Co-Debruijn: Everybody's Got To Be Somewhere[2]

From Debruijn to co-Debruijn using Category Theory

Marius Weidner

Chair of Programming Languages, University of Freiburg

January 15, 2024

Outline

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The Category of Scopes: Δ_+^X

Definition

Let Δ_+^X be the category of scopes.

- Objects: $\bar{x} \in |\Delta_+^X| = X^*$
- Morphisms: $f \in \Delta_+^X(\bar{x}, \bar{y})$ for $\bar{x}, \bar{y} \in X^*$ are inductively defined:

$$\frac{}{\varepsilon \sqsubseteq \varepsilon}\ \cdot \qquad \frac{\bar{x} \sqsubseteq \bar{y}}{\bar{x}x \sqsubseteq \bar{y}x}\ 1 \qquad \frac{\bar{x} \sqsubseteq \bar{y}}{\bar{x} \sqsubseteq \bar{y}y}\ 0$$

Remark

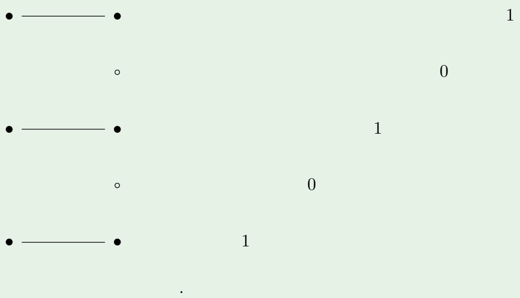
Morphisms in Δ_+^X can be represented by *bit vectors* $\bar{b} \in \{0,1\}^*$ with one bit per variable of the target scope telling whether it has been mapped to or skipped by the source scope.

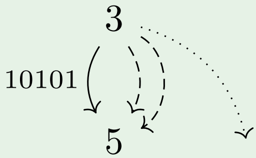
Objects & Morphisms in $\Delta_+^\top$

Example

Let $X = \top$ (where $\top$ is the set with exactly one element $\langle\rangle$).
Thus, Objects $\bar{x} \in X^*$ represents numbers.

$$3 \xrightarrow{10101} 5$$

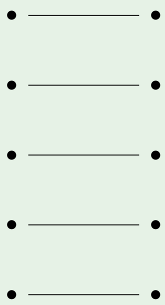




Identity and Composition in $\Delta_+^\top$

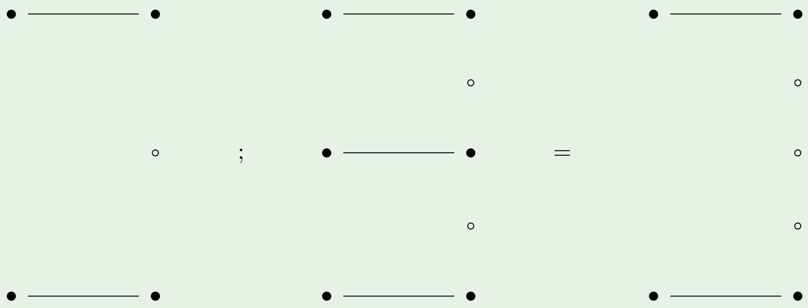
Example

$$5 \xrightarrow{id\ 5} 5$$



Example

$$2 \xrightarrow{101} 3 \qquad 3 \xrightarrow{10101} 5 \qquad 2 \xrightarrow{10001} 5$$



Δ_+^X is in Fact a Category

Lemma

In Δ_+^X every object $\bar{x} \in X^*$ has an identity morphism, i.e. we can construct an identity morphism for $\bar{x}$ using the inference rules.

Proof.

$$
\begin{array}{lcl}
id & : & (\bar{x} : X^*) \to \bar{x} \sqsubseteq \bar{x} \\
id\ \varepsilon & = & \cdot \\
id\ \bar{x}x & = & (\text{id}\ \bar{x})1
\end{array}
$$

□

Corollary

$$
\begin{array}{lcl}
id - l & : & id; f = f \\
id - r & : & f; id = f
\end{array}
$$

Lemma

In Δ_+^X two morphisms $f : \bar{x} \sqsubseteq \bar{y}$ and $g : \bar{y} \sqsubseteq \bar{z}$ compose to a morphism $f;g : \bar{x} \sqsubseteq \bar{z}$, i.e. we can construct a morphism $f;g$ from f and g using the inference rules.

Proof.

$$
\begin{array}{lcl}
\_;\_ & : & \bar{x} \sqsubseteq \bar{y} \to \bar{y} \sqsubseteq \bar{z} \to \bar{x} \sqsubseteq \bar{z} \\
\cdot\ ;\ \cdot & = & \cdot \\
f1\ ;\ g1 & = & (f;g)1 \\
f0\ ;\ g1 & = & (f;g)0 \\
f\ ;\ g0 & = & (f;g)0
\end{array}
$$

□

Corollary

$$
\begin{array}{lcl}
\textit{assoc} & : & f;(g;h) = (f;g);h \\
\textit{antisym} & : & (f : \bar{x} \sqsubseteq \bar{y}) \to (g : \bar{y} \sqsubseteq \bar{z}) \to \bar{x} = \bar{y} \wedge f = g = id\ \bar{x}
\end{array}
$$

Intrinsically Scoped Debruijn Syntax via $\Delta_+^\top$

Definition

Let $Tm : |\Delta_+^\top| \to Set$ be inductively defined:

$$\frac{\langle\rangle \sqsubseteq \bar{x}}{Tm\ \bar{x}}\ Var \qquad \frac{Tm\ \bar{x} \quad Tm\ \bar{x}}{Tm\ \bar{x}}\ App \qquad \frac{Tm\ \bar{x}\langle\rangle}{Tm\ \bar{x}}\ Abs$$

Example

Lifting Scope Indexed Terms using Composition in $\Delta_+^\top$

Lemma

Given an intrinsically scoped term $t \in Tm\ \bar{x}$ we can lift t to a $Tm\ \bar{y}$, if there exists a morphism $\bar{x} \sqsubseteq \bar{y} \in \Delta_+^\top(\bar{x}, \bar{y})$, i.e. $\bar{x}$ is a subscope of $\bar{y}$.

