# From De Bruijn to co-De Bruijn using Category Theory

## Everybody's Got To Be Somewhere[coDe Bruijn]

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# Outline

# The Category of Scopes: $\Delta_+^X$

## Definition

Let $\Delta_+^X$ be the category of scopes.

- Objects: $\bar{x}, \bar{y}, \bar{s} \in |\Delta_+^X| = X^*$
- Morphisms: $f, g \in \Delta_+^X(\bar{x}, \bar{y})$ for $\bar{x}, \bar{y} \in X^*$ are inductively defined:

$$\frac{}{\varepsilon \sqsubseteq \varepsilon}\ \cdot \qquad\qquad \frac{\bar{x} \sqsubseteq \bar{y}}{\bar{x}x \sqsubseteq \bar{y}x}\ 1 \qquad\qquad \frac{\bar{x} \sqsubseteq \bar{y}}{\bar{x} \sqsubseteq \bar{y}y}\ 0$$

## Corollary

*The initial object of the $\Delta_+^X$ category is the empty scope $\varepsilon$ with the $\bar{0}$ as the unique morphism.*

## Remark

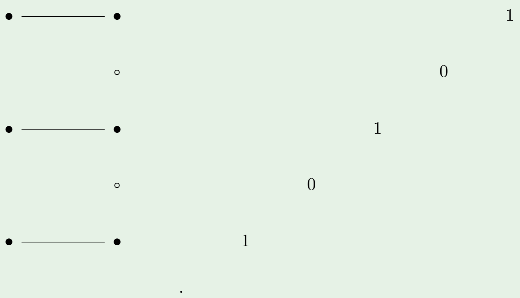
Morphisms in $\Delta_+^X$ can be represented by *bit vectors* $\bar{b} \in \{0,1\}^*$ with one bit per variable of the target scope telling whether it has been mapped to or skipped by the source scope.

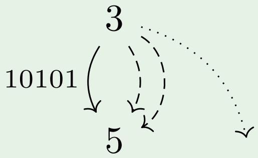
# Objects & Morphisms in $\Delta_+^\top$

## Example

Let $X = \top$ (where $\top$ is the set with exactly one element $\langle\rangle$).
Thus, Objects $n \in X^*$ represents numbers.

$$3 \xrightarrow{10101} 5$$

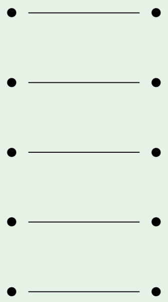


# Identity and Composition in $\Delta_+^\top$

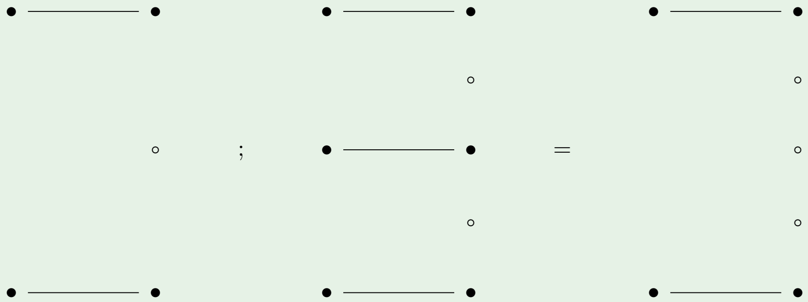
## Example

$$5 \xrightarrow{id\ 5} 5$$



## Example

$$2 \xrightarrow{101} 3 \qquad 3 \xrightarrow{10101} 5 \qquad 2 \xrightarrow{10001} 5$$



# $\Delta_+^X$ is in Fact a Category

**Lemma**

*In $\Delta_+^X$ every object $\bar{x} \in X^*$ has an identity morphism, i.e. we can construct an identity morphism for $\bar{x}$ using the inference rules.*

