Co-Debruijn: Everybody's Got To Be Somewhere[2]

From Debruijn to co-Debruijn using Category Theory

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Outline

The Category of Scopes: Δ_+^X

Definition

Let Δ_+^X be the category of scopes.

- Objects: $\bar{x}, \bar{y}, \bar{s} \in |\Delta_+^X| = X^*$
- Morphisms: $f \in \Delta_+^X(\bar{x}, \bar{y})$ for $\bar{x}, \bar{y} \in X^*$ are inductively defined:

$$\frac{}{\varepsilon \sqsubseteq \varepsilon}\ \cdot \qquad\qquad \frac{\bar{x} \sqsubseteq \bar{y}}{\bar{x}x \sqsubseteq \bar{y}x}\ 1 \qquad\qquad \frac{\bar{x} \sqsubseteq \bar{y}}{\bar{x} \sqsubseteq \bar{y}y}\ 0$$

Corollary

The initial object of the Δ_+^X category is the empty scope ε with the $\bar{0}$ as the unique morphism.

Remark

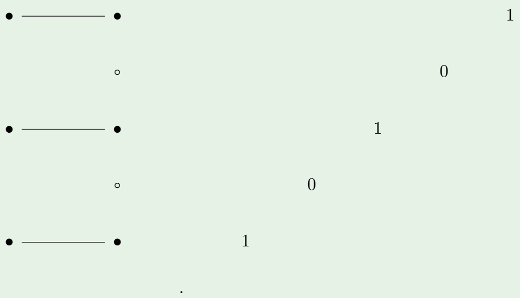
Morphisms in Δ_+^X can be represented by *bit vectors* $\bar{b} \in \{0, 1\}^*$ with one bit per variable of the target scope telling whether it has been mapped to or skipped by the source scope.

Objects & Morphisms in $\Delta_+^\top$

Example

Let $X = \top$ (where $\top$ is the set with exactly one element $\langle\rangle$).
Thus, Objects $\bar{x} \in X^*$ represents numbers.

$$3 \xrightarrow{10101} 5$$

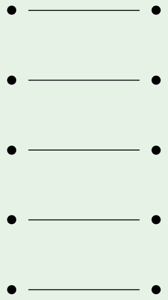


$$\begin{tikzcd} 3 \arrow[d, bend right, "10101"'] \\ 5 \end{tikzcd}$$

Identity and Composition in $\Delta_{+}^{\top}$

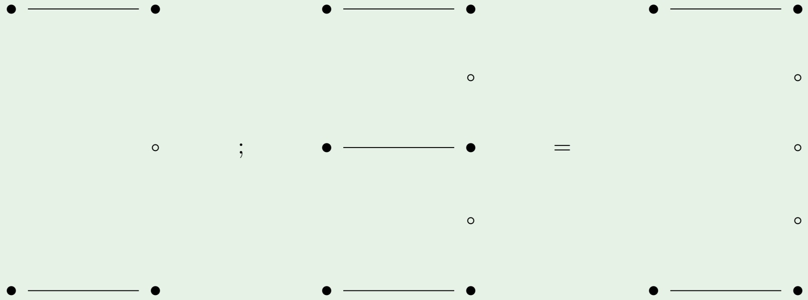
Example

$5 \xrightarrow{id\ 5} 5$



Example

$2 \xrightarrow{101} 3 \quad ; \quad 3 \xrightarrow{10101} 5 \quad = \quad 2 \xrightarrow{10001} 5$



Δ_+^X is in Fact a Category

Lemma

In Δ_+^X every object $\bar{x} \in X^*$ has an identity morphism, i.e. we can construct an identity morphism for $\bar{x}$ using the inference rules.

Proof.

$$
\begin{aligned}
id \quad &: (\bar{x} : X^*) \to \bar{x} \sqsubseteq \bar{x} \\
id\ \varepsilon \quad &= \cdot \\
id\ \bar{x}x \quad &= (\text{id}\ \bar{x})1
\end{aligned}
$$

□

Corollary

$$
\begin{aligned}
id - l \quad &: id; f = f \\
id - r \quad &: f; id = f
\end{aligned}
$$

Lemma

In Δ_+^X two morphisms $f : \bar{x} \sqsubseteq \bar{y}$ and $g : \bar{y} \sqsubseteq \bar{z}$ compose to a morphism $f;g : \bar{x} \sqsubseteq \bar{z}$, i.e. we can construct a morphism $f;g$ from f and g using the inference rules.

