

Co-Debruijn: Everybody's Got To Be Somewhere^[2]

From Debruijn to co-Debruijn using Category Theory

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3 Wrapping Up: What I've (Not) Told You

- This Is Actually an Agda Paper (!)
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The Category of Scopes: Δ_+^X

Definition

Let Δ_+^X be the category of scopes.

- Objects: $\bar{x}, \bar{y}, \bar{s} \in |\Delta_+^X| = X^*$
- Morphisms: $f, g \in \Delta_+^X(\bar{x}, \bar{y})$ for $\bar{x}, \bar{y} \in X^*$ are inductively defined:

$$\frac{}{\varepsilon \sqsubseteq \varepsilon} \cdot \qquad \frac{\bar{x} \sqsubseteq \bar{y}}{\bar{x}x \sqsubseteq \bar{y}x} 1 \qquad \frac{\bar{x} \sqsubseteq \bar{y}}{\bar{x} \sqsubseteq \bar{y}y} 0$$

Corollary

The initial object of the Δ_+^X category is the empty scope ε with the $\bar{0}$ as the unique morphism.

Remark

Morphisms in Δ_+^X can be represented by *bit vectors* $\bar{b} \in \{0, 1\}^*$ with one bit per variable of the target scope telling whether it has been mapped to or skipped by the source scope.

Objects & Morphisms in Δ_{+}^{\top}

Example

Let $X = \top$ (where \top is the set with exactly one element $\langle \rangle$).

Thus, Objects $\bar{x} \in X^*$ represents numbers.

$$3 \xrightarrow{10101} 5$$

• ————— •

◦

• ————— •

◦

• ————— •

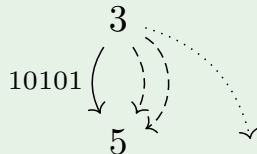
1

1

0

1

0



Identity and Composition in Δ_+^\top

Example

$$5 \xrightarrow{id\ 5} 5$$

• ————— •

• ————— •

• ————— •

• ————— •

• ————— •

Example

$$2 \xrightarrow{101} 3$$

• ————— •

◦ ;

• ————— •

$$3 \xrightarrow{10101} 5$$

• ————— •

◦

• ————— •

◦

• ————— •

=

$$2 \xrightarrow{10001} 5$$

• ————— •

◦

• ————— •

◦

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Δ_+^X is in Fact a Category

Lemma

In Δ_+^X every object $\bar{x} \in X^*$ has an identity morphism, i.e. we can construct an identity morphism for \bar{x} using the inference rules.

Proof.

id : $(\bar{x} : X^*) \rightarrow \bar{x} \sqsubseteq \bar{x}$
 $id \varepsilon = \cdot$
 $id \bar{x}x = (id \bar{x})1$ □

Corollary

$id - l$: $id; f = f$
 $id - r$: $f; id = f$

Lemma

In Δ_+^X two morphisms $f : \bar{x} \sqsubseteq \bar{y}$ and $g : \bar{y} \sqsubseteq \bar{z}$ compose to a morphism $f; g : \bar{x} \sqsubseteq \bar{z}$, i.e. we can construct a morphism $f; g$ from f and g using the inference rules.

Proof.

$_ ; _$: $\bar{x} \sqsubseteq \bar{y} \rightarrow \bar{y} \sqsubseteq \bar{z} \rightarrow \bar{x} \sqsubseteq \bar{z}$
 $\cdot ; \cdot = \cdot$
 $f1 ; g1 = (f; g)1$
 $f0 ; g1 = (f; g)0$
 $f ; g0 = (f; g)0$ □

Corollary

$assoc$: $f; (g; h) = (f; g); h$
 $antisym$: $(f : \bar{x} \sqsubseteq \bar{y}) \rightarrow (g : \bar{y} \sqsubseteq \bar{z}) \rightarrow \bar{x} = \bar{y} \wedge f = g = id \bar{x}$

Intrinsically Scoped Debruijn Syntax via Δ_+^\top

Definition

Let $Tm : |\Delta_+^\top| \rightarrow Set$ be inductively defined:

$$\frac{\langle \rangle \sqsubseteq \bar{x}}{Tm \bar{x}} \#$$

$$\frac{Tm \bar{x} \quad Tm \bar{x}}{Tm \bar{x}} \$$$

$$\frac{Tm \bar{x} \langle \rangle}{Tm \bar{x}} \lambda$$

Example

Lifting Scope Indexed Terms using Composition in Δ_+^\top

Lemma

Given an intrinsically scoped term $t \in Tm \bar{x}$ we can lift t to a $Tm \bar{y}$, if there exists a morphism $\bar{x} \sqsubseteq \bar{y} \in \Delta_+^\top(\bar{x}, \bar{y})$, i.e. \bar{x} is a subscope of \bar{y} .

