $Ref(T) = \bar{x} \mapsto T \Uparrow \bar{x} \in \bar{X}$
- $Ref(f_.) = (t, h) \mapsto (f_.(t), h) \in T \overset{.}{\to} S$

Remark

The set $T \Uparrow \bar{x}$ packs an set $T \in \bar{X}$ indexed by $\bar{x} \in X^*$ applied to a subscope $\bar{s}$ of $\bar{x}$, together with a selection $h \in |\Delta_+^X \setminus \bar{x}|$ of the variables of T.

Δ_+^X makes Ref a Monad!

Theorem

The functor $Ref : Set_X \to Set_X$ gives rise to a monad with the two natural transformations

- $unit : Id(T) \overset{\cdot}{\to} Ref(T) = t \mapsto (t, id)$
- $mult : Ref(Ref(T)) \overset{\cdot}{\to} Ref(T) = ((t, h_1)h_2) \mapsto (t, h_1; h_2)$

Example

$$Tm \dashrightarrow^{unit} \bar{x} \mapsto (Tm\ \bar{x}, \bar{x} \sqsubseteq \bar{x})$$

$$_{Ref}$$

$$\bar{y} \mapsto ([\bar{x} \mapsto (Tm\ \bar{x}, \bar{x} \sqsubseteq \bar{x})]\bar{s}, \bar{s} \sqsubseteq \bar{y}) \dashrightarrow^{mult} \bar{y} \mapsto (Tm\ \bar{s}, \bar{s} \sqsubseteq \bar{y})$$

The Notion of Relevant Pairs

Definition

Let $\_ \times_R \_ : \bar{X} \to \bar{X} \to \bar{X} = (T, S, \bar{x}) \mapsto ((\_, \bar{b}_1 : T \Uparrow \bar{x}) \times (\_, \bar{b}_2 : S \Uparrow \bar{x}) \times \exists \bar{b}(\bar{b} \equiv \bar{b}_1 + \bar{b}_2))$.
We write
$\_,_R \_ : T \Uparrow \bar{x} \to S \Uparrow \bar{x} \to (T \times_R S) \Uparrow \bar{x} = ((\_, \bar{b}_1), (\_, \bar{b}_2)) \mapsto (L(\bar{b}_1 + \bar{b}_2), R(\bar{b}_1 + \bar{b}_2), \bar{b}_1 + \bar{b}_2)$
to construct a $T \times_R S$.

Example

We'll look at Tm
Let $..1..2.. : Tm5$

References

[1] Conor McBride. *Cats and types: Best friends?* Aug. 2021. URL:
https://www.youtube.com/watch?v=05IJ3YL8p0s.

[2] Conor McBride. "Everybody's Got To Be Somewhere". In: *Electronic Proceedings in
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