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# From De Bruijn to co-De Bruijn using Category Theory

## Everybody's Got To Be Somewhere[2]

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January 30, 2024

# Outline

# The Category of Scopes: $\Delta_+^X$

## Definition

Let $\Delta_+^X$ be the category of scopes.

- Objects: $\bar{x}, \bar{y}, \bar{s} \in |\Delta_+^X| = X^*$
- Morphisms: $f, g \in \Delta_+^X(\bar{x}, \bar{y})$ for $\bar{x}, \bar{y} \in X^*$ are inductively defined:

$$\frac{}{\varepsilon \sqsubseteq \varepsilon}\,\cdot \qquad \frac{\bar{x} \sqsubseteq \bar{y}}{\bar{x}x \sqsubseteq \bar{y}x}\,1 \qquad \frac{\bar{x} \sqsubseteq \bar{y}}{\bar{x} \sqsubseteq \bar{y}y}\,0$$

## Corollary

*The initial object of the $\Delta_+^X$ category is the empty scope $\varepsilon$ with the $\bar{0}$ as the unique morphism.*

## Remark

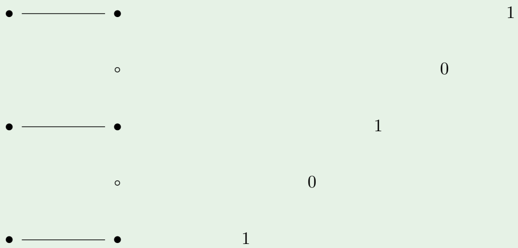
Morphisms in $\Delta_+^X$ can be represented by *bit vectors* $\bar{b} \in \{0,1\}^*$ with one bit per variable of the target scope telling whether it has been mapped to or skipped by the source scope.

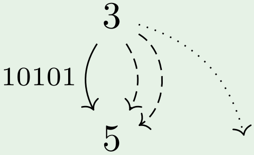
# Objects & Morphisms in $\Delta_+^\top$

## Example

Let $X = \top$ (where $\top$ is the set with exactly one element $\langle\rangle$).
Thus, Objects $n \in X^*$ represents numbers.

$$3 \xrightarrow{10101} 5$$

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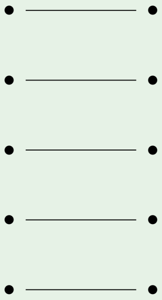
# Identity and Composition in $\Delta_{+}^{\top}$

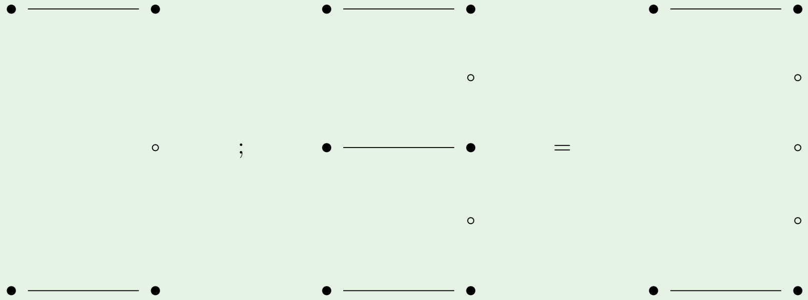
## Example

$$5 \xrightarrow{id\ 5} 5$$



## Example

$$2 \xrightarrow{101} 3 \qquad ; \qquad 3 \xrightarrow{10101} 5 \qquad = \qquad 2 \xrightarrow{10001} 5$$



## $\Delta_+^X$ is in Fact a Category

### Lemma

*In $\Delta_+^X$ every object $\bar{x} \in X^*$ has an identity morphism, i.e. we can construct an identity morphism for $\bar{x}$ using the inference rules.*

### Proof.

$$
\begin{aligned}
id &: (\bar{x} : X^*) \to \bar{x} \sqsubseteq \bar{x} \\
id\ \varepsilon &= \cdot \\
id\ \bar{x}x &= (\text{id}\ \bar{x})1
\end{aligned}
$$