Proof.

$$\begin{aligned} _ \uparrow _ & : Tm \bar{x} \rightarrow \bar{x} \sqsubseteq \bar{y} \rightarrow Tm \bar{y} \\ (\# v) \uparrow f &= \# (v; f) \\ (t_1 \$ t_2) \uparrow f &= (t_1 \uparrow f) \$ (t_2 \uparrow f) \\ (\lambda t) \uparrow f &= \lambda (t \uparrow S f) \end{aligned}$$

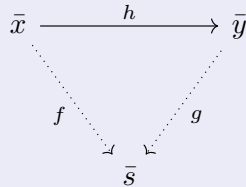


The Slice Category of Subscopes: $\Delta_+^X \setminus \bar{s}$

Definition

Let $\Delta_+^X \setminus \bar{s}$ be the category of subscopes for a given $\bar{s} \in X^*$.

- Objects: $\bar{b}, (\bar{x}, f) \in |\Delta_+^X \setminus \bar{s}| = (\bar{x} : X^* \times \Delta_+^X(\bar{x}, \bar{s}))$
- Morphisms: $h \in [\Delta_+^X \setminus \bar{s}]((\bar{x}, f), (\bar{y}, g))$ such that $f = h; g$



Corollary

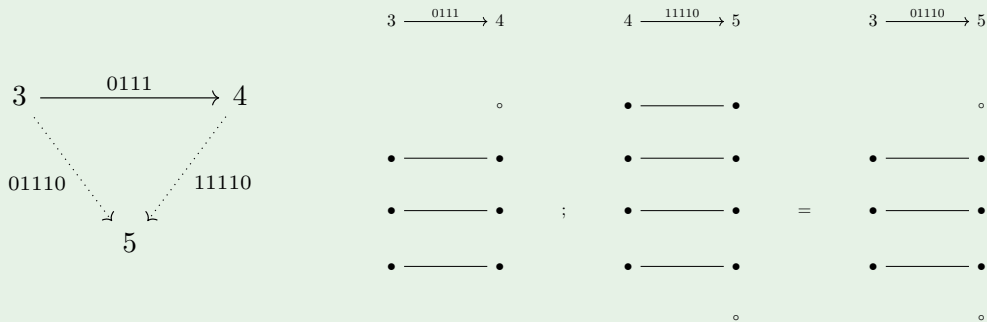
The initial object of the $\Delta_+^X \setminus \bar{s}$ category is the empty subscope $(\varepsilon, \bar{0})$.

Remark

Objects in $\Delta_+^X \setminus \bar{s}$ can be represented by *bit vectors* $\bar{b} \in \{0, 1\}^*$ with one bit per variable of scope \bar{s} , telling whether it has been selected.

Objects & Morphisms in $\Delta_+^T \setminus 5$

Example



Alternatively:

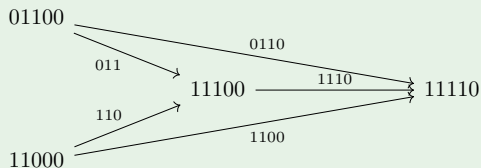
- $(3, 01110) \xrightarrow{0111} (4, 11110)$
- $01110 \xrightarrow{0111} 11110$

Coproducts in $\Delta_+^X \setminus \bar{s}$

Theorem

Objects in the slice category $\bar{b}_1, \bar{b}_2 \in |\Delta_+^X \setminus \bar{s}|$ have a coproduct object $\bar{b}_1 + \bar{b}_2$ if there exist morphisms $l \in [\Delta_+^X \setminus \bar{s}](\bar{b}_1, \bar{b}_1 + \bar{b}_2)$ and $r \in [\Delta_+^X \setminus \bar{s}](\bar{b}_2, \bar{b}_1 + \bar{b}_2)$. Then for every $f \in [\Delta_+^X \setminus \bar{s}](\bar{b}_1, \bar{b}_3)$ and $g \in [\Delta_+^X \setminus \bar{s}](\bar{b}_2, \bar{b}_3)$, there exists a unique $h \in [\Delta_+^X \setminus \bar{s}](\bar{b}_1 + \bar{b}_2, \bar{b}_3)$ such that $f = l; h$ and $g = r; h$.