Proof.

$$
\begin{aligned}
\_;\_ \quad &: \bar{x} \sqsubseteq \bar{y} \to \bar{y} \sqsubseteq \bar{z} \to \bar{x} \sqsubseteq \bar{z} \\
\cdot\ \ ;\ \cdot \quad &= \cdot \\
f1\ ;\ g1 \quad &= (f;g)1 \\
f0\ ;\ g1 \quad &= (f;g)0 \\
f\ \ ;\ g0 \quad &= (f;g)0
\end{aligned}
$$

□

Corollary

$$
\begin{aligned}
\textit{assoc} \quad &: f;(g;h) = (f;g);h \\
\textit{antisym} \quad &: (f : \bar{x} \sqsubseteq \bar{y}) \to (g : \bar{y} \sqsubseteq \bar{z}) \to \bar{x} = \bar{y} \wedge f = g = id\ \bar{x}
\end{aligned}
$$

Intrinsically Scoped Debruijn Syntax via $\Delta_+^\top$

Definition

Let $Tm : |\Delta_+^\top| \to Set$ be inductively defined:

$$\frac{\langle\rangle \sqsubseteq \bar{x}}{Tm\ \bar{x}}\ \# \qquad \frac{Tm\ \bar{x} \quad Tm\ \bar{x}}{Tm\ \bar{x}}\ \$ \qquad \frac{Tm\ \bar{x}\langle\rangle}{Tm\ \bar{x}}\ \lambda$$

Example

Lifting Scope Indexed Terms using Composition in $\Delta_+^\top$

Lemma

Given an intrinsically scoped term $t \in Tm\ \bar{x}$ we can lift t to a $Tm\ \bar{y}$, if there exists a morphism $\bar{x} \sqsubseteq \bar{y} \in \Delta_+^\top(\bar{x}, \bar{y})$, i.e. $\bar{x}$ is a subscope of $\bar{y}$.

Proof.

$$
\begin{array}{lrcl}
\_ \uparrow \_ & & : & Tm\ \bar{x} \to \bar{x} \sqsubseteq \bar{y} \to Tm\ \bar{y} \\
(\#\ v) & \uparrow f & = & \#\ (v; f) \\
(t_1\ \$\ t_2) & \uparrow f & = & (t_1 \uparrow f)\ \$\ (t_2 \uparrow f) \\
(\lambda\ t) & \uparrow f & = & \lambda\ (t \uparrow S f)
\end{array}
$$

□

The Slice Category of Subscopes: $\Delta_+^X \setminus \bar{s}$

Definition

Let $\Delta_+^X \setminus \bar{s}$ be the category of subscopes for a given $\bar{s} \in X^*$.

- Objects: $(\bar{x}, f) \in |\Delta_+^X \setminus \bar{s}| = (X^* \times \Delta_+^X(\bar{x}, \bar{s}))$
- Morphisms: $h \in [\Delta_+^X \setminus \bar{s}]((\bar{x}, f), (\bar{y}, g))$ such that $f = h; g$

$$\begin{tikzcd} \bar{x} \arrow[rr, "h"] \arrow[dr, dotted, "f"'] & & \bar{y} \arrow[dl, dotted, "g"] \\ & \bar{s} & \end{tikzcd}$$

Corollary

The initial object of the $\Delta_+^X \setminus \bar{s}$ category is the empty subscope $(\varepsilon, \bar{0})$.

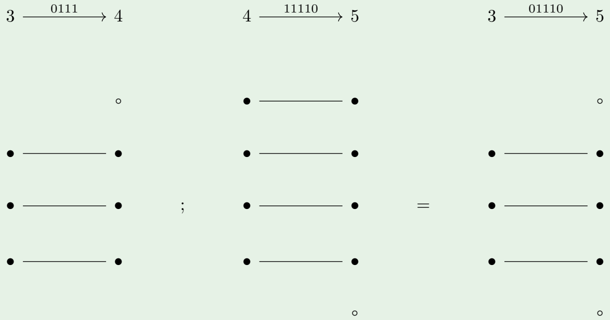
Remark

Objects in $\Delta_+^X \setminus \bar{s}$ can be represented by *bit vectors* $\bar{b} \in \{0, 1\}^*$ with one bit per variable telling whether it has been selected.

Objects & Morphisms in $\Delta_+^T \setminus 5$

Example

$$\begin{tikzcd}
3 \arrow[rr, "0111"] \arrow[dr, dotted, "01110"'] & & 4 \arrow[dl, dotted, "11110"] \\
& 5 &
\end{tikzcd}$$



Alternatively:

- $(3, 01110) \xrightarrow{\quad 0111 \quad} (4, 11110)$
- $01110 \xrightarrow{\quad 0111 \quad} 11110$

Category of Sets Indexed by Scopes

Definition

Let Set_X be the category of sets indexed by scopes $\bar{x} \in X^*$.

- Objects: $T, S \in |Set_X| = X^* \to Set = \bar{X}$
- Morphisms: $f_. \in Set_X(T, S) = \forall\{\bar{x} \in X^*\} \to T\ \bar{x} \to S\ \bar{x} = T \stackrel{.}{\to} S$

Definition

Let $Ref : \bar{X} \to \bar{X}$ be the endofunctor induced by the mapping

- $Ref(T) = \bar{x} \mapsto (T\ \bar{s}, \bar{s} \sqsubseteq \bar{x}) \in X^* \to Set$
- $Ref(f_.) = (t, h) \mapsto (f_.\ t, h) \in T \stackrel{.}{\to} S$

Remark

The functor Ref packs a set $T \in \bar{X}$ indexed by $\bar{x} \in X^*$ together with a selection $h \in \Delta_+^X \setminus \bar{x}$ of the variables that T actually uses.

Ref functor in action

Example

$$Tm \xrightarrow{f.} T \quad \overset{Ref}{\dashrightarrow} \quad \bar{x} \mapsto (Tm\ \bar{s}, \bar{s} \sqsubseteq \bar{x}) \xrightarrow{(tm,h)\mapsto(f.\ \ tm,h)} \bar{x} \mapsto (T\ \bar{s}, \bar{s} \sqsubseteq \bar{x})$$

Δ_+^X makes Ref a Monad!

Theorem

The functor $Ref : \bar{X} \to \bar{X}$ gives rise to a monad with the two natural transformations

- $unit : T \xrightarrow{\cdot} Ref(T) = t \mapsto (t, id)$
- $mult : Ref(Ref(T)) \xrightarrow{\cdot} Ref(T) = ((t, h_1)h_2) \mapsto (t, h_1; h_2)$

Example

$$
\begin{tikzcd}
 & Tm \arrow[r, "unit"] & \bar{x} \mapsto (Tm\ \bar{x}, \bar{x} \sqsubseteq \bar{x}) \arrow[dl, "Ref"'] \\
\bar{y} \mapsto ([\bar{x} \mapsto (Tm\ \bar{x}, \bar{x} \sqsubseteq \bar{x})]\bar{s}, \bar{s} \sqsubseteq \bar{y}) \arrow[rr, "mult"] & & \bar{y} \mapsto (Tm\ \bar{s}, \bar{s} \sqsubseteq \bar{y})
\end{tikzcd}
$$

References

[1] Conor McBride. *Cats and types: Best friends?* Aug. 2021. URL: https://www.youtube.com/watch?v=05IJ3YL8p0s.

[2] Conor McBride. “Everybody’s Got To Be Somewhere”. In: *Electronic Proceedings in Theoretical Computer Science* 275 (July 2018), pp. 53–69. ISSN: 2075-2180. DOI: 10.4204/eptcs.275.6. URL: http://dx.doi.org/10.4204/EPTCS.275.6.