Proof.

$$
\begin{array}{lll}
\_ \uparrow \_ & : & Tm\ \bar{x} \to \bar{x} \sqsubseteq \bar{y} \to Tm\ \bar{y} \\
(Var\ v) \quad\ \uparrow f & = & Var\ (i; f) \\
(App\ t_1\ t_2) \uparrow f & = & App\ (t_1 \uparrow f)\ (t_2 \uparrow f) \\
(Abs\ t) \quad\ \uparrow f & = & Abs\ (t \uparrow S f)
\end{array}
$$

□

The Slice Category of Subscopes: $\Delta_+^X \setminus \bar{s}$

Definition

Let $\Delta_+^X \setminus \bar{s}$ be the category of subscopes for a given $\bar{s} \in X^*$.

- Objects: $(\bar{x}, f) \in |\Delta_+^X \setminus \bar{s}| = \left(X^* \times \Delta_+^X(\bar{x}, \bar{s})\right)$
- Morphisms: $h \in [\Delta_+^X \setminus \bar{s}]((\bar{x}, f), (\bar{y}, g))$ such that $f = h; g$

$$\begin{tikzcd} \bar{x} \arrow[rr, "h"] \arrow[dr, dotted, "f"'] & & \bar{y} \arrow[dl, dotted, "g"] \\ & \bar{s} & \end{tikzcd}$$

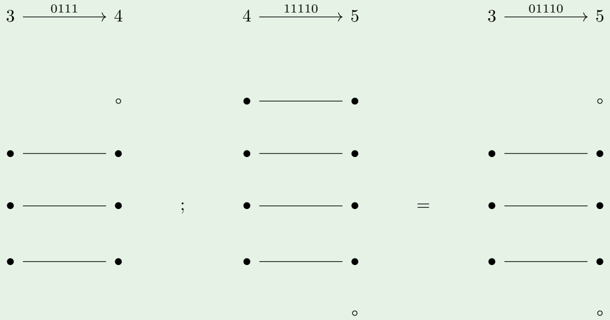
Remark

Objects in $\Delta_+^X \setminus \bar{s}$ can be represented by *bit vectors* $\bar{b} \in \{0,1\}^*$ with one bit per variable telling whether it has been selected.

Objects & Morphisms in $\Delta_+^T \setminus 5$

Example

$$\begin{tikzcd}
3 \arrow[rr, "0111"] \arrow[dr, dotted, "01110"'] & & 4 \arrow[dl, dotted, "11110"] \\
& 5 &
\end{tikzcd}$$



Alternatively:

- $(3, 01110) \xrightarrow{0111} (4, 11110)$
- $01110 \xrightarrow{0111} 11110$

Category of Sets Indexed by Scopes

Definition

Let Set_X be the category of sets indexed by scopes $\bar{x} \in X^*$.

- Objects: $T, S \in |Set_X| = X^* \to Set = \bar{X}$
- Morphisms: $f_. \in Set_X(T, S) = \forall\{\bar{x} \in X^*\} \to T\ \bar{x} \to S\ \bar{x} = T \dot{\to} S$

Definition

Let $Ref : \bar{X} \to \bar{X}$ be the endofunctor induced by the mapping

- $Ref(T) = \bar{x} \mapsto (T\ \bar{s}, \bar{s} \sqsubseteq \bar{x}) \in X^* \to Set$
- $Ref(f_.) = (t, h) \mapsto (f_.\ t, h) \in T \dot{\to} S$

Remark

The Functor Ref packs a set $T \in \bar{X}$ indexed by $\bar{x} \in X^*$ together with a selection $h \in \Delta_+^X \setminus \bar{x}$ of the variables that T actually uses.

Refining Things to only their Subscipe using the Ref Functor

Example

$$\begin{tikzcd}
Tm \arrow[r, "f."] \arrow[rr, bend left, dashed, "Ref"] & T & \bar{x} \mapsto (Tm\ \bar{s}, \bar{s} \sqsubseteq \bar{x}) \arrow[r, "(tm{,}h) \mapsto (f.\ tm{,}h)"] & \bar{x} \mapsto (T\ \bar{s}, \bar{s} \sqsubseteq \bar{x})
\end{tikzcd}$$

Ref is actually a Monad!

Corollary

The functor $Ref : \bar{X} \to \bar{X}$ *gives rise to a monad, i.e. we define two natural transformations*

- $unit : T \xrightarrow{\cdot} Ref(T) = t \mapsto (t, id)$
- $mult : Ref(Ref(T)) \xrightarrow{\cdot} Ref(T) = ((t, h_1)h_2) \mapsto (t, h_1; h_2)$

Example

References

[1] Conor McBride. *Cats and types: Best friends?* Aug. 2021. URL: https://www.youtube.com/watch?v=05IJ3YL8p0s.

[2] Conor McBride. "Everybody's Got To Be Somewhere". In: *Electronic Proceedings in Theoretical Computer Science* 275 (July 2018), pp. 53–69. ISSN: 2075-2180. DOI: 10.4204/eptcs.275.6. URL: http://dx.doi.org/10.4204/EPTCS.275.6.