□

### Corollary

$$
\begin{aligned}
id-l &: id; f = f \\
id-r &: f; id = f
\end{aligned}
$$

### Lemma

*In $\Delta_+^X$ two morphisms $f : \bar{x} \sqsubseteq \bar{y}$ and $g : \bar{y} \sqsubseteq \bar{z}$ compose to a morphism $f;g : \bar{x} \sqsubseteq \bar{z}$, i.e. we can construct a morphism $f;g$ from $f$ and $g$ using the inference rules.*

### Proof.

$$
\begin{aligned}
\_;\_ &: \bar{x} \sqsubseteq \bar{y} \to \bar{y} \sqsubseteq \bar{z} \to \bar{x} \sqsubseteq \bar{z} \\
\cdot \ ; \ \cdot &= \cdot \\
f1 \ ; \ g1 &= (f;g)1 \\
f0 \ ; \ g1 &= (f;g)0 \\
f \ ; \ g0 &= (f;g)0
\end{aligned}
$$

□

### Corollary

$$
\begin{aligned}
\textit{assoc} &: f;(g;h) = (f;g);h \\
\textit{antisym} &: (f : \bar{x} \sqsubseteq \bar{y}) \to (g : \bar{y} \sqsubseteq \bar{z}) \to \bar{x} = \bar{y} \wedge f = g = \textit{id}\ \bar{x}
\end{aligned}
$$

# Intrinsically Scoped De Bruijn Syntax via $\Delta_+^\top$

## Definition

Let $Tm : |\Delta_+^\top| \to Set$ be inductively defined:

$$\frac{1 \sqsubseteq n}{Tm\ n}\ \# \qquad \frac{Tm\ n \quad Tm\ n}{Tm\ n}\ \$ \qquad \frac{Tm\ (n+1)}{Tm\ n}\ \lambda$$

## Example

# The Slice Category of Subscopes: $\Delta_+^X \setminus \bar{s}$

## Definition

Let $\Delta_+^X \setminus \bar{s}$ be the category of subscopes for a given $\bar{s} \in X^*$.

- Objects: $\bar{b}, (\bar{x}, f) \in |\Delta_+^X \setminus \bar{s}| = (\bar{x} : X^* \times \Delta_+^X(\bar{x}, \bar{s}))$
- Morphisms: $h \in [\Delta_+^X \setminus \bar{s}]((\bar{x}, f), (\bar{y}, g))$ such that $f = h; g$

$$\begin{tikzcd} \bar{x} \arrow[rr, "h"] \arrow[dr, dotted, "f"'] & & \bar{y} \arrow[dl, dotted, "g"] \\ & \bar{s} & \end{tikzcd}$$

## Corollary

*The initial object of the $\Delta_+^X \setminus \bar{s}$ category is the empty subscope $(\varepsilon, \bar{0})$.*

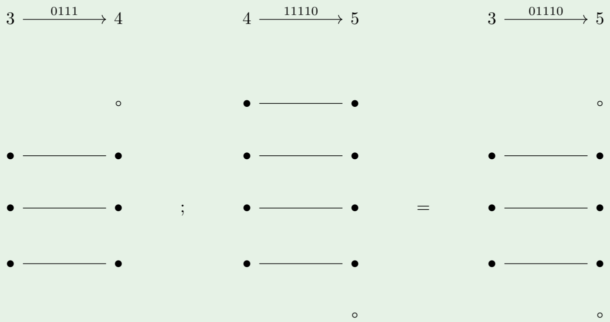
## Remark

Objects in $\Delta_+^X \setminus \bar{s}$ can be represented by *bit vectors* $\bar{b} \in \{0, 1\}^*$ with one bit per variable of scope $\bar{s}$, telling whether it has been selected.