Example



Remark

The coproduct $\bar{b}_1 + \bar{b}_2$ of two subscoopes \bar{b}_1, \bar{b}_2 corresponds to the minimal subscope covering both \bar{b}_1 and \bar{b}_2 . The coproduct $\bar{b}_1 + \bar{b}_2$ can be computed by pointwise disjunction of \bar{b}_1 and \bar{b}_2 .

Category of Sets Indexed by Scopes

Definition

Let Set_X be the category of sets indexed by scopes $\bar{x} \in X^*$.

- Objects: $T, S \in |Set_X| = X^* \rightarrow Set = \bar{X}$
- Morphisms: $f. \in Set_X(T, S) = (\bar{x} \in X^*) \rightarrow T(\bar{x}) \rightarrow S(\bar{x}) = T \dot{\rightarrow} S$

Definition

Let $_ \uparrow _ : \bar{X} \rightarrow \bar{X} = (T, \bar{x}) \mapsto (T(\bar{s}) \times \bar{s} \sqsubseteq \bar{x})$.

We define $Ref : Set_X \dot{\rightarrow} Set_X$ to be the endofunctor induced by the mapping

- $Ref(T) = \bar{x} \mapsto T \uparrow \bar{x} \in \bar{X}$
- $Ref(f.) = (t, h) \mapsto (f.(t), h) \in T \dot{\rightarrow} S$

Remark

The set $T \uparrow \bar{x}$ packs an set $T \in \bar{X}$ indexed by $\bar{x} \in X^*$ applied to a subscope \bar{s} of \bar{x} , together with a selection $h \in |\Delta_+^X \setminus \bar{x}|$ of the variables of T .

Δ_+^X makes Ref a Monad!

Theorem

The functor $Ref : Set_X \rightarrow Set_X$ gives rise to a monad with the two natural transformations

- $unit : Id(T) \rightarrow Ref(T) = t \mapsto (t, id)$
- $mult : Ref(Ref(T)) \rightarrow Ref(T) = ((t, h_1)h_2) \mapsto (t, h_1; h_2)$

Example

$$\begin{array}{ccc} Tm & \xrightarrow{\quad unit \quad} & \bar{x} \mapsto (Tm \bar{x}, \bar{x} \sqsubseteq \bar{x}) \\ & \nwarrow \scriptstyle Ref & \\ \bar{y} \mapsto ([\bar{x} \mapsto (Tm \bar{x}, \bar{x} \sqsubseteq \bar{x})] \bar{s}, \bar{s} \sqsubseteq \bar{y}) & \xrightarrow{\quad mult \quad} & \bar{y} \mapsto (Tm \bar{s}, \bar{s} \sqsubseteq \bar{y}) \end{array}$$

The Notion of Relevant Pairs

Definition

Let $_ \times_R _ : \bar{X} \rightarrow \bar{X} \rightarrow \bar{X} = (T, S, \bar{x}) \mapsto ((_, \bar{b}_1 : T \uparrow \bar{x}) \times (_, \bar{b}_2 : S \uparrow \bar{x}) \times (???))$.
We write $_,R_ : T \uparrow \bar{x} \rightarrow S \uparrow \bar{x} \rightarrow T \times_R S = (T, S) \mapsto \dots, \dots$,

$A, B : \text{Set } ((a : A) \times B \rightarrow a) : \text{Set } X : \text{Set } \text{List } X \rightarrow \text{Set}$

References

- [1] Conor McBride. *Cats and types: Best friends?* Aug. 2021. URL: <https://www.youtube.com/watch?v=05IJ3YL8p0s>.
- [2] Conor McBride. “Everybody’s Got To Be Somewhere”. In: *Electronic Proceedings in Theoretical Computer Science* 275 (July 2018), pp. 53–69. ISSN: 2075-2180. DOI: 10.4204/eptcs.275.6. URL: <http://dx.doi.org/10.4204/EPTCS.275.6>.