**Proof.**

$$
\begin{array}{lcl}
id & : & (\bar{x} : X^*) \to \bar{x} \sqsubseteq \bar{x} \\
id\ \varepsilon & = & \cdot \\
id\ \bar{x}x & = & (\mathrm{id}\ \bar{x})1
\end{array}
$$

□

**Corollary**

$$
\begin{array}{lcl}
id - l & : & id; f = f \\
id - r & : & f; id = f
\end{array}
$$

**Lemma**

*In $\Delta_+^X$ two morphisms $f : \bar{x} \sqsubseteq \bar{y}$ and $g : \bar{y} \sqsubseteq \bar{z}$ compose to a morphism $f;g : \bar{x} \sqsubseteq \bar{z}$, i.e. we can construct a morphism $f;g$ from $f$ and $g$ using the inference rules.*

**Proof.**

$$
\begin{array}{lcl}
\_;\_ & : & \bar{x} \sqsubseteq \bar{y} \to \bar{y} \sqsubseteq \bar{z} \to \bar{x} \sqsubseteq \bar{z} \\
\cdot\ \ ;\ \cdot & = & \cdot \\
f1\ ;\ g1 & = & (f;g)1 \\
f0\ ;\ g1 & = & (f;g)0 \\
f\ \ ;\ g0 & = & (f;g)0
\end{array}
$$

□

**Corollary**

$$
\begin{array}{lcl}
\textit{assoc} & : & f;(g;h) = (f;g);h \\
\textit{antisym} & : & (f : \bar{x} \sqsubseteq \bar{y}) \to (g : \bar{y} \sqsubseteq \bar{z}) \to \bar{x} = \bar{y} \wedge f = g = id\ \bar{x}
\end{array}
$$

# Intrinsically Scoped De Bruijn Syntax via $\Delta_+^\top$

## Definition

Let $Tm : |\Delta_+^\top| \to Set$ be inductively defined:

$$\frac{1 \sqsubseteq n}{Tm\ n}\ \# \qquad \frac{Tm\ n \quad Tm\ n}{Tm\ n}\ \$ \qquad \frac{Tm\ (n+1)}{Tm\ n}\ \lambda$$

## Example

# The Slice Category of Subscopes: $\Delta_+^X \setminus \bar{s}$

## Definition

Let $\Delta_+^X \setminus \bar{s}$ be the category of subscopes for a given $\bar{s} \in X^*$.

- Objects: $\bar{b}, (\bar{x}, f) \in |\Delta_+^X \setminus \bar{s}| = (\bar{x} : X^* \times \Delta_+^X(\bar{x}, \bar{s}))$
- Morphisms: $h \in [\Delta_+^X \setminus \bar{s}]((\bar{x}, f), (\bar{y}, g))$ such that $f = h; g$

$$\begin{tikzcd} \bar{x} \arrow[rr, "h"] \arrow[dr, dotted, "f"'] & & \bar{y} \arrow[dl, dotted, "g"] \\ & \bar{s} & \end{tikzcd}$$

## Corollary

*The initial object of the $\Delta_+^X \setminus \bar{s}$ category is the empty subscope $(\varepsilon, \bar{0})$.*

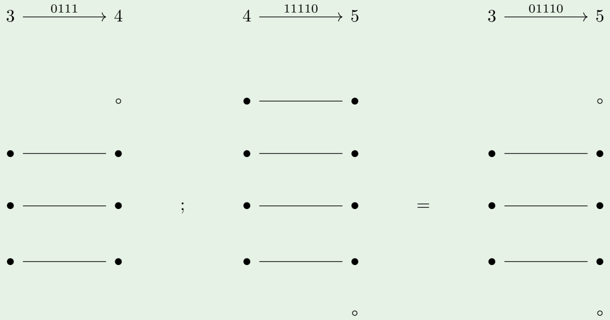
## Remark

Objects in $\Delta_+^X \setminus \bar{s}$ can be represented by *bit vectors* $\bar{b} \in \{0, 1\}^*$ with one bit per variable of scope $\bar{s}$, telling whether it has been selected.

