HM(X): Type Inference with Constraint Types [1]

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Outline

Hindley Milner Basics

- Let Polymorphism
- Full Type Inference
- Principal Type Property
  - most general type is inferred

Example: Polymorphic Identity Function

```
let id = λx. x in        : ∀α.α → α
id 42                    : Int
id "Hello World"         : Str
```

Example: Polymorphic List Constructors

```
let nil = ..    : ∀α. List α
let cons = ..   : ∀α.α → List α → List α
```

Hindley Milner Syntax

$$
\begin{aligned}
e \;::=\; & x \\
\mid\; & \lambda x.e \\
\mid\; & e\ e \\
\mid\; & \textbf{let}\ x = e\ \textbf{in}\ e
\end{aligned}
$$

$$
\begin{aligned}
\tau \;::=\; & \alpha \\
\mid\; & \tau \rightarrow \tau
\end{aligned}
$$

$$
\begin{aligned}
\sigma \;::=\; & \tau \\
\mid\; & \forall \vec{\alpha}.\tau
\end{aligned}
$$

Hindley Milner Typing

$$\frac{\Gamma \vdash e : \sigma \quad \Gamma, x : \sigma \vdash e^{'} : \tau^{'}}{\Gamma \vdash \textbf{let } x = e \textbf{ in } e^{'} : \tau^{'}} \text{ Let}$$

$$\frac{\Gamma \vdash e : \tau \quad \vec{\alpha} \notin free(\Gamma)}{\Gamma \vdash e : \forall \vec{\alpha}.\tau} \;\forall \text{ Intro}$$

$$\frac{\Gamma \vdash e : \forall \vec{\alpha}.\tau}{\Gamma \vdash e : [\vec{\tau}/\vec{\alpha}]\tau} \;\forall \text{ Elim}$$

HM($\mathcal{R}$) Syntax

$$
\begin{aligned}
e \;::=\;& \ldots \\
&\mid \{l_i = e_i\} \text{ for } i \in \mathbb{N} \\
&\mid e.l
\end{aligned}
$$

$$
\begin{aligned}
\tau \;::=\;& \ldots \\
&\mid \{l_i : \tau_i\} \text{ for } i \in \mathbb{N}
\end{aligned}
$$

$$
\begin{aligned}
C \;::=\;& \ldots \\
&\mid \tau \leq \{l : \tau'\}
\end{aligned}
$$

A Program with Constraint Types

Example: Record Subtyping Constraints

```
let gte = λr₁. λr₂.                 : ∀α, β.(α ≤{key : Int} ∧ β ≤{key : Int}) ⇒
  r₁.key ≥ r₂.key in                    α → β → Bool
gte {key = 17, foo = "bar"}         : Bool
    {key = 42, bar = "foo"}
```

$$\texttt{let gte} = \lambda \texttt{r}_1.\ \lambda \texttt{r}_2.\ \texttt{r}_1\texttt{.key} \geq \texttt{r}_2\texttt{.key}\ \texttt{in} \quad : \forall \alpha, \beta.(\alpha \leq \{\texttt{key} : \texttt{Int}\} \wedge \beta \leq \{\texttt{key} : \texttt{Int}\}) \Rightarrow \alpha \to \beta \to \texttt{Bool}$$

$$\texttt{gte}\ \{\texttt{key} = 17, \texttt{foo} = \texttt{"bar"}\}\ \{\texttt{key} = 42, \texttt{bar} = \texttt{"foo"}\} \quad : \texttt{Bool}$$

Constraint Syntax

$$
\begin{aligned}
C \;::=\; & \bot \\
\mid\; & \top \\
\mid\; & C \wedge C \\
\mid\; & \exists \alpha . C \\
\mid\; & \tau = \tau \\
\mid\; & \ldots
\end{aligned}
$$

Constraint Conditions

Entailment

$$C \vdash D \text{ iff } C \text{ implies } D$$

$$C = D \text{ iff } C \vdash D \wedge D \vdash C$$

$$\vdash C \text{ iff } \top \vdash C$$

Substitution

$$[\tau/\alpha]C = \exists\alpha.C \wedge (\alpha = \tau) \text{ for } \alpha \text{ not free in } \tau$$

Existential Quantification

$$C \vdash \exists\alpha.C$$

$$C \vdash D \text{ implies } \exists\alpha.C \vdash \exists\alpha.D$$

$$\exists\alpha.(C \wedge \exists\alpha.D) = \exists\alpha.C \wedge \exists\alpha.D$$

$$\exists\alpha.\exists\beta.C = \exists\beta.\exists\alpha.C$$

Solved Form

Satisfiability

$$sat(C) \text{ iff } \vdash \exists\vec{\alpha}.C \text{ for } \vec{\alpha} \text{ free in } C$$

Constraints in Solved Form

$C \in \mathcal{S}$ implies $sat(C)$

$C \in \mathcal{S}$ and $C \vdash (\tau = \tau')$ implies $\vdash (\tau = \tau')$