# Objects & Morphisms in $\Delta_+^T \setminus 5$

## Example

$$
\begin{tikzcd}
3 \arrow[rr, "0111"] \arrow[dr, dotted, "01110"'] & & 4 \arrow[dl, dotted, "11110"] \\
& 5 &
\end{tikzcd}
$$




Alternatively:

- $(3, 01110) \xrightarrow{0111} (4, 11110)$
- $01110 \xrightarrow{0111} 11110$

# Coproducts in $\Delta_+^X \setminus \bar{s}$

## Theorem

*Objects in the slice category* $\bar{b}_1, \bar{b}_2 \in |\Delta_+^X \setminus \bar{s}|$ *have a coproduct object* $\bar{b}_1 + \bar{b}_2$ *if there exist morphisms* $l \in [\Delta_+^X \setminus \bar{s}](\bar{b}_1, \bar{b}_1 + \bar{b}_2)$ *and* $r \in [\Delta_+^X \setminus \bar{s}](\bar{b}_2, \bar{b}_1 + \bar{b}_2)$. *Then for every* $f \in [\Delta_+^X \setminus \bar{s}](\bar{b}_1, \bar{b}_3)$ *and* $g \in [\Delta_+^X \setminus \bar{s}](\bar{b}_2, \bar{b}_3)$, *there exists a unqiue* $h \in [\Delta_+^X \setminus \bar{s}](\bar{b}_1 + \bar{b}_2, \bar{b}_3)$ *such that* $f = l; h$ *and* $g = r; h$.

## Example

$$\begin{tikzcd}
01100 \arrow[dr, "011"'] \arrow[drr, "0110"] & & \\
& 11100 \arrow[r, "1110"] & 11110 \\
11000 \arrow[ur, "110"] \arrow[urr, "1100"'] & &
\end{tikzcd}$$

## Remark

The coproduct $\bar{b}_1 + \bar{b}_2$ of two subscopes $\bar{b}_1, \bar{b}_2$ corresponds to the minimal subscope covering both $\bar{b}_1$ and $\bar{b}_2$. The coproduct $\bar{b}_1 + \bar{b}_2$ can be computed by pointswise disjunction of $\bar{b}_1$ and $\bar{b}_2$.

# Category of Sets Indexed by Scopes

## Definition

Let $Set_X$ be the category of sets indexed by scopes $\bar{x} \in X^*$.

- Objects: $T, S \in |Set_X| = X^* \to Set = \bar{X}$
- Morphisms: $f_. \in Set_X(T, S) = (\bar{x} \in X^*) \to T(\bar{x}) \to S(\bar{x}) = T \dot{\to} S$

## Definition

Let $\_ \Uparrow \_ : \bar{X} \to \bar{X} = (T, \bar{x}) \mapsto (T(\bar{s}) \times \bar{s} \sqsubseteq \bar{x})$.

We define $Ref : Set_X \dot{\to} Set_X$ to be the endofunctor induced by the mapping

- $Ref(T) = \bar{x} \mapsto T \Uparrow \bar{x} \in \bar{X}$
- $Ref(f_.) = (t, h) \mapsto (f_.(t), h) \in T \dot{\to} S$

## Remark

The set $T \Uparrow \bar{x}$ packs an set $T \in \bar{X}$ indexed by $\bar{x} \in X^*$ applied to a subscope $\bar{s}$ of $\bar{x}$, together with a selection $h \in |\Delta_+^X \setminus \bar{x}|$ of the variables of $T$.

# $\Delta_+^X$ makes $Ref$ a Monad!

**Theorem**

*The functor $Ref : Set_X \to Set_X$ gives rise to a monad with the two natural transformations*

- $unit : Id(T) \stackrel{.}{\to} Ref(T) = t \mapsto (t, id)$
- $mult : Ref(Ref(T)) \stackrel{.}{\to} Ref(T) = ((t, h_1)h_2) \mapsto (t, h_1; h_2)$

**Example**

$$\begin{tikzcd}
 & Tm \arrow[r, dashed, "unit"] & \bar{x} \mapsto (Tm\ \bar{x}, \bar{x} \sqsubseteq \bar{x}) \arrow[dl, dashed, "Ref"'] \\
\bar{y} \mapsto ([\bar{x} \mapsto (Tm\ \bar{x}, \bar{x} \sqsubseteq \bar{x})]\bar{s}, \bar{s} \sqsubseteq \bar{y}) \arrow[rr, dashed, "mult"] & & \bar{y} \mapsto (Tm\ \bar{s}, \bar{s} \sqsubseteq \bar{y})
\end{tikzcd}$$