# Objects & Morphisms in $\Delta_+^T \setminus 5$

## Example

$$\begin{tikzcd}
3 \arrow[rr, "0111"] \arrow[dr, dotted, "01110"'] & & 4 \arrow[dl, dotted, "11110"] \\
& 5 &
\end{tikzcd}$$




Alternatively:

- $(3, 01110) \xrightarrow{0111} (4, 11110)$
- $01110 \xrightarrow{0111} 11110$

# Coproducts in $\Delta_+^X \setminus \bar{s}$

## Theorem

*Objects in the slice category* $\bar{b}_1, \bar{b}_2 \in |\Delta_+^X \setminus \bar{s}|$ *have a coproduct object* $\bar{b}_1 + \bar{b}_2$ *if there exist morphisms* $l \in [\Delta_+^X \setminus \bar{s}](\bar{b}_1, \bar{b}_1 + \bar{b}_2)$ *and* $r \in [\Delta_+^X \setminus \bar{s}](\bar{b}_2, \bar{b}_1 + \bar{b}_2)$. *Then for every* $f \in [\Delta_+^X \setminus \bar{s}](\bar{b}_1, \bar{b}_3)$ *and* $g \in [\Delta_+^X \setminus \bar{s}](\bar{b}_2, \bar{b}_3)$, *there exists a unqiue* $h \in [\Delta_+^X \setminus \bar{s}](\bar{b}_1 + \bar{b}_2, \bar{b}_3)$ *such that* $f = l; h$ *and* $g = r; h$.

## Example

$$\begin{tikzcd}
01100 \arrow[dr, "011"'] \arrow[drr, "0110"] & & \\
& 11100 \arrow[r, "1110"] & 11110 \\
11000 \arrow[ur, "110"] \arrow[urr, "1100"'] & &
\end{tikzcd}$$

## Remark

The coproduct $\bar{b}_1 + \bar{b}_2$ of two subscopes $\bar{b}_1, \bar{b}_2$ corresponds to the minimal subscope covering both $\bar{b}_1$ and $\bar{b}_2$. The coproduct $\bar{b}_1 + \bar{b}_2$ can be computed by pointswise disjunction of $\bar{b}_1$ and $\bar{b}_2$.

# Category of Sets Indexed by Scopes

## Definition

Let $Set_X$ be the category of sets indexed by scopes $\bar{x} \in X^*$.

- Objects: $T, S \in |Set_X| = X^* \to Set = \bar{X}$
- Morphisms: $f_. \in Set_X(T, S) = (\bar{x} \in X^*) \to T(\bar{x}) \to S(\bar{x}) = T \dot{\to} S$

## Definition

Let $\_ \Uparrow \_ : \bar{X} \to \bar{X} = (T, \bar{x}) \mapsto (T(\bar{s}) \times \bar{s} \sqsubseteq \bar{x})$.

We define $Ref : Set_X \dot{\to} Set_X$ to be the endofunctor induced by the mapping

- $Ref(T) = \bar{x} \mapsto T \Uparrow \bar{x} \in \bar{X}$
- $Ref(f_.) = (t, h) \mapsto (f_.(t), h) \in T \dot{\to} S$

## Remark

The set $T \Uparrow \bar{x}$ packs an set $T \in \bar{X}$ indexed by $\bar{x} \in X^*$ applied to a subscope $\bar{s}$ of $\bar{x}$, together with a selection $h \in |\Delta_+^X \setminus \bar{x}|$ of the variables of $T$.