$C \in \mathcal{S}$ implies $\exists\alpha.C \in \mathcal{S}$

HM(X) Syntax

$$
\begin{aligned}
e \quad ::= &\; x \\
\mid &\; \lambda x.e \\
\mid &\; e\ e \\
\mid &\; \textbf{let}\ x = e\ \textbf{in}\ e
\end{aligned}
$$

$$
\begin{aligned}
\tau \quad ::= &\; \alpha \\
\mid &\; \tau \rightarrow \tau
\end{aligned}
$$

$$
\begin{aligned}
\sigma \quad ::= &\; \tau \\
\mid &\; \forall \vec{\alpha}.C \Rightarrow \tau \text{ for } C \in \mathcal{S}
\end{aligned}
$$

HM(X) Typing

$$\frac{C,\, \Gamma \vdash e : \sigma \quad C,\, (\Gamma, x : \sigma) \vdash e' : \tau'}{C,\, \Gamma \vdash \textbf{let } x = e \textbf{ in } e' : \tau'}\ \text{Let}$$

$$\frac{C \wedge D,\, \Gamma \vdash e : \tau \quad \vec{\alpha} \notin free(\Gamma) \cup free(C)}{C \wedge \exists\vec{\alpha}.D,\, \Gamma \vdash e : \forall\vec{\alpha}.D \Rightarrow \tau}\ \forall\ \text{Intro}$$

$$\frac{C,\ \Gamma \vdash e : \tau \quad \vdash (\tau = \tau')}{C,\ \Gamma \vdash e : \tau'}\ \text{Eq}$$

$$\frac{C,\, \Gamma \vdash e : \forall\vec{\alpha}.D \Rightarrow \tau \quad C \vdash^{e} [\vec{\tau}/\vec{\alpha}]D}{C,\, \Gamma \vdash e : [\vec{\tau}/\vec{\alpha}]\tau}\ \forall\ \text{Elim}$$

HM($\mathcal{R}$) Constraint Conditions

Subtyping

$$\vdash \{l_1 : \tau_1, .., l_n : \tau_n\} \leq \{l_i : \tau_i\}$$
$$\tau \leq \{l : \tau_1\} \wedge \tau \leq \{l : \tau_2\} \vdash \tau_1 = \tau_2$$
$$\{.., l : \tau_1, ..\} \leq \{l : \tau_2\} \vdash \tau_1 = \tau_2$$
$$\{l_1 : \tau_1, .., l_n : \tau_n\} \leq \{l : \tau\} \vdash \bot$$

Restrict Recusion

$\exists \alpha . \alpha \leq \{l : \tau\} = \top$ for α not free in τ

$\exists \alpha . \alpha \leq \{l : \tau\} = \bot$ for α free in τ

Ordering

$$\vdash \{l_i : \tau_i\} = \{l_{\pi(i)} : \tau_{\pi(i)}\}$$

where π is a permutation

Solved Form

$\alpha \leq \{l : \tau\} \in \mathcal{S}$

where all $\alpha \leq \{l : \tau\}$ non-recursive

HM($\mathcal{R}$) Example

Example: Nested Record Subtyping Constraints

```
let unwrap = λr.                 : ∀α.∃β(α ≤{outer:β}∧β ≤{inner:Int}) ⇒
  r.outer.inner in                 α → Int
unwrap {                         : Int
  outer = {
    inner = 42,
    bar = "foo"
  },
  foo = "bar"
}
```

Meta Theoretical Properties & Outlook

- Full Type Inference: HM(X) comes with a type inference algorithm that
  - solves typing problems by translating them to constraint problems
  - preserves the *principal type property*, if X fulfills the *principal constraint property*
- Type Soundness: HM(X) is sound, i.e. all valid typing judgements $C, \Gamma \vdash e : \sigma$ where $sound(C)$ imply $e \hookrightarrow e' \vee val(e)$ (*progress*) and $e \hookrightarrow e' \rightarrow C, \Gamma \vdash e' : \sigma$ (*subject reduction*) [3]
- SHM(X): extension with subtyping constraint $\tau \leq \tau'$, small extension to type inference algorithm required

References

[1] Martin Odersky, Martin Sulzmann, and Martin Wehr. “Type Inference with Constrained Types”. In: *TAPOS* 5 (Jan. 1999), pp. 35–55. DOI: 10.1002/(SICI)1096-9942(199901/03)5:1<35::AID-TAPO4>3.0.CO;2-4.

[2] Benjamin C. Pierce. *Advanced Topics in Types and Programming Languages*. The MIT Press, 2004. ISBN: 0262162288.

[3] Christian Skalka and François Pottier. “Syntactic Type Soundness for HM(X)”. In: *Electronic Notes in Theoretical Computer Science* 75 (2003). TIP’02, International Workshop in Types in Programming, pp. 61–74. ISSN: 1571-0661. DOI: https://doi.org/10.1016/S1571-0661(04)80779-5. URL: https://www.sciencedirect.com/science/article/pii/S1571066104807795.