Co-Debruijn: Everybody's Got To Be Somewhere[2]

From Debruijn to co-Debruijn using Category Theory

Marius Weidner

Chair of Programming Languages, University of Freiburg

January 14, 2024

Outline

The Category of Scopes: Δ_+^X

Definition

Let Δ_+^X be the category of scopes.

- Objects: $\bar{x} \in |\Delta_+^X| = X^*$
- Morphisms: $f \in \Delta_+^X(\bar{x}, \bar{y})$ for $\bar{x}, \bar{y} \in |\Delta_+^X|$ are inductively defined:

$$\frac{}{\varepsilon \sqsubseteq \varepsilon}\ . \qquad\qquad \frac{\bar{x} \sqsubseteq \bar{y}}{\bar{x}x \sqsubseteq \bar{y}x}\ 1 \qquad\qquad \frac{\bar{x} \sqsubseteq \bar{y}}{\bar{x} \sqsubseteq \bar{y}y}\ 0$$

Remark

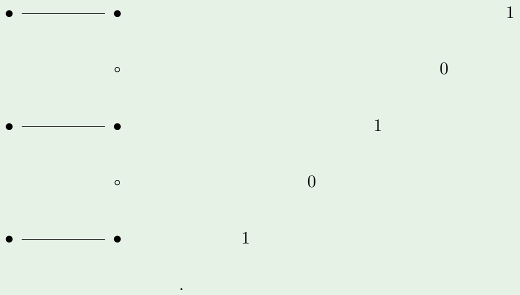
Morphisms in Δ_+^X can be represented by *bit vectors* $\bar{b} \in \{0,1\}^*$ with one bit per variable of the target scope telling whether it has been mapped to or skipped by the source scope.

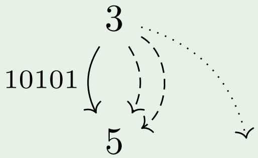
Objects & Morphisms in $\Delta_+^\top$

Example

Let $X = \top$ (where $\top$ is the set with exactly one element $\langle\rangle$).
Thus, Objects $\bar{x} \in X^*$ represents numbers.

$$3 \xrightarrow{10101} 5$$

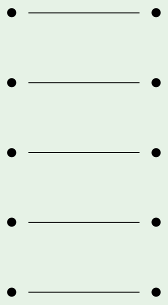




Identity and Composition in $\Delta_+^\top$

Example

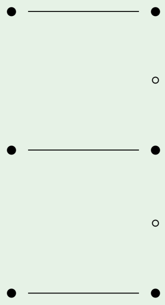
$$5 \xrightarrow{id\ 5} 5$$



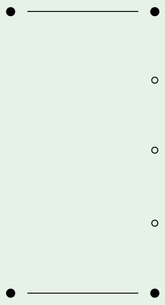
Example

$$2 \xrightarrow{101} 3 \qquad 3 \xrightarrow{10101} 5 \qquad 2 \xrightarrow{10001} 5$$

;



=



Δ_+^X is in Fact a Category

Lemma

In Δ_+^X every object $\bar{x} \in X^*$ has an identity morphism, i.e. we can construct an identity morphism for $\bar{x}$ using the inference rules.

Proof.

$$\begin{array}{lcl} id & : & (\bar{x} : X^*) \to \bar{x} \sqsubseteq \bar{x} \\ id\ \varepsilon & = & \cdot \\ id\ \bar{x}x & = & (\text{id}\ \bar{x})1 \end{array}$$

□

Corollary

$$\begin{array}{lcl} id - l & : & id; f = f \\ id - r & : & f; id = f \end{array}$$

Lemma

In Δ_+^X two morphisms $f : \bar{x} \sqsubseteq \bar{y}$ and $g : \bar{y} \sqsubseteq \bar{z}$ compose to a morphism $f; g : \bar{x} \sqsubseteq \bar{z}$, i.e. we can construct a morphism $f; g$ from f and g using the inference rules.

Proof.

$$\begin{array}{lcl} \_;\_ & : & \bar{x} \sqsubseteq \bar{y} \to \bar{y} \sqsubseteq \bar{z} \to \bar{x} \sqsubseteq \bar{z} \\ \cdot\ ;\ \cdot & = & \cdot \\ f1\ ;\ g1 & = & (f; g)1 \\ f0\ ;\ g1 & = & (f; g)0 \\ f\ \ ;\ g0 & = & (f; g)0 \end{array}$$

□

Corollary

$$\begin{array}{lcl} \textit{assoc} & : & f; (g; h) = (f; g); h \\ \textit{antisym} & : & (f : \bar{x} \sqsubseteq \bar{y}) \to (g : \bar{y} \sqsubseteq \bar{z}) \to \bar{x} = \bar{y} \wedge f = g = \textit{id}\ \bar{x} \end{array}$$

Intrinsically Scoped Debruijn Syntax via $\Delta_+^\top$

Definition

Let $Tm : |\Delta_+^\top| \to Set$ be inductively defined:

$$\frac{\langle\rangle \sqsubseteq \bar{x}}{Tm\ \bar{x}}\ Var \qquad \frac{Tm\ \bar{x} \quad Tm\ \bar{x}}{Tm\ \bar{x}}\ App \qquad \frac{Tm\ \bar{x}\langle\rangle}{Tm\ \bar{x}}\ Abs$$

Example