# $\Delta_+^X$ makes $Ref$ a Monad!

**Theorem**

*The functor $Ref : Set_X \to Set_X$ gives rise to a monad with the two natural transformations*

- $unit : Id(T) \xrightarrow{\cdot} Ref(T) = t \mapsto (t, id)$
- $mult : Ref(Ref(T)) \xrightarrow{\cdot} Ref(T) = ((t, h_1)h_2) \mapsto (t, h_1; h_2)$

**Example**

$$\begin{tikzcd}[column sep=small]
& Tm \arrow[r, dashed, "unit"] & \bar{x} \mapsto (Tm\ \bar{x}, \bar{x} \sqsubseteq \bar{x}) \arrow[dl, dashed, "Ref"'] \\
\bar{y} \mapsto ([\bar{x} \mapsto (Tm\ \bar{x}, \bar{x} \sqsubseteq \bar{x})]\bar{s}, \bar{s} \sqsubseteq \bar{y}) \arrow[rr, dashed, "mult"] & & \bar{y} \mapsto (Tm\ \bar{s}, \bar{s} \sqsubseteq \bar{y})
\end{tikzcd}$$

# The Notion of Relevant Pairs

## Definition

We define the set of relevant pairs as:

- $\_ \times_R \_ : \bar{X} \to \bar{X} \to \bar{X} = (T, S, \bar{x}) \mapsto ((T \Uparrow \bar{x}) \times (S \Uparrow \bar{x}))$
- $\_ ,_R \_ : T \Uparrow \bar{x} \to S \Uparrow \bar{x} \to (T \times_R S) \Uparrow \bar{x} = ((t_1, \bar{b}_1), (t_2, \bar{b}_2)) \mapsto (((t_1, \bar{b}'_1), (t_2, \bar{b}'_2)), \bar{b}')$
  where $\bar{b}' = \bar{b}_1 | \bar{b}_2$ and $\bar{b}'_{1,2} = ((\bar{b}_1 | \bar{b}_2) \& \bar{b}_{1,2})$

## Example

We again look at $Tm$ indexed by $\Delta_+^T$. Let
$(t_1, \bar{b}_1) = (.., 01100) : Tm \Uparrow 5 = Tm\ 2 \times 2 \sqsubseteq 5$,
$(t_2, \bar{b}_2) = (.., 11000) : Tm \Uparrow 5 = Tm\ 2 \times 2 \sqsubseteq 5$.
Then $\bar{b} = \bar{b}_1 | \bar{b}_2 = 11100$ and $\bar{b}'_1 = 011$, $\bar{b}'_2 = 110$.
Thus $(t_1, \bar{b}_1) ,_R (t_1, \bar{b}_2) = (((t_1, \bar{b}'_1), (t_2, \bar{b}'_2)), (\bar{b}_1 | \bar{b}_2))$
$: (Tm \times_R Tm) \Uparrow 5$
$= (Tm\ 2 \times 2 \sqsubseteq 3) \times (Tm\ 2 \times 2 \sqsubseteq 3) \times (3 \sqsubseteq 5)$

$$\begin{tikzcd}
01100 \arrow[dr, "011"'] \arrow[drr, "01100"] & & \\
& 11100 \arrow[r, "11100"] & 11111 \\
11000 \arrow[ur, "110"] \arrow[urr, "11000"'] & &
\end{tikzcd}$$

# Intrinsically Scoped co-De Bruijn Syntax

## Definition

Let $Tm : |\Delta_+^\top| \to Set$ be inductively defined:

$$\frac{}{Tm\ 1}\ \# \qquad \frac{(Tm \times_R Tm)\ n}{Tm\ n}\ \$ \qquad \frac{Tm\ (n+1)}{Tm\ n}\ \lambda$$

## Example

## What I've (not) told you: This is actually an Agda Paper

- All categorical concepts formalized in Agda
- Suitable representations of objects and morphisms
- Comes with a *universe of metasyntaxes-with-binding*
- Categorical concepts applied to formally reason over programming languages

# Using co-De Bruijn is not hard, category theory is!

# References


[1] Conor McBride. *Cats and types: Best friends?* Aug. 2021. URL: https://www.youtube.com/watch?v=05IJ3YL8p0s.