# The Notion of Relevant Pairs

## Definition

bla

## Definition

Let the set of relevant pairs be defined as

- $\_ \times_R \_ : \bar{X} \to \bar{X} \to \bar{X} = (T, S, \bar{x}) \mapsto ((\_, \bar{b}_1 : T \Uparrow \bar{x}) \times (\_, \bar{b}_2 : S \Uparrow \bar{x}) \times Cov\ \bar{b}_1\ \bar{b}_2)$.
- $\_ ,_R \_ : T \Uparrow \bar{x} \to S \Uparrow \bar{x} \to (T \times_R S)\ \bar{x} = ((t_1, \bar{b}_1), (t_2, \bar{b}_2)) \mapsto ((t_1, \bar{b}_1), (t_2, \bar{b}_2), \bar{b}_1 \oplus \bar{b}_2)$

## Example

$$
\begin{tikzcd}
01100 \arrow[dr, "011"'] \arrow[drr, "01100"] & & \\
& 1{\color{red}2}100 \arrow[r, "11100"] & 11111 \\
11000 \arrow[ur, "110"] \arrow[urr, "11000"'] & &
\end{tikzcd}
$$

# Intrinsically Scoped co-De Bruijn Syntax

## Definition

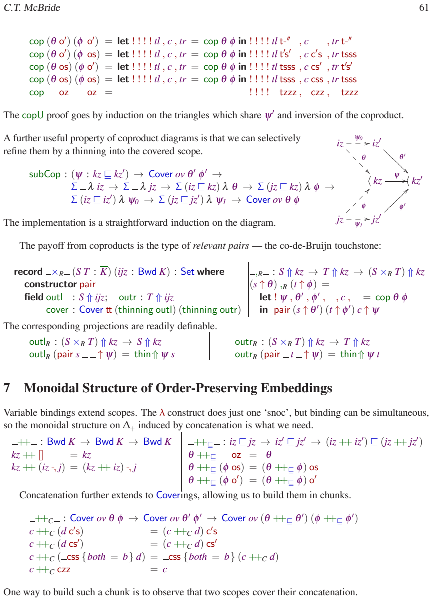
Let $Tm : |\Delta_+^\top| \to Set$ be inductively defined:

$$\frac{}{Tm\ 1}\ \# \qquad \frac{(Tm \times_R Tm)\ n}{Tm\ n}\ \$ \qquad \frac{Tm\ (n+1)}{Tm\ n}\ \lambda$$

## Example

# This is actually an Agda Paper

- All categorical concepts formalized in Agda
- Suitable representations of objects and morphisms
- Comes with a *universe of metasyntaxes-with-binding*
- Categorical concepts applied to formally reason over programming languages



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```
cop (θ o′) (ϕ o′) = let ! ! ! ! tl , c , tr = cop θ ϕ in ! ! ! ! tl t-″  , c     , tr t-″
cop (θ o′) (ϕ os) = let ! ! ! ! tl , c , tr = cop θ ϕ in ! ! ! ! tl t′s′ , c c′s  , tr tsss
cop (θ os) (ϕ o′) = let ! ! ! ! tl , c , tr = cop θ ϕ in ! ! ! ! tl tsss , c cs′  , tr t′s′
cop (θ os) (ϕ os) = let ! ! ! ! tl , c , tr = cop θ ϕ in ! ! ! ! tl tsss , c css  , tr tsss
cop   oz     oz   =                                      ! ! ! !  tzzz ,   czz ,    tzzz
```

The copU proof goes by induction on the triangles which share $\psi'$ and inversion of the coproduct.

A further useful property of coproduct diagrams is that we can selectively refine them by a thinning into the covered scope.

```
subCop : (ψ : kz ⊑ kz′) → Cover ov θ′ ϕ′ →
         Σ _ λ iz → Σ _ λ jz → Σ (iz ⊑ kz) λ θ → Σ (jz ⊑ kz) λ ϕ →
         Σ (iz ⊑ iz′) λ ψ₀ → Σ (jz ⊑ jz′) λ ψ₁ → Cover ov θ ϕ
```

$$\begin{array}{ccccc} iz & \xrightarrow{\psi_0} & iz' & & \\ & \searrow^{\theta} & & \searrow^{\theta'} & \\ & & kz & \xrightarrow{\psi} & kz' \\ & \nearrow^{\phi} & & \nearrow^{\phi'} & \\ jz & \xrightarrow{\psi_1} & jz' & & \end{array}$$

The implementation is a straightforward induction on the diagram.

The payoff from coproducts is the type of *relevant pairs* — the co-de-Bruijn touchstone:

```
record _×R_ (S T : K̄) (ijz : Bwd K) : Set where
  constructor pair
  field outl  : S ⇑ ijz;   outr : T ⇑ ijz
        cover : Cover tt (thinning outl) (thinning outr)

_,R_ : S ⇑ kz → T ⇑ kz → (S ×R T) ⇑ kz
(s ↑ θ) ,R (t ↑ ϕ) =
  let ! ψ , θ′ , ϕ′ , _ , c , _ = cop θ ϕ
  in  pair (s ↑ θ′) (t ↑ ϕ′) c ↑ ψ
```

The corresponding projections are readily definable.

```
outlR : (S ×R T) ⇑ kz → S ⇑ kz
outlR (pair s _ _ ↑ ψ) = thin⇑ ψ s

outrR : (S ×R T) ⇑ kz → T ⇑ kz
outrR (pair _ t _ ↑ ψ) = thin⇑ ψ t
```

## 7 Monoidal Structure of Order-Preserving Embeddings

Variable bindings extend scopes. The λ construct does just one ‘snoc’, but binding can be simultaneous, so the monoidal structure on $\Delta_+$ induced by concatenation is what we need.

```
_++_ : Bwd K → Bwd K → Bwd K
kz ++ []       = kz
kz ++ (iz -, j) = (kz ++ iz) -, j

_++⊑_ : iz ⊑ jz → iz′ ⊑ jz′ → (iz ++ iz′) ⊑ (jz ++ jz′)
θ ++⊑   oz   = θ
θ ++⊑ (ϕ os) = (θ ++⊑ ϕ) os
θ ++⊑ (ϕ o′) = (θ ++⊑ ϕ) o′
```

Concatenation further extends to Coverings, allowing us to build them in chunks.

```
_++C_ : Cover ov θ ϕ → Cover ov θ′ ϕ′ → Cover ov (θ ++⊑ θ′) (ϕ ++⊑ ϕ′)
c ++C (d c′s)                 = (c ++C d) c′s
c ++C (d cs′)                 = (c ++C d) cs′
c ++C (_css {both = b} d)     = _css {both = b} (c ++C d)
c ++C czz                     = c
```

One way to build such a chunk is to observe that two scopes cover their concatenation.

# Using co-De Bruijn is not hard, category theory is!

```
data Cover : (k l m : ℕ) → Set where
  done  : Cover           0     0 0
  left  : Cover k l m → Cover (suc k) l (suc m)
  right : Cover k l m → Cover k (suc l) (suc m)
  both  : Cover k l m → Cover (suc k) (suc l) (suc m)

data Term : ℕ → Set where
  var : Term 1
  lam : Term (suc n) → Term n
  app : Cover k l m → Term k → Term l → Term m

_ = lam {-f-} (lam {-x-} (app (right (left done)) (var {-f-}) (var {-x-})))
```

# References


[1] Conor McBride. *Cats and types: Best friends?* Aug. 2021. URL: https://www.youtube.com/watch?v=05IJ3YL8p0s.

[2] Conor McBride. "Everybody's Got To Be Somewhere". In: *Electronic Proceedings in Theoretical Computer Science* 275 (July 2018), pp. 53–69. ISSN: 2075-2180. DOI: 10.4204/eptcs.275.6. URL: http://dx.doi.org/10.4204/EPTCS